PHD THESIS:
HOMOTOPY REPRESENTATIONS OF SIMPLY CONNECTED p-COMPACT GROUPS OF RANK 1 OR 2

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Abstract. In this thesis we show that the restriction map $R_p(X) \to R_p(T)^W$ is an isomorphism for all simply connected p-compact groups of rank at most 2. Here T is a maximal torus of X with Weyl group W, $R_p(X)$ is the complex homotopy representation ring of X and $R_p(T)^W$ is the invariants under W of the representation ring of T.

(In Danish:) I denne afhandling viser vi at restriktionsafbildningen $R_p(X) \to R_p(T)^W$ er en isomorfi for all enkeltsammenhængende p-kompakte grupper af rang højst 2. Her er T en maksimal torus for X med Weyl-gruppe W, $R_p(X)$ er den komplekse homotopi-repræsentationsring for X og $R_p(T)^W$ er invarianterne under W af repræsentationsringen for T.

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1. Introduction

The goal of this thesis is to understand the complex homotopy representation ring (henceforth just called the representation ring) of simply connected p-compact groups of small rank. The p-completion of any connected compact Lie group is a p-compact group and the major part of the thesis can be understood without knowing about p-compact groups. The representation ring $R_p(X)$ of a p-compact group X is defined as

$$R_p(X) := \mathrm{Gr}(\coprod_{n\geq 0}[BX, BU(n)\hat{_p}])$$

where Gr is the Grothendieck group/group completion and $BU(n)\hat{_p}$ is the p-completion of $BU(n)$ as defined in [5]. Our main theorem is:

Theorem 1.1. *Let X be a simply connected p-compact group of rank 1 or 2. Let T be a maximal torus of X with Weyl group W. Then the restriction map*

$$R_p(X) \to R_p(T)^W$$

is an isomorphism. Here $R_p(T)^W$ is the invariants under W of the representation ring of T.

The above theorem is analogous the following classical result: Let G be a connected compact Lie group and let $T \leq G$ be a maximal torus of G with Weyl group $W = W_G(T)$. Then the restriction map $R(G) \to R(T)$ is an isomorphism onto the invariant subring $R(T)^W$ [1, theorem 6.20]. Here $R(G)$ is the classical representation ring of G. We also have the following integral result, shown in [21]:

$$\mathrm{Gr}([BG, \coprod_{n\geq 0} BU(n)]) \cong R(G)$$

This result motivates the definition of the homotopy representations $[BG, \coprod_{n\geq 0} BU(n)\hat{_p}]$ of a compact Lie group G at a prime p or more generally the definition of $R_p(X)$.

In lemma 5.1 our main theorem is shown for all connected p-compact groups X where p does not divide the order of the Weyl group. Since connected p-compact groups have been classified (see [3] and [4]) we can list all the remaining p-compact groups of rank 1 or 2 and we show our main theorem case by case for all these. By factoring the map $\Phi\colon R_p(X) \to R_p(T)^W$ into two maps

$$\Phi = (R_P(G) \xrightarrow{\mathrm{Gr}(\Phi_1)} \mathrm{Gr}(\lim \mathrm{Rep}_p(P)) \xrightarrow{\mathrm{Gr}(\Phi_2)} R_p(T)^W)$$

(with definitions and notation to be given later) the proof for each case is divided into two parts: To show that $\mathrm{Gr}(\Phi_1)$ is an isomorphism we use a certain obstruction theory whereas showing that $\mathrm{Gr}(\Phi_2)$ is an isomorphism is more combinatorial. For both parts we need to understand $\lim \mathrm{Rep}_p(P)$, but luckily this limit can be described purely algebraically and we can use classical representation theory (that is character theory) to understand it.

In section 3 and 4 we survey the necessary background material needed for our proofs of the main theorem, first for compact Lie groups and then for general p-compact groups. In section 5 we give general proofs not related to one specific p-compact group. The remainder of the thesis consists of all the case by case proofs of our main theorem. We have tried to order the proofs in terms of similarity: For example the proof for G_2 at $p = 2$ has similarities with the proof for $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ at $p = 2$ and the proof for $\mathrm{DI}(2)$ has similarities with the proof for G_2 at $p = 3$. With regards to proving that $\mathrm{Gr}(\Phi_1)$ is an isomorphism it is best to read the proof for $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ first; this is the simplest non-trivial proof, and it has been written in much greater detail than the later proofs.

2. Notation

Let X be a p-compact group. We denote the (complex) n-dimensional homotopy representations of X by

$$\mathrm{Rep}_p^n(X) := [BX, BU(n)\hat{_p}]$$

and define

$$\mathrm{Rep}_p(X) := \coprod_{n\geq 0} \mathrm{Rep}_p^n(X)$$

which is a semiring with addition and product induced by direct sum and tensor product in $\coprod_{n\geq 0} U(n)$. So we can write

$$R_p(X) := \mathrm{Gr}(\mathrm{Rep}_p(X))$$

Let $D_1 \colon G \to U(n_1)$ and $D_2 \colon H \to U(n_2)$ be representations of the groups G and H. Then $D_1 \times D_2 \colon G \times H \to U(n_1 n_2)$ is the outer tensor product of D_1 and D_2. That is $(D_1 \times D_2)(g, h) = D_1(g) \otimes D_2(h)$.

W will always denote the Weyl group of the particular p-compact group we are discussing, that is the Weyl group of the chosen maximal torus.

3. Background for compact Lie groups

Let G be a connected compact Lie group with maximal torus T and Weyl group W.

3.1. **Factorisation of** $\mathrm{Rep}_p(G) \to \mathrm{Rep}_p(T)^W$**.** A p-toral subgroup $P \leq G$ is a closed subgroup such that its one-component P_1 is a torus, and $\pi_0(P)$ is a finite p-group. A p-toral subgroup P is called a p-radical subgroup if and only if its Weyl group $W(P) := N_G(P)/P$ is a finite group and $O_p(W(P)) = 1$. Let $\mathcal{O}_p(G)$ be the category with objects G/P for P a p-toral subgroup of G and with $\mathrm{Mor}(G/P, G/Q)$ being the G-equivariant maps. Here $G/P = \{Pg \mid g \in G\}$ on which G acts on the right. Let $\mathcal{R}_p(G)$ be the full subcategory of $\mathcal{O}_p(G)$ with objects G/P for P p-radical. In [17] it is shown that

$$\operatorname{hocolim}_{G/P\in\mathcal{R}_p(G)} EG\times_G G/P \to BG$$

is an $\mathbb{F}_p$ homology isomorphism, so that

$$\operatorname{Rep}_p^k(G) \to [\operatorname{hocolim}_{G/P\in\mathcal{R}_p(G)} EG\times_G G/P, BU(k)\hat{_p}]$$

is a bijection (since BG is p-good and $BU(k)\hat{_p}$ is p-complete, cf. [5]). We have a map $[\operatorname{hocolim}_{G/P\in\mathcal{R}_p(G)} EG/P, BU(k)\hat{_p}] \to \lim_{G/P\in\mathcal{R}_p(G)}[EG\times_G G/P, BU(k)] \cong \lim_{G/P\in\mathcal{R}_p(G)} \operatorname{Rep}_p^k(P)$. So we get a factorization

$$\operatorname{Rep}_p(G) \xrightarrow{\Phi_1} \lim_{G/P\in\mathcal{R}_p(G)} \operatorname{Rep}_p(P) \xrightarrow{\Phi_2} \operatorname{Rep}_p(T)^W$$

For descriptions of $\mathcal{R}_p(G)$ for the classical compact simple Lie groups, see [23].

3.2. **Describing** $\lim_{R_p(G)} \operatorname{Rep}_p^n(P)$**.** Let $N_p \leq G$ be a maximal p-toral subgroup of G. One such can be constructed by taking the preimage of a Sylow-p-subgroup of W in $N_G(T)$. Then

$$\lim_{\mathcal{R}_p(G)} \operatorname{Rep}_p(P) \hookrightarrow \operatorname{Rep}_p(N_p)$$

is injective, since any p-toral subgroup of G conjugates into N_p (see [18, lemma A.1]).

We have an isomorphism $\lim_{\mathcal{O}_p(G)} \operatorname{Rep}_p(P) \cong \lim_{\mathcal{R}_p(G)} \operatorname{Rep}_p(P)$ induced by the natural inclusion (see [17]).

Let $\phi\colon G/P \to G/Q$ and let $g\in G$ be such that $\phi(Px) = Pgx$. Let $c_g\colon P \to Q$ be conjugation from the left. Then the following square commutes up to homotopy:

$$\begin{tikzcd}
BP \arrow[r, "Bc_g"] \arrow[d] & BQ \arrow[d] \\
EG\times_G G/P \arrow[r, "1\times\phi"] & EG\times_G G/Q
\end{tikzcd}$$

Because of this square we use the following language:

Definition 3.1. An element of $\operatorname{Rep}_p(N_p)$ is called *fusion invariant* if it comes from $\lim_{\mathcal{R}_p(G)} \operatorname{Rep}_p(P) \cong \lim_{\mathcal{O}_p(G)} \operatorname{Rep}_p(P)$.

Definition 3.2. Let P be p-toral. $\breve{P} \leq P$ is a *p-discrete approximation of* P if $\breve{P}$ is dense in P and

$$\breve{P}_1 := \breve{P}\cap P_1 = \{x\in P_1 \mid x^{p^r} = 1 \text{ for some } r\in\mathbb{N}\}$$

A p-discrete approximation of P always exists. And $\breve{P}_1$ is unique up to conjugation with an element of P_1 (see [20, theorem 1.1]). $\breve{P}$ should be regarded as a discrete group. $\breve{P}$ is called a p-discrete toral group.

For a group K define $\operatorname{Rep}(K, U(k)) := \operatorname{Hom}(K, U(k))/\operatorname{Inn}(U(k))$. We have isomorphisms

$$\operatorname{Rep}(\breve{P}, U(k)) \xrightarrow{B} [B\breve{P}, BU(k)\hat{_p}] \leftarrow \operatorname{Rep}_p^k(P)$$

For the proof of this see [20, theorem 1.1]: The proof is based on [10] which shows it for finite p-groups, and on the fact that $B\breve{P} \to BP$ is a mod p equivalence. Thus $\lim_{R_p(G)} \operatorname{Rep}_p^n(P)$ can be given a quite algebraic description.

A p-discrete toral group is a special type of a countable locally finite group. That G is countable locally finite means that there exists an ascending sequence

$$1 = G^0 \leq G^1 \leq G^2 \leq \cdots \leq G$$

of finite groups such that $G = \bigcup_n G^n$. The representation theory of a countable locally finite group G is very similar to the representation theory of finite groups, as explained in [29, Appendix B]: Among other things, any representation of G splits uniquely (up to permutation) into a finite sum of irreducible representations, and any representation of G i determined by its character. Also Schur's lemma holds.

3.3. **Obstruction theory for Φ_1.** Let ρ be a k-dimensional fusion invariant representation of $\breve{N}_p$ (or equivalently let $\rho \in \lim_{\mathcal{R}_p(G)} \operatorname{Rep}_p^k(P)$). For $i \geq 1$ define $\Pi_i^\rho \colon \mathcal{R}_p(G)^{\mathrm{op}} \to \mathrm{Grp}$ as

$$\Pi_i^\rho(G/P) := \pi_i(\operatorname{Map}(EG \times_G G/P, BU(k)\hat{_p})_{B\rho}$$

If $H^{i+1}(\mathcal{R}_p(G); \Pi_i^\rho) = 0$ for all $i \geq 1$ then Φ_1 hits ρ. And if $H^i(\mathcal{R}_p(G); \Pi_i^\rho) = 0$ for all $i \geq 1$ then the element hitting ρ is unique (see [27]).

We have a natural weak equivalence

$$BC_{U(k)}(\rho(\breve{P}))\hat{_p} \simeq \operatorname{Map}(EG \times_G G/P, BU(k)\hat{_p})_{B\rho}$$

This is shown in [20, theorem 1.1] and the proof is based on [10]. Say $\rho|\breve{P} \cong \rho_1^{k_1} \oplus \cdots \oplus \rho_r^{k_r}$ where the ρ_i's are non-isomorphic irreducible representations. Then by Schur's lemma

$$C_{U(k)}(\rho(\breve{P})) \cong U(k_1) \times \cdots \times U(k_r)$$

In particular, since $\pi_0(U(l)) = \pi_2(U(l)) = 0$ for all $l \geq 0$ we get that

$$\Pi_1^\rho = \Pi_3^\rho = 0$$

3.3.1. *Understanding* Π_2^ρ. Let k be the dimension of ρ. We have natural isomorphisms

$$\begin{aligned}\Pi_2^\rho(G/P) &= \pi_2(\operatorname{Map}(EG \times_G G/P, BU(k)\hat{_p})_{B\rho}) \\ &\cong \pi_2(BC_{U(k)}(\rho(\breve{P}))) \otimes \mathbb{Z}_p \\ &\cong \pi_1(C_{U(k)}(\rho(\breve{P}))) \otimes \mathbb{Z}_p\end{aligned}$$

Write $\rho|\breve{P} = \rho_1^{a_1} \oplus \cdots \oplus \rho_r^{a_r}$ where the ρ_i's are pairwise non-isomorphic irreducible representations of $\breve{P}$. Say ρ_i has dimension k_i. Then

$$C_{U(k)}(\rho(\breve{P})) \cong U(a_1) \otimes I_{k_1} \oplus \cdots \oplus U(a_r) \otimes I_{k_r}$$

where I_{k_i} is the identity matrix of rank k_i. Since $\pi_1(U(l)) \cong \mathbb{Z}$ for all $l \geq 1$ we get

$$\Pi_2^\rho(G/P) \cong \mathbb{Z}_p\{\rho_1, \dots, \rho_r\}$$

This is a $W(P)^{\mathrm{op}}$-permutation representation of rank r.

Now assume $\breve{Q} \leq \breve{P}$. For simplicity say $\rho|\breve{P} = \rho_1^{a_1}$. Assume $\rho_1|\breve{Q} = \sigma_1^{b_1} \oplus \cdots \oplus \sigma_s^{b_s}$, where the σ_i's are non-isomorphic irreducible representations, so that $\rho|\breve{Q} = (\sigma_1^{b_1} \oplus \cdots \oplus \sigma_s^{b_s})^{a_1}$. Say σ_i has dimension l_i. We then want to calculate the map $\Pi_2^\rho(P) \to \Pi_2^\rho(Q)$ as a map

$$\mathbb{Z}_p\{\rho_1\} \to \mathbb{Z}_p\{\sigma_1, \dots, \sigma_s\}$$

The element $\rho_1 \in Z_p\{\rho_1\}$ corresponds to the element in $\pi_1(C_{U(k)}(\rho(\breve{P})))$ with representative $f\colon S^1 \to U(a_1) \otimes I_{k_1}$ where $f(z) = \mathrm{diag}(z \cdot I_{k_1}, I_{k_1}, \dots, I_{k_1})$. Postcomposing with the inclusion $C_{U(k)}(\rho(\breve{P})) \to C_{U(k)}(\rho(\breve{Q}))$ we get a map

$$\begin{aligned} S_1 &\to U(a_1) \otimes (U(b_1) \otimes I_{l_1} \oplus \cdots \oplus U(b_s) \otimes I_{l_s}) \\ z &\mapsto \mathrm{diag}(z \cdot I_{k_1}, I_{k_1}, \cdots, I_{k_1}) \end{aligned}$$

This map represents the element $b_1\sigma_1 + \cdots + b_s\sigma_s$. So we get

$$\rho_1 \mapsto b_1\sigma_1 + \cdots + b_s\sigma_s$$

3.3.2. *Spectral sequence for $H^*(\mathcal{R}_p(G); F)$.* Fix a *height* function $\mathrm{ht}\colon \mathrm{Ob}(\mathcal{R}_p(G)) \to \mathbb{Z}_{\geq 0}$ satisfying $G/P \cong G/Q \Rightarrow \mathrm{ht}(G/P) = \mathrm{ht}(G/Q)$ and ($G/P \ncong G/Q$ and $\mathrm{Mor}(G/P, G/Q) \neq 0) \Rightarrow \mathrm{ht}(G/P) > \mathrm{ht}(G/Q)$.

Theorem 3.3. [13, theorem 1.3] *There is a cohomological spectral sequence converging to $H^*(\mathcal{R}_p(G); F)$ with E_1 page given by*

$$E_1^{s,t} = \bigoplus_{\mathrm{ht}(G/P)=s} \Lambda^{s+t}(W(P); F(G/P))$$

Here $\Lambda^*(\Gamma, M) := H^*(\mathcal{R}_p(\Gamma); F_M)$ where $F_M(\Gamma/1) = M$ (M is a Γ^{op}-module) and $F_M(\Gamma/P) = 0$ for $P \neq 1$.

See [13] and [18] for methods for calculating the Λ^* groups.

The *height* $\mathrm{ht}(\mathcal{C})$ of a category $\mathcal{C}$ is the maximal length of a chain of inclusions in the category.

3.4. **Describing $\mathrm{Rep}_p(T)$.** Say $\breve{T} = (\mathbb{Z}/p^\infty)^r \leq U(1)^r$. Since $\breve{T}$ is abelian all irreducible representations of $\breve{T}$ are 1-dimensional (see [24, exercise 3.1]). And

$$\begin{aligned}\mathrm{Rep}(\breve{T}, U(1)) &= \mathrm{Hom}(\breve{T}, U(1)) \\ &\cong \mathrm{Hom}(\mathbb{Z}/p^\infty, U(1))^r \\ &\cong \mathrm{Hom}(\operatorname{colim} \mathbb{Z}/p^n, U(1))^r \\ &\cong (\lim \mathrm{Hom}(\mathbb{Z}/p^n, U(1)))^r \\ &\cong (\lim \mathbb{Z}/p^n)^r \\ &\cong \mathbb{Z}_p^r\end{aligned}$$

An element $(\alpha_1, \dots, \alpha_r) \in \mathbb{Z}_p^r$ corresponds to the map

$$(t_1, \dots, t_r) \mapsto t_1^{\alpha_1 \mod p^n} \cdots t_r^{\alpha_r \mod p^n} \qquad t_i \in \mathbb{Z}/p^n \leq U(1)$$

The element $(\alpha_1, \dots, \alpha_r)$ is called a *weight*.

We will write $\mathrm{Rep}_p(T) = \mathbb{Z}_{\geq 0}[x_1, \dots, x_r]$ (called the *character lattice*) where an exponent of x_i is in $\mathbb{Z}_p$. As an element of $\mathrm{Rep}_p(T)$ the weight $(\alpha_1, \dots, \alpha_r)$ is written as $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. Given a $\breve{N}_p$ representation ρ, its restriction $\rho|\breve{T}$, as an element of $\mathbb{Z}_{\geq 0}[x_1, \dots, x_r]$, is called the *Lie character* of ρ (not to be confused with the *character* χ of ρ, that is $\chi = \mathrm{trace} \circ \rho$).

We will define $[x_1^{\alpha_1} \cdots x_r^{\alpha_r}]$ to be the orbit sum of $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ under the action of the Weyl group W, that is

$$[x_1^{\alpha_1} \cdots x_r^{\alpha_r}] = \sum_{y \in W.(x_1^{\alpha_1} \cdots x_r^{\alpha_r})} y$$

4. Background for p-compact groups

A p-compact group (where p is a prime) is a triple (X, BX, e) where BX is a connected pointed p-complete space, X is a space with finite-dimensional $\mathbb{F}_p$-homology and $e \colon X \to \Omega BX$ is a homotopy equivalence. p-compact groups were first defined in [11]. See [11] and [12] for basic definitions and facts about p-compact groups, which will not be repeated here. Recall though that P is called a p-compact toral group if BP is the total space of a fibration with fiber the p-completed classifying space of a torus and with base the classifying space of a finite p-group.

Now assume X is a p-compact group. Define the category $\mathcal{O}(X)$ as follows: The objects are all pairs (P, i_P) where P is a p-compact toral group and $i_P \colon P \to X$ is a monomorphism (in the sense of p-compact groups). A morphism $\alpha \colon (P, i_p) \to (Q, i_Q)$ is a free (i.e. non-pointed) homotopy class of maps $B\alpha \colon BP \to BQ$ such that $Bi_Q \circ B\alpha$ is freely homotopic to Bi_P. Define the category $\mathcal{R}(X)$ to be the full subcategory of $\mathcal{O}(X)$ with objects (P, i_P) where P is p-radical and centric. Here p-radical is defined in terms of the Weyl group $W(P) := \mathrm{Aut}_{O(X)}(P)$ in the same way as for compact Lie groups and centric means that the natural map $BZ(P) \to BC_X(P)$ is a weak equivalence (here the centralizers are to be understood in the sense of p-compact groups). In [8] it is shown that there exists a functor $\Phi\colon \mathcal{R}(X) \to \mathrm{Top}$ such that $\Phi(P, i_P) \simeq BP$ for all $(P, i_P) \in \mathcal{R}(X)$ and such that there there exists a natural F_p-homology equivalence

$$\operatorname{hocolim}_{\mathcal{R}(X)} \Phi \to BX$$

Let G be a connected compact Lie group. In [8, Appendix B] it is furthermore shown that the natural map $\mathcal{R}_p(G) \to \mathcal{R}(G\hat{_p})$ is a equivalence of categories. And it is shown that via this equivalence the above homology decomposition is equivalent to the homology decomposition

$$\operatorname{hocolim}_{G/P \in \mathcal{R}_p(G)} EG \times_G G/P \to BG$$

of the previous section up to p-completion.

As in the previous section the above homology decomposition gives a factorization

$$\mathrm{Rep}_p(X) \xrightarrow{\Phi_1} \lim_{(P,i_P)\in\mathcal{R}(X)} \mathrm{Rep}_p(P) \xrightarrow{\Phi_2} \mathrm{Rep}_p(T)^W$$

where T is a maximal torus of X with Weyl group W. The rest of the previous section more or less generalizes to this context:

- Any p-compact toral group has a discrete approximation, and any homomorphism between p-compact toral groups lifts uniquely to a homomorphism of the chosen discrete approximation (see [12, proposition 3.2]). So we get the same algebraic description for $\lim_{(P,i_P)\in\mathcal{R}(X)} \mathrm{Rep}_p(P)$ as we did for compact Lie groups.
- The obstruction theory for Φ_1, including the spectral sequence for calculating the obstruction groups, is the same as for compact Lie groups.
- Choose a maximal p-compact toral subgroup N_p of X (say the p-normalizer of the chosen maximal torus T). Then $\lim_{(P,i_P)\in\mathcal{R}(X)} \mathrm{Rep}_p(P) \to \mathrm{Rep}_p(N_p)$ is injective: This is because any morphism $i_P\colon P \to X$ in $\mathcal{R}(X)$ lifts to a morphism $P \to N_p$ by [12, proposition 2.14] where we use the fact that p does not divide the Euler characteristic $\chi(X/N_p)$ (see [12, proposition 2.10]).

  Now we again say that an element of $\mathrm{Rep}_p(N_p)$ is fusion invariant if and only if it lies in $\lim_{\mathcal{R}(X)} \mathrm{Rep}_p(P) \cong \lim_{\mathcal{O}(X)} \mathrm{Rep}_p(P)$ (cf. lemma 5.2).

Connected p-compact groups have been completely classified in [3] and [4]. If X is a connected 2-compact group not isomorphic to the 2-completion of compact Lie group then X contains DI(4) as factor by [4, theorem 1.1] and since DI(4) has rank 3 X has rank at least 3. In the case of odd primes p the classification says that there is a one to one correspondence between isomorphism classes of simply connected p-compact groups and isomorphism classes of finite $\mathbb{Q}_p$-reflection groups (see [3, theorem 1.1] and [4, theorem 8.13(2)]). Now if X is a connected p-compact group with p not dividing the order of the Weyl group, the main theorem (theorem 1.1) is true for X by lemma 5.1. By inspecting a table of the irreducible $\mathbb{Q}_p$-reflection groups (see [14, page 4]) we see that the remaining cases we need to prove the main theorem for are $\mathrm{Sp}(1)$ at $p = 2$, $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ at $p = 2$, $\mathrm{SU}(3)$ at $p = 2, 3$, $\mathrm{Sp}(2)$ at $p = 2$, G_2 at $p = 2, 3$ and $\mathrm{DI}(2)$. Here $\mathrm{DI}(2)$ is a 3-compact group – the only simple exotic p-compact group of rank at most 2 where p divides the order of the Weyl group.

5. General results

5.1. When p does not divide the order of the Weyl group.

Lemma 5.1. *Let X be a connected compact Lie group with maximal torus T and Weyl group W. Let p be a prime such that $p \not| |W|$. Then*

$$\mathrm{Rep}_p(X) \xrightarrow{\cong} \mathrm{Rep}_p(T)^W$$

is an isomorphism.

Proof. In this case T is a p-radical subgroup and it is a maximal one since the p-normalizer of T is T itself. Choose any $P \in \mathcal{O}(X)$ not isomorphic to T. Then we have a monomorphism $P \to T$ and P has rank strictly less that T. On the other hand $C_X(P)$ has the same rank as T by [12, proposition 4.3]. It follows that P is not centric, so $P \notin \mathcal{R}(X)$.

From this calculation of $R(X)$ we get that $\lim_{\mathcal{R}(X)} \mathrm{Rep}_p(P) = \mathrm{Rep}_p(T)^W$. Since $\mathcal{R}_p(T)$ has height 0, we also get that all obstructions for the map $\Phi_1 \colon \mathrm{Rep}_p(X) \to \lim_{\mathcal{R}(X)} \mathrm{Rep}_p(P)$ vanish by lemma 5.5, so that Φ_1 is an isomorphism. □

5.2. Condition for fusion invariance.

Lemma 5.2. *Let X be a p-compact group and let $\rho \in \mathrm{Rep}_p(N_p)$ where N_p is a maximal p-compact toral subgroup of X. Then ρ is fusion invariant if and only if $\alpha^*(\rho|P) \cong \rho|P$ for all $\alpha \in W(P)$ for all $(P, i_P) \in \mathcal{R}(X)$.*

Proof. This is an application of Alperin's Fusion Theorem (AFT): AFT is proven in [6, theorem 3.6] for p-local compact groups, and thus also holds for $\mathcal{O}(X)$ since any p-compact group is a p-local compact group by [6, chapter 10].

Let $\alpha \colon P \to P'$ be a morphism in $\mathcal{O}(X)$. We can assume that α is an isomorphism, since any morphism factors as an isomorphism followed by an inclusion.

By AFT we have objects $P = P_0, \dots, P_k = P'$ in $\mathcal{O}(X)$, objects $Q_1, \dots, Q_k$ in $\mathcal{R}(X)$ and morphisms $\alpha_i \in W(Q_i)$ such that $P_{i-1}, P_i \leq$

Q_i, $\alpha_i\colon P_{i-1} \to P_i$ is an isomorphism and $\alpha = \alpha_k \circ \cdots \circ \alpha_1$. Now $\alpha_i^*(\rho|P_i) \cong \rho|P_{i-1}$ for all i implying that $\alpha^*(\rho|P') \cong \rho|P$. So ρ lies in $\lim_{\mathcal{O}(X)} \operatorname{Rep}_p(P)$ and hence in $\lim_{\mathcal{R}(X)} \operatorname{Rep}_p(P)$, that is ρ is fusion invariant. ☐

5.3. **Injectivity of Φ_2.**

Lemma 5.3. *Let X be a connected p-compact group with maximal torus T. Then $\Phi_2\colon \lim_{\mathcal{R}_p(G)} \operatorname{Rep}_p^n(P) \to \operatorname{Rep}_p^n(T)$ is injective.*

Proof. Let $\breve{N}_p \leq N_p$ be a p-discrete approximation of a maximal p-toral subgroup and let $\breve{T}$ be a p-discrete approximation of T. Let ρ_1 and ρ_2 be k-dimensional fusion invariant $\breve{N}_p$-representations. Assume $\rho_1|\breve{T} \simeq \rho_2|\breve{T}$.

Remember that ρ_i is determined by its character χ_i. So let $n \in \breve{N}_p$; we then have to show that $\chi_1(n) = \chi_2(n)$. The map $B\langle n\rangle \to BN_p \to X$ is a monomorphism and there exists a map $B\phi\colon B\langle n\rangle \to BT$ in $\mathcal{O}(X)$: This follows by a proof almost identical to the proof of [11, proposition 8.11] except that one uses theorem 4.6 instead of theorem 4.7 in the proof. Now since ρ_i is fusion invariant we have that $\chi_i(n) = \chi_i(\phi(n))$. So we get

$$\chi_1(n) = \chi_1(\phi(n)) = \chi_2(\phi(n)) = \chi_2(n)$$

☐

Corollary 5.4. $\operatorname{Gr}(\Phi_2)$ *is injective.*

Proof. This follows from the previous lemma and the fact that $R_p(T)$ satisfies additive cancellation. ☐

5.4. **Bound on non-zero obstruction groups.**

Lemma 5.5. *Let X be p-compact group. Then $H^n(\mathcal{R}(X); F) = 0$ for $n > \operatorname{ht}(\mathcal{R}(X))$ for all functors F.*

Proof. We want to use proposition 17.31 in [22]. First we note that a skeleton of $\mathcal{R}(X)$ is a finite EI-category. Let $M\colon \mathcal{R}(X)^{\mathrm{op}} \to \mathbb{Z}_p$-mod be the constant functor $M(P) = \mathbb{Z}_p$. Obviously $M(P)$ is projective over $\mathbb{Z}_p$. Now by [13, theorem 1.1] $\Lambda^n(W(P), F(P)) = 0$ for $n > \operatorname{ht}(\mathcal{R}_p(W(P)))$ so by the spectral sequence converging to $H^*(\mathcal{R}(X); F)$ we have that there exists an N such that $\operatorname{Ext}^n_{\mathbb{Z}_p\mathcal{R}(X)^{\mathrm{op}}}(M, F) = H^n(\mathcal{R}(X); F) = 0$ for $n > N$, and this N is independent of F. This implies that M has a finite projective resolution. Now by [22, proposition 17.31] the projective dimension of M is less than or equal to $\operatorname{ht}(\mathcal{R}(X))$. This implies that $H^n(\mathcal{R}(X); F) = \operatorname{Ext}^n_{\mathbb{Z}_p\mathcal{R}(X)^{\mathrm{op}}}(M, F) = 0$ for $n > \operatorname{ht}(\mathcal{R}(X))$. ☐

5.5. **Using unstable Adams operations.**

Lemma 5.6. *Let G be a connected compact Lie group with maximal torus T and Weyl group W. Let $R_p(T) \cong \mathbb{Z}[x_1, \dots, x_r]$ be any isomorphism, and let $\alpha \in \mathbb{Z}_p$. Then $[x_i^\alpha] \in R_P(T)^W$ is hit by $\Phi \colon R_p(G) \to R_p(T)^W$.*

Proof. Write $\alpha = kp^i$ with $i \geq 0$ and $k \in \mathbb{Z}_p^*$. Since we have an isomorphism $R(G) \to R(T)^W$ the "integral" orbit sum $[x_i^{p^i}]$ is hit by Φ. Furthermore for all $k \in \mathbb{Z}_p^*$ there exists an *unstable Adams operation* $\psi^k \colon BG\hat{_p} \to BG\hat{_p}$ and precomposing a representation with ψ^k corresponds on the character lattice to multiplying each exponential by k. So by precomposing with ψ^k we see that also $[x_i^\alpha]$ is hit by Φ. ☐

Remark 5.7. Notice that by tensoring virtual representations we see that Φ also hits products of the above orbit sums; for example $[x_1^\alpha] \cdot [x_2^\beta]$ for $\alpha, \beta \in \mathbb{Z}_p$ is also hit.

6. Proof of case: Sp(1)

Let $G = \mathrm{Sp}(1)$. Here $R_2(T) = \mathbb{Z}[x_1]$ and the Weyl group Σ_2 acts by $x_1^\alpha \mapsto x_1^{-\alpha}$. By lemma 5.6 any orbit sum $[x_1^\alpha]$ is hit by $R_2(G) \to R_2(T)^W$. A skeleton of $\mathcal{R}_2(G)$ is $Q \hookrightarrow N$. So $\mathcal{R}_2(G)$ has height 1, so Φ_1 is an isomorphism. So

$$R_2(\mathrm{Sp}(1)) \xrightarrow{\cong} R_2(T)^W$$

is an isomorphism.

7. Proof of case: SU(3)

Here we can use lemma 5.6 to show surjectivity. We have $R_p(T) = \mathbb{Z}[x_1, x_2, x_3]/(x_1^\alpha x_2^\alpha x_3^\alpha, \alpha \in \mathbb{Z}_p)$ and the Weyl group Σ_3 acts by permuting $x_1, \dots, x_3$. We already know that any orbit sum of the form $[x_1^\alpha]$ is hit. Then, for $\alpha, \beta \neq 0$ and $\alpha \neq \beta$,

$$[x_1^\alpha] \cdot [x_1^\beta] = [x_1^{\alpha+\beta}] + [x_1^\alpha x_2^\beta]$$

so also $[x_1^\alpha x_2^\beta]$ is hit.

Both $\mathcal{R}_2(\mathrm{SU}(3))$ and $\mathcal{R}_3(\mathrm{SU}(3))$ have height 1 (see [23]), so Φ_1 is always an isomorphism. So

$$R_p(\mathrm{SU}(3)) \xrightarrow{\cong} R_p(T)^W$$

is an isomorphism for all primes p.

8. Proof of case: Sp(1) × Sp(1) at $p = 2$

Let $G = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. By using lemma 5.6 and its following remark it is easy to see that $R_2(G) \to R_2(T)^W$ is surjective.

Representatives for the conjugacy classes of groups in $\mathcal{R}_2(G) \cong \mathcal{R}_2(\mathrm{Sp}(1)) \times \mathcal{R}_2(\mathrm{Sp}(1))$ are given by the following table:

P	$W(P)$	$\mathrm{ht}(P)$
$P_1 = N \times N = N_2(T)$	1	0
$P_2 = N \times Q$	$1 \times \Sigma_3$	1
$P_3 = Q \times N$	$\Sigma_3 \times 1$	1
$P_4 = Q \times Q$	$\Sigma_3 \times \Sigma_3$	2

Here $N = \langle U(1), j\rangle$ and $Q = \langle i, j\rangle$, the quaternion group.

$$\begin{tikzcd}
 & N \times N & \\
N \times Q \arrow[ur, hook] & & Q \times N \arrow[ul, hook] \\
 & Q \times Q \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

The morphisms between the P_i's are generated by the automorphisms and the inclusions: This follows by the following lemma by noting that $\mathcal{R}_2(G) \cong \mathcal{R}_2(\mathrm{Sp}(1)) \times \mathcal{R}_2(\mathrm{Sp}(1))$:

Lemma 8.1. *Let $H \leq N$, $H \cong Q$ (for example $H = {}^{x}Q$ for $x \in \mathrm{Sp}(1)$). Then there exists $n \in N$ such that $H = {}^{n}Q$.*

Proof. Assume $\langle x, y\rangle = H \leq N$ is isomorphic to the quaternion group via $x \mapsto i$ and $y \mapsto j$. Since not all elements of order 4 in H can lie in $j \cdot U(1)$ we must have one of the elements equal to i, say $x = i$. Then $y \in j \cdot U(1)$. Write $y = y'j$ for $y' \in U(1)$, choose $n' \in U(1)$ such that $(n')^2 = y'$ and put $n = n'j$. Then ${}^{n}i = -i$ and ${}^{n}j = (n')^2 j = y$, so ${}^{n}Q = H$. $\square$

Let $\breve{N} = \langle \breve{U}(1), j\rangle \subseteq \mathrm{Sp}(1)$ (a p-discrete approximation of N). Choose discrete approximations $\breve{P}_1, \dots, \breve{P}_4$ of $P_1, \dots, P_4$ by replacing any factor N by $\breve{N}$.

As $\mathcal{R}_2(G)$ has height 2 there is just one potential obstruction group to deal with, namely $H^2(\mathcal{R}_2(G), \Pi_2^{\rho})$.

8.1. Representations of $\breve{P}_1, \cdots, \breve{P}_4$.

Lemma 8.2. *The irreducible representations of $\breve{N}$ are given as follows:*

- *Two 1-dimensional representations ϕ_ϵ for $\epsilon \in \{1, -1\}$ given by $\phi_\epsilon(\breve{U}(1)) = 1$ and $\phi_\epsilon(j) = \epsilon$.*
- *For all $\alpha \in \mathbb{Z}_2 - \{0\}$ a representation $\psi_\alpha = \mathrm{Ind}_{\breve{U}(1)}^{\breve{N}}(\alpha)$. Then*

$$\psi_\alpha(t) = \begin{pmatrix} t^\alpha & 0 \\ 0 & t^{-\alpha} \end{pmatrix}$$

for all $t \in \breve{U}(1)$ *and*

$$\psi_\alpha(j) = \begin{pmatrix} 0 & (-1)^\alpha \\ 1 & 0 \end{pmatrix}$$

Proof. Any irreducible representation of $\breve{N}$ is contained in a representation induced from $\breve{U}(1)$ by [24, exercise 3.4]. Let $\alpha \in \mathbb{Z}_2$ be an irreducible representation of $\breve{U}(1)$. Then $\mathrm{Ind}_{\breve{U}(1)}^{\breve{N}}(\alpha)$ is irreducible if and only if the action of $\breve{N}/\breve{U}(1)$ on α has trivial stabilizer (see [15, problem 6.1] which can be proven using theorem 6.11), that is if and only if $\alpha^j \neq \alpha$. Since $\alpha^j = -\alpha$, this is if and only if $\alpha \neq 0$. And $\mathrm{Ind}_{\breve{U}(1)}^{\breve{N}}(0) \cong \phi_1 \oplus \phi_{-1}$. □

The irreducible representations of $\breve{P}_1$ are exactly the products of an irreducible representation of $\breve{N}$ with an irreducible representation of $\breve{N}$ (see [29, Appendix B]). So they are given as follows:

- 1-dimensional representations $\tau_{\epsilon_1,\epsilon_2} = \phi_{\epsilon_1} \times \phi_{\epsilon_2}$, $\epsilon_i \in \{\pm 1\}$.
- 2-dimensional representations $\theta^1_{\alpha,\epsilon} = \psi_\alpha \times \phi_\epsilon$ with character $x_1^{\pm\alpha}$ and $\theta^2_{\beta,\epsilon} = \phi_\epsilon \times \psi_\beta$ with character $x_2^{\pm\beta}$. Here $\alpha, \beta \neq 0$ and $\epsilon \in \{\pm 1\}$.
- 4-dimensional representations $\rho_{\alpha,\beta} = \psi_\alpha \times \psi_\beta$ with character $x_1^{\pm\alpha} x_2^{\pm\beta}$. Here $\alpha, \beta \neq 0$.

Q has five irreducible representations. Four 1-dimensional representations $\chi_{\epsilon_1,\epsilon_1}$, $\epsilon_i \in \{1, -1\}$ given by

$$\chi_{\epsilon_1,\epsilon_2}(i) = \epsilon_1$$
$$\chi_{\epsilon_1,\epsilon_2}(j) = \epsilon_2$$

and one 2-dimensional representation ζ. Then $\breve{P}_4 = Q \times Q$ has 25 irreducible representations given by products of these.

To determine how representations of $\breve{P}_1$ restrict to $\breve{P}_2, \dots, \breve{P}_4$ we use the fact that, for ρ_1, ρ_2 representations of R, S, the character of $\rho_1 \times \rho_2$ (a representation of $R \times S$) can be calculated as

$$\chi_{\rho_1 \times \rho_2}(x, y) = \chi_{\rho_1}(x) \cdot \chi_{\rho_2}(y)$$

And then we use the following table

Representation of $\breve{N}$	Restriction to Q	
$\psi_\alpha,\ \alpha \equiv 1\ (2)$	ζ	
$\psi_\alpha,\ \alpha \equiv 0\ (2)$	$\chi_{\delta,1} \oplus \chi_{\delta,-1}$	$\delta = \begin{cases} 1 & \alpha \equiv 0\ (4) \\ -1 & \alpha \equiv 2\ (4) \end{cases}$
ϕ_ϵ	$\chi_{1,\epsilon}$	

So for example $\rho_{1,2}|Q \times Q \cong \zeta \times \chi_{-1,1} \oplus \zeta \times \chi_{-1,-1}$.

8.2. **Determining fusion invariance.** We will now determine fusion invariance (see definition 3.1 and theorem 5.2).

Lemma 8.3. *A representation of the discrete 2-normalizer $\breve{P}_1$ is fusion invariant if and only if its character χ satisfies $\chi(x,i) - \chi(x,j) = 0$ and $\chi(i,x) - \chi(j,x) = 0$ for all $x \in \breve{N}$.*

Proof. An $\breve{P}_1$-representation with character χ is fusion invariant if and only its restriction to $\breve{P}_i$ is invariant under the action of $W(P_i)$ for $i = 1, \dots, 4$. It is invariant under $W(P_2)$ if and only if $\chi(x,i) = \chi(x,j) = \chi(x,k)$ and invariant under $W(P_3)$ if and only if $\chi(i,x) = \chi(j,x) = \chi(k,x)$ for all $x \in N$. Since $\chi(x,j) = \chi(x,k)$ and $\chi(j,x) = \chi(k,x)$ for all representations of $\breve{P}_1$ this is equivalent to $\chi(x,i) - \chi(x,j) = 0$ and $\chi(i,x) - \chi(j,x) = 0$. By straightforward calculation one sees the representation is also invariant under $W(P_4)$ if these two equations are satisfied. ☐

For fusion invariance the values of χ for the irreducible representations are (where $t \in \breve{U}(1)$ and congruences are module 4):

	Rep.	Value
$\chi(t,i) - \chi(t,j)$	$\tau_{\pm,-}$	2
	$\theta^1_{\alpha,-}$	$2t^{\pm\alpha}$
	$\theta^2_{\beta,\pm}$	$\beta \equiv 2\colon -2,\ \beta \equiv 0\colon 2$
	$\rho_{\alpha,\beta}$	$\beta \equiv 2\colon -2t^{\pm\alpha},\ \beta \equiv 0\colon 2t^{\pm\alpha}$
$\chi(jt,i) - \chi(jt,j)$	$\tau_{+,-}$	2
	$\tau_{-,-}$	-2
	$\theta^2_{\beta,+}$	$\beta \equiv 2\colon -2,\ \beta \equiv 0\colon 2$
	$\theta^2_{\beta,-}$	$\beta \equiv 2\colon 2,\ \beta \equiv 0\colon -2$
$\chi(i,t) - \chi(j,t)$	$\tau_{-,\pm}$	2
	$\theta^2_{\beta,-}$	$2t^{\pm\beta}$
	$\theta^1_{\alpha,\pm}$	$\alpha \equiv 2\colon -2,\ \alpha \equiv 0\colon 2$
	$\rho_{\alpha,\beta}$	$\alpha \equiv 2\colon -2t^{\pm\beta},\ \alpha \equiv 0\colon 2t^{\pm\beta}$
$\chi(i,jt) - \chi(j,jt)$	$\tau_{-,+}$	2
	$\tau_{-,-}$	-2
	$\theta^1_{\alpha,+}$	$\alpha \equiv 2\colon -2,\ \alpha \equiv 0\colon 2$
	$\theta^1_{\alpha,-}$	$\alpha \equiv 2\colon 2,\ \alpha \equiv 0\colon -2$

For representations not listed the values are 0. Here for example $\tau_{\pm,-}$ means either $\tau_{+,-}$ or $\tau_{-,-}$. And $2t^{\pm\alpha} = 2t^{\alpha} + 2t^{-\alpha}$

Studying the above table we see that the representation is fusion invariant if and only if the following equations are all satisfied:

Equation no.	Equation
(1a)	For each $\alpha \in \mathbb{Z}_2 - \{0\}$: $\underset{\beta\equiv 0}{\#}\,\rho_{\alpha,\beta} + \#\theta^1_{\alpha,-} = \underset{\beta\equiv 2}{\#}\,\rho_{\alpha,\beta}$
(1b)	$\#\tau_{\pm,-} + \underset{\beta\equiv 0}{\#}\,\theta^2_{\beta,\pm} = \underset{\beta\equiv 2}{\#}\,\theta^2_{\beta,\pm}$
(2)	$\#\tau_{+,-} + \underset{\beta\equiv 0}{\#}\,\theta^2_{\beta,+} + \underset{\beta\equiv 2}{\#}\,\theta^2_{\beta,-} = \#\tau_{-,-} + \underset{\beta\equiv 2}{\#}\,\theta^2_{\beta,+} + \underset{\beta\equiv 0}{\#}\,\theta^2_{\beta,-}$
(3a)	For each $\beta \in \mathbb{Z}_2 - \{0\}$: $\underset{\alpha\equiv 0}{\#}\,\rho_{\alpha,\beta} + \#\theta^2_{\beta,-} = \underset{\alpha\equiv 2}{\#}\,\rho_{\alpha,\beta}$
(3b)	$\#\tau_{-,\pm} + \underset{\alpha\equiv 0}{\#}\,\theta^1_{\alpha,\pm} = \underset{\alpha\equiv 2}{\#}\,\theta^1_{\alpha,\pm}$
(4)	$\#\tau_{-,+} + \underset{\alpha\equiv 0}{\#}\,\theta^1_{\alpha,+} + \underset{\alpha\equiv 2}{\#}\,\theta^1_{\alpha,-} = \#\tau_{-,-} + \underset{\alpha\equiv 2}{\#}\,\theta^1_{\alpha,+} + \underset{\alpha\equiv 0}{\#}\,\theta^1_{\alpha,-}$

Here, for example, $\underset{\beta\equiv 0}{\#}\,\rho_{\alpha,\beta}$ means the number of irreducible summands in the representation of the form $\rho_{\alpha',\beta}$ with $\alpha' = \alpha$ and $\beta \equiv 0$ (4).

For example the representation $\rho_{2,2} \oplus \rho_{2,4} \oplus \rho_{4,2} \oplus \rho_{4,4}$ is fusion invariant.

8.3. Injectivity of $\mathrm{Gr}(\Phi_1)\colon R_2(G) \to \mathrm{Gr}(\lim \mathrm{Rep}_2(P))$.

Injectivity is governed by the uniqueness obstruction group $H^2(\mathcal{R}_2(G); \Pi_2^\rho)$. We will construct a specific fusion invariant representation ρ such that this group is 0. For this particular representation the following holds: For any other fusion invariant representation $\widetilde{V}$ also $H^2(\mathcal{R}_2(G); \Pi_2^{\widetilde{V}\oplus\rho}) = 0$ (see below). Injectivity will then follow by the following lemma (noting that Φ_1 is surjective, since all existence obstruction groups are 0, since the height of $\mathcal{R}_2(G)$ is 2)

Lemma 8.4. *Let p be a prime and let G be a connected compact Lie group. Assume $\mathrm{ht}(\mathcal{R}_p(G)) \le 3$, assume Φ_1 is surjective and assume that for all $\widetilde{V} \in \lim \mathrm{Rep}_p(P)$ there exists a representation $\widetilde{X} \in \lim \mathrm{Rep}_p(P)$ such that $H^2(\mathcal{R}_p(G); \Pi_2^{\widetilde{V}\oplus\widetilde{X}}) = 0$. Then $\mathrm{Gr}(\Phi_1)$ is injective.*

Proof. As $\mathrm{ht}(\mathcal{R}_p(G)) \le 3$ all uniqueness obstruction groups except H^2 vanish (remember $\Pi_1^\rho = \Pi_3^\rho = 0$).

Assume $\mathrm{Gr}(\Phi_1)([V_1 - V_2]) = 0$ that is $\Phi_1(V_1) \oplus \widetilde{W} = \Phi_1(V_2) \oplus \widetilde{W}$ for some $\widetilde{W}$. Then by assumption there exists $\widetilde{X}$ such that $H^2(\mathcal{R}_p(G); \Pi_2^{\Phi_1(V_1)\oplus\widetilde{W}\oplus\widetilde{X}}) = 0$. Choose W and X such that $\widetilde{W} = \Phi_1(W)$ and $\widetilde{X} = \Phi_1(X)$. Since the obstruction group vanishes for $\Phi_1(V_1 \oplus W \oplus X) = \Phi_1(V_2 \oplus W \oplus X)$ we get $V_1 \oplus W \oplus X = V_2 \oplus W \oplus X$. So $[V_1 - V_2] = 0$. ☐

We will show that $H^2(\mathcal{R}_2(G); \Pi_2^\rho) = 0$ by showing that the differential in the spectral sequence

$$\Lambda^1(1 \times \Sigma_3; \Pi_2^\rho(P_2)) \oplus \Lambda^1(\Sigma_3 \times 1; \Pi_2^\rho(P_3)) \xrightarrow{\partial} \Lambda^2(\Sigma_3 \times \Sigma_3; \Pi_2^\rho(P_4))$$

is surjective.

Let $\mathbb{Z}_2\mathcal{R}_2(G)^{\mathrm{op}} \to \mathbb{Z}_2W(P_4)^{\mathrm{op}}$ be the functor $T \mapsto T(P_4)$. This functor has a right adjoint, a right Kan extension, which we will call Ran. Let $M = \Pi_2^{\rho}(P_4)$ and put $F = \mathrm{Ran}(M)$. The unit of the adjunction gives a natural transformation $\Pi_2^{\rho} \to F$. This induces a natural transformation of spectral sequences giving a commutative square

(8.1)

$$\begin{tikzcd}
\Lambda^1(1 \times \Sigma_3; \Pi_2^{\rho}(P_2)) \oplus \Lambda^1(\Sigma_3 \times 1; \Pi_2^{\rho}(P_3)) \arrow[r, "\partial"] \arrow[d] & \Lambda^2(\Sigma_3 \times \Sigma_3; \Pi_2^{\rho}(P_4)) \arrow[d, equal] \\
\Lambda^1(1 \times \Sigma_3; F(P_2)) \oplus \Lambda^1(\Sigma_3 \times 1; F(P_3)) \arrow[r, "\widetilde{\partial}"] & \Lambda^2(\Sigma_3 \times \Sigma_3; M)
\end{tikzcd}$$

giving a factorization of ∂. First we will show that $\widetilde{\partial}$ is surjective. To do this we need to understand the category $\mathcal{R}_2(G)$ and the functor F a little better:

$Q < N$ are 2-toral groups, so $Q < N_N(Q)$ by [18, lemma A.2]. Since $\mathrm{Out}(Q) = \Sigma_3$ (identifying i with 1, j with 2 and k with 3) we must have $N_N(Q)/Q \cong C_2$. In fact $N_N(Q)/Q = \langle \frac{(1+i)}{\sqrt{2}} \rangle$: Putting $x = \frac{(1+i)}{\sqrt{2}}$ we see ${}^{x}i = i$, ${}^{x}j = k$ and ${}^{x}k = -j$. So $N_N(Q)/Q = \langle \tau \rangle \leq \Sigma_3$ where $\tau = (2\ 3)$. Let $C_2 = \langle \tau \rangle$ denote this particular subgroup of Σ_3.

Let $\mathcal{O}$ denote the following full subcategory of $\Sigma_3 \times \Sigma_3 = W(P_4)$:

$$\begin{tikzcd}
& N_{N\times N}(Q \times Q)/(Q \times Q) & \\
N_{N\times Q}(Q \times Q)/(Q \times Q) \arrow[ur, hook] & & N_{Q\times N}(Q \times Q)/(Q \times Q) \arrow[ul, hook'] \\
& 1 \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

that is the subcategory

$$\begin{tikzcd}
& C_2 \times C_2 & \\
C_2 \times 1 \arrow[ur, hook] & & 1 \times C_2 \arrow[ul, hook'] \\
& 1 \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

Notice that this is a skeleton of $\mathcal{O}_2(\Sigma_3 \times \Sigma_3)$. Let $\mathcal{R}_G$ denote the full subcategory (skeleton) of $\mathcal{R}_2(G)$ with objects $P_1, \ldots, P_4$. We want to show that the obvious map $\mathcal{O} \to \mathcal{R}_G$ on objects gives an isomorphism of categories:

First notice that the Weyl groups are isomorphic. To determine $\mathrm{Mor}_{\mathcal{R}_2(G)}(G/P_4, G/P_2) = N(P_4, P_2)/P_2$ we use that any such map is given by an automorphism of G/P_4 followed by the projection $G/P_4 \to$

G/P_2 followed by an automorphism of G/P_2. And by the description of the Weyl groups of P_4 and P_2 we see that any composition of the projection with an automorphism of G/P_2 is equal to an automorphism of G/P_4 followed by the projection. This implies that $W(P_4) \to N(P_4, P_2)/P_2$ (mapping $\bar{x}$ to $\bar{x}$) is surjective. So

$$N(P_4, P_2)/P_2 = W(P_4)/(N_{P_2}(P_4)/P_4) = \Sigma_3 \times \Sigma_3/C_2 \times 1$$

By similar calculations we get

$$\begin{aligned}
N(P_4, P_3)/P_3 &= \Sigma_3 \times \Sigma_3/1 \times C_2\\
N(P_4, P_1)/P_1 &= \Sigma_3 \times \Sigma_3/C_2 \times C_2\\
N(P_2, P_1)/P_1 &= 1 \times \Sigma_3/1 \times C_2\\
N(P_3, P_1)/P_1 &= \Sigma_3 \times 1/C_2 \times 1
\end{aligned}$$

This shows that $\mathcal{O} \cong \mathcal{R}_G$.

Now,

$$\begin{aligned}
F(P_2) &= \left(\prod_{N(P_4,P_2)/P_2} M\right)^{W(P_4)}\\
&= \left(\prod_{\Sigma_3\times\Sigma_3/C_2\times 1} M\right)^{\Sigma_3\times\Sigma_3}\\
&= \mathrm{Hom}_{\Sigma_3\times\Sigma_3}(\mathbb{Z}_2, \mathrm{Ind}_{C_2\times 1}^{\Sigma_3\times\Sigma_3}(M))\\
&= \mathrm{Hom}_{C_2\times 1}(\mathbb{Z}_2, M)\\
&= M^{C_2\times 1}
\end{aligned}$$

where we in the first equality write up a concrete expression for F. Similarly we calculate $F(P_3)$ and $F(P_1)$. All in all

$$\begin{aligned}
F(P_1) &= M^{C_2\times C_2}\\
F(P_2) &= M^{C_2\times 1}\\
F(P_3) &= M^{1\times C_2}\\
F(P_4) &= M
\end{aligned}$$

Restricting F to a functor $\mathcal{O} \to \mathbb{Z}_2$-mod we see that F is a fixed point functor. Such functors are known to be acyclic by [18, proposition 5.2] (the proof is to show that F is a proto-Mackey functor and then use [16, proposition 5.14]). Looking at the spectral sequence converging to $H^*(\mathcal{O}; F)$ we see that the map

$$\Lambda^1(1 \times \Sigma_3; F(P_2)) \oplus \Lambda^1(\Sigma_3 \times 1; F(P_3)) \xrightarrow{\widetilde{\partial}} \Lambda^2(\Sigma_3 \times \Sigma_3; M)$$

is surjective. Returning to the square 8.1 this implies that ∂ is surjective if the left vertical map is surjective. We now want to find a representation ρ where this is the case:

Put

$$\begin{aligned}\rho\ =&(\rho_{2,2}\oplus\rho_{2,4}\oplus\rho_{4,2}\oplus\rho_{4,4})\oplus(\rho_{1,2}\oplus\rho_{1,4})\oplus(\rho_{2,1}\oplus\rho_{4,1})\oplus\rho_{1,1}\\ &\oplus(\tau_{-,+}\oplus\theta^1_{2,+})\oplus(\tau_{+,-}\oplus\theta^2_{2,+})\oplus\theta^1_{1,+}\oplus\theta^2_{1,+}\end{aligned}$$

Then ρ is fusion invariant (check that the equations in section 8.2 are satisfied).

For this ρ we have that $\rho|P_4$ contains all irreducible representations of P_4, so M has as basis all the irreducible representations. Let $M_2 = \Pi_2^\rho(P_2)$. Then M_2 has basis

$$\begin{aligned}&\{\psi_2\times\chi_{\epsilon_1,\epsilon_2}\mid\epsilon_i\in\{\pm1\}\}\cup\{\psi_4\times\chi_{\epsilon_1,\epsilon_2}\mid\epsilon_i\in\{\pm1\}\}\cup\\ &\{\psi_1\times\chi_{\epsilon_1,\epsilon_2}\mid\epsilon_i\in\{\pm1\}\}\cup\{\psi_2\times\zeta,\psi_4\times\zeta,\psi_1\times\zeta\}\cup\\ &\{\phi_{-1}\times\chi_{1,1},\phi_1\times\chi_{1,-1}\}\cup\{\phi_1\times\chi_{-1,\epsilon}\mid\epsilon\in\{\pm1\}\cup\{\phi_1\times\zeta\}\end{aligned}$$

The map $M_2\to M$ induced by the inclusion $P_4\hookrightarrow P_2$ is easily determined from the table above detailing how representations of $\breve{P}_2$ restrict to $\breve{P}_4$.

We will now show that $\Lambda^1(1\times\Sigma_3;\Pi_2^\rho(P_2))\to\Lambda^1(1\times\Sigma_3;F(P_2))$ is surjective. In general $\Lambda^1(1\times\Sigma_3;L) = L^{1\times C_2}/L^{1\times\Sigma_3}$. So it is enough to show that $\Pi_2^\rho(P_2)^{1\times C_2}\to F(P_2)^{1\times C_2}$ is surjective. That is, that $M_2^{1\times C_2}\to M^{C_2\times C_2}$ is surjective.

$M_2^{1\times C_2}$ has basis

$$\begin{aligned}&\{\psi_2\times\chi_{1,\epsilon_2},\psi_4\times\chi_{1,\epsilon_2},\psi_1\times\chi_{1,\epsilon_2}\mid\epsilon_2\in\{\pm1\}\}\cup\\ &\{\psi_2\times\chi_{-1,\pm1},\psi_4\times\chi_{-1,\pm1}\psi_1\times\chi_{-1,\pm1}\}\cup\\ &\{\psi_2\times\zeta,\psi_4\times\zeta,\psi_1\times\zeta\}\cup\\ &\{\phi_{-1}\times\chi_{1,1},\phi_1\times\chi_{1,-1},\phi_1\times\chi_{-1,\pm1}\}\cup\\ &\{\phi_1\times\zeta\}\end{aligned}$$

Here we are using the summing convention that for example $\psi_2\times\chi_{-1,\pm1} = \psi_2\times\chi_{-1,1}+\psi_2\times\chi_{-1,-1}$. $M^{C_2\times C_2}$ has basis

$$\begin{aligned}&\{\chi_{1,\epsilon_2}\times\chi_{1,\epsilon_4}\mid\epsilon_i\in\{\pm1\}\}\cup\\ &\{\chi_{1,\epsilon_2}\times\chi_{-1,\pm1}\mid\epsilon_2\in\{\pm1\}\}\cup\\ &\{\chi_{-1,\pm1}\times\chi_{1,\epsilon_4}\mid\epsilon_4\in\{\pm1\}\}\cup\\ &\{\chi_{-1,\pm1}\times\chi_{-1,\pm1}\}\cup\\ &\{\zeta\times\chi_{1,\epsilon},\chi_{1,\epsilon}\times\zeta\mid\epsilon\in\{\pm1\}\}\cup\\ &\{\zeta\times\chi_{-1,\pm1},\chi_{-1,\pm1}\times\zeta\}\cup\\ &\{\zeta\times\zeta\}\end{aligned}$$

We can now calculate that $\Gamma\colon M_2^{1\times C_2} \to M^{C_2\times C_2}$ is surjective. Explicitly:

$$\begin{aligned}
\chi_{1,1}\times\chi_{1,1} &= \Gamma(\psi_4\times\chi_{1,1} - \phi_{-1}\times\chi_{1,1})\\
\chi_{1,1}\times\chi_{1,-1} &= \Gamma(\phi_1\times\chi_{1,-1})\\
\chi_{1,-1}\times\chi_{1,1} &= \Gamma(\phi_{-1}\times\chi_{1,1})\\
\chi_{1,-1}\times\chi_{1,-1} &= \Gamma(\psi_4\times\chi_{1,-1} - \phi_1\times\chi_{1,-1})\\
\chi_{1,1}\times\chi_{-1,\pm1} &= \Gamma(\phi_1\times\chi_{-1,\pm1})\\
\chi_{1,-1}\times\chi_{-1,\pm1} &= \Gamma(\psi_4\times\chi_{-1,\pm1} - \phi_1\times\chi_{-1,\pm1})\\
\chi_{-1,\pm1}\times\chi_{1,\epsilon} &= \Gamma(\psi_2\times\chi_{1,\epsilon})\\
\chi_{-1,\pm1}\times\chi_{-1,\pm1} &= \Gamma(\psi_2\times\chi_{-1,\pm1})\\
\zeta\times\chi_{1,\epsilon} &= \Gamma(\psi_1\times\chi_{1,\epsilon})\\
\zeta\times\chi_{-1,\pm1} &= \Gamma(\psi_1\times\chi_{-1,\pm1})\\
\chi_{1,1}\times\zeta &= \Gamma(\phi_1\times\zeta)\\
\chi_{1,-1}\times\zeta &= \Gamma(\psi_4\times\zeta - \phi_1\times\zeta)\\
\chi_{-1,\pm1}\times\zeta &= \Gamma(\psi_2\times\zeta)\\
\zeta\times\zeta &= \Gamma(\psi_1\times\zeta)
\end{aligned}$$

By the symmetry in the definition of ρ we get by symmetrical calculations that also $\Lambda^1(\Sigma_3\times 1;\Pi_2^\rho(P_3)) \to \Lambda^1(\Sigma_3\times 1; F(P_3))$ is surjective. So we conclude, for this particular ρ, that the left vertical map in diagram 8.1 is surjective.

To finish the argument we note that this vertical map is surjective for the representation $\widetilde{V}\oplus\rho$ for any other fusion invariant representation $\widetilde{V}$. This follows because the basis of M consists of *all* irreducible representations of $\breve{P}_4$. We conclude that $H^2(\mathcal{R}_2(G);\Pi_2^{\widetilde{V}\oplus\rho}) = 0$.

In conclusion

$$R_2(\mathrm{Sp}(1)\times\mathrm{Sp}(1)) \xrightarrow{\cong} R_2(T)^W$$

is an isomorphism.

9. PROOF OF CASE: G_2 AT $p = 2$

9.1. **2-radical subgroups of G_2.** Let $G = G_2$. Following [19] let $z\in G$ be an element of order 2 (all these are conjugate in G). Then $C_G(z)$ is isomorphic to $\mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1)$ where $C_2 = \langle(-1,-1)\rangle$. Then $T = U(1)\times_{C_2} U(1) \subseteq \mathrm{Sp}(1)\times_{C_2}\mathrm{Sp}(1)$ is a maximal torus of G and $\breve{T} = \mathbb{Z}/2^\infty\times_{C_2}\mathbb{Z}/2^\infty \subseteq T$ is a 2-discrete approximation. The weight lattice of $\breve{T}$ is $\{(\alpha,\beta)\in\mathbb{Z}_2\times\mathbb{Z}_2 \mid \alpha+\beta\equiv 0\ (2)\}$. The Weyl group of G acts on the weight lattice by the two generating matrices

$$D = \frac{1}{2}\begin{pmatrix} 1 & -3\\ 1 & 1\end{pmatrix}$$

(rotation) and

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(reflection).

Each conjugacy class of 2-radical subgroups of G has a representative in $\mathrm{Sp}(1) \times_{C_2} \mathrm{Sp}(1)$. The representatives are given in the following list (copied from [19]), where $N = \langle U(1), j\rangle$ and $Q = \langle i, j\rangle$:

P	$W(P)$	$\mathrm{ht}(P)$
$P_1 = N \times_{C_2} N = N_2(T)$	1	0
$P_2 = N \times_{C_2} Q$	$1 \times \Sigma_3$	1
$P_3 = Q \times_{C_2} N$	$\Sigma_3 \times 1$	1
$P_4 = Q \times_{C_2} Q$	$\Sigma_3 \times \Sigma_3$	2
$P_5 = \langle T, (j, j)\rangle$	Σ_3	1
$P_6 = \langle (i, i), (j, j), (1, -1)\rangle$	$\mathrm{GL}_3(\mathbb{F}_2)$	3

The morphisms between the P_i's are generated by the automorphisms and the inclusions (see [19]).

$$\begin{tikzcd}
 & P_1 & & \\
P_2 \arrow[ur, hook] & & P_3 \arrow[ul, hook] & P_5 \arrow[ull, hook] \\
 & P_4 \arrow[ul, hook] \arrow[ur, hook] & & \\
 & & P_6 \arrow[ul, hook] \arrow[uur, hook] &
\end{tikzcd}$$

9.2. **Surjectivity of $\Phi_1\colon \mathrm{Rep}_2(G_2) \to \lim \mathrm{Rep}_2(P)$.** Let ρ be a fusion invariant representation of $\breve{P}_1$. Surjectivity follows if we show that $H^3(\mathcal{R}_2(G); \Pi_2^\rho) = 0$. By the spectral sequence this follows if we show that $\Lambda^3(\mathrm{GL}_3(\mathbb{F}_2); M) = 0$ where $M = \Pi_2^\rho(P_6)$. Since $\mathrm{GL}_3(\mathbb{F}_2)$ is a finite group of Lie type we have that $\Lambda^3(\mathrm{GL}_3(\mathbb{F}_2); M) \cong \mathrm{Hom}_{\mathrm{GL}_3(\mathbb{F}_2)}(\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Z}_2, M)$ (see [13]). Here $\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)}$ is the *Steinberg* $\mathbb{Z}\,\mathrm{GL}_3(\mathbb{F}_2)^{op}$*-module*, a module which is free over $\mathbb{Z}$ of rank 8.

Now M is a permutation module on the isomorphism classes of irreducible summands of $\rho|P_6$. We have $\breve{P}_6 \cong C_2^3$ and the irreducible representations can also be identified with the elements of C_2^3.

Assume $\rho|P_6$ contains all irreducible representations of P_6 as summand. Then $M \cong \mathbb{Z}_p^{C_2^3}$ where $\mathrm{GL}_3(\mathbb{F}_2)$ acts on the basis C_2^3 in the canonical way (that is, C_2^3 is identified with the vector space $\mathbb{F}_2^3$).

$\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{F}_2$ belongs to a block with trivial defect group (see [9, remark 67.13]). By standard modular representations theory this implies that $\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Q}_2$ is simple, and that its restriction to a Sylow-2-subgroup

$$S = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

is isomorphic to the regular module $\mathbb{Q}_2 S$. But this is not true for the restriction of M to S: The value on its character on

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is 4 whereas the value of the regular representation's character on this element is 0. Since $\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Q}_2 = \mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Z}_2 \otimes \mathbb{Q}$ is simple this implies that $\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{F}_2)}(\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Q}_2, M \otimes \mathbb{Q}) = 0$. Now by the following diagram

$$\begin{tikzcd}
\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Z}_2 \arrow[r] \arrow[d, hook] & M \arrow[d, hook] \\
\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Q}_2 \arrow[r, "0"] & M \otimes \mathbb{Q}
\end{tikzcd}$$

where the horizontal maps are inclusions, since $\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Z}_2$ and M are free $\mathbb{Z}_2$-modules, we see that also

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{F}_2)}(\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Z}_2, M) = 0$$

If $\rho|P_6$ does not contain all irreducible representations of P_6 as summand then M has rank strictly less that 8, and again, by similar arguments as above, we get

$$\mathrm{Hom}_{\mathrm{GL}_3(\mathbb{F}_2)}(\mathrm{St}_{\mathrm{GL}_3(\mathbb{F}_2)} \otimes \mathbb{Z}_2, M) = 0$$

9.3. Representations of $\breve{P}_1, \cdots, \breve{P}_4$.

The description for $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ above gives the necessary information for this case as well. The irreducible representations of $\breve{P}_1$ are the irreducible representations of $\breve{N} \times \breve{N}$ that factor through C_2. That is: For $\rho_{\alpha,\beta}$ we require that $\alpha \equiv \beta \ (2)$ and for $\theta^i_{\alpha,\epsilon}$ we require that $\alpha \equiv 0 \ (2)$. And for $\breve{P}_4$ there are 17 irreducible representations: The 16 1-dimensional representations $\chi_{\epsilon_1,\epsilon_2} \times \chi_{\epsilon_3,\epsilon_4}$, $\epsilon_i \in \{\pm 1\}$ and the 4-dimensional representation $\zeta \times \zeta$.

9.4. Determining fusion invariance.

Lemma 9.1. *A representation of the discrete 2-normalizer $\breve{P}_1$ is fusion invariant if and only if*

1. *it is invariant under the action of the Weyl group W and*
2. *its character χ satisfies $\chi(x, i) - \chi(x, j) = 0$ and $\chi(i, x) - \chi(j, x) = 0$ for all $x \in N$.*

Proof. The condition on χ comes from being invariant under $W(P_2)$ and $W(P_3)$. See the proof for $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ above. If this condition is satisfied then the representation is also invariant under $W(P_4)$ and $W(P_6)$.

Regarding invariance under $W(P_5)$: In [19] it is shown that $W(P_5) = W/\langle[(j,j)]\rangle$, so invariance at P_5 implies invariance at T (that is invariance under W). So we just have to worry about how $W(P_5)$ acts on (j,j). A representative of an element of $W(P_5)$ maps (j,j) to $(j,j)t$ for some $t \in \breve{T}$. Now similar to the proof of lemma 8.2 one can show that any irreducible representation (with character χ') of $\breve{P}_5$ is either induced from $\breve{T}$ or is trivial on $\breve{T}$. In both cases $\chi'(j,j) = \chi'((j,j)t)$. So invariance under W also implies invariance at P_5. □

The tables for the case $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ used to determine when the condition on χ is satisfied (see above) are the same for this case.

9.5. **Example: The adjoint representation.** The adjoint representation of G is a 14-dimensional representation with character $2 + [x_1^2] + [x_2^2]$. It restricts to the $\breve{P}_1$-representation

$$\tau_{+,-} \oplus \tau_{-,+} \oplus \theta^1_{2,+} \oplus \theta^2_{2,+} \oplus \rho_{1,1} \oplus \rho_{3,1}$$

This can be seen by noting, that this is the only way to make a fusion invariant representation with the given character. The adjoint representation splits as a sum of 2 fusion invariant representations, namely

$$\tau_{-,+} \oplus \theta^1_{2,+} \oplus \rho_{1,1}$$

and

$$\tau_{+,-} \oplus \theta^2_{2,+} \oplus \rho_{3,1}$$

9.6. **Injectivity of** $\mathrm{Gr}(\Phi_1)\colon R_2(G) \to \mathrm{Gr}(\lim \mathrm{Rep}_2(P))$**.** The proof is basically the same as for the case $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Hence we will not repeat all the arguments. Only the fusion invariant $\breve{P}_1$-representation used to stabilize with has to be changed:

Again we define $F = \mathrm{Ran}(M)$, $M = \Pi_2^{\rho}(P_4)$, where Ran is the same right Kan extension as for $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. And again we get a natural transformation of spectral sequence. Then we note that the full subcategory of $\mathcal{R}_2(G)$ with objects $P_1, \ldots, P_4$ is isomorphic to the skeleton $\mathcal{O}$ of $\mathcal{O}(\Sigma_3 \times \Sigma_3)$. Then we note that the restriction $F\colon \mathcal{O} \to \mathbb{Z}_2$-mod is a fixpoint functor and hence acyclic.

Now put

$$\rho' = (\rho_{2,2} \oplus \rho_{2,4} \oplus \rho_{4,2} \oplus \rho_{4,4}) \oplus (\tau_{-,+} \oplus \theta^1_{2,+}) \oplus (\tau_{+,-} \oplus \theta^2_{2,+}) \oplus \rho_{1,1}$$

ρ' is invariant at $P_1, \ldots, P_4$ and P_6 but is not invariant under the action of the Weyl group. We note that $\rho'|\breve{P}_4$ contains all irreducible representations of $\breve{P}_4$, which is exactly what we need to be able to use ρ' for stabilizing.

Let

$$\begin{aligned}\rho \ &= \rho' \oplus \theta^1_{4,+} \oplus \big(\rho_{D\cdot(2,4)} \oplus \rho_{D^2\cdot(2,4)}\big) \oplus \big(\rho_{D\cdot(4,2)} \oplus \rho_{D^2\cdot(4,2)}\big) \oplus \theta^1_{8,+} \\ &\qquad \oplus \rho_{3,1} \oplus 2(\theta^1_{2,+} \oplus \rho_{1,1})\end{aligned}$$

See above for the definition of the matrix D. Then ρ is fusion invariant. One can ignore the extra representations of ρ not in ρ' as adding extra representations does not hurt the argument.

Now one can check that $\Pi^\rho_2(P_2)^{1\times C_2} \to M^{C_2\times C_2}$ is surjective, so that $\Lambda^1(1\times\Sigma_3; \Pi^\rho_2(P_2)) \to \Lambda^1(1\times\Sigma_3; F(P_2))$ is surjective. And by the symmetry in the definition of ρ' also $\Lambda^1(\Sigma_3\times 1; \Pi^\rho_2(P_3)) \to \Lambda^1(\Sigma_3\times 1; F(P_3))$ is surjective.

The rest of the proof of injectivity of $\mathrm{Gr}(\Phi_1)$ is the same as for $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$.

9.7. **Surjectivity of** $\mathrm{Gr}(\Phi_2)\colon \mathrm{Gr}(\lim \mathrm{Rep}_2(P)) \to R_2(T)^W$**.** I write $R_2(T)^W = \mathbb{Z}[x_1, x_2]^W$ similarly to how I wrote characters above. Given a free orbit sum $x_1^{\pm\alpha_1}x_2^{\pm\beta_1} + x_1^{\pm\alpha_2}x_2^{\pm\beta_2} + x_1^{\pm\alpha_3}x_2^{\pm\beta_3}$ then, calculating modulo 4, $(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3)$ equals one of (after possibly changing sign on α_i or β_i and permuting the i's)

$$\begin{aligned}&(0,0),(0,0),(0,0) && (1)\\ &(0,0),(2,2),(2,2) && (2)\\ &(0,2),(1,1),(1,1) && (3)\\ &(2,0),(1,1),(1,1) && (4)\end{aligned}$$

Let (α,β) be a weight with $\alpha,\beta \neq 0$. I will define the following families of $\breve{P}_1$-representations: For $\alpha,\beta \equiv 2\ (4)$:

$$\Psi_{\alpha,\beta} = \rho_{\alpha,\beta} \oplus (\theta^1_{\alpha,-} \oplus \rho_{\alpha/2,\alpha/2}) \oplus (\theta^2_{\beta,-} \oplus \rho_{\beta/2,\beta/2}) \oplus \tau_{-,-}$$

This is almost fusion invariant, except for missing the rest of the orbit of (α,β). The parentheses indicate which subrepresentations make up a Weyl group invariant orbit sum.

For $\alpha,\beta \equiv 0\ (4)$ let

$$\begin{aligned}\Psi_{\alpha,\beta} = \rho_{\alpha,\beta} &\oplus \big(\rho_{\alpha,2} \oplus \rho_{D\cdot(\alpha,2)} \oplus \rho_{D^2\cdot(\alpha,2)}\big)\\ &\oplus \big(\rho_{2,\beta} \oplus \rho_{D\cdot(2,\beta)} \oplus \rho_{D^2\cdot(2,\beta)}\big)\\ &\oplus (\rho_{2,2} \oplus \theta^1_{4,+}) \oplus (\theta^1_{2,+} \oplus \rho_{1,1})\end{aligned}$$

Here D is the matrix defined above. This is almost fusion invariant, except for missing the rest of the orbit of (α,β). In this regard notice that $(\alpha,2)$ and $(2,\beta)$ always generate free orbits.

Now assume that (α, β) generates a free orbit. For $\alpha \equiv 0\ (4)$ and $\beta \equiv 2\ (4)$ let

$$\begin{aligned}\Psi_{\alpha,\beta} = {} & (\rho_{\alpha,\beta} \oplus \rho_{D\cdot(\alpha,\beta)} \oplus \rho_{D^2\cdot(\alpha,\beta)}) \oplus (\rho_{\beta,\beta} \oplus \theta^1_{2\beta,+}) \\ & \oplus (\theta^1_{\alpha,-} \oplus \Psi_{\alpha/2,\alpha/2}) \oplus (\theta^1_{\beta,-} \oplus \rho_{\beta/2,\beta/2}) \oplus (\theta^1_{2,+} \oplus \rho_{1,1})\end{aligned}$$

This is fusion invariant.

For $\alpha \equiv 2\ (4)$ and $\beta \equiv 0\ (4)$ let

$$\begin{aligned}\Psi_{\alpha,\beta} = {} & (\rho_{\alpha,\beta} \oplus \rho_{D\cdot(\alpha,\beta)} \oplus \rho_{D^2\cdot(\alpha,\beta)}) \oplus (\rho_{\alpha,\alpha} \oplus \theta^1_{2\alpha,+}) \\ & \oplus (\theta^2_{\alpha,-} \oplus \rho_{3\alpha/2,\alpha/2}) \oplus (\theta^2_{\beta,-} \oplus \Psi_{3\beta/2,\beta/2}) \oplus (\theta^1_{2,+} \oplus \rho_{1,1})\end{aligned}$$

This is fusion invariant.

Lemma 9.2. $\mathrm{Gr}(\Phi_2)$ *is surjective.*

Proof. We have to show that all orbit sums $[x_1^\alpha x_2^\beta]$ in $R_2(T)^W$ are hit by $\mathrm{Gr}(\Phi_2)$.

(1) All non-free orbit sums are hit. This follows by lemma 5.6.

(2) Assume $\nu(\alpha) = 2$ and $\nu(\beta) = 1$ (here ν is the valuation of the 2-adic number). The orbit sum $[x_1^\alpha x_2^\beta]$ is hit since all the other orbit sums in $\Psi_{\alpha,\beta}$ are hit (check the definition of $\Psi_{\alpha,\beta}$ and $\Psi_{\alpha/2,\alpha/2}$). Similarly for the case $\nu(\alpha) = 1$ and $\nu(\beta) = 2$.

(3) Assume $\nu(\alpha) > 2$ and $\nu(\beta) = 1$. Here we use that by induction we can assume that the orbit sums $[x_1^{\alpha/2} x_2^2]$ and $[x_1^2 x_2^{\alpha/2}]$ are hit. Similarly for $\nu(\alpha) = 1$ and $\nu(\beta) > 2$. So all orbit sums of type (3) or (4) above are hit.

(4) Assume $[x_1^\alpha x_2^\beta]$ is a free orbit sum of type (1) or type (2) above. Then $\Psi_{\alpha,\beta} \oplus \Psi_{D\cdot(\alpha,\beta)} \oplus \Psi_{D^2\cdot(\alpha,\beta)}$ only consists of the orbit sum $[x_1^\alpha x_2^\beta]$ plus orbit sums of type (3) and (4) plus non-free orbit sums. So also the orbit sum $[x_1^\alpha x_2^\beta]$ is hit.

□

We conclude that

$$R_2(G_2) \xrightarrow{\cong} R_2(T)^W$$

is an isomorphism.

10. Proof of case: Sp(2) at $p = 2$

Let $G = \mathrm{Sp}(2)$. We have $R_2(T) = \mathbb{Z}[x_1, x_2]$ and the Weyl group, the dihedral group of order 8, acts by transposing x_1 and x_2 and by changing the sign of the exponent on x_1 and on x_2. Now by lemma 5.6 the map $\Phi\colon R_2(G) \to R_2(T)^W$ hits any non-free orbit sum (that is $[x_1^\alpha]$ or $[(x_1x_2)^\alpha]$). Let $\alpha, \beta \in \mathbb{Z}_2 - \{0\}$ such that $\alpha \neq \pm\beta$. Then

$$[x_1^\alpha] \cdot [x_1^\beta] = [x_1^\alpha x_2^\beta] + [x_1^{\alpha+\beta}] + [x_1^{\alpha-\beta}]$$

showing that Φ also hits any free orbit sum $[x_1^\alpha x_2^\beta]$. So Φ is surjective.

So we just have to show that $\mathrm{Gr}(\Phi_1)\colon R_2(G) \to \mathrm{Gr}(\lim \mathrm{Rep}_2(P))$ is injective. We will use the same general method as we did for $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$.

Representatives for the conjugacy classes of groups in $\mathcal{R}_2(G)$ are given by the following table (see [23]):

P	$W(P)$	$\mathrm{ht}(P)$
$P_1 = N \wr C_2$	1	0
$P_2 = N \times Q$	$1 \times \Sigma_3$	1
$P_3 = Q \wr C_2$	Σ_3	1
$P_4 = \overline{\Gamma}_2$	Σ_3	1
$P_5 = Q \times Q$	$\Sigma_3 \wr C_2$	2
$P_6 = \Gamma_2$	$O_4^-(\mathbb{F}_2)$	2

$$\begin{tikzcd}
 & & N \wr C_2 & & \\
N \times Q \arrow[urr, hook] & & Q \wr C_2 \arrow[u, hook] & & \overline{\Gamma}_2 \arrow[ull, hook] \\
 & Q \times Q \arrow[ul, hook] \arrow[ur, hook] & & \Gamma_2 \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Here

$$\begin{aligned}
N &= \langle U(1), j\rangle \leq \mathrm{Sp}(1)\\
Q &= \langle i, j\rangle \leq \mathrm{Sp}(1)\\
\overline{\Gamma}_2 &= \langle N \cdot I, A, B\rangle\\
\Gamma_2 &= \langle Q \cdot I, A, B\rangle
\end{aligned}$$

where I is the identity matrix and

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\\
B &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

The morphisms between the P_i's are generated by the automorphisms and the inclusions: This follows by the following lemma:

Lemma 10.1. *Let $P_j < P_i$, $P_i, P_j \in \{P_1, \dots, P_6\}$. Let $A \in \mathrm{Sp}(2)$ such that ${}^{A}P_j \leq P_i$. Then there exists $n \in P_i$ such that ${}^{A}P_j = {}^{n}P_j$.*

To show this lemma we need the following lemma:

Lemma 10.2. *Let $K = \mathrm{Sp}(1)$. Let $A \in \mathrm{Sp}(2)$ such that ${}^{A}(Q \times Q) \leq K \wr C_2$. Then ${}^{A}(Q \times Q) \leq K \times K$.*

Proof. Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $X \in Q \times Q$, $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Notice that if X is a square in $Q \times Q$ then ${}^{A}X$ is diagonal. We calculate

$$ {}^{A}X = \begin{pmatrix} ax\overline{a} + by\overline{b} & \cdots \\ \cdots & cx\overline{c} + dy\overline{d} \end{pmatrix} $$

By proof of contradiction assume ${}^{A}X$ is in $\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$. Then all of $a, \dots, d$ must be non-zero. Also $({}^{A}X)^*$ is in $\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$ so we get

$$ a(x + \overline{x})\overline{a} + b(y + \overline{y})\overline{b} = 0 $$

Now assume $x \in \{\pm 1\}$ and $y \in \{\pm i, \pm j, \pm k\}$. Then we see that $a = 0$, a contradiction. Similarly if $y \in \{\pm 1\}$ and $x \in \{\pm i, \pm j, \pm k\}$.

We conclude that ${}^{A}X$ is diagonal if $x \in \{\pm 1\}$ or $y \in \{\pm 1\}$. And if neither of x and y is in $\{\pm 1\}$ then

$$ {}^{A}X = {}^{A}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} {}^{A}\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} $$

is also diagonal. □

Proof of lemma 10.1. Case $Q \times Q < Q \wr C_2$: This follows immediately by lemma 10.2.

Case $Q \times Q < N \wr C_2$. By lemma 10.2 ${}^{A}(Q \times Q) \leq N \times N$. And it is easy to see that if $\phi \colon Q \times Q \hookrightarrow N \times N$ is a monomorphism then $\phi(Q \times Q) \cong H_1 \times H_2$ with $H_i \leq N$, $H_i \cong Q$ by using that $\phi(x, 1)$ commutes with $\phi(1, y)$ for all $x, y \in Q$. Now use lemma 8.1.

Case $Q \times Q < N \times Q$: Here ${}^{A}(Q \times Q) = H \times Q$ with $H \leq N$, $H \cong Q$. Now use lemma 8.1.

Case $N \times Q < N \wr C_2$: Since ${}^{A}(Q \times Q) \leq N \times N$ also ${}^{A}(N \times Q) \leq N \times N$ (since any element of N can be written as a linear combination of elements of Q). So ${}^{A}(N \times Q) = N \times H$ or ${}^{A}(N \times Q) = H \times N$ with $H \leq N$, $H \cong Q$. Now use lemma 8.1.

Case $Q \wr C_2 < N \wr C_2$: Let $\phi \colon Q \wr C_2 \hookrightarrow N \wr C_2$ be a monomorphism. We have $\phi(Q \times Q) \cong H_1 \times H_2$ with $H_i \leq N$, $H_i \cong Q$ like in the case $Q \times Q < N \wr C_2$. Now $\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & n \\ \overline{n} & 0 \end{pmatrix}$ since the image has to be antidiagonal and have order 2. Now by replacing ϕ with ϕ postcomposed with conjugation by $\begin{pmatrix} \overline{n} & 0 \\ 0 & 1 \end{pmatrix}$ we can assume that $\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we must have $H_1 = H_2$ so that $\phi(Q \wr C_2) = H \wr C_2$ for $H \leq N$, $H \cong Q$. Now use lemma 8.1.

Case $\Gamma_2 < Q \wr C_2$: Let $\phi \colon \Gamma_2 \hookrightarrow N \wr C_2$ be a monomorphism. By checking the possible subgroups of $Q \wr C_2$ isomorphic to $D_4 = \langle A, B \rangle$ (first list all elements of order 2 and then find all pairs whose product has order 4) we see that by precomposing ϕ with an automorphism of Γ_2

(look closely at $O_4^-(\mathbb{F}_2)$) we can assume that $\phi(A) = A$ and $\phi(B) = B$. Then for $x \in Q$ we must have

$$\phi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha(x) & 0 \\ 0 & \alpha(x) \end{pmatrix}$$

for $\alpha\colon Q \hookrightarrow N$ a monomorphism: The two diagonal entries of the image have to be equal, since the element commutes with B. Also we cannot have $\phi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & \alpha(x) \\ \alpha(x) & 0 \end{pmatrix}$ since $\phi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} A$ has order 4. Now use lemma 8.1.

Case $\Gamma_2 < N \wr C_2$: Consider $\mathrm{Sp}(2)$ modulo its center that is $\mathrm{Sp}(2)/\{\pm I\} \cong \mathrm{SO}(5)$. A 2-normalizer of the standard maximal torus in $\mathrm{SO}(5)$ is $O(2) \wr C_2 \leq O(4) \leq \mathrm{SO}(5)$ and $\Gamma_2/\{\pm I\} \cong C_2^4$. We will show that all elementary abelian subgroups of rank 4 in $O(2) \wr C_2$ are conjugate via an element of $O(2) \wr C_2$. Then by lifting such an element to $\mathrm{Sp}(2)$ we see that any subgroup of $N \wr C_2$ isomorphic to Γ_2 is conjugate to Γ_2 via an element in $N \wr C_2$. Let $L \leq N \wr C_2$, $L \cong C_2^4$. If $L \leq O(2) \times O(2)$ then we are done, since all elementary abelian subgroups of rank 2 in $O(2)$ are conjugate by an element of $O(2)$. By proof of contradiction assume $L \not\leq O(2) \times O(2)$, say $X \in L$, $X = \begin{pmatrix} 0 & N \\ N^{-1} & 0 \end{pmatrix} \in M_2(O(2))$ (X must have this form since X has order 2). Then by conjugating L by $\begin{pmatrix} N^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ we can assume that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in L$. But since all other elements in L have to commute with this element, we see that L can have rank at most 3 (since we can think of L as lying in $O(2) \times C_2$), a contradiction.

Case $\Gamma_2 < \overline{\Gamma}_2$: As in the case of $\Gamma_2 < N \wr C_2$ we reduce this to a question in $\mathrm{SO}(5)$, namely: Are all elementary abelian subgroups of rank 4 in $\overline{\Gamma}_2/\{\pm I\} \cong O(2) \times C_2^2$ conjugate by an element of $O(2) \times C_2^2$? And the answer to this question is yes.

Case $\overline{\Gamma}_2 < N \wr C_2$: Similar to the case $\Gamma_2 < N \wr C_2$: Here we use that all subgroups of $O(2) \wr C_2$ isomorphic to $O(2) \times C_2^2$ are conjugate by an element of $O(2) \wr C_2$: Let $\phi\colon O(2) \times C_2^2 \hookrightarrow O(2) \wr C_2$ be a monomorphism. By changing ϕ by an automorphism of $O(2) \times C_2^2$ and by conjugation with an element of $O(2) \wr C_2$ we can assume $\phi(N, x, y) = (N, \alpha(N) \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) \leq O(2) \times O(2)$ where $\alpha\colon O(2) \to O(2)$ is a homomorphism. Since $\alpha(N)$ has to commute with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ we see that the image of $\alpha(N)$ has to be a diagonal matrix. Now use that $O(2)$ is generated by its elements of order 2 to see that $\operatorname{Im} \alpha \leq \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\}$. $\square$

10.1. **Representations of $\breve{P}_1, \ldots, \breve{P}_6$.** First we will describe the irreducible representations for each group:

10.1.1. *$\breve{N} \times Q$ and $Q \times Q$.* These have already been described in the section on $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$.

10.1.2. *$\breve{N} \wr C_2$ and $Q \wr C_2$.* Let G be a countable locally finite group (such as a discrete approximation of a p-toral group). Then the irreducible representations of $G \wr C_2$ come in 2 families:

1. Let D_1 be an irreducible representation of G. We define the representations $(D_1 \times D_1)^{\sim}_{\eta} := (D_1 \times D_1)^{\sim} \otimes E_\eta$, $\eta \in \{\pm 1\}$. Here $(D_1 \times D_1)^{\sim}$ is the representation equaling $D_1 \times D_1$ on $G \times G$ and where the generator of C_2 acts by transposing the first and the second D_1. And E_η is a representation of $C_2 = (G \wr C_2)/(G \times G)$: E_1 is the trivial one and E_{-1} is the nontrivial one.
2. Let D_1 and D_2 be non-isomorphic irreducible representations of G. We define the representation $(D_1 \times D_2)\!\uparrow := \mathrm{Ind}_{G \times G}^{G \wr C_2}(D_1 \times D_2)$.

Lemma 10.3. *With notation as above, letting D_1 run through all the irreducible representations in family 1 and letting $\{D_1, D_2\}$ run though all unordered pairs of irreducible representations in family 2 gives all the irreducible representations of $G \wr C_2$ and they are pairwise non-isomorphic.*

Proof. Compare with the proof of 8.2. Any irreducible representation of $G \wr C_2$ is contained in a representation induced from $G \times G$. $(D_1 \times D_2)\!\uparrow$ is irreducible if and only if $D_1 \not\cong D_2$. Also $(D_1 \times D_2)\!\uparrow \cong (D_2 \times D_1)\!\uparrow$. And $(D_1 \times D_1)\!\uparrow \cong (D_1 \times D_1)^{\sim}_{1} \oplus (D_1 \times D_1)^{\sim}_{-1}$. ☐

For $\breve{N} \wr C_2$ we will define

$$\widetilde{\rho}_{\alpha,\beta} := \rho_{\alpha,\beta}\!\uparrow = (\psi_\alpha \times \psi_\beta)\!\uparrow$$

10.1.3. *Γ_2.* Γ_2 is an extraspecial group, so it has 16 1-dimensional representations and 1 irreducible representation of rank 4 (see [26]).

We will denote the 1-dimensional representations by $\chi_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, $\epsilon_i \in \{\pm 1\}$. This denotes the representation of $\Gamma_2/\{\pm I\} \cong C_2^4$ with

$$\begin{aligned} i &\mapsto \epsilon_1 \\ j &\mapsto \epsilon_2 \\ A &\mapsto \epsilon_3 \\ B &\mapsto \epsilon_4 \end{aligned}$$

We will denote the last irreducible representation by ζ (like we denoted the higher dimensional irreducible representation of the extraspecial group Q by ζ). The character χ of ζ satisfies $\chi(I) = 4$, $\chi(-I) = -4$ and $\chi(x) = 0$ for $x \neq \pm I$. We can define ζ by the map

$$\Gamma_2 \hookrightarrow \mathrm{Sp}(2) \hookrightarrow U(4)$$

10.1.4. $\breve{\bar{\Gamma}}_2$. We have the following irreducible representations:

The representation ζ defined as

$$\breve{\bar{\Gamma}}_2 \hookrightarrow \mathrm{Sp}(2) \hookrightarrow U(4)$$

This is irreducible as its restriction to Γ_2 is irreducible.

Let $A = \breve{\bar{\Gamma}}_2/\{\pm I\} \cong N/\{\pm I\} \times C_2^2$. Then we have all the irreducible representations of A. We will denote these by $(D, \epsilon_1, \epsilon_2)$ where D is an irreducible representation of N factoring through $\{\pm I\}$ (that is $D = \phi_\epsilon$ or $D = \psi_\alpha$, $\alpha \equiv 0$ (2)) and $\epsilon_i \in \{\pm 1\}$. By $(D, \epsilon_1, \epsilon_2)$ we mean that $A \mapsto \epsilon_1$ and $B \mapsto \epsilon_2$.

We haven't proven that these are all the irreducible representations of $\breve{\bar{\Gamma}}_2$ though we suspect that this is the case.

10.1.5. *Restricting representations.* In general we determine how a representation restricts to a subgroup by calculating its character, and determining how this character decomposes into irreducible characters of the subgroup. In this section we will notice some facts that will help us later determine how representations restrict.

Let χ be a representation of $G \times G$, where $G = \breve{N}$ or $G = Q$. Then

$$(\chi\uparrow)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \chi\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \chi\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

In particular, if $\chi = \chi_1 \times \chi_2$ we get

$$(\chi\uparrow)|(G \times G) = \chi_1 \times \chi_2 + \chi_2 \times \chi_1$$

This allows us to determine how χ restricts to $\breve{N} \times Q$ or to $Q \times Q$.

Also from the above, when $G = \breve{N}$ we get

$$(\chi\uparrow)|(Q \wr C_2) = (\chi|Q \times Q)\uparrow$$

since both sides are equal on $Q \times Q$ and both are equal to 0 outside $Q \times Q$.

The next tables explain how $\widetilde{\rho}_{\alpha,\beta}$ restricts to $\breve{\bar{\Gamma}}_2$ and to Γ_2

Condition	Restriction to $\breve{\bar{\Gamma}}_2$
$\alpha \not\equiv \beta$ (2)	2ζ
$\alpha \equiv \beta$ (2), $\alpha \neq \pm\beta$	$(\psi_{\alpha+\beta}, \delta, \pm 1) \oplus (\psi_{\alpha-\beta}, \delta, \pm 1) \quad \delta = \begin{cases} 1 & \alpha \equiv 0\ (2) \\ -1 & \alpha \equiv 1\ (2) \end{cases}$
$\alpha = \beta$	$(\psi_{2\alpha}, \delta, \pm 1) \oplus (\phi_{\pm 1}, \delta, \pm 1) \quad \delta = \begin{cases} 1 & \alpha \equiv 0\ (2) \\ -1 & \alpha \equiv 1\ (2) \end{cases}$

Condition	Restriction to Γ_2	
$\alpha \not\equiv \beta$ (2)	2ζ	
$\alpha \equiv \beta \equiv 1$ (2)	$\chi_{\pm 1,\pm 1,-1,\pm 1}$	
$\alpha \equiv \beta \equiv 0$ (2)	$2\chi_{\delta,\pm 1,1,\pm 1}$	$\delta = \begin{cases} 1 & \alpha+\beta \equiv 0 \ (4) \\ -1 & \alpha+\beta \equiv 2 \ (4) \end{cases}$

Remember our summing convention that ±1 means summing over all combinations of 1 and −1: For example $\chi_{\pm 1,\pm 1,-1,\pm 1} = \oplus_{\epsilon_1,\epsilon_2,\epsilon_4 \in \{\pm 1\}} \chi_{\epsilon_1,\epsilon_2,-1,\epsilon_4}$.

10.2. Relating $\mathcal{R}_2(G)$ to orbit categories of finite groups.

10.2.1. *The $Q \times Q$-interval.* Define $\mathcal{R}_{\Sigma_3 \wr C_2}$ to be the subcategory of $\mathcal{R}_2(W(Q \times Q))$ with objects

$$\begin{tikzcd}
 & N_{N \wr C_2}(Q \times Q)/(Q \times Q) & \\
N_{N \times Q}(Q \times Q)/(Q \times Q) \arrow[ur, hook] & & N_{Q \wr C_2}(Q \times Q)/(Q \times Q) \arrow[ul, hook'] \\
 & 1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

that is $\mathcal{R}_{\Sigma_3 \wr C_2}$ equals

$$\begin{tikzcd}
 & C_2 \wr C_2 & \\
C_2 \times 1 \arrow[ur, hook] & & 1 \wr C_2 \arrow[ul, hook'] \\
 & 1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Here $C_2 \leq \Sigma_3$ equals $\langle (2\ 3) \rangle$ that is the generator transposes j and k (up to signs) in Q. Compare with the calculation of $N_N(Q)/Q$ in the section on $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. It is easy to see that $\mathcal{R}_{\Sigma_3 \wr C_2}$ is a skeleton of $\mathcal{R}_2(\Sigma_3 \wr C_2)$.

We claim that $\mathcal{R}_{\Sigma_3 \wr C_2}$ is isomorphic to the full subcategory

$$\begin{tikzcd}
 & N \wr C_2 & \\
N \times Q \arrow[ur, hook] & & Q \wr C_2 \arrow[ul, hook'] \\
 & Q \times Q \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

of $\mathcal{R}_2(\mathrm{Sp}(2))$: First it is easy to see that the Weyl groups of both categories agree. Then use the same calculation as in the section on $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ for the remaining morphism sets to see that these also agree.

10.2.2. *The Γ_2-interval.* We claim that the full subcategory

$$\begin{tikzcd}
 & N \wr C_2 & \\
Q \wr C_2 \arrow[ur, hook] & & \overline{\Gamma}_2 \arrow[ul, hook'] \\
 & \Gamma_2 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

of $\mathcal{R}_2(\mathrm{Sp}(2))$ is isomorphic the the subcategory $\mathcal{R}_{O_4^-(\mathbb{F}_2)}$ of $\mathcal{R}_2(O_4^-(\mathbb{F}_2))$ where $\mathcal{R}_{O_4^-(\mathbb{F}_2)}$ is

$$\begin{tikzcd}
 & N_{N \wr C_2}(\Gamma_2)/\Gamma_2 & \\
N_{Q \wr C_2}(\Gamma_2)/\Gamma_2 \arrow[ur, hook] & & N_{\overline{\Gamma}_2}(\Gamma_2)/\Gamma_2 \arrow[ul, hook'] \\
 & 1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

To show this, as above, it is enough to check that the Weyl groups of the two categories agree.

To understand $\mathcal{R}_{O_4^-(\mathbb{F}_2)}$ category first we will describe $W(\Gamma_2)$: We have

$$W(\Gamma_2) = \mathrm{Out}(\Gamma_2) = \{\phi \in \mathrm{Aut}(\Gamma_2/\{\pm I\}) \mid \phi(x)^2 = x^2\}$$

where the second equality follows because Γ_2 is an extraspecial group. We have $\Gamma_2/\{\pm I\} \cong C_2^4$ and by choosing the basis (i, j, A, B) we can identify $\mathrm{Aut}(\Gamma_2/\{\pm I\})$ with $\mathrm{GL}_4(\mathbb{F}_2)$. Via this identification we get

$$W(\Gamma_2) \cong O_4^-(\mathbb{F}_2) = \{\phi \in \mathrm{GL}_4(\mathbb{F}_2) \mid Q(\phi(x)) = Q(x)\}$$

where Q is the quadratic form

$$Q(x_1, \ldots, x_4) = x_1 + x_2 + x_1x_2 + x_3x_4$$

Now we get

$$N_{\overline{\Gamma}_2}(\Gamma_2)/\Gamma_2 = \langle \left[\frac{1+i}{\sqrt{2}}\right] \rangle = \langle y \rangle \cong C_2$$

where

$$y = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O_4^-(\mathbb{F}_2)$$

And we get

$$\begin{aligned} N_{Q\wr C_2}(\Gamma_2)/\Gamma_2 &= \left\{ \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}\right], \left[\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\right] \right\} \\ &= \{I, x^2, xy, x^3y\} \cong C_2^2 \end{aligned}$$

where

$$x^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O_4^-(\mathbb{F}_2)$$

$$xy = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O_4^-(\mathbb{F}_2)$$

$$x^3y = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in O_4^-(\mathbb{F}_2)$$

Now define $x := (xy)y$. Then

$$N_{N\wr C_2}(\Gamma_2)/\Gamma_2 = \langle x, y\rangle = \langle x, y \mid x^4 = y^2 = 1, yxy = x^3\rangle \cong D_4$$

So we get that $\mathcal{R}_{\mathcal{O}_4^-(\mathbb{F}_2)}$ equals

$$\begin{tikzcd}
& \langle x, y\rangle & \\
\langle x^2, xy\rangle \arrow[ur, hook] & & \langle y\rangle \arrow[ul, hook'] \\
& 1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

Using that $O_4^-(\mathbb{F}_2) \cong \Sigma_5$ is is easy to see that $\mathcal{R}_{O_4^-(\mathbb{F}_2)}$ is a skeleton of $\mathcal{R}_2(O_4^-(\mathbb{F}_2))$.

To check that the Weyl groups of the above two categories agree we will describe $W(\overline{\Gamma})$ and $W(Q \wr C_2)$ as subquotients of $O_4^-(\mathbb{F}_2)$.

$W(\overline{\Gamma})$: In [23] it is shown that $W(\overline{\Gamma}_2) \cong \Sigma_3$ fixes $U(1) \leq \overline{\Gamma}_2$ and acts as $\mathrm{GL}_2(\mathbb{F}_2) \cong \Sigma_3$ on $\overline{\Gamma}_2/N \cong C_2^2$. From this information we can determine $W(\overline{\Gamma}_2)$ as a subquotient of $O_4^-(\mathbb{F}_2)$. The three involutions are given by the 3 matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$W(Q \wr C_2)$: This just corresponds to the subgroup $\mathrm{Out}(Q) \times 1 = \mathrm{GL}_2(\mathbb{F}_2) \times 1 \leq O_4^-(\mathbb{F}_2)$.

We see that the above Weyl groups agree with the Weyl groups of $\mathcal{R}_{O_4^-(\mathbb{F}_2)}$.

10.3. **Fix point functors and spectral sequences.** We want to compare the functor Π_2^ρ to 2 fix point functors. Similar to the section on $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ we define Ran_i to be the right Kan extension of the functor $\mathbb{Z}_2\mathcal{R}_2(G)^{\mathrm{op}} \to \mathbb{Z}_2 W(P_i)^{\mathrm{op}}$, $T \mapsto T(P_i)$, let $M_i = \Pi_2^\rho(P_i)$ and let $F_5 = \mathrm{Ran}_5(M_5)$ and $F_6 = \mathrm{Ran}_6(M_6)$. Then we get a natural transformation $\Pi_2^\rho \to F_5 \oplus F_6$. As a functor $F_5 \colon \mathcal{R}_{\Sigma_3 \wr C_2} \to \mathbb{Z}_2$-mod we have

$$F_5(H) = M_5^H$$

and as a functor $F_6 \colon \mathcal{R}_{O_4^-(\mathbb{F}_2)} \to \mathbb{Z}_2$-mod we have

$$F_6(H) = M_6^H$$

In particular F_5 and F_6 are acyclic. We now get a natural transformation of spectral sequences: As we are only interested in calculating $H^2(\mathcal{R}_2(G); \Pi_2^\rho)$ we will just write the parts of the E_1-pages relevant for doing this:

$$\begin{tikzcd}
 & \begin{matrix} \Lambda^1(1 \times \Sigma_3; M_2) \\ \oplus \\ \Lambda^1(\Sigma_3; M_3) \\ \oplus \\ \Lambda^1(\Sigma_3; M_4) \end{matrix} \arrow[r, "\partial_1"] \arrow[d, "\eta"] & \begin{matrix} \Lambda^2(\Sigma_3 \wr C_2; M_5) \\ \oplus \\ \Lambda^2(O_4^-(\mathbb{F}_2); M_6) \end{matrix} \arrow[d, equal] & \\
\begin{matrix} \Lambda^0(1; F_5(P_1)) \\ \oplus \\ \Lambda^0(1; F_6(P_1)) \end{matrix} \arrow[r, "\widetilde{\partial}_0"] & \begin{matrix} \Lambda^1(1 \times \Sigma_3; F_5(P_2)) \\ \oplus \\ \Lambda^1(\Sigma_3; F_5(P_3)) \\ \oplus \\ \Lambda^1(\Sigma_3; F_6(P_3)) \\ \oplus \\ \Lambda^1(\Sigma_3; F_6(P_4)) \end{matrix} \arrow[r, "\widetilde{\partial}_1"] & \begin{matrix} \Lambda^2(\Sigma_3 \wr C_2; F_5(P_5)) \\ \oplus \\ \Lambda^2(O_4^-(\mathbb{F}_2); F_6(P_6)) \end{matrix} \arrow[r] & 0
\end{tikzcd}$$

Here the lower sequence is exact. From this it follows that ∂_1 is surjective if and only $\mathrm{Im}\, \widetilde{\partial}_0 + \mathrm{Im}\, \eta$ equals the whole lower middle module.

10.3.1. *Describing* $\widetilde{\partial}_0$. The following theorem can be applied to the case of $G = \mathrm{Sp}(2)$, $p = 2$, $P = P_2, P_3, P_4$ and $F = F_5, F_6$.

Theorem 10.4. *Let G be a compact Lie group, let p be a prime and let $F \colon \mathcal{R}_p(G)^{op} \to \mathbb{Z}_p$-mod. Fix a height function* ht *on $\mathcal{R}_p(G)$. Let $S \leq G$ be a maximal p-toral subgroup and assume $P < S$ is a p-radical subgroup with* $\mathrm{ht}(P) = 1$*. Assume p divides $|W(P)|$ exactly once. Let $C_p = N_S(P)/P \leq W(P)$ (this is a Sylow-p-subgroup of $W(P)$ since $N_S(P) >$*

P). If we identify $\Lambda^1(W(P);F(P)) \cong F(P)^{N_{W(P)}(C_p)}/F(P)^{W(P)}$ in the natural way then the differential

$$\Lambda^0(1;F(S)) \to \Lambda^1(W(P);F(P))$$

coming from the spectral sequence converging to $H^*(\mathcal{R}_p(G);F)$ is identified with the map

$$\begin{aligned} F(S) &\to F(P)^{N(C_p)}/F(P)^{W(P)} \\ m &\mapsto [F(P \hookrightarrow S)(m)] \end{aligned}$$

Proof. All the results used in the following proof can be found in [13].

For a compact Lie group G and a p-toral subgroup $P \le G$ let $\mathcal{B}_p(G)_{\ge P}$ (respectively $\mathcal{B}_p(G)_{>P}$) denote the poset of p-radical subgroups of G containing (respectively strictly containing) P. Let $\mathcal{B}_p(G) = \mathcal{B}_p(G)_{>1}$.

We have

$$\Lambda^1(W(P);F(P)) \cong H^0(\mathrm{Hom}_{W(P)}(\mathrm{St}_*(W(P)),F(P)))$$

Here $\mathrm{St}_*(W) = \widetilde{C}_*(|\mathcal{B}_p(W)|;\mathbb{Z}_p)$, the reduced normalized chain complex. In our case $\mathcal{B}_p(W(P))$ is just the discrete set of Sylow-p-subgroups of $W(P)$. $W(P)$ acts transitively on this (via conjugation) and the stabilizer of C_p is $N(C_p)$. So $\mathrm{St}_*(W)$ looks like

$$\cdots \to 0 \to \mathbb{Z}_p[W(P)/N(C_p)] \to \mathbb{Z}_p \to 0 \to \cdots$$

where $\mathbb{Z}_p[W(P)/N(C_p)]$ is in degree 0. Now

$$\mathrm{Hom}_{W(P)}(\mathbb{Z}_p[W(P)/N(C_p)],F(P)) \cong F(P)^{N(C_p)}$$

and

$$\mathrm{Hom}_{W(P)}(\mathbb{Z}_p,F(P)) \cong F(P)^{W(P)}$$

so $\mathrm{Hom}_{W(P)}(\mathrm{St}_*(W(P)),F(P))$ is isomorphic to

$$\cdots \leftarrow 0 \leftarrow F(P)^{N(C_p)} \leftarrow F(P)^{W(P)} \leftarrow 0 \leftarrow \cdots$$

giving the isomorphism

$$\Lambda^1(W(P);F(P)) \cong F(P)^{N(C_p)}/F(P)^{W(P)}$$

The map

$$\begin{aligned} \mathcal{B}_p(G)_{>P} &\to \mathcal{B}_p(W(P)) \\ Q &\mapsto N_Q(P)/P \end{aligned}$$

induces an $N_G(P)$-homotopy equivalence $|\mathcal{B}_p(G)_{>P}| \to |\mathcal{B}_p(W(P))|$ which in fact is an isomorphism since both sides are discrete sets. So

$$\Lambda^1(W(P);F(P)) \cong H^0(\mathrm{Hom}_{N(P)}(\widetilde{C}_*(|\mathcal{B}_p(G)_{>P}|;\mathbb{Z}_p),F(P)))$$

We have

$$\Lambda^0(1;F(S)) \cong \mathrm{Hom}(C_0(|\{S\}|;\mathbb{Z}_p),F(S)) \cong F(S)$$

Now the differential $d\colon \Lambda^0(1; F(S)) \to \Lambda^1(W(P); F(P))$ can be described as follows:

Let $\alpha \in \mathrm{Hom}(C_0(|\{S\}|; \mathbb{Z}_p), F(S))$. Extend α to $\widetilde{\alpha} \in \mathrm{Hom}(C_0(|\mathcal{B}_p(G)_{>P}|; \mathbb{Z}_p), F(S))$ via conjugation. Then $d(\alpha)$ corresponds to the map

$$C_0(|\mathcal{B}_p(G)_{>P}|; \mathbb{Z}_p) \overset{\widetilde{\alpha}}{\to} F(S) \xrightarrow{F(P \hookrightarrow S)} F(P)$$

Now going through the above isomorphisms and using that $S \in \mathcal{B}_p(G)_{>P}$ maps to $C_p \in \mathcal{B}_p(W(P))$ the result of the theorem follows. ☐

10.4. **Injectivity of** $\mathrm{Gr}(\Phi_1)\colon R_2(G) \to \mathrm{Gr}(\lim \mathrm{Rep}_2(P))$**.** The following lemma is actually true in the more general setting of p-local compact groups:

Lemma 10.5. *Let G be a compact Lie group and let ρ' be a representation of a discrete approximation $\breve{N}_p(T)$ of a maximal p-toral subgroup. Then there exists a fusion invariant $\breve{N}_p(T)$-representation ρ containing ρ'.*

Proof. Let $\breve{T} = \breve{N}_p(T)_1$ be a discrete approximation of a maximal torus T. In [7] they contruct a $\breve{T}$-representation called ψ. This representation satisfies that if ϕ is any representation of $\breve{T}$ which is invariant under the Weyl group W, then $\rho = \mathrm{Ind}_{\breve{T}}^{\breve{N}_p(T)}(\phi \otimes \psi)$ is fusion invariant. Also ρ contains $\mathrm{Ind}_{\breve{T}}^{\breve{N}_p(T)}(\phi)$ since ψ contains the trivial representation. Now just choose ϕ' such that $\mathrm{Ind}_{\breve{T}}^{\breve{N}_p(T)}(\phi')$ contains ρ' (e.g. choose $\phi' = \rho'|\breve{T}$) and let $\phi = \oplus_{w \in W} w^* \phi'$. ☐

Define

$$\begin{aligned} \rho' =& \widetilde{\rho}_{2,2} \oplus \widetilde{\rho}_{2,4} \oplus \widetilde{\rho}_{4,2} \oplus \widetilde{\rho}_{4,4} \oplus \widetilde{\rho}_{1,2} \oplus \widetilde{\rho}_{1,4} \oplus \widetilde{\rho}_{1,1} \\ & \oplus (\psi_2 \times \phi_1)\uparrow \oplus (\psi_2 \times \phi_{-1})\uparrow \oplus (\psi_4 \times \phi_1)\uparrow \oplus (\psi_4 \times \phi_{-1})\uparrow \end{aligned}$$

By the previous lemma we can choose a fusion invariant representation ρ containing ρ'. We will show that ρ can be used a a stabilizing representation.

First we see that a basis of M_5 consists of all irreducible representations of $Q \times Q$ and a basis of M_6 consists of all irreducible representations of Γ_2. Thus if we show that $H^2(\mathcal{R}_2(G); \Pi_2^{\rho}) = 0$ (which we will) then also $H^2(\mathcal{R}_2(G); \Pi_2^{(\rho \oplus \widetilde{V})}) = 0$ for any fusion invariant representation $\widetilde{V}$. This implies that $\mathrm{Gr}(\Phi_1)$ is injective (lemma 8.4).

10.4.1. *Surjectivity of* $\Gamma\colon \Lambda^1(1\times\Sigma_3;M_2)\to\Lambda^1(1\times\Sigma_3;F_5(P_2))$. A basis for $\Lambda^1(1\times\Sigma_3;F_5(P_2))\cong M^{C_2\times C_2}/M^{C_2\times\Sigma_3}$ is

$$\begin{aligned}\chi_{1,\epsilon}\times\chi_{1,-1}&=-\chi_{1,\epsilon}\times\chi_{-1,\pm1} &\qquad \epsilon\in\{\pm1\}\\ \chi_{-1,\pm1}\times\chi_{1,-1}&=-\chi_{-1,\pm1}\times\chi_{-1,\pm1}\\ \zeta\times\chi_{1,-1}&=-\zeta\times\chi_{-1,\pm1}\end{aligned}$$

$M_2^{C_2}$ (of which $\Lambda^1(1\times\Sigma_3;M_2)$ is a quotient) contains the elements $\psi_2\times\chi_{1,-1}$, $\psi_1\times\chi_{1,-1}$ and $\phi_\epsilon\times\chi_{1,-1}$, $\epsilon\in\{\pm1\}$. Then

$$\begin{aligned}\chi_{1,\epsilon}\times\chi_{1,-1}&=\Gamma(\phi_\epsilon\times\chi_{1,-1})\\ \chi_{-1,\pm1}\times\chi_{1,-1}&=\Gamma(\psi_2\times\chi_{1,-1})\\ \zeta\times\chi_{1,-1}&=\Gamma(\psi_1\times\chi_{1,-1})\end{aligned}$$

showing that Γ is surjective.

10.4.2. *Surjectivity of* $\Gamma\colon \Lambda^1(\Sigma_3;M_4)\to\Lambda^1(\Sigma_3;F_6(P_4))$. For $N=M_4$ or $N=F_6(P_4)$ we can write $\Lambda^1(\Sigma_3;N)\cong N^{\langle z\rangle}/N^{\Sigma_3}$ where $\langle z\rangle$ is any choice of Sylow-2-subgroup in $W(\overline{\Gamma}_2)$. So let us choose

$$z=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

We will then show that $M_4^{\langle z\rangle}\to F_6(P_4)^{\langle z\rangle}$ is surjective. A basis for $F_6(P_4)^{\langle z\rangle}=M_6^{\langle y,z\rangle}$ is

$$\begin{aligned}&\{\chi_{1,\epsilon_2,\epsilon_3,\epsilon_3}\mid\epsilon_i\in\{\pm1\}\}\cup\\ &\{\chi_{1,\epsilon_2,1,-1}\oplus\chi_{1,\epsilon_2,-1,1}\mid\epsilon_2\in\{\pm1\}\}\cup\\ &\{\chi_{-1,\pm1,\epsilon_3,\epsilon_3}\mid\epsilon_3\in\{\pm1\}\}\cup\\ &\{\chi_{-1,\pm1,1,-1}\oplus\chi_{-1,\pm1,-1,1}\}\end{aligned}$$

$M_4^{\langle z\rangle}$ contains the elements $(\phi_\epsilon,\epsilon_3,\epsilon_3)$, $(\phi_\epsilon,1,-1)\oplus(\phi_\epsilon,-1,1)$, $(\psi_2,\epsilon_3,\epsilon_3)$ and $(\psi_2,1,-1)\oplus(\psi_2,-1,1)$ with $\epsilon,\epsilon_3\in\{\pm1\}$ and we get

$$\begin{aligned}\chi_{1,\epsilon_2,\epsilon_3,\epsilon_3}&=\Gamma(\phi_{\epsilon_2},\epsilon_3,\epsilon_3)\\ \chi_{1,\epsilon_2,1,-1}\oplus\chi_{1,\epsilon_2,-1,1}&=\Gamma((\phi_{\epsilon_2},1,-1)\oplus(\phi_{\epsilon_2},-1,1))\\ \chi_{-1,\pm1,\epsilon_3,\epsilon_3}&=\Gamma(\psi_2,\epsilon_3,\epsilon_3)\\ \chi_{-1,\pm1,1,-1}\oplus\chi_{-1,\pm1,-1,1}&=\Gamma((\psi_2,1,-1)\oplus(\psi_2,-1,1))\end{aligned}$$

So Γ is surjective.

At this point we are reduced to showing that

$$\Lambda^0(1;F_5(P_1))\oplus\Lambda^0(1;F_6(P_1))\oplus\Lambda^1(\Sigma_3;M_3)\to\Lambda^1(\Sigma_3;F_5(P_3))\oplus\Lambda^1(\Sigma_3;F_6(P_3))$$

is surjective.

10.4.3. *Surjectivity of* $\Gamma\colon \Lambda^0(1; F_6(P_1)) \to \Lambda^1(\Sigma_3; F_6(P_3))$. We have $\Lambda^0(1; F_6(P_1)) = F_6(P_1) = M_6^{\langle x,y\rangle}$ and we have that $\Lambda^1(\Sigma_3; F_6(P_3))$ is a quotient of $M_6^{\langle x^2,xy,C_2\rangle}$ where $C_2 = N_{P_1}(P_3)/P_3 = \langle y\rangle$. So $\langle x^2, xy, C_2\rangle = \langle x^2, xy, y\rangle = \langle x, y\rangle$. So Γ is surjective.

10.4.4. *Surjectivity of* $\Gamma\colon \Lambda^0(1; F_5(P_1)) \to \Lambda^1(\Sigma_3; F_5(P_3))$. We have $\Lambda^0(1; F_5(P_1)) = F_5(P_1) = M_5^{C_2\wr C_2}$ and

$$\Lambda^1(\Sigma_3; F_5(P_3)) = F_5(P_3)^{C_2}/F_5(P_3)^{\Sigma_3} = M_3^{\langle C_2, 1\wr C_2\rangle}/M_3^{\langle \Sigma_3, 1\wr C_2\rangle}$$

We calculate that $M_5^{\langle C_2,1\wr C_2\rangle}/M_5^{C_2\wr C_2}$ has a basis consisting of 1 element, namely

$$\chi_{-1,1}\times\chi_{-1,1}\oplus\chi_{-1,-1}\times\chi_{-1,-1} = -(\chi_{-1,1}\times\chi_{-1,-1}\oplus\chi_{-1,-1}\times\chi_{-1,1})$$

And since $M_5^{\langle\Sigma_3,1\wr C_2\rangle}$ contains the element $(\chi_{1,-1}\oplus\chi_{1,-1})\oplus(\chi_{-1,1}\times\chi_{-1,1}\oplus\chi_{-1,-1}\times\chi_{-1,-1})$ we see that Γ is surjective.

10.4.5. *Conclusion.* All in all we have shown that ∂_1 in diagram 10.3 is surjective. It turned out that we didn't even need the group $\Lambda^1(\Sigma_3; M_3)$ for showing this. Hence $H^2(\mathcal{R}_2(G); \Pi_2^\rho) = 0$ and hence $\mathrm{Gr}(\Phi_1)$ is injective. We conclude that

$$R_2(\mathrm{Sp}(2)) \xrightarrow{\cong} R_2(T)^W$$

is an isomorphism.

11. Proof of case: G_2 at $p = 3$

Let $G = G_2$. Each conjugacy class of 2-radical subgroups of G has a representative in $\mathrm{SU}(3) \le G$ (here $\mathrm{SU}(3)$ is a centralizer in G, see [19]). The representatives are given in the following list (copied from [19])

P	$W(P)$	$\mathrm{ht}(P)$
$N_3 = \langle T, B\rangle$	$C_2\times C_2$	0
$\Gamma = \langle A, B\rangle$	$\mathrm{GL}_2(\mathbb{F}_3)$	1

Here T is the standard maximal torus in $\mathrm{SU}(3)$,

$$A = \begin{pmatrix} 1 & 0 & 0\\ 0 & \zeta & 0\\ 0 & 0 & \zeta^2\end{pmatrix}, \qquad \zeta = e^{2\pi i/3},$$

and

$$B = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$$

Let $\breve{T} \le T$ be the 3-discrete approximation of T, that is

$$T = \left\{ \begin{pmatrix} t_1 & 0 & 0\\ 0 & t_2 & 0\\ 0 & 0 & t_3\end{pmatrix} \mid t_i \in \mathbb{Z}/3^\infty, t_1t_2t_3 = 1\right\}$$

and let $\breve{N}_3 = \langle \breve{T}, B\rangle$ be a 3-discrete approximation of N_3.

Since $\mathcal{R}_3(G)$ has height 1 there are no obstructions, that is $\mathrm{Gr}(\Phi_1)$ is an isomorphism. So we just have to show that $\mathrm{Gr}(\Phi_2)\colon \mathrm{Gr}(\lim \mathrm{Rep}_3(P)) \to R_3(T)^W$ is surjective.

We have $R_3(T) = \mathbb{Z}[x_1, x_2, x_3]/(x_1^\alpha x_2^\alpha x_3^\alpha, \alpha \in \mathbb{Z}_3)$. The Weyl group of T is the dihedral group of order 12, generated by

$$x_1 \mapsto x_3^{-1}$$
$$x_2 \mapsto x_1^{-1}$$

(rotation) and

$$x_1 \mapsto x_2$$
$$x_2 \mapsto x_1$$

(reflection).

11.1. Representations of $\breve{N}_3$.

There are two types of irreducible $\breve{N}_3$-representations: Three 1-dimensional ones τ_0, τ_1, τ_2 with $\tau_i(\breve{T}) = 1$ and $\tau_i(B) = \zeta^i$. And one 3-dimensional one $\rho_{\alpha_1,\alpha_2,\alpha_3} = \mathrm{Ind}_{\breve{T}}^{\breve{N}_3}(\alpha_1, \alpha_2, \alpha_3)$ for each non-zero weight $(\alpha_1, \alpha_2, \alpha_3)$. $\rho_{\alpha_1,\alpha_2,\alpha_3}$ has Lie character $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} + x_1^{\alpha_3}x_2^{\alpha_1}x_3^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_3}x_3^{\alpha_1}$.

11.2. Fusion invariance.

Lemma 11.1. *Let ρ be a representation of $\breve{N}_3$ with character χ. Then ρ is fusion invariant if and only if*

1. *it is invariant under the action of the Weyl group W,*
2. *it satisfies that $\chi(A) - \chi(B) = 0$ and*
3. *the number of τ_1's in ρ equals the number of τ_2's in ρ.*

Proof. ρ is fusion invariant if and only if it is invariant at N_3 and at Γ. Let χ be the character of ρ.

Regarding N_3: Since $W(N_3) = W/\langle[B]\rangle$ if ρ is invariant at N_3 then it is invariant under W. Assume condition 3 (the condition on the τ's). Then χ takes the same value on all elements in $N_3 - T$. So invariance under W implies invariance at N_3.

Regarding Γ: The conjugacy classes of Γ are the elements of the center $Z(\Gamma) = \langle \zeta \cdot I\rangle$ and the sets $A^iB^jZ(\Gamma)$, $(A, B) \not\equiv (0, 0)$ (3). The Weyl group $W(\Gamma)$ fixes $Z(\Gamma)$ and acts transitively on the remaining conjugacy classes. Assuming ρ is invariant under the Weyl group we have $\chi(A) = \chi(A^2)$. And assuming condition 3 we have $\chi(A^iB) = \chi(B) = \chi(A^jB^2)$ for all i and j. So we just need to require that $\chi(A) = \chi(B)$ to get invariance at Γ. □

The following table gives values of $\chi(A) - \chi(B)$ for different Weyl group invariant $\breve{N}_3$-representations. When specifying a Lie character, we mean the (unique) $\breve{N}_3$-representation with the given character. For example $[x_1^\alpha]$ means $\rho_{\alpha,0,0} \oplus \rho_{-\alpha,0,0}$. In the following table we assume that the Weyl group acts freely on $x_1^\alpha x_2^\beta$. The congruences are modulo 3.

Type	Representation or Lie character	Condition	$\chi(A) - \chi(B)$
1	τ_0		0
2	$\tau_1 \oplus \tau_2$		3
3	$[x_1^\alpha x_2^\beta]$	$\alpha \equiv \beta \equiv 0$	12
4a		$\alpha \not\equiv 0$, $\alpha + \beta \not\equiv 0$	0
4b		$\beta \not\equiv 0$, $\alpha + \beta \not\equiv 0$	0
5		$\alpha \not\equiv 0$, $\alpha + \beta \equiv 0$	-6

11.3. **Surjectivity of** $\mathrm{Gr}(\Phi_2)$.

Lemma 11.2. $\mathrm{Gr}(\Phi_2)\colon \mathrm{Gr}(\lim \mathrm{Rep}_3(P)) \to R_3(T)^W$ *is surjective.*

Proof.

(1) By the argument of unstable Adams operations (section 5.6), if the Weyl group does not act freely on $x_1^\alpha x_2^\beta$ then $[x_1^\alpha x_2^\beta] \in R_3(T)^W$ is hit by $\mathrm{Gr}(\Phi_2)$.
(2) All orbit sums of type 4a and 4b (see above) are hit.
(3) Let ρ have Lie character $[x_1^\alpha x_2^\beta]$ of type 5. Then $\rho \oplus 2(\tau_1 \oplus \tau_2)$ is fusion invariant with Lie character $[x_1^\alpha x_2^\beta] + 2$. Since the Lie character 2 is hit, also $[x_1^\alpha x_2^\beta]$ is hit.
(4) By taking the sum of 2 representations of type 5 with a representation of type 3 we see that all orbit sums of type 3 are hit.

As all elements of $R_3(T)^W$ are dealt with above, we conclude that $\mathrm{Gr}(\Phi_2)$ is surjective. ☐

In conclusion

$$R_3(G_2) \xrightarrow{\cong} R_3(T)^W$$

is an isomorphism.

12. PROOF OF CASE: DI(2)

Let $X = \mathrm{DI}(2)$. This is a 3-compact group of rank 2 with Weyl group the 3-adic reflection group $W = G_{12}$. Up to conjugacy G_{12} equals the 3-adic reflection group $G_{12}' = \langle A, B, S, T\rangle \subseteq \mathrm{GL}_2(\mathbb{Z}_3)$ of order 48 where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} -z^{-1} & z^{-1} \\ z^{-1} & z^{-1} \end{pmatrix}$$

$$S = \begin{pmatrix} w & 1/2 \\ -1/2 & \overline{w} \end{pmatrix}$$

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $z \in \mathbb{Z}_3$ satisfies $z^2 = -2$ and $w = (-1+z)/2$ (see [25, page 201 ff]. The map $G'_{12} \to \mathrm{GL}_2(\mathbb{F}_3)$ reducing modulo 3 is a bijection. And since $G_{12} = (G'_{12})^X$ for some $X \in \mathrm{GL}_2(\mathbb{Z}_3)$ also $G_{12} \to \mathrm{GL}_2(\mathbb{F}_3)$ is a bijection.

X was first constructed in [28] and later reconstructed in [2] together with 3 other exotic p-compact groups. Let us refresh the construction in [2]:

Let $\mathbb{I}$ be the category with two objects 0 and 1 and with

$$\begin{aligned} \mathrm{Mor}(0,0) &= W \\ \mathrm{Mor}(0,1) &= \emptyset \\ \mathrm{Mor}(1,0) &= W/D_6 \\ \mathrm{Mor}(0,0) &= 1 \end{aligned}$$

Here D_6 is the Weyl group of G_2. Define the functor $F' \colon \mathbb{I} \to \mathrm{HoTop}$ by $F'(0) = BT\hat{_3}$ where T is a maximal torus in G_2 and $F'(1) = (BG_2)\hat{_3}$. This functor lifts to a functor $F \colon \mathbb{I} \to \mathrm{Top}$ and we define

$$BX = (\mathrm{hocolim}_{\mathbb{I}} F)\hat{_p}$$

To see that this is the same construction as in [2] notice that $(BG_2)\hat{_3} \simeq (B\,\mathrm{SU}(3)_{hC_2})\hat{_3}$. where C_2 acts on $\mathrm{SU}(3)$ by conjugation.

We have a homomorphism

$$(BG_2)\hat{_3} \simeq F(1) \to X$$

and this is a monomorphism. In [7] it is shown that X has the same centric 3-radical subgroups as $\mathrm{SU}(3)$. So we have the following list of representatives of the conjugacy classes of centric 3-radical subgroups in X.

P	$W(P)$	$\mathrm{ht}(P)$
$N_3 = \langle \breve{T}, B \rangle$	$C_2 \times C_2$	0
$\Gamma = \langle A, B \rangle$	$\mathrm{GL}_2(\mathbb{F}_3)$	1
T	G_{12}	1

Here N_3 and Γ are the same groups as for G_2 at $p = 3$ and T is the maximal torus in G_2. $W(\Gamma) = \mathrm{GL}_2(\mathbb{F}_3)$ since it has to contain the Weyl group of Γ in G_2 (which is also $\mathrm{GL}_2(\mathbb{F}_3)$) and it can't be any bigger since $\mathrm{GL}_2(\mathbb{F}_3) = \mathrm{Out}(\Gamma)$, since Γ is an extra-special group. Regarding this notice that $W(\Gamma) \to \mathrm{Out}(\Gamma)$ is injective since Γ is centric. Regarding $W(N_3)$ notice that it has to contain $W_{G_2}(N_3) = W_{G_2}(T)/\langle [B] \rangle$. In fact it has to equal $W_{G_2}(N_3)$: As abstract groups $G_{12} \cong \mathrm{GL}_2(\mathbb{F}_3)$ and the normalizer of a Sylow-3-subgroup in $\mathrm{GL}_2(\mathbb{F}_3)$ is isomorphic to $D_6 \cong W_{G_2}(T)$ (the normalizer of the Sylow-3-subgroup $\left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}$ is $\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}$).

Again, since $\mathcal{R}(X)$ has height 1 there are no obstructions, that is $\mathrm{Gr}(\Phi_1)$ is an isomorphism. So we just have to show that $\mathrm{Gr}(\Phi_2)\colon \mathrm{Gr}(\lim \mathrm{Rep}_3(P)) \to R_3(T)^W$ is surjective.

12.1. **Fusion invariance.**

Lemma 12.1. *Let ρ be a representation of $\breve{N}_3$ with character χ. Then ρ is fusion invariant if and only if*

1. *it is invariant under the action of the Weyl group G_{12},*
2. *it satisfies that $\chi(A) - \chi(B) = 0$ and*
3. *the number of τ_1's in ρ equals the number of τ_2's in ρ.*

Proof. This is the same proof as for the case of G_2 at the prime 3. □

Let ρ be an $\breve{N}_3$-representation with Lie character $[x_1^\alpha x_2^\beta]$. If $(\alpha,\beta) \neq (0,0)$ then ρ is uniquely determined by this Lie character (namely ρ is a sum of $\rho_{\alpha_1,\alpha_2,\alpha_3}$'s), and $\chi(B) = 0$. First assume G_{12} acts freely on (α,β), then

$$\chi(A) = \sum_{w \in G_{12}} (\alpha,\beta)(w(A))$$

where $(\alpha,\beta)\colon \breve{T} \to U(1)$ is the map $\mathrm{diag}(t_1,t_2,t_3) \mapsto t_1^\alpha t_2^\beta$. Now since $\zeta \in \mathbb{Z}_3 \subseteq U(1)$ and $A = \mathrm{diag}(1,\zeta,\zeta^2)$ actually $w(A)$ only depends on w modulo 3, that is on the image of w in $\mathrm{GL}_2(\mathbb{F}_3)$. Remember that $G_{12} \to \mathrm{GL}_2(\mathbb{F}_3)$ is a bijection.

First assume $(\alpha,\beta) \not\equiv (0,0)$ (3). Then we calculate

$$\begin{aligned}
\chi(A) &= \sum_{w \in G_{12}} (\alpha,\beta)(w(A)) \\
&= \sum_{w \in \mathrm{GL}_2(\mathbb{F}_3)} (\alpha,\beta)(w(A)) \\
&= 48/8 \sum_{(a,b) \in \mathbb{F}_3^2 - \{(0,0)\}} 1^a \zeta^b \\
&= 6(3(1+\zeta+\zeta^2) - 1) \\
&= -6
\end{aligned}$$

This calculation uses that $\mathrm{GL}_2(\mathbb{F}_3)$ acts transitively on the set $\mathbb{F}_3^2 - \{(0,0)\}$ of 8 elements, so that $\{(\alpha,\beta)w \mid w \in G_{12}\}$ modulo 3 equals $\mathbb{F}_3^2 - \{(0,0)\}$.

Second, if $(\alpha,\beta) \equiv (0,0)$ (3) it is clear that $\chi(A) = 48$.

Now consider the general case where G_{12} does not necessarily act freely on (α,β), but still $(\alpha,\beta) \neq (0,0)$. Then the action has an isotropy group, call it I, and we get

$$\begin{aligned}\chi(A) &= \sum_{w\in G_{12}/I} (\alpha,\beta)(w(A)) \\ &= \left(\sum_{w\in G_{12}} (\alpha,\beta)(w(A))\right)/|I|\end{aligned}$$

Lemma 12.2. $|I| \in \{1,2\}$.

Proof. G_{12} is generated by pseudoreflections. Since ± 1 are the only roots of unity in $\mathbb{Z}_3$ any pseudoreflection over $\mathbb{Z}_3$ is determined by its hyperplane. Now (α,β) can lie in at most 1 such hyperplane since we are in dimension 2. If (α,β) lies in a hyperplane of one of the pseudoreflections, then $|I| = 2$, otherwise $|I| = 1$. □

This means that $\chi(A) = \chi(A) - \chi(B) \in \{24, 48, -3, -6\}$.

12.2. **Surjectivity of** $\mathrm{Gr}(\Phi_2)$.

Lemma 12.3. $\mathrm{Gr}(\Phi_2)\colon\ \mathrm{Gr}(\lim \mathrm{Rep}_3(P)) \to R_3(T)^W$ *is surjective.*

Proof. τ_0 is fusion invariant so its Lie character 1 is hit by $\mathrm{Gr}(\Phi_2)$. Now let $[(\alpha,\beta)]$ be a non-trivial Lie character corresponding to the $\breve{N}_3$-representation ρ. Say ρ has character χ. If $\chi(A) = -3$ then $\rho\oplus\tau_1\oplus\tau_2$ is fusion invariant with character $[(\alpha,\beta)]+2$ and since 2 is hit by $\mathrm{Gr}(\Phi_2)$ also $[(\alpha,\beta)]$ is hit. If $\chi(A) = -6$ one uses that $\rho\oplus 2(\tau_1\oplus\tau_2)$ is fusion invariant. If $\chi(A) \in \{24, 48\}$ one adds some orbits whose character on A is negative (using that 3 divides 24 and 48), to get fusion invariance and again one gets that $[(\alpha,\beta)]$ is hit. □

We conclude that

$$R_3(X) \xrightarrow{\cong} R_3(T)^W$$

is an isomorphism.

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