MASTER'S THESIS
COMPACT CLASSICAL LIE GROUPS G FOR WHICH $H^*(BG;\mathbb{F}_2)$ IS POLYNOMIAL

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Abstract. The goal of this thesis is the calculate $H^*(BG;\mathbb{F}_2)$, for G a compact classical Lie group, in cases where this is a polynomial algebra. I calculate $H^*(BG;\mathbb{F}_2)$ for G equal to $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$. I start by introducing the classical Lie groups and giving some of their elementary properties. I then construct the classifying space BG via simplicial spaces and prove that we get a principal G-bundle $G \to EG \to BG$ with EG contractible. The calculations of the thesis are done by using the Serre spectral sequence on this and other fiber bundles, so I introduce this spectral sequence and review its basic properties. I also introduce the Steenrod squares which play a role in the calculations since they commute with the transgressions of the Serre spectral sequence.

(In Danish:) Målet med dette speciale er at udregne $H^*(BG;\mathbb{F}_2)$, for G en kompakt klassisk Lie group, i tilfælde hvor dette er en polynomiel algebra. Jeg udregner $H^*(BG;\mathbb{F}_2)$ for G lig $\mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ og $\mathrm{Spin}(n)$. Jeg starter med at introducere de klassiske Lie grupper og give nogle af deres grundlæggende egenskaber. Jeg fortsætter ved at introducere det klassificerende rum BG via simplicielle rum og vise at vi får et principalt G-bundt $G \to EG \to BG$, hvor EG er kontraktibelt. Udregningerne i dette speciale foretages ved at anvende Serres spektralfølge på dette og andre fiberbundter, så jeg introducerer denne spektralfølge og gennemgår dens grundlæggende egenskaber. Jeg introducerer også Steenrod-kvadraterne, der spiller en rolle i udregningerne, fordi de kommuterer med transgressionerne i Serres spektralfølge.

Date: 22. january 2010.

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Contents

1. Introduction

This section discusses concepts which are only introduced later in the thesis.

Two closely related questions:

(1) For which compact connected Lie groups G is $H^*(BG; \mathbb{F}_2)$ a polynomial algebra.
(2) Which polynomial algebras appear as $H^*(BG; \mathbb{F}_2)$ for some compact connected Lie group G.

Question (1) is the one I have focused on in this thesis. However by [1] theorem 1.4 an answer to question (2) would give an answer to which polynomial algebras in finitely many variables can appear as $H^*(X; \mathbb{F}_2)$ for *any space* X.

We have the following classification of the compact connected Lie groups (see [10]):

Theorem 1.1. *Any compact connected Lie group G is isomorphic to a quotient of the form*

$$(G_1 \times \cdots \times G_n \times U(1)^k)/A$$

where G_i is a compact simply-connected simple Lie group, $k \geq 0$, and A is a finite subgroup of the center of the product.

Furthermore, any compact simply-connected simple Lie group is diffeomorphic to one of $\mathrm{Spin}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ *(the classical ones), or* G_2, F_4, E_6, E_7 *or* E_8 *(the exceptional ones).*

I will not mention the exceptional Lie groups in the rest of my thesis.

There are two ways to try to find an answer to the questions above.

1: Look for G for which $H^*(BG; \mathbb{F}_2)$ is polynomial. However the calculation of $H^*(BG; \mathbb{F}_2)$ quickly becomes complicated – even for simple Lie groups G. See for example [3] which calculates $\mathrm{PSU}(4n+2)$ and $\mathrm{PO}(4n+2)$ and [4] which calculates $\mathrm{PSp}(4n+2)$.

2: Look for an easy way to show that $H^*(BG; \mathbb{F}_2)$ can not be polynomial. Even if this is not possible for all G, it might be possible to show that $H^*(BG; \mathbb{F}_2)$ is not polynomial for large families of G's. Maybe this can be done by relating $H^*(BG; \mathbb{F}_2)$ to $H^*(G; \mathbb{F}_2)$ – [2] for example mentions that if $H^*(BG; \mathbb{F}_2)$ is a polynomial algebra then $H^*(G; \mathbb{F}_2)$ has a simple system of primitive generators. Besides the Serre spectral sequence, there are two other spectral sequences relating G to BG. One converging (if it converges) to $H^*(G; \mathbb{F}_2)$ with $E_2 = \mathrm{Tor}_{H^*(BG;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$, and one converging to $H^*(BG; \mathbb{F}_2)$ with $E_2 = \mathrm{Cotor}_{H^*(G;\mathbb{F}_2)}(\mathbb{F}_2, \mathbb{F}_2)$.

2. Conventions and notation

- The projection maps of a product $X \times Y$ are denoted by $\mathrm{pr}_1 \colon X \times Y \to X$ (or pr_X) and $\mathrm{pr}_2 \colon X \times Y \to Y$ (or pr_Y).

- The interval $[0,1] \subseteq \mathbb{R}$ is written I.
- A *manifold* means a smooth manifold, that is a manifold with a smooth structure.
- By a map between topological spaces (including for example a group action) I mean a *continuous* map. A map between manifolds is not assumed to be smooth unless explicitly written.
- If nothing else is written, a subgroup H of a group G is assumed to act by *right* translation, such that $G/H := \{gH\}$.
- $H^*(X)$ means $H^*(X; R)$ where R is a commutative ring with 1-element.
- A map between graded or bigraded modules means a *degree preserving* map.

3. Classical Lie groups

In this section I will define the *compact classical Lie groups*. I will give basic results regarding these, without giving full details. See [8] chapter 1 for more details.

Definition 3.1. A *topological group* G is a topological space G with maps $\mu\colon G \times G \to G$ (the multiplication) and $\iota\colon G \to G$ (the inverse map) and an element $e \in G$ (the identity), that satisfy the axioms of a group. Often $\mu(g,h)$ is written gh and $\iota(g)$ is written g^{-1}.

Definition 3.2. A *Lie group* G is a topological group, which is also a manifold, such that the maps μ and ι are smooth. A smooth group homomorphism between Lie groups is called a *Lie homomorphism*.

In what follows let F denote either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $d := \dim_{\mathbb{R}} F$. Here $\mathbb{H}$ denotes the *quaternions* – that is, $\mathbb{H}$ is the algebra over $\mathbb{R}$ with basis $1, i, j, k$ and with multiplication determined by 1 being the identity and by $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. We have a conjugation map $\mathbb{H} \to \mathbb{H}$ given by $\overline{a + bi + cj + dk} = a - bi - cj - dk$, and conjugation induces an inner product on $\mathbb{H}$.

Let $M(n,F)$ denote the set of $n \times n$-matrices with entries in F. $M(n,F)$ becomes a manifold by identifying it with $\mathbb{R}^{dn^2}$. The general linear group $\mathrm{GL}(n,F) := \{A \in M(n,F) \mid A \text{ invertible}\}$ is an open subset, and hence a submanifold, of $M(n,F)$. So $\mathrm{GL}(n,F)$ is a Lie group.

Let $\phi\colon \mathrm{GL}(n,F) \to \mathrm{GL}(n,F)$ be given by $\phi(A) = AA^*$, where A^* is the conjugate transpose of A. Define $\mathrm{U}(n,F) := \phi^{-1}(I)$, where I is the identity matrix. Since ϕ is continuous and $(AB)^* = B^*A^*$, it follows that $\mathrm{U}(n,F)$ is a closed subgroup of $\mathrm{GL}(n,F)$. When $F = \mathbb{R}$ or $F = \mathbb{C}$ we can use the determinant to define $\mathrm{SU}(n,F) := \{A \in \mathrm{U}(n,F) \mid \det(A) = 1\}$. This is also a closed subgroup of $\mathrm{GL}(n,F)$. It follows from the the next theorem, that $\mathrm{U}(n,F)$ and $\mathrm{SU}(n,F)$ are Lie groups:

Theorem 3.3. *Let G be an Lie group and let $H \subseteq G$ be a closed subgroup. Then H is a submanifold of G. As a submanifold H becomes a Lie group.*

Proof. Omitted. See [8] chapter 1 theorem 5.13. The result is proved by using the exponential map of G. $\square$

It is easily seen that $U(n, F)$ is a bounded subset of $M(n, F)$, so $\mathrm{U}(n, F)$ and $\mathrm{SU}(n, F)$ are compact.

Let $n \geq 1$. We have a Lie monomorphism $i\colon \mathrm{U}(n-1, F) \hookrightarrow \mathrm{U}(n, F)$ given by

$$i(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

I will suppress the i in my notation, and consider $\mathrm{U}(n-1, F)$ to be a subgroup of $\mathrm{U}(n, F)$. Similarly I will consider $\mathrm{SU}(n-1, F)$ to be a subgroup of $\mathrm{SU}(n, F)$.

Theorem 3.4. *The orbit space $\mathrm{U}(n, F)/\,\mathrm{U}(n-1, F)$ is homeomorphic to the sphere S^{dn-1} (remember $d = \dim_{\mathbb{R}}(F)$). When $n \geq 2$ also $\mathrm{SU}(n, F)/\,\mathrm{SU}(n-1, F)$ is homeomorphic to S^{dn-1}.*

Proof. $\mathrm{U}(n, F)$ acts on $F^n \cong \mathbb{R}^{dn}$ by left multiplication. Since this action preserves the norm on $\mathbb{R}^{dn}$ it restricts to an action on S^{dn-1}. Put $e_n = (0, \dots, 0, 1) \in F^n$ and let $\phi\colon \mathrm{U}(n, F) \to S^{dn-1}$ be given by $\phi(A) = Ae_n$. This map is surjective: Let $v_n \in F^n$, $||v_n|| = 1$. By extending v_n to an orthonormal basis $v_1, \dots, v_n$ we get an element $A = \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \in \mathrm{U}(n, F)$ and $Ae_n = v_n$. Furthermore the isotropy group of e_n is $\mathrm{U}(n-1, F)$, so ϕ induces a continuous bijection $\overline{\phi}\colon \mathrm{U}(n, F)/\,\mathrm{U}(n-1, F) \to S^{dn-1}$. As $\mathrm{U}(n, F)$ is compact and S^{dn-1} is Hausdorff, $\overline{\phi}$ is a homeomorphism.

That also $\mathrm{SU}(n, F)/\,\mathrm{SU}(n-1, F)$ is homeomorphic to S^{dn-1} is shown the same way. However in this case, $\mathrm{SU}(n, F)$ only acts transitively on S^{dn-1} when $n \geq 2$. $\square$

We'll use the notation

$$\begin{aligned}
\mathrm{O}(n) &= \mathrm{U}(n, \mathbb{R}) && \text{The orthogonal group.} \\
\mathrm{SO}(n) &= \mathrm{SU}(n, \mathbb{R}) && \text{The special orthogonal group.} \\
\mathrm{U}(n) &= \mathrm{U}(n, \mathbb{C}) && \text{The unitary group.} \\
\mathrm{SU}(n) &= \mathrm{SU}(n, \mathbb{C}) && \text{The special unitary group.} \\
\mathrm{Sp}(n) &= \mathrm{U}(n, \mathbb{H}) && \text{The symplectic group.}
\end{aligned}$$

4. Fiber bundles and fibrations

Definition 4.1. A *fiber bundle* consists of a map $p\colon E \to B$ of spaces E and $B \neq \emptyset$ and a space $F \neq \emptyset$ satisfying the following: There exists a covering $\bigcup_\alpha U_\alpha = B$ and homeomorphisms $h_\alpha \colon U_\alpha \times F \to p^{-1}(U_\alpha)$ such that $p \circ h_\alpha = \mathrm{pr}_{U_\alpha}$.

F, E and B are called the *fiber*, *total space* and *base space*, respectively. The h_α's are called *local trivializations.*

A fiber bundle is often written in short form as $F \to E \xrightarrow{p} B$.

Some texts have a slightly more complicated definition of a fiber bundle, since they include the notions of structure groups and equivalence classes of sets of local trivializations – see for example [8] pages 55-57.

$\mathrm{pr}_B \colon B \times F \to B$ is called the *trivial fiber bundle.* A fiber bundle can be interpreted as a generalized, or "twisted", product.

A map $f \colon (F \to E \xrightarrow{p} B) \to (F' \to E' \xrightarrow{p'} B')$ of fiber bundles is a pair of maps $f \colon E \to E'$ and $\overline{f} \colon B \to B'$ that satisfies $p'f = \overline{f}p$.

Theorem 4.2. *Let $F \to E \xrightarrow{p} B$ be a fiber bundle with F and B compact. The also E is compact.*

Proof. This can be shown by a simple generalization of the proof in [9] (Theorem 26.7) which shows the result for the trivial bundle.

Let $\mathcal{U}$ be an open covering of E. For each $b \in B$ there exists an open neighborhood $B \in U_b$ such that finitely many sets in $\mathcal{U}$ cover $p^{-1}(U_b)$ (use compactness of F). By compactness of B, finitely many $p^{-1}(U_b)$ cover E. The result follows. $\square$

Fiber bundles have an important property now to be described:

A map $p \colon E \to B$ of spaces is said to have the *homotopy lifting property* for a pair $A \subseteq X$ of spaces if the following lift/extension problem can always be solved: Given a square of maps as below, there exists a map $\widetilde{G}$ such that the resulting two triangles commute:

$$\begin{tikzcd}
X \times 0 \cup A \times I \arrow[r, "G_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "G"] \arrow[ur, dotted, "\widetilde{G}"] & B
\end{tikzcd}$$

Definition 4.3. A *Serre fibration* is a map $p \colon E \to B$ of spaces which has the homotopy lifting property for all CW pairs (X, A). A Serre fibration is often written $F \to E \xrightarrow{p} B$ where $F = p^{-1}(b_0)$ for a chosen base point $b_0 \in B$.

A *relative Serre fibration* is a map $p \colon \widetilde{E} \to B$ of spaces together with a subspace $E \subseteq \widetilde{E}$ such that both p and $p|E \colon E \to B$ are Serre fibrations. A relative Serre fibration is written $(\widetilde{F}, F) \to (\widetilde{E}, E) \xrightarrow{p} B$ where $(\widetilde{F}, F) = (p^{-1}(b), (p|E)^{-1}(b))$.

Theorem 4.4. *A fiber bundle $p \colon E \to B$ is a Serre fibration.*

Proof. See [5] page 379 proposition 4.48. $\square$

A Serre fibration $F \to E \to B$ gives a long exact sequence of homotopy groups

$$\cdots \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \cdots$$

A map of fibrations induces a (chain) map of these long-exact sequences.

Definition 4.5. Let G be a topological group. A *principal G-bundle* is a map of spaces $p\colon E \to B$, where E has a a right G-action, that satisfies the following: There exists an open covering $\bigcup U_\alpha = B$ and local sections $s_\alpha\colon U_\alpha \to p^{-1}(U_\alpha)$ of p (that is $p \circ s_\alpha = 1$), such that the maps $\phi_\alpha\colon U_\alpha \times G \to p^{-1}(U_\alpha)$ given by $\phi_\alpha(u,g) = s_\alpha(u)g$ are homeomorphisms.

Notice that a principal G-bundle $p\colon E \to B$ induces a bijection $E/G \to B$ which is a homeomorphism – locally the inverse is induced by the local sections.

It is clear that a principal G-bundle is a fiber bundle, but many more fiber bundles can be produced from a principal G-bundle:

Theorem 4.6. *Let $p\colon E \to E/G$ be a principal G-bundle and let F be a space with a left G-action. Then $F \to E \times_G F \xrightarrow{q} E/G$ is a fiber bundle. Here $E \times_G F := E \times F/((eg,f) \sim (e,gf))$, $e \in E$, $g \in G$, $f \in F$ and $q(e,f) = [e]$.*

Proof. q is clearly well defined.

A local trivialization $h_\alpha\colon U_\alpha \times F \to q^{-1}(U_\alpha)$ can be given by $h_\alpha(b,f) = [s_\alpha(b), f]$ (here s_α is as in definition 4.5). This has a well-defined inverse $k_\alpha\colon q^{-1}(U_\alpha) \to U_\alpha \times F$ given by $k_\alpha([e,f]) = (b, gf)$ where $(b,g) = \phi_\alpha^{-1}(e)$. $\square$

The following result, which I will not prove, builds upon theorem 3.3. See [8] theorem 5.17.

Theorem 4.7. *Let H be a closed subgroup of a Lie group G. Then G/H can be given a manifold structure such that $p\colon G \to G/H$ and the action $G \times G/H \to G/H$ of G are smooth maps. Furthermore p has a local section around $\overline{e} \in G/H$ – that is there exists an open neighborhood U around $\overline{e}$ and a smooth map $s\colon U \to p^{-1}(U)$ such that $p \circ s = 1$.*

With this result we can prove

Theorem 4.8. *Let H be a closed subgroup of a Lie group G. Then $H \to G \xrightarrow{p} G/H$ is a principal H-bundle.*

Proof. Choose U and $s\colon U \to p^{-1}(U)$ as in theorem 4.7. For any $x \in G$ the set $xU = \{xgH \mid gH \in U\}$ is open and we can define a local section $s_x\colon xU \to p^{-1}(xU)$ by $s_x(gH) = xs(x^{-1}gH)$: we have $s(x^{-1}gH) = x^{-1}gh$ for some $h \in H$, so $p(s_x(gH)) = p(xx^{-1}gh) = gH$. Now $\phi_x\colon U_x \times$

$H \to p^{-1}(U_x)$ from definition 4.5 has $\psi_x\colon p^{-1}(U_x) \to U_x \times H$ given by $\psi_x(g) = (p(g), s_x(p(g))^{-1}g)$ as inverse. So ϕ_x is a homeomorphism. $\square$

A particularly nice kind of fiber bundle is given in the next definition:

Definition 4.9. Let B be a connected and locally path connected space. A *covering* of B is a fiber bundle $F \to E \xrightarrow{p} B$ where E is connected and F is discrete. E is called a *covering space* of B.

This definition of a covering space is stricter than in some texts (compare for example with the definition in [5]).

Lemma 4.10. *Let $F \to E \xrightarrow{p} B$ be a covering of B. Assume B is a manifold. Let $h_\alpha\colon U_\alpha \times F \to B$, $\bigcup U_\alpha = B$ be a choice of trivializations. Then E gets a unique smooth structure by declaring the h_α's to be diffeomorphisms. Furthermore this smooth structure does not depend on the choice of trivializations. With regards to this smooth structure p is smooth.*

Proof. A chart of $U_\alpha \times F$ can be assumed to be given as $\sigma_\alpha^f\colon \mathbb{R}^n \to V_\alpha \times f$ where $\sigma_\alpha^f(x) = (\sigma_\alpha(x), f)$ for $f \in F$ and $\sigma_\alpha(x)\colon \mathbb{R}^n \to V_\alpha \subseteq U_\alpha$ a chart of B. Given two such charts $\sigma_{\alpha_1}^{f_1}$ and $\sigma_{\alpha_2}^{f_2}$ the transition $(h_{\alpha_2} \circ \sigma_{\alpha_2}^{f_2})^{-1} \circ (h_{\alpha_1} \circ \sigma_{\alpha_1}^{f_1})$ is given as the following composition – with the domains and codomains appropriately restricted:

$$\mathbb{R}^n \xrightarrow{\sigma_{\alpha_1}} V_{\alpha_1} \hookrightarrow V_{\alpha_1} \times f_1 \xrightarrow{h_{\alpha_1}} E \xrightarrow{h_{\alpha_2}^{-1}} V_{\alpha_2} \times f_2 \xrightarrow{\mathrm{pr}_2} V_{\alpha_2} \xrightarrow{\sigma_{\alpha_2}^{-1}} \mathbb{R}^n$$

In this composition $\mathrm{pr}_2 \circ h_{\alpha_2}^{-1} \circ h_{\alpha_1}$ is just projection onto the second coordinate. So the transition is just a restriction of the transition $\sigma_{\alpha_2}^{-1} \circ \sigma_{\alpha_1}$ which is smooth. So E gets a smooth atlas. Furthermore the calculation shows that a different choice of trivializations (and different choice of atlas for each $U_\alpha \times F$) gives an equivalent atlas. $\square$

A universal covering $\widetilde{B} \xrightarrow{p} B$ is a covering where $\widetilde{B}$ is simply connected. A universal covering is unique up to an isomorphism of fiber bundles (c.f. [8] theorem 1.12 page 54).

Theorem 4.11. *Let B is connected, locally path connected, and locally semisimply connected (meaning that all $b \in B$ has an open neighborhood U such that $\pi_1(U, b) \hookrightarrow \pi_1(X, b)$ induced by the inclusion is the zero map). Then B has a universal covering space.*

Proof. See [8] theorem 1.13 page 54. $\square$

In particular all Lie groups have universal covering spaces.

5. Covering groups

Theorem 5.1. *Let $p\colon \widetilde{X} \to X$ be a covering of the space X. Let $f\colon (Y, y_0) \to (X, x_0)$ be a map of spaces with Y connected and locally path connected. Let $\widetilde{x}_0 \in p^{-1}(x_0)$. Then a lift $\widetilde{f}\colon (Y, y_0) \to (\widetilde{X}, \widetilde{x}_0)$ of f*

exists if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. *Such a lift is uniquely determined up to the choice of* $\widetilde{x}_0$.

Proof. See [8] chapter 2 lemma 1.7. $\square$

Remark 5.2. If X and Y in 5.1 are manifolds, f is smooth, and $\widetilde{X}$ is given the smooth structure from lemma 4.10, then a lift $\widetilde{f}$ will also be smooth: Let $y \in Y$. Choose a trivialization $h\colon F\colon U \to p^{-1}(U)$ where $f(y) \in U$ and write $h^{-1}(\widetilde{f}(y)) = (z, f(y))$. Choose a path connected open neighborhood V of y with $f(V) \subseteq U$. Then $\widetilde{f}(v) = h(z, f(v))$ for all $v \in V$ – that is, $\widetilde{f}$ is locally a composition of smooth maps.

Let G be a topological group with multiplication $\mu\colon G \times G \to G$, inverse $\iota\colon G \to G$ and unit $e \in G$. Let $p\colon \widetilde{G} \to G$ be a covering. Let $\widetilde{e} \in p^{-1}(e)$. Define for $u, v \in \Omega(G)$ (here $\Omega(G)$ is the space of loops based in e) $u \cdot v$ as $(u \cdot v)(t) = \mu(u(t), v(t))$. Let $u * v$ denote the standard product in $\Omega(G)$, that is, $u * v$ first traverses u and then traverses v. Then $[u \cdot v] = [u][v]$ in $\pi_1(G)$. This follows from the next calculation (where I use a standard trick):

$$u * v = (u \cdot e) * (e \cdot v) = (u * e) \cdot (e * v) \simeq u \cdot v$$

Here e denotes the constant loop and $\simeq$ means path homotopic. Using this result we see that if $[(u, v)] \in (p \times p)_*(\pi_1(\widetilde{G} \times \widetilde{G}, (\widetilde{e}, \widetilde{e})))$, then $\mu_*([(u, v)]) = [u \cdot v] = [u][v] \in p_*(\pi_1(\widetilde{G}, \widetilde{e}))$. So by theorem 5.1 a lift $\widetilde{\mu}$ exists in the following diagram:

$$\begin{tikzcd} (\widetilde{G} \times \widetilde{G}, (\widetilde{e}, \widetilde{e})) \arrow[r, "\widetilde{\mu}"] \arrow[d, "p \times p"'] & (\widetilde{G}, \widetilde{e}) \arrow[d, "p"'] \\ (G \times G, (e, e)) \arrow[r, "\mu"] & (G, e) \end{tikzcd} \tag{5.1}$$

Furthermore a lift $\widetilde{\iota}$ exists in the next diagram:

$$\begin{tikzcd} (\widetilde{G}, \widetilde{e}) \arrow[r, "\widetilde{\iota}"] \arrow[d, "p"'] & (\widetilde{G}, \widetilde{e}) \arrow[d, "p"'] \\ (G, e) \arrow[r, "\iota"] & (G, e) \end{tikzcd}$$

By using the uniqueness part of theorem 5.1, it is now easy to show that $\widetilde{\mu}$ and $\widetilde{\iota}$ give $\widetilde{G}$ the structure of a topological group with unit $\widetilde{e}$. Associativity, for example, follows from the fact that both $\widetilde{\mu} \circ (\widetilde{\mu} \times 1)$ and $\widetilde{\mu} \circ (1 \times \widetilde{\mu})$ are lifts of $\mu \circ (\mu \times 1) \circ (p \times p \times p) = \mu \circ (1 \times \mu) \circ (p \times p \times p)$ mapping $(\widetilde{e}, \widetilde{e}, \widetilde{e})$ to $\widetilde{e}$.

Notice that diagram (5.1) says that p is a group homomorphism. This means that we have given $\widetilde{G}$ the structure of a *covering group* of G:

Definition 5.3. Let $q\colon \widetilde{H} \to H$ be a covering af a topological group H. Then $\widetilde{H}$ is a *covering group* of H if $\widetilde{H}$ is a topological group and q is a homomorphism.

If we had used another lift $\widetilde{e}_2 \in p^{-1}(e)$ to construct $\widetilde{\mu}$ and $\widetilde{\iota}$ we would have constructed a different group $\widetilde{G}$, this time with unit $\widetilde{e}_2$. But this new group is isomorphic to $\widetilde{G}$ with unit $\widetilde{e}$; the isomorphism $\widetilde{p}$ is given by lifting p in the following diagram:

$$\begin{tikzcd} & (\widetilde{G}, \widetilde{e}_2) \arrow[d, "p"] \\ (\widetilde{G}, \widetilde{e}) \arrow[ur, "\widetilde{p}"] \arrow[r, "p"] & (G, e) \end{tikzcd}$$

If G is a manifold, then $\widetilde{G}$ also becomes a manifold in a unique way, as explained in lemma 4.10. Furthermore $\widetilde{\mu}$ and $\widetilde{\iota}$ in the construction above become smooth – cf. remark 5.2. So $\widetilde{G}$ becomes a Lie group, and p becomes a Lie homomorphism.

So the universal covering group of a Lie group is essentially unique.

Theorem 5.4. *Let G be a connected, locally path-connected and locally semisimply connected topological group. Let $\widetilde{G} \xrightarrow{p} G$ be the universal covering. Then* $\operatorname{Ker} p \subseteq Z(G)$ *where $Z(G)$ is the center of G. Furthermore any covering is isomorphic to a covering of the form* $\operatorname{Ker} p/H \to \widetilde{G}/H \to G$.

Proof. See [8] theorem 4.8 page 71. □

6. Classical Lie groups continued

6.1. **Homotopy groups.**

6.1.1. *Symplectic groups.* Clearly $\mathrm{Sp}(1)$ is diffeomorphic to S^3, so $\mathrm{Sp}(1)$ is 2-connected. Assume inductively that $\mathrm{Sp}(n-1)$ is 2-connected. In the fiber bundle $\mathrm{Sp}(n-1) \to \mathrm{Sp}(n) \to \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$ the base, which is homeomorphic to S^{4n-1}, is also 2-connected, so from the long exact sequence of homotopy groups it follows that also $\mathrm{Sp}(n)$ is 2-connected.

6.1.2. *Unitary groups.* We have $SU(1) = 1$. For all $n \geq 1$ we have an injective Lie-homomorphism $r_n\colon \mathrm{Sp}(n) \to U(2n)$ given as follows: Let $X \in \mathrm{Sp}(n)$. We can write $X = A + Bj$ for $A, B \in U(n)$ in a unique way. Then

$$r_n(A) = \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

Let $Y \in \mathrm{SU}(2)$, $Y = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. Then $Y^{-1} = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$, so $Y^* = Y^{-1}$ if and only if $d = \overline{a}$ and $c = -\overline{b}$. It follows that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \right\}$$

and we see that r_2 gives a Lie-isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. In particular $\mathrm{SU}(2)$ is 2-connected. From the long exact sequence of homotopy groups for the fiber bundle $\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$ we see that $\mathrm{SU}(n)$ is 2-connected for all $n \geq 1$.

For all $n \geq 1$ we have a Lie-epimorphism $\phi\colon\ \mathrm{SU}(n) \times \mathbb{R} \to \mathrm{U}(n)$ given by $\phi(A,t) = e^{2\pi i t}A$ with kernel isomorphic to $\mathbb{Z}$. This is a universal covering, so

$$\begin{aligned} \pi_0(\mathrm{U}(n)) &= 0 \\ \pi_1(\mathrm{U}(n)) &= \mathbb{Z} \\ \pi_2(\mathrm{U}(n)) &= 0 \end{aligned}$$

6.1.3. *Orthogonal groups.* $\mathrm{O}(n)$ is not connected since $\mathrm{Im}(\mathrm{O}(n) \xrightarrow{\text{determinant}} \mathbb{R}) = \{\pm 1\}$ is not connected.

We have $\mathrm{SO}(1) = 1$ og $\mathrm{SO}(2) \cong SU(1)$. Regarding $\mathrm{SO}(3)$ we have a Lie-homomorphism $\phi\colon\ \mathrm{Sp}(1) \to \mathrm{GL}(3,\mathbb{R})$ given as follows: We can identify $\mathbb{R}^3$ with the subspace $I = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ of $\mathbb{H}$ and since $hIh^{-1} = I$ for all $h \in \mathbb{H}$ we can define $\phi(h)$ to be the conjugation action $i \mapsto hih^{-1}$. One can now check that $\mathrm{Im}\,\phi = \mathrm{SO}(3)$ and $\mathrm{Ker}\,\phi = \mathbb{Z}_2$, so ϕ induces a universal covering $\mathbb{Z}_2 \to \mathrm{Sp}(1) \to \mathrm{SO}(3)$. It follows that $\pi_0(\mathrm{SO}(3)) = 0$, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$ and $\pi_2(\mathrm{SO}(3)) = 0$. From the long exact sequence in homotopy groups for the fiber bundle $\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}$, $n \geq 3$, we see that

$$\begin{aligned} \pi_0(\mathrm{SO}(n)) &= 0 \\ \pi_1(\mathrm{SO}(n)) &= \mathbb{Z}_2 \\ \pi_2(\mathrm{SO}(n)) &= 0 \end{aligned}$$

Since $\mathbb{Z}_2$ has exactly one non-trivial subgroup, up to isomorphism there exists exactly one non-trivial covering of $\mathrm{SO}(n)$ – the universal one. The universal covering group of $\mathrm{SO}(n)$, $n \geq 3$, is called $\mathrm{Spin}(n)$.

One immediate way the homotopy groups say something about the homology is by the *Hurewicz theorem* (see [5] theorem 4.37) of which the following theorem is a corollary:

Theorem 6.1. *Let X be an n-connected space. Then the reduced homology $\widetilde{H}^i(X) = 0$ for $i \leq n$.*

6.2. **Centers.** In [8] page 72-74 the following centers of classical groups are calculated:

$\mathrm{U}(n)$	$\mathrm{SU}(n)$	$\mathrm{Sp}(n)$	$\mathrm{Spin}(2n+1)$	$\mathrm{Spin}(4n+2)$	$\mathrm{Spin}(4n)$
$\mathrm{U}(1)$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Here the upper row are the Lie groups and the lower row are the groups that the centers are isomorphic to. All the centers are closed subgroups. The quotient of one of the groups above by it's center is called a *projective classical group* and is written by putting a P in front of the group's name.

7. The classifying space

Let $\mathcal{C}$ be a small category. The structure of $\mathcal{C}$ is given by the so-called *stucture maps*:

$$\begin{aligned} i\colon & \operatorname{Ob}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{C}) \\ s,t\colon & \operatorname{Mor}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{C}) \\ m\colon & \{(\beta,\alpha) \in \operatorname{Mor}(\mathcal{C})^2 \mid s(\beta) = t(\alpha)\} \to \operatorname{Mor}(\mathcal{C}) \end{aligned}$$

Here $i(x) = 1_x$, the identity morphism on x, for $\alpha\colon x \to y$ a morphism $s(\alpha) = x$ and $t(\alpha) = y$, and $m(\beta,\alpha) = \beta \circ \alpha$ is the composition in $\mathcal{C}$.

Definition 7.1. Let $\mathcal{C}$ be a small category. If $\operatorname{Ob}(\mathcal{C})$ and $\operatorname{Mor}(\mathcal{C})$ are topological spaces, and if all the structure maps are continuous, then we say that $\mathcal{C}$ is a *topological category.* We denote the category of topological categories by TopCat: The morphisms in TopCat are the functors $F\colon \mathcal{C} \to \mathcal{D}$ which are continuous as maps $\operatorname{Mor}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$ (and thus also as maps $\operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$). The set of functors $\operatorname{Hom}_{\mathrm{TopCat}}(\mathcal{C},\mathcal{D})$ is given a topology by considering it to be a subspace of $\operatorname{Hom}_{\mathrm{Top}}(\operatorname{Mor}(\mathcal{C}),\operatorname{Mor}(\mathcal{D}))$ which has the compact-open topology. A natural transformation $\eta\colon F \Rightarrow G$ is required to be continuous as a map $\operatorname{Ob}(\mathcal{C}) \to \operatorname{Mor}(\mathcal{D})$.

We will consider a group G to be a category as follows: $\operatorname{Ob}(G) = *$ (a one-point set), $\operatorname{Mor}(G) = G$ and $g \circ h = gh$. If G is a topological group, then the category G becomes a topological category.

For $n \in \mathbb{N}_0$, define the ordered set $\underline{n} := \{0,\dots,n\}$. Let Δ denote the category with $\operatorname{Ob}(\Delta) = \{\underline{n} \mid n \in \mathbb{N}_0\}$ and with $\operatorname{Hom}(\underline{n},\underline{m})$ being the order preserving maps from $\underline{n}$ to $\underline{m}$.

Definition 7.2. A *simplicial space* is a functor $\Delta^{\mathrm{op}} \to \mathrm{Top}$. We denote the category of these functors by sTop (the morphisms of this category are natural transformations). A *cosimplicial space* is a functor $\Delta \to Top$. The category of these functors is denoted by cTop.

A partially ordered set P can be considered to be a category with $\operatorname{Ob}(P) = P$ and $\operatorname{Mor}(P) = \{(a,b) \in P^2 \mid a \leq b\}$. By giving $\operatorname{Ob}(P)$ and $\operatorname{Mor}(P)$ the discrete topology P thus becomes an object of TopCat. In this way we can consider Δ to be a full subcategory of TopCat. Let $I\colon \Delta \hookrightarrow \mathrm{TopCat}$ be the inclusion functor.

Definition 7.3. Let $\mathcal{C}$ be a topological category. Then we define the *nerve functor* to be the functor $N\colon \mathrm{TopCat} \to \mathrm{sTop}$ with $N\mathcal{C} := \operatorname{Hom}_{\mathrm{TopCat}}(I(\_),\mathcal{C})$ and $NF := \operatorname{Hom}_{\mathrm{TopCat}}(I(\_),F)$. We will use the notation $N\mathcal{C}_n := N(\mathcal{C})(\underline{n})$. Note that $N\mathcal{C}_n$ can be seen as a subspace of $\operatorname{Mor}(\mathcal{C})^n$; for $n = 0$ this identification is $N\mathcal{C}_0 = \operatorname{Ob}(\mathcal{C})$. We call $N\mathcal{C}$ the *nerve of* $\mathcal{C}$.

Let $\Delta^\bullet$ denote the cosimplicial space defined as follows: $\Delta^\bullet(\underline{n})$ will be written as Δ^n, and

$$\Delta^n := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

Given a morphism $\theta \colon \underline{n} \to \underline{m}$ we let $\theta_* = \Delta^\bullet(\theta)$ be the linear extension of θ, that is

$$\theta_*(t_0, \dots, t_n) = \left(\sum_{i \in \theta^{-1}(0)} t_i, \dots, \sum_{i \in \theta^{-1}(n)} t_i \right)$$

For a singular simplex $\sigma \colon \Delta^n \to X$ I will write $\sigma|[v_r, \dots, v_{r+k}]$ to mean $\sigma \circ \theta_*$ where $\theta \colon \underline{k} \to \underline{n}$ is given by $\theta(i) = r + i$.

Definition 7.4. Let A be a simplicial space. We define the *geometric realization* $|A|$ of A as

$$|A| := (\coprod_{n \geq 0} A_n \times \Delta^n)/((A(\theta)(a), t) \sim (a, \theta_*(t)))$$

where θ spans the morphisms of Δ.

The geometric realization is a functor: For a morphism $f \colon A \to B$ we define $|f|(a, t) = (f(a), t)$. By naturality of f this is well defined.

$|A|$ has a *skeleton filtration*

$$|A|_0 \subseteq \cdots \subseteq |A|_n \subseteq \cdots$$

where $|A|_n$ is the image of $\coprod_{k=0}^{n} A_k \times \Delta^k$ in $|A|$. Notice that $|A| = \operatorname{colim} |A|_n$.

Definition 7.5. Let G be a topological group. We define the *classifying space* BG of G as $BG = |NG|$.

7.1. **Products.** As a category of functors sTop inherits the product from Top: For $A, B \in \mathrm{Ob}(\mathrm{sTop})$ we put $(A \times B)_n = A_n \times B_n$ and $(A \times B)(\underline{n} \xrightarrow{\theta} \underline{m}) = A(\theta) \times B(\theta)$.

Lemma 7.6. *The nerve functor preserves products.*

Proof. This is easy to show directly using the fact that $\mathrm{Hom}_{\mathrm{Top}}(\underline{n}, \mathcal{C} \times \mathcal{D}) \cong \mathrm{Hom}_{\mathrm{Top}}(\underline{n}, \mathcal{C}) \times \mathrm{Hom}_{\mathrm{Top}}(\underline{n}, \mathcal{D})$ as topological spaces, for topological categories $\mathcal{C}$ and $\mathcal{D}$. □

Lemma 7.7. *The geometric realization preserves products.*

Proof. (Sketch.) For $A, B \in \mathrm{sTop}$ it's easy to define a map $\phi \colon |A \times B| \to |A| \times |B|$, namely, for $(a, b) \in (A \times B)_n$ and $t \in \Delta^n$

$$\phi((a, b), t) = ((a, t), (b, t))$$

Finding the inverse to ϕ is harder - it's a question of subdividing products $\Delta^n \times \Delta^m$ of simplices into Δ^{n+m}-simplices in the correct way. □

Given a natural transformation $\eta\colon F \Rightarrow G$, $F, G\colon \mathcal{C} \to \mathcal{D}$ we can define a functor $E\colon \mathcal{C} \times \underline{1} \to \mathcal{D}$ as

$$\begin{aligned} E(a,0) &= F(a) \\ E(a,1) &= G(a) \\ E(a \xrightarrow{\alpha} b, 0 \le 1) &= \eta_b \circ F(\alpha) \qquad\qquad (= G(\alpha) \circ \eta_a) \end{aligned}$$

which is clearly continuous. Since B preserves products E induces a map $B\mathcal{C} \times \Delta^1 \to B\mathcal{D}$, and this is a homotopy from F to G.

Lemma 7.8. *Let $\mathcal{C}$ be a topological category with an initial object i. Define $\eta\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ as $\eta(a) = (i \to a)$ where $i \to a$ is the unique morphism. If η is continuous then $B\mathcal{C}$ is contractible.*

Proof. η is a natural transformation from the constant functor $(a \xrightarrow{\alpha} b) \mapsto (i \xrightarrow{1} i)$ to the identity $1_{\mathcal{C}}$. So η induces a homotopy from a constant map to the identity on $B\mathcal{C}$. $\square$

7.2. **Fiber bundles involving the classifying space.** Let G be a topological group. We define a new topological category $\overline{G}$ as follows:

$$\begin{aligned} \mathrm{Ob}(\overline{G}) &= G \\ \mathrm{Mor}(\overline{G}) &= G \times G \\ s(g,h) &= g \\ t(g,h) &= h \\ 1_g &= (g,g) \\ (h,k) \circ (g,h) &= (g,k) \end{aligned}$$

This is the "under category" $(* \downarrow G)$ under the unique object in G – the morphism (g,h) should be thought of as the commutative triangle

$$\begin{tikzcd} & * \arrow[dl, "g"'] \arrow[dr, "h"] & \\ * \arrow[rr, "hg^{-1}"'] & & * \end{tikzcd}$$

where the horizontal morphism is uniquely determined by g and h. We then have a canonical functor $P\colon \overline{G} \to G$ given by $P(g,h) = hg^{-1}$. For each $g \in G$ we get af functor $R_g\colon \overline{G} \to \overline{G}$ given by $R_g(g_0, g_1) = (g_0 g, g_1 g)$, that is R_g maps

$$\begin{tikzcd} & * \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & \\ * \arrow[rr, "g_1 g_0^{-1}"'] & & * \end{tikzcd}$$

to

$$\begin{tikzcd}
& * \arrow[d, "g"] & \\
& * \arrow[dl, "g_0"'] \arrow[dr, "g_1"] & \\
* \arrow[rr, "g_1 g_0^{-1}"'] & & *
\end{tikzcd}$$

It's clear that $P \circ R_g = P$.

We can identify an element $((h_0, h_1), (h_1, h_2), \dots, (h_{n-1}, h_n)) \in N\overline{G}_n$ with the element $(h_0, P(h_0, h_1), \dots, P(h_{n-1}, h_n)) \in \overline{G}^{n+1}$ – this also works for $n = 0$. Thus we identify $N\overline{G}_n = \overline{G}^{n+1}$. We define $EG := B\overline{G}$.

Lemma 7.9. *EG is contractible.*

Proof. $\overline{G}$ has (e, e) as initial element and η from lemma 7.8 is given by $\eta(g) = (e, g)$ which is clearly continuous. □

The collection of R_g's above induce a right action of G on EG: For $g \in G$ and $x = ((g_0, \dots, g_n), t) \in EG$ we get $xg = ((g_0 g, g_1, \dots, g), t)$. The map $p = B(P)$ is given as

$$p((g_0, \dots, g_n), t) = ((g_1, \dots, g_n), t)$$

Theorem 7.10. *Let G be a Lie group. Then $G \to EG \xrightarrow{p} BG$ is a principal G-bundle.*

Proof. We have

$$\begin{aligned} BG_n - BG_{n-1} &= (G^n - W) \times (\Delta^n - \partial\Delta^n) \\ EG_n - EG_{n-1} &= G \times (G^n - W) \times (\Delta^n - \partial\Delta^n) \end{aligned}$$

where

$$W = (e \times G^{n-1}) \cup (G \times e \times G^{n-2}) \times \cup \dots \cup (G^{n-1} \times e)$$

Now assume that for all $n \geq 1$ there exists a retraction $r_n \colon X \to (W \times \Delta^n) \cup (G^n \times \partial\Delta^n)$ where $X \subseteq G^n \times \Delta^n$ is open. Let $\iota_B \colon G^n \times \Delta^n \to BG_n$ and $\iota_E \colon G^{n+1} \times \Delta^n \to EG_n$ be induced by the inclusions. Then we can construct local trivializations for $G \to EG_n \xrightarrow{p} BG_n$ by induction over n as follows:

Notice that $p^{-1}(BG_n) = EG_n$.

$n = 0$: We have $BG_0 = *$ and $EG_0 = G$, so putting $U = *$ we have $p^{-1}(U) = G$ and can define a local trivialization $h \colon p^{-1}(U) \to U \times G$ as $h(g) = (*, g)$.

Assume $U_{n-1} \subseteq BG_{n-1}$ is an open set with local trivialization $h_{n-1}\colon p^{-1}(U_{n-1}) \to U_{n-1} \times G$. Put

$$\begin{aligned} V_{n-1} &= \iota_B^{-1}(U_{n-1}) \subseteq (W \times \Delta^n) \cup (G^n \times \partial\Delta^n) \\ V_n &= r_n^{-1}(V_{n-1}) \\ U_n &= \iota_B(V_n) \end{aligned}$$

Notice that $BG_{n-1} \cap U_n = U_{n-1}$ and that $\iota_B^{-1}(U_n) = V_n$. It follows that U_n is open in BG_n. Furthermore $p^{-1}(U_n) = \iota_E(G \times V_n)$. Define a retraction $\widetilde{r}\colon p^{-1}(U_n) \to p^{-1}(U_{n-1})$ as

$$\widetilde{r}(x) = \iota_E(g, r_n(v)) \qquad\qquad \text{for } \iota_E(g,v) = x$$

This is well-defined: If $x \in EG_n - EG_{n-1}$ then there is only 1 choice of (g, v). Otherwise we have $x \in p^{-1}(U_{n-1})$ which implies $(g, v) \in G \times V_{n-1}$ and hence $r_n(v) = v$. We can now define a local trivialization $h_n\colon p^{-1}(U_n) \to U_n \times G$ as

$$h_n(x) = (p(x), (\mathrm{pr}_G \circ h_{n-1} \circ \widetilde{r})(x))$$

It's easy to see that if h_{n-1} commutes with the action of G (which we may assume inductively) then so does h_n. h_n is clearly a continuous bijection (again assuming h_{n-1} is). Now if $h_n([e, v]) = ([v], g)$ then $h_n([g^{-1}, v]) = ([v], e)$, so the section of definition 4.5 corresponding to h_n is given by

$$s_n([v]) := h^{-1}([v], e) = [(\mathrm{pr}_G\, h_{n-1}([e, r_n(v)]))^{-1}, v]$$

We see that $s_n \circ \iota_B$ is continuous and since $s_n|BG_{n-1}$ may be assumed inductively to be continuous we conclude that s_n is continuous.

Extending all the local trivializations of $G \to EG_{n-1} \to BG_{n-1}$ in this way gives local trivializations of $G \to EG_n \to BG_n$ which cover the whole of BG_{n-1}. The rest of BG_n can be covered by the local section $s\colon (BG_n - BG_{n-1}) \to (EG_n - EG_{n-1}) = G \times (BG_n - BG_{n-1})$ given by the inclusion.

The retraction $r_n\colon X \to (W \times \Delta^n) \cup (G^n \times \partial\Delta^n)$ can be constructed as follows: Let $\sigma\colon \mathbb{R}^n \to U \subseteq G$ be a chart with $\sigma(0) = e$, and identify U with $\mathbb{R}^n$ via σ. Let $B_\epsilon = \{u \in U \mid ||u|| < \epsilon\}$ and define $R\colon G \times I \to G$ as the homotopy which (1) moves the elements of B_1 towards e at constant speed (and then becomes constant when they get there), (2) stretches $(B_2 - B_1)$ linearly towards the center, and (3) is the identity outside B_2. Then $R|B_1$ is a deformation retract onto e. R can first be combined with itself to give a deformations retract of $(B_1 \times G^{n-1}) \cup \cdots \cup (G^{n-1} \times B_1)$ onto $(e \times G^{n-1}) \cup \cdots \cup (G^{n-1} \times e)$. This deformation retract can second be combined with a deformation retract of $(\Delta^n - (1/n, \ldots, 1/n))$ onto $\partial\Delta$ to give r_n. $\square$

$G \to EG \to BG$ is called the *universal G-bundle*. According to [6] (page 128) $G \to EG \to BG$ is a G-bundle for any topological group

G for which $e \hookrightarrow G$ is a cofibration (i.e. for which $G \times 0 \cup e \times I$ is a retract of $G \times I$).

From the long-exact sequence associated to $G \to EG \to BG$ we see that $\pi_{n+1}(BG) \cong \pi_n(G)$

Corollary 7.11. *Let G be a Lie group with closed subgroup H. Then BH is weakly equivalent to EG/H and there exists a fiber bundle $G/H \to EG/H \to BG$.*

Proof. We have $BH \cong EH/H$ and we have a fiber bundle $H \to EG \to EG/H$. Now the following commutative diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d, "1_H"] & EH \arrow[r] \arrow[d, "f"] & EH/H \arrow[d, "\overline{f}"] \\
H \arrow[r] & EG \arrow[r] & EG/H
\end{tikzcd}$$

where f is induced by the inclusion $H \hookrightarrow G$, gives a map of long exact sequences of homotopy groups. Since $EH \simeq EG \simeq *$ we see that $\pi_k(\overline{f})$ is an isomorphism for all k – also for $k = 1$.

Using theorem 4.6 we produce the bundle $G/H \to EG \times_G (G/H) \to BG$ from the universal bundle. And it is clear that the inclusion $EG \hookrightarrow EG \times (G/H)$ induces a homeomorphism $EG/H \cong EG \times_G (G/H)$. $\square$

Theorem 7.12. *Let G be a Lie group with closed subgroup H. Then we have a fiber bundle $BH \to BG \to B(G/H)$.*

8. The Serre spectral sequence

Let $(\widetilde{F}, F) \to (\widetilde{E}, E) \xrightarrow{p} B$ be a relative Serre fibration. For each $b \in B$ let $\widetilde{F}_b := p^{-1}(b)$ and $F_b := (p|E)^{-1}(b)$ be the fibers over b with $i_b \colon F_b \hookrightarrow E$ and $\widetilde{i}_b \colon \widetilde{F}_b \hookrightarrow E$ the inclusions. Assume $(\widetilde{F}_b, F_b)$ is a CW pair for all b. Given a path γ in B from a to b we can construct a map $L_\gamma \colon (\widetilde{F}_a, F_a) \to (\widetilde{F}_b, F_b)$ as follows: (See [5] page 405 for details in the case of an absolute (Hurewich) fibration.)

First choose a lift Γ in the next diagram

$$\begin{tikzcd}
F_a \times 0 \arrow[r, "i_a"] \arrow[d, hook] & E \arrow[d, "p"] \\
F_a \times I \arrow[r, "\gamma \circ \mathrm{pr}_I"'] \arrow[ur, dotted, "\Gamma"] & B
\end{tikzcd}$$

Then choose a lift $\widetilde{\Gamma}$ in

$$\begin{tikzcd}
\widetilde{F}_a \times 0 \cup F_a \times I \arrow[r, "\widetilde{i}_a \cup \Gamma"] \arrow[d, hook] & \widetilde{E} \arrow[d, "p"] \\
\widetilde{F}_a \times I \arrow[r, "\gamma \circ \mathrm{pr}_I"'] \arrow[ur, dotted, "\Gamma"] & B
\end{tikzcd}$$

and let $L_\gamma(f) = \widetilde{\Gamma}(f,1)$.

If γ_1 and γ_2 are path-homotopic then L_{γ_1} and L_{γ_2} are homotopic. Furthermore given composable paths γ and δ we have $L_{\gamma*\delta} \simeq L_\delta \circ L_\gamma$. This implies that the L_γ's induce a so called *local coefficient system (LCS) of algebras* $\underline{H^*(\widetilde{F},F)} := \{H^*(\widetilde{F}_b, F_b) \mid b \in B\}$ on B (see [8] page 103 for a definition). The assumption that $(\widetilde{F}_b, F_b)$ is a CW pair is actually not needed as $\underline{H^*(\widetilde{F},F)}$ alternatively can be constructed by lifting from each singular simplex instead of from the whole fiber (this is for example done in [8] (page 107)).

In this thesis I'm only interested in the case where this system of local coefficients is trivial:

Definition 8.1. The local coefficient system $\underline{H^*(\widetilde{F},F)}$ is *trivial* if $(L_\gamma)^* = 1_{H^*(\widetilde{F},F)}$ for all $[\gamma] \in \pi_1(B)$.

Lemma 8.2. *(1) If B is simply connected, then $\underline{H^*(\widetilde{F},F)}$ is trivial. (2) If $H^k(\widetilde{F},F;\mathbb{F}_2) \in \{0, \mathbb{F}_2\}$ for all k then $\underline{H^*(\widetilde{F},F;\mathbb{F}_2)}$ is trivial.*

Proof. (1) This follows from the fact that $L_c \simeq 1_{\widetilde{F}}$ for c the constant loop. (2) This is obvious since in this case $1_{H^*(\widetilde{F},F;\mathbb{F}_2)}$ is the only automorphism of $H^*(\widetilde{F},F;\mathbb{F}_2)$. □

Theorem 8.3 (The Serre spectral sequence)**.** *Let $(\widetilde{F},F) \to (\widetilde{E},E) \xrightarrow{p} B$ be a relative Serre fibration with trivial LCS $\underline{H^*(\widetilde{F},F)}$. Then there exists a cohomological, first quadrant spectral sequence converging to $H^*(\widetilde{E},E)$ with page $E_2^{p,q} = H^p(B; H^q(\widetilde{F},F))$. The spectral sequence is natural.*

The case where $F = E = \emptyset$ gives the absolute version of the Serre spectral sequence.

Explanation of expressions in the above theorem:

- *Spectral sequence:* A sequence $\{(E_r, d_r) \mid r \geq 1\}$ of bigraded (i.e. $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$) R-modules (called *pages*) with differentials $d_r \colon E_r \to E_r$ such that $E_{r+1} = H(E_r, d_r)$. *First quadrant* means that $E_r^{p,q} = 0$ when $p < 0$ or $q < 0$ and *cohomological* means that d_r has bidegree $(r, -(r-1))$ (i.e. $d_r(E_r^{p,q}) \subseteq E_r^{p+r,q-(r-1)}$).
- *...converging to $H^*(\widetilde{E},E)$...:* Since the spectral sequence is a first quadrant one, each sequence $\{E_r^{p,q} \mid r \geq 1\}$ eventually becomes constant (up to isomorphism), and we denote this constant value by $E_\infty^{p,q}$. This gives a bigraded module $E_\infty = \bigoplus_{p,q\in\mathbb{Z}} E_\infty^{p,q}$ – the E_∞ page. The expression means that $H = H^*(\widetilde{E},E)$ has a filtration (a sequence of submodules)

$$\cdots \subseteq F^k H \subseteq \cdots \subseteq F^1 H \subseteq F^0 H = H$$

This gives a bigraded module $E_0(H)$ with $E_0^{p,q}(H) = (F^pH \cap H^{p+q}(\widetilde{E}, E))/(F^{p+1}H \cap H^{p+q}(\widetilde{E}, E))$ and the theorem says that we have an isomorphism $E_0(H) \cong E_\infty$.
- *The spectral sequence is natural:* Consider the next diagram

$$\begin{tikzcd}
(\widetilde{F}, F) \arrow[r] \arrow[d, "f_0"] & (\widetilde{E}, E) \arrow[r] \arrow[d, "f"] & B \arrow[d, "\overline{f}"] \\
(\widetilde{F}', F') \arrow[r] & (\widetilde{E}', E') \arrow[r] & B'
\end{tikzcd}$$

where the rows are Serre fibrations with trivial LCSs giving spectral sequences $\{E_n\}$ and $\{E'_n\}$ respectively. The expression means that for any such diagram there exists a sequence of maps $\{f_n : E_n \to E'_n \mid n \geq 2\}$ satisfying the following:
  - f_2 is induced by f_0 and $\overline{f}$.
  - f_n commutes with the differentials and $H(f_n) = f_{n+1}$
  - f induces a map $E_0(H(\widetilde{E}, E)) \to E_0(H(\widetilde{E}', E'))$ such that the following diagram commutes

$$\begin{tikzcd}
E_0(H(\widetilde{E}, E)) \arrow[r, phantom, "\cong"] \arrow[d, "f_*"] & E_\infty \arrow[d, "f_\infty"] \\
E_0(H(\widetilde{E}', E')) \arrow[r, phantom, "\cong"] & E'_\infty
\end{tikzcd}$$

Here $f_\infty : E_\infty \to E'_\infty$ is the map arising from the sequence $\{f_n\}$.

Also, a Serre spectral sequence $\{(E_r, d_r)\}$ is said to collapse if $d_r = 0$ for all $r \geq 2$.

Proof. (Sketch.) The spectral sequence is constructed by filtering the chain complex $C^*(\widetilde{E}, E)$: Assuming B is a CW complex (otherwise use a CW approximation of B) let $(\widetilde{E}^p, E^p) = (p^{-1}(B^p), (p|E)^{-1}(B^p))$ where B^p is the p-skeleton of B. Then we put $F^pC^n(\widetilde{E}, E) = \mathrm{Ker}(C^n(\widetilde{E}, E) \xrightarrow{i_p^*} C^n(\widetilde{E}^p, E^p))$ where $i_p : \widetilde{E}^p \hookrightarrow \widetilde{E}$ is the inclusion, and thus we get a filtration

$$\cdots \subseteq F^pC^n(\widetilde{E}, E) \subseteq \cdots \subseteq F^0C^n(\widetilde{E}, E) \subseteq C^n(\widetilde{E}, E)$$

Any such filtration gives a spectral sequences, and because this particular filtration is what is called *bounded* the spectral sequence converges to the homology of the total chain complex $C^*(\widetilde{E}, E)$ – see [7] theorem 2.6.

The central part of the proof is to now identify the E_2 page of this spectral sequence with $H^*(B; H^*(\widetilde{F}, F))$ – see for example [7] section 5.3. □

Let $F \to E \xrightarrow{p} B$ be a Serre fibration. Consider the following diagram of homomorphisms

$$H^{r-1}(F) \xrightarrow{\partial} H^r(E,F) \xleftarrow{p^*} H^r(B,b_0) \xrightarrow{j^*} H^r(B)$$

where ∂ comes from the long exact sequence in cohomology for the pair (E,F), b_0 is the base point of B (remember $F = p^{-1}(b_0)$) and j^* is induced by the identity on B. From this diagram we get an induced map

$$\tau_r \colon \partial^{-1}(\operatorname{Im} p^*) \xrightarrow{\partial} \operatorname{Im} p^* \xrightarrow{(p^*)^{-1}} H^r(B,b_0)/\operatorname{Ker} p^* \xrightarrow{j^*} H^r(B)/j^*(\operatorname{Ker} p^*)$$

τ_r is called the *transgression*. An element $f \in H^{r-1}(F)$ is called *transgressive* if $f \in \partial^{-1}(\operatorname{Im} p^*)$. The transgression is important because of the next theorem:

Theorem 8.4. *Assume the fibration $F \to E \xrightarrow{p} B$ has trivial LCS. In the Serre spectral sequence for the fibration we have $E_r^{0,r-1} = \partial^{-1}(\operatorname{Im} p^*)$ (or strictly speaking only canonically isomorphic), $E_r^{r,0} = H^r(B)/j^*(\operatorname{Ker} p^*)$ and $d_r \colon E_r^{0,r-1} \to E_r^{r,0}$ equals the transgression τ_r.*

Lemma 8.5. *Let $(\widetilde{F}, F) \xrightarrow{i} (\widetilde{E}, E) \xrightarrow{p} B$ be a Serre fibration with trivial LCS. Assume $H^0(B) = H^0(\widetilde{F}, F) = R$. Then $p^* \colon H^k(B) \to H^k(\widetilde{E}, E)$ equals the composition*

$$H^k(B) \cong H^k(B; H^0(\widetilde{F}, F) = E_2^{k,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{k,0} \hookrightarrow H^k(\widetilde{E}, E)$$

and $i^* \colon H^k(\widetilde{E}, E) \to H^k(\widetilde{F}, F)$ equals the composition

$$H^k(\widetilde{E}, E) \twoheadrightarrow E_\infty^{0,k} \hookrightarrow \cdots \hookrightarrow E_2^{0,k} = H^0(B; H^k(\widetilde{F}, F)) \cong H^k(\widetilde{F}, F)$$

Proof. This follows by using the naturality of the Serre spectral sequences on

$$\begin{tikzcd} (\widetilde{F}, F) \arrow[r] \arrow[d] & (\widetilde{E}, E) \arrow[r] \arrow[d, "p"] & B \arrow[d, "1"] \\ * \arrow[r] & B \arrow[r, "1"] & B \end{tikzcd}$$

and

$$\begin{tikzcd} (\widetilde{F}, F) \arrow[r, "1"] \arrow[d, "1"] & (\widetilde{F}, F) \arrow[r] \arrow[d, "i"] & * \arrow[d] \\ (\widetilde{F}, F) \arrow[r] & (\widetilde{E}, E) \arrow[r] & B \end{tikzcd}$$

respectively. □

For spaces $A \subseteq X$ there exists a so called cup product

$$H^*(X,A) \otimes H^*(X) \xrightarrow{\smile} H^*(X,A)$$

induced by the product

$$\phi \smile \psi(\sigma) = \phi(\sigma|[v_0, \ldots, v_{n_1}])\psi(\sigma|[v_{n_1}, \ldots, v_{n_1+n_2}])$$

where $\phi \in C^{n_1}(X, A)$, $\psi \in C^{n_2}(X)$ and $\sigma \in C_{n_1+n_2}$. See [5] chapter 3.2 for basic properties of the cup product.

Let $(\widetilde{F}, F) \to (\widetilde{E}, E) \to B$ be a Serre fibration with trivial LCS. Then the cup products give a product $H^*(B; H^*(\widetilde{F}, F)) \otimes H^*(B; H^*(\widetilde{F})) \to H^*(B; H^*(\widetilde{F}, F))$, namely the product $(\phi \cdot \psi)(\sigma) = \phi(\sigma|[v_0, \dots, v_{n_1}]) \smile \psi(\sigma|[v_{n_1}, \dots, v_{n_1+n_2}])$.

Let $\{E_r\}$ and $\{G_r\}$ be the Serre spectral sequences associated to $(\widetilde{F}, F) \to (\widetilde{E}, E) \to B$ and $\widetilde{F} \to \widetilde{E} \to B$ respectively. We can the define a product $E_2 \otimes G_2 \xrightarrow{\cdot_2} E_2$ by letting $E_2^{p_1,q_1} \otimes G_2^{p_2,q_2} \to E_2^{p_1+p_2,q_1+q_2}$ equal $(-1)^{q_1 p_2}$ times the product on $H^*(B; H^*(\widetilde{F}, F))$.

If we inductively assume we have a product $E_r \otimes G_r \xrightarrow{\cdot_r} E_r$ which commutes with the differentials (i.e. $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^{p_1+q_1} x \cdot d_r(y)$ for $x \in E_r^{p_1,q_1}$) we get a well defined product $E_{r+1} \otimes G_{r+1} \to E_{r+1}$ as the composition $H(E_r) \otimes H(G_r) \xrightarrow{f} H(E_r \otimes G_r) \xrightarrow{H(\cdot_r)} H(E_r)$ where $f([x] \otimes [y]) = [x \otimes y]$.

So we get a sequence of products, and this sequence induces a product $E_\infty \otimes G_\infty \to E_\infty$. We have the following theorem:

Theorem 8.6 (Product on the Serre spectral sequence)**.** *The cup product $H^*(\widetilde{E}, E) \otimes H^*(\widetilde{E}) \to H^*(\widetilde{E}, E)$ induces a well defined product $E_0(H^*(\widetilde{E}, E)) \otimes E_0(H^*(\widetilde{E})) \to E_0(H^*(\widetilde{E}, E))$. The products $E_r \otimes G_r \to E_r$ defined recursively above commute with the differentials and thus exist for all $r = 2, \dots, \infty$. The products agree, in the sense that the following diagram commutes:*

$$\begin{tikzcd}
E_0(H^*(\widetilde{E}, E)) \otimes E_0(H^*(\widetilde{E})) \arrow[r] \arrow[d, "\phi \otimes \psi"] & E_0(H^*(\widetilde{E}, E)) \arrow[d, "\phi"] \\
E_\infty \otimes G_\infty \arrow[r] & E_\infty
\end{tikzcd}$$

Here ϕ and ψ are the isomorphisms from theorem 8.3.

We can define a chain map $\alpha\colon C^*(B) \otimes H^*(\widetilde{F}, F) \to C^n(B; H^*(\widetilde{F}, F))$ by $\alpha_n(\phi \otimes f) = (\sigma \mapsto \phi(\sigma) f)$ and this chain maps commutes with the products, that is $\alpha((\phi_1 \smile \phi_2) \otimes (f_1 \smile f_2)) = \alpha(\phi_1 \otimes f_1) \smile \alpha(\phi_2 \otimes f_2)$. In the case that $H^q := H^q(\widetilde{F}, F)$ is a free module of rank d the map $C^n(B) \otimes H^q \xrightarrow{\alpha} C^n(B; H^q)$ is an isomorphism – it equals the following composition of isomorphisms:

$$\begin{aligned}
C^n(B; R) \otimes H^q \cong C^n(B; R) \otimes R^d &\cong \bigoplus_{i=1}^{d} C^n(B; R) \otimes R \\
&\cong \bigoplus_{i=1}^{d} C^n(B; R) \cong C^n(B; R^d) \cong C^n(B; H^q)
\end{aligned}$$

So assume H^q is a free module of finite rank for all q; then α is a chain isomorphism and induces an isomorphism of algebras

$$H(\alpha)\colon H(C^*(B) \otimes H^*(\widetilde{F}, F)) \to H^*(B; H^q(\widetilde{F}, F))$$

We further more have a product preserving map

$$\beta\colon H(C^*(B)) \otimes H^*(\widetilde{F}, F) \to H(C^*(B) \otimes H^*(\widetilde{F}, F))$$

given by $\beta([x] \otimes f) = [x \otimes f]$, and this map is also an isomorphism: The map $H(C^*(B)) \otimes H^q \xrightarrow{\beta} H(C^*(B) \otimes H^q)$ equals the composition

$$H(C^*(B)) \otimes H^q \cong \bigoplus_{i=1}^{d} H(C^*(B)) \cong H(\bigoplus_{i=1}^{d} C^*(B)) \cong H(C^*(B) \otimes H^q)$$

Using these isomorphisms on the Serre spectral sequence gives the following lemma:

Lemma 8.7. *(Same notation as in theorem 8.6.) Assume $H^q(\widetilde{F}, F)$ is a free module of finite rank for all q. Then the map*

$$\zeta := H(\alpha) \circ \beta\colon H^*(B) \otimes H^*(\widetilde{F}, F) \to E_2$$

is an isomorphism. If also $H^q(\widetilde{F})$ is a free module of finite rank for all q we get a similar isomorphism $H^*(B) \otimes H^*(\widetilde{F}) \cong G_2$ and via these isomorphisms the product $E_2 \otimes G_2 \to E_2$ corresponds to the product $(b_1 \otimes f_1) \cdot (b_2 \otimes f_2) = (-1)^{q_1 p_2}(b_1 \smile b_2) \otimes (f_1 \smile f_2)$ where $f_1 \in H^{q_1}(\widetilde{F}, F)$ and $b_2 \in H^{p_2}(B)$.

I will also be using the following result ([5] theorem 3.16) which I will call *The Künneth isomorphism* for easy reference later.

Theorem 8.8. *Let X and Y be spaces with $H^i(Y)$ a free module of finite rank for all i. Then we have an isomorphism of graded algebras $\times\colon H^*(X) \otimes H^*(Y) \xrightarrow{\sim} H^*(X \times Y)$. Here $\times$, called the* cross product*, is defined as $x \times y := \mathrm{pr}_X^*(x) \smile \mathrm{pr}_Y^*(y)$.*

9. Calculating $H^*(B\,\mathrm{U}(n, F); \mathbb{F}_2)$ and $H^*(B\,\mathrm{SU}(n, F); \mathbb{F}_2)$

9.1. **The Gysin sequence.** The plan is to calculate $H^*(B\,\mathrm{U}(n, F))$ by induction by using the fiber bundle $\mathrm{U}(n, F)/\mathrm{U}(n-1, F) \to B\,\mathrm{U}(n-1, F) \to B\,\mathrm{U}(n, F)$ (cf. corollary 7.11). The fiber of this bundle is a sphere (cf. 3.4), so we need to look at this kind of fiber bundle:

Let $F \to E \xrightarrow{p} B$ be a fiber bundle with $F = S^{n-1}$. Let $\widetilde{E} = (B \sqcup E \times I)/(p(e) \simeq (e, 0)$ be the mapping cylinder of p and let $\pi\colon \widetilde{E} \to B$ be the retraction onto B, that is $\pi(e, t) = p(e)$. Identifying E with $E \times 1 \subseteq \widetilde{E}$ we have $\pi|E = p$. And the fibers of π are homeomorphic to $\widetilde{F} = (F \times I)/(F \times 0) \cong D^n$.

Lemma 9.1. *Assume either B is locally compact Hausdorff or $B = BG$ for G a compact Hausdorff group. Then $\widetilde{F} \to \widetilde{E} \to B$ is a fiber bundle. So $(\widetilde{F}, F) \to (\widetilde{E}, E) \to B$ is a relative Serre fibration.*

Proof. Let $h_\alpha\colon U_\alpha \times F \to p^{-1}(U_\alpha)$, $B = \bigcup U_\alpha$, be the local trivializations, with inverses $\phi_\alpha = h_\alpha^{-1}$. Notice that $\pi^{-1}(U_\alpha) = \{[e,t] \in \widetilde{E} \mid e \in p^{-1}(U_\alpha)\}$.

Define extensions $\widetilde{h}_\alpha\colon U_\alpha \times \widetilde{F} \to \pi^{-1}(U_\alpha)$ and $\widetilde{\phi}_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \widetilde{F}$ as

$$\begin{aligned} \widetilde{h}_\alpha(u,[f,t]) &= [h_\alpha(u,f),t] \\ \widetilde{\phi}_\alpha[e,t] &= (u,[f,t]) \qquad\qquad \text{where } (u,f) = \phi_\alpha(e) \end{aligned}$$

These maps are easily checked to be well-defined maps which are inverse to each other. It is straigtforward to check that $\widetilde{\phi}_\alpha$ is continuous.

$\widetilde{h}_\alpha$ is continuous: This follows by showing that the canonical bijection

$$\theta\colon (U_\alpha \times F \times I)/\sim\; \to U_\alpha \times \widetilde{F}$$

is a homeomorphism, since $\widetilde{h}_\alpha = \zeta \circ \theta^{-1}$ where $\zeta\colon (U_\alpha \times F \times I)/\sim\; \to \pi^{-1}(U_\alpha)$ is induced by $h_\alpha \times 1_I$.

In the case B is locally compact Hausdorff U_α is also locally compact Hausdorff (cf. [9] corollary 29.3) and so the map $U_\alpha \times F \times I \to U_\alpha \times \widetilde{F}$ inducing θ is a quotient map (cf. [5] A.17). Hence θ is a homeomorphism.

Now consider the case $B = BG$. In this case the skeleton BG_n is compact Hausdorff for all n so arguing as above we get that θ_n in the following commutative diagram is a homeomorphism

$$\begin{tikzcd} ((U_\alpha \cap BG_n) \times F \times I)/\sim \arrow[r, "\theta_n"] \arrow[d, "\iota_1"] & (U_\alpha \cap BG_n) \times \widetilde{F} \arrow[d, "\iota_2"] \\ (U_\alpha \times F \times I)/\sim \arrow[r, "\theta"] & U_\alpha \times \widetilde{F} \end{tikzcd}$$

All the maps in the diagram are the canonical ones. Let $V \subseteq (U_\alpha \times F \times I)/\sim$ be open. Then $\theta_n(\iota_n^{-1}(V))$ is open for all n. Now since $BG = \operatorname{colim} BG_n$ also $U_\alpha = \operatorname{colim}(U_\alpha \cap BG_n)$. It's a basic category theoretic fact, that functors that have right adjoints commute with the colimit, and $(\_) \times \widetilde{F}$ has the right adjoint $\operatorname{Hom}_{\mathrm{Top}}(I, \_)$ since $\widetilde{F}$ is locally compact (cf. [5] prop A.14). So $U_\alpha \times \widetilde{F} = \operatorname{colim}((U_\alpha \cap BG_n) \times \widetilde{F})$ which means that $\theta(V)$ is open.

Thus the $\widetilde{h}_\alpha$'s are local trivializations.

The second part of the lemma is true since fiber bundles are Serre fibrations (theorem 4.4). $\square$

Let $F \to E \xrightarrow{p} B$ be a fiber bundle with $F = S^{n-1}$ and B one of the cases of lemma 9.1. Then the lemma gives a relative fibration $(D^n, S^{n-1}) \to (\widetilde{E}, E) \xrightarrow{\pi} B$. Assume the local coefficient system $\underline{H^*(D^n, S^{n-1})}$ is trivial (such as if $R = \mathbb{F}_2$, cf. lemma 8.2). Since $H^k(D^n, S^{n-1}) = R$ for $k = n$ and 0 otherwise we get the following commutative diagram

$$\begin{tikzcd}
H^*(B) \otimes H^*(\widetilde{F}, F) \otimes H^*(B) \otimes H^*(\widetilde{F}) \arrow[r, "\cdot"] \arrow[d, "\cong"', "\zeta\otimes\xi"] & H^*(B) \otimes H^*(\widetilde{F}, F) \arrow[d, "\cong"', "\zeta"] \\
H^*(B; H^*(\widetilde{F}, F)) \otimes H^*(B; H^*(\widetilde{F})) \arrow[r, "\cdot"] \arrow[d, "\cong"', "\phi\otimes\psi"] & H^*(B; H^*(\widetilde{F}, F)) \arrow[d, "\cong"', "\phi"] \\
H^*(\widetilde{E}, E) \otimes H^*(\widetilde{E}) \arrow[r, "\smile"] & H^*(\widetilde{E}, E)
\end{tikzcd}$$

Here the top two horizontal maps are the products from lemma 8.7 and theorem 8.6. All the vertical maps are isomorphisms; the top vertical maps come from lemma 8.7 and the bottom vertical maps come from the Serre spectral sequences which collapse. We have an isomorphism $H^*(B) \cong H^*(B) \otimes H^*(\widetilde{F}, F)$ given by $x \mapsto (-1)^{n\cdot\deg(x)} x \otimes f$ where f is a generator of $H^n(\widetilde{F}, F)$. Since $(-1)^{n\cdot\deg(x)} x \otimes f = (1 \otimes f) \cdot (x \otimes 1)$, by the above diagram we get the isomorphism

$$\theta\colon H^*(B) \simeq H^{*+n}(\widetilde{E}, E)$$

given by $\theta(x) = \phi\zeta(1 \otimes f) \smile \psi\xi(x \otimes 1)$. Since $\psi\xi(x \otimes 1) = \pi^*(x)$, and putting $U := \phi\zeta(1 \otimes f)$ we have

$$\theta(x) = U \smile \pi^*(x)$$

Combining the isomorphisms θ and π^* with the long exact sequence in cohomology for the pair $(\widetilde{E}, E)$ results in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i+n-1}(E) \arrow[r, "\partial"] & H^{i+n}(\widetilde{E}, E) \arrow[r, "j^*"] & H^{i+n}(\widetilde{E}) \arrow[r, "i^*"] & H^{i+n}(E) \arrow[r] & \cdots \\
& & H^i(B) \arrow[u, "\cong", "\theta"'] & H^{i+n}(B) \arrow[u, "\cong", "\pi^*"'] & &
\end{tikzcd}$$

which reduces to the so called *Gysin sequence* for the fiber bundle $S^{n-1} \to E \xrightarrow{p} B$:

$$\cdots \to H^{i+n-1}(E) \xrightarrow{\lambda} H^i(B) \xrightarrow{\chi\smile(\_)} H^{i+n}(B) \xrightarrow{p^*} H^{i+n}(E) \to \cdots$$

where $\chi \in H^n(B)$ equals $(\pi^*)^{-1} j^*(U)$. χ is called the *Euler class* for the fiber bundle.

9.2. **The calculations.** Now consider the fiber bundle $S^{dn-1} \to B\,\mathrm{U}(n-1, F) \to B\,\mathrm{U}(n, F)$, $d = \dim_{\mathbb{R}}(F)$, mentioned in the previous subsection.

Lemma 9.2. *Assume $F = \mathbb{R}$. Then the Euler class $\chi \in H^n(B\,\mathrm{O}(n); \mathbb{F}_2)$ associated to the above fiber bundle is non-zero.*

Proof. See the proof of theorem 3.19 in [8] (page 124).

The proof involves constructing a sphere bundle $S^{n-1} \to E \to B$ (by use of the bundle $O(1) \to S^1 \to \mathbb{R}P^1 = S^1$) whose Euler class χ_n is non-zero. Then since $S^{n-1} \to B\,\mathrm{U}(n-1,F) \to B\,\mathrm{U}(n,F)$ is produced (using lemma 4.6) from the universal $\mathrm{O}(n)$-bundle there exists a map $f\colon B \to B\,\mathrm{U}(n,F)$ such that $S^{n-1} \to E \to B$ is the pullback bundle via f of $S^{n-1} \to B\,\mathrm{U}(n-1,F) \to B\,\mathrm{U}(n,F)$. By a property of the Euler class we then have $f^*(\chi) = \chi_n \neq 0$ and hence $\chi \neq 0$. □

Theorem 9.3. *Assume $R = \mathbb{F}_2$. Assume $H^*(B\,\mathrm{U}(n-1,F)) = \mathbb{F}_2[x_1,\dots,x_{n-1}]$ with $\deg(x_i) = di$. Then $H^*(B\,\mathrm{U}(n,F)) = \mathbb{F}_2[y_1,\dots,y_{n-1},\chi]$ with $\deg(y_i) = di$ and $\chi \in H^{dn}(B\,\mathrm{U}(n,F))$ the Euler class of the above fiber bundle.*

Proof. The Gysin sequence for the fiber bundle looks like

$$\cdots \to H^i(B\,\mathrm{U}(n,F)) \xrightarrow{\chi\smile(\_)} H^{i+dn}(B\,\mathrm{U}(n,F)) \xrightarrow{p^*} H^{i+dn}(B\,\mathrm{U}(n-1,F)) \xrightarrow{\lambda} H^{i+1}(B\,\mathrm{U}(n,F)) \to \cdots$$

We have $\lambda(x_i) \in H^{d(i-n)+1}$ so $\lambda(x_i) = 0$ for all $i \leq n-2$ and also $\lambda(x_{n-1}) = 0$ when $d \geq 2$. In the case $d = 1$, i.e. $F = \mathbb{R}$, lemma 9.2 plus the fact that $H^0(B\,\mathrm{U}(0,F)) = \mathbb{F}_2$ gives that $\chi \smile (\_)\colon H^0(B\,\mathrm{U}(0,F)) \to H^{dn}(B\,\mathrm{U}(0,F))$ is injective. Hence $\lambda(x_{n-1}) \in \mathrm{Ker}(\chi \smile (\_))$ equals 0.

So in all cases all the generators $x_1,\dots,x_{n-1}$ are in the image of p^* and so p^* is surjective and hence $\lambda = 0$. This means that the Gysin sequence gives a short exact sequence

$$0 \to H^*(B\,\mathrm{U}(n,F)) \xrightarrow{\chi\smile(\_)} H^{*+dn}(B\,\mathrm{U}(n,F)) \xrightarrow{p^*} H^{*+dn}(B\,\mathrm{U}(n-1,F)) \to 0$$

This sequence splits; let s be a section for p^*. Put $y_i := s(x_i)$. Then the result follows. □

From the above theorem we conclude:

$$H^*(B\,\mathrm{O}(n);\mathbb{F}_2) = \mathbb{F}_2[x_1,\dots,x_n]$$
$$H^*(B\,\mathrm{U}(n);\mathbb{F}_2) = \mathbb{F}_2[x_2,\dots,x_{2n}]$$
$$H^*(B\,\mathrm{Sp}(n);\mathbb{F}_2) = \mathbb{F}_2[x_4,\dots,x_{4n}]$$

where $\deg x_i = i$. (Actually I have shown the last 2 equalities for more general coefficient rings.)

Theorem 9.4. *Assume $R = \mathbb{F}_2$. Let $F = \mathbb{R}$ or $F = \mathbb{C}$ and let $d = \dim_{\mathbb{R}} F$. Then p in the fiber bundle*

$$\mathrm{U}(n,F)/\,\mathrm{SU}(n,F) \to B\,\mathrm{SU}(n,F) \xrightarrow{p} B\,\mathrm{U}(n,F)$$

induces an epimorphism

$$p^*\colon H^*(B\,\mathrm{U}(n,F)) \to H^*(B\,\mathrm{SU}(n,F))$$

with kernel the ideal (x_d).

Proof. By the short exact sequence $0 \to \mathrm{SU}(n,F) \to \mathrm{U}(n,F) \xrightarrow{\text{determinant}} U(1,F) \to 0$ we see that the fiber of the fiber bundle is isomorphic to

$\mathrm{U}(n,1) \cong S^{d-1}$. Thus the bundle gives a Gysin sequence

$$\cdots \xrightarrow{\lambda} H^i(B\,\mathrm{U}(n,F)) \xrightarrow{\chi\smile(\_)} H^{i+d}(B\,\mathrm{U}(n,F)) \xrightarrow{p^*} H^{i+d}(B\,\mathrm{SU}(n-1,F)) \xrightarrow{\lambda} H^{i+1}(B\,\mathrm{U}(n,F)) \to \cdots$$

By the Hurewicz theorem $H^d(B\,\mathrm{SU}(n,F)) = 0$ so $\chi \smile (\_)\colon H^0(B\,\mathrm{U}(n,F)) \to H^d(B\,\mathrm{U}(n,F))$ is surjective, and hence $\chi = x_d$. It follows that $\chi \smile (\_)\colon H^*(B\,\mathrm{U}(n,F)) \to H^{*+2}(B\,\mathrm{U}(n,F))$ is injective, so that $\lambda = 0$ and p^* is surjective. Clearly $\operatorname{Ker} p^* = (x_d)$. □

10. Calculating $H^*(B\,\mathrm{Spin}(n); \mathbb{F}_2)$

In this section $R = \mathbb{F}_2$

The stategy in this section is to calculate $H^*(B\,\mathrm{Spin}(n))$ by using the Serre spectral sequence on the fiber bundle $B\mathbb{Z}_2 \to B\,\mathrm{Spin}(n) \to B\,\mathrm{SO}(n)$. To help us determine the differentials (specifically the transgressions) in this spectral sequence I will introduce another structure that exists on $H^*(X;\mathbb{F}_2)$: the so called *Steenrod squares*. The Steenrod squares on $H^*(B\mathbb{Z}_2^{n-1})$ are known, and by finding an injection $H^*(B\,\mathrm{SO}(n)) \hookrightarrow H^*(B\mathbb{Z}^{n-1})$ they can then be determined on $H^*(B\,\mathrm{SO}(n))$.

10.1. **The injection.** Let G be a Lie group and let $A \subseteq G$ be a closed abelian subgroup. Let $N := \{n \in G \mid nA = An\}$ be the normalizer of A in G. N acts on A by conjugation: $a.n := n^{-1}an$. The map $a \mapsto a.n$ (a continuous map) induces a map on cohomology and in this way the right action on A gives a left (left because of the contravariance of H^*) action of N on $H^*(A)$.

Consider the fiber bundle $G/A \xrightarrow{i} EG/A \xrightarrow{p} BG$. Here the right action of G on EG induces an action of N on EG/A: $[e].n := [en]$. This action induces a left action of N on $H^*(EG/A)$.

Let $H^*(EG/A)^N := \{x \in H^*(EG/A) \mid x.n = x \text{ for all } n \in N\}$ – this is a subalgebra.

Lemma 10.1. *For the above fiber bundle we have* $\operatorname{Im} p^* \subseteq H^*(EG/A)^N$.

Proof. For $\phi \in C^n(EG/A)$ we get

$$\begin{aligned}(n.p^*(\phi))(\sigma) &= p^*(\phi)(t \mapsto \sigma(t).n) = \phi(t \mapsto p(\sigma(t).n)) \\ &= \phi(t \mapsto p(\sigma(t))) = p^*(\phi)(\sigma)\end{aligned}$$

□

Now let $G = \mathrm{SO}(n)$ and let $A = A^n$ be the diagonal matrices in G. A is closed since $A \cong \mathbb{Z}_2^{n-1}$ is finite.

Lemma 10.2. $\dim H^1(\mathrm{SO}(n)/A) = n-1$ *and* $H^*(\mathrm{SO}(n)/A)$ *is multiplicatively generated by* $H^1(\mathrm{SO}(n)/A)$.

Proof. We have $\mathrm{SO}(n)/A^n \cong \mathrm{O}(n)/B^n$ where B^n are the diagonal matrices in $\mathrm{O}(n)$. The result is proved by induction by using the Serre spectral sequence on the fiber bundle $(\mathrm{O}(n) \times \mathrm{O}(1))/B^{n+1} \xrightarrow{j} \mathrm{O}(n+1)/B^{n+1} \xrightarrow{\pi} \mathrm{O}(n+1)/(\mathrm{O}(n) \times \mathrm{O}(1))$ which can be shown to have trivial LCS. The fiber of this bundle is homeomorphic to $\mathrm{O}(n)/B^n$ and the base is homeomorphic to $\mathbb{R}P^n$. We have $H^*(\mathbb{R}P^n) = \mathbb{F}_2[z]/(z^{n+1})$ with $\deg z = 1$. One can now show that $(j^*)^1$ is surjective with kernel homeomorphic to $H^1(\mathbb{R}P^n)$, which implies that j^* is surjective and that $\dim H^1(\mathrm{SO}(n+1)/B^{n+1}) = n$. Furthermore one can show that $H^*(\mathrm{SO}(n+1)/B^{n+1})$ is a free $\operatorname{Im} \pi^* = \pi^*(t)$-module and the basis elements for this module can be chosen as elements of $H^1(\mathrm{SO}(n+1)/B^{n+1})$. The result follows. □

Lemma 10.3. *$i^* \colon H^*(E\,\mathrm{SO}(n)/A) \to H^*(\mathrm{SO}(n)/A)$ is surjective.*

Proof. By lemma 10.2 it is enough to show that $(i^*)^1 \colon H^1(EG/A) \to H^1(G/A)$ is surjective. In the factorization of $(i^*)^1$ through the Serre spectral sequence of the fiber bundle (lemma 8.5)

$$(i^*)^1 \colon H^1(EG/A) \twoheadrightarrow E_\infty^{0,1} \hookrightarrow \cdots \hookrightarrow E_2^{0,1} \cong H^1(G/A)$$

the first map is an isomorphism, since $E_\infty^{1,0} \cong H^1(BG) = 0$. So $(i^*)^1$ is injective. Now $H^*(EG/A) \cong H^*(BA) \cong H^*(B\mathbb{Z}_2)^{\otimes(n-1)}$ by corollary 7.11 and the Künneth isomorphism so $\dim H^1(EG/A) = n-1$ – the same dimension as $H^1(G/A)$. Hence $(i^*)^1$ is an isomorphism. □

Lemma 10.4. *$p^* \colon H^*(B\,\mathrm{SO}(n)) \to H^*(E\,\mathrm{SO}(n)/A)$ is injective.*

Proof. Consider the Serre spectral sequence for $\mathrm{SO}(n)/A \xrightarrow{i} E\,\mathrm{SO}(n)/A \xrightarrow{p} B\,\mathrm{SO}(n)$. Since i^* is surjective (lemma 10.3), then in the factorization of i^* from lemma 8.5 we must have $E_\infty^{0,k} = E_2^{0,k}$. So $d_r^{0,k} = 0$ for all r and k. It follows from lemma 10.2 that $H^k(\mathrm{SO}(n)/A)$ has finite dimension for all k, so by lemma 8.7 $E_2 \cong H^*(B\,\mathrm{SO}(n)) \otimes H^*(\mathrm{SO}(n)/A)$. By this isomorphism d_2 is determined by it's values on $E_2^{0,*}$, so $d_2 = 0$ and $E_3 = E_2$. Repeating this argument inductively we get $d_r = 0$ for all r – that is, the spectral sequence collapses.

In the factorization of p^* from lemma 8.5 we therefore have $E_2^{p,0} = \cdots = E_\infty^{p,0}$, so p^* is injective. □

Lemma 10.5. *The action of an $n \in N$ on $H^*(E\,\mathrm{SO}(n)/A) = \mathbb{F}_2[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$ is given by a permutations of $x_1, \ldots, x_n$ and all such permutations occur.*

Proof. See [8] corollary 4.19. □

Definition 10.6. A polynomial $f \in R[x_1, \ldots, x_n]$ that satisfies $f(x_1, \ldots, x_n) = f(x_{\tau(1)}, \ldots, x_{\tau(n)})$ for all permutations $tau \in \Sigma_n$ is called a *symmetric polynomial*. Of these we have the *elementary symmetric polynomials*

$$\sigma_i = \sigma_i(x_1, \ldots, x_n) := \sum_{\{k_1, \ldots, k_i\} \subseteq \{1, \ldots, n\}} x_{k_1} \cdots x_{k_n}$$

Lemma 10.7. *The algebra A of symmetric polynomials in $R[x_1, \dots, x_n]$ equals $R[\sigma_1, \dots, \sigma_n]$.*

Proof. More precisely we should show that the homomorphism $\phi\colon R[y_1, \dots, y_n] \to R[x_1, \dots, x_n]$ given by $\phi(g) = g(\sigma_1, \dots, \sigma_n)$ satisfies $\operatorname{Im}\phi = A$ and $\operatorname{Ker}\phi = 0$.

To show this we introduce an ordering on the monomials in $R[x_1, \dots, x_n]$: We define $x_1^{k_1} \cdots x_n^{k_n} > x_1^{l_1} \cdots x_n^{l_n}$ if

(1) $k_1 + \cdots + k_n > l_1 + \cdots + l_n$, or
(2) $k_1 + \cdots + k_n = l_1 + \cdots + l_n$ and $(k_1, \dots, k_n)$ comes before $(l_1, \dots, l_n)$ in the lexicographical ordering.

$\operatorname{Ker}\phi = 0$: Assume $g \in R[y_1, \dots, y_n]$ is non-zero. Write $g = \sum g_j$ as a sum of distinct monomials. Choose g_{j_0} such that $\phi(g_{j_0})$ contains the largest monomonial m of all the monomials of all the $\phi(g_j)$'s. Then $\phi(g_{j_0})$ is actually the only polynomial of the $\phi(g_j)$'s which contains m. So m is a non-zero monomial of $\phi(g)$, which therefore is non-zero.

Clearly $\operatorname{Im}\phi \subseteq A$. Conversely let $f \in A$. The fact that $f \in \operatorname{Im}\phi$ can be shown by giving a recursive algorithm which returns a g such that $f = \phi(g)$:

If $f = 0$ we are done. Otherwise let $rx_1^{k_1} \cdots x_n^{k_n}$ be the largest monomial of f. By symmetry we must have $k_1 > \cdots > k_n$. Let $h = ry_n^{k_n} y_{n-1}^{k_{n-1}-k_n} \cdots y_1^{k_1-k_2}$. Repeat the procedure on $f - \phi(h)$ to get $f - \phi(h) = \phi(g')$. Then $f = \phi(h + g')$.

The above algorithm terminates since the largest monomial of $\phi(h)$ is also $rx_1^{k_1} \cdots x_n^{k_n}$ and hence the largest monomial of $f - h$ is strictly smaller than that of f. □

Theorem 10.8. *p^* induces an algebra isomorphism $H^*(B\,\mathrm{SO}(n)) \xrightarrow{\sim} \mathbb{F}_2[\sigma_1, \dots, \sigma_n]/(\sigma_1)$.*

Proof. Combining lemma 10.5 and lemma 10.7 we get $H^*(EG/A)^N = \mathbb{F}_2[\sigma_1, \dots, \sigma_n]/(\sigma_1)$.

We see that $\dim H^*(E\,\mathrm{SO}(n)/A)^N = \dim H^*(B\,\mathrm{SO}(n))$ so the injection $p^*\colon H^*(B\,\mathrm{SO}(n)) \to H^*(E\,\mathrm{SO}(n)/A)^N$ (c.f. lemmas 10.1 and 10.4) is actually an isomorphism. □

Let $w_i \in H^i(B\,\mathrm{SO}(n))$ be the element corresponding to σ_i under the above isomorphism. w_i is called the i'th *Stiefel-Whitney class* ([8] page 140).

10.2. **The Steenrod Squares.** I repeat the following result from [8] page 102:

Theorem 10.9. *There exists a family $\mathrm{Sq}^i\colon H^n(X, A) \to H^{n+i}(X, A)$, $n, r \in \mathbb{N}_0$, of natural transformations called the* Steenrod Squares *(remember here that $H^n(\_) = H^n(\_;\mathbb{F}_2)$). They satisfy the following:*

(1) Stability: *For any pair (X, A) of spaces and $\partial\colon H^n(A) \to H^n(X, A)$ the boundary map we have $\mathrm{Sq}^i\,\partial = \partial\,\mathrm{Sq}^i$.*

(2) $\mathrm{Sq}^0(x) = x$ *and* $\mathrm{Sq}^n(x) = x^2$ *for* $x \in H^n(X, A)$.
(3) Instability: *For* $x \in H^n(X, A)$ *and* $i > n$ *we have* $\mathrm{Sq}^i(x) = 0$.
(4) The Cartan formula: $\mathrm{Sq}^i(xy) = \sum_{j=0}^{i} \mathrm{Sq}^j(x)\,\mathrm{Sq}^{n-j}(y)$.

The Steenrod Squares are characterized by properties (1) and (2) above.

Since the isomorphism $H^*(B\,\mathrm{SO}(n)) \cong \mathbb{F}_2[\sigma_1, \ldots, \sigma_n]/(\sigma_1)$ from theorem 10.8 is induced by a continuous map, by naturality $\mathrm{Sq}^j(w_i)$ can be calculated by calculating $\mathrm{Sq}^j(\sigma_i)$. Since $\deg x_k = 1$ for all k we get

$$\begin{aligned}
\mathrm{Sq}^j(\sigma_i) &= \sum_{\{k_1,\ldots,k_i\}\subseteq\{1,\ldots,n\}} \mathrm{Sq}^j(x_{k_1}\cdots x_{k_i}) \\
&= \sum_{\{k_1,\ldots,k_i\}\subseteq\{1,\ldots,n\}} \left(\sum_{(\epsilon_1,\ldots,\epsilon_i)\in\{0,1\}^i,\ \epsilon_1+\cdots+\epsilon_i=j} x_{k_1}^{\epsilon_1+1}\cdots x_{k_i}^{\epsilon_i+1} \right)
\end{aligned}$$

Now we want to apply the algorithm in the proof of lemma 10.7. We see that the largest monomial of $\mathrm{Sq}^j(\sigma_i)$ is $x_1^2\cdots x_j^2 x_{j+1}\cdots x_i$ (assuming $j \le i$). So the first term in the expression of $\mathrm{Sq}^j(\sigma_i)$ in terms of $\sigma_1, \ldots, \sigma_n$ is $\sigma_i\sigma_j$ which I therefore want to calculate. We get

$$\sigma_i\sigma_j = \sum_{k=0}^{j} \binom{i-j+2k}{k} \mathrm{Sq}^{j-k}(\sigma_{i+k})$$

Explanation for the coefficients: When multiplying out $\sigma_i\sigma_j$ exactly $\binom{i-j+2k}{k}$ of the terms will equal $x_1^2\cdots x_{j-k}^2 x_{j-k+1}\cdots x_{i+k}$: One term for each monomial $x_1\cdots x_{j-k}x_{l_1}\cdots x_{l_k}$, $\{l_1,\ldots,l_k\} \subseteq \{j-k+1,\ldots,i+k\}$ of σ_j.

This equation for $\sigma_i\sigma_j$ can be used to determine $\mathrm{Sq}^j(\sigma_i)$ – and hence $\mathrm{Sq}^j(w_i)$. However the general expression might not be pretty – a nice expression for $\mathrm{Sq}^j(w_i)$ is proven in [8] (page 141) by induction on the number of variables n:

Lemma 10.10. *For the Stiefel Whitney classes $w_i \in H^i(B\,\mathrm{SO}(n))$ we have for $j = 0, \ldots, i$ the formula*

$$\mathrm{Sq}^j(w_i) = \sum_{k=0}^{j} \binom{i-k-1}{j-k} w_{i+j-k} w_k$$

with the conventions that $w_k = 0$ for $k > n$ and $\binom{-1}{0} = 1$.

Lemma 10.11. *Let $F \to E \to B$ be a Serre fibration. Let $f \in H^{r-1}(F)$ be transgressive. Then also $\mathrm{Sq}^j(f)$ is transgressive, and we have $\tau_{r+j}(\mathrm{Sq}^j(f)) = \mathrm{Sq}^j(\tau_r(f))$ (the right hand side is well-defined) where τ_r and τ_{r+j} are the transgressions.*

Proof. This follows from considering the commutative diagram

$$\begin{tikzcd}
H^{r-1}(F) \arrow[r, "\partial"] \arrow[d, "\mathrm{Sq}^j"] & H^r(E,F) \arrow[d, "\mathrm{Sq}^j"] & H^r(B,b_0) \arrow[l, "p^*"'] \arrow[r, "j^*"] \arrow[d, "\mathrm{Sq}^j"] & H^r(B) \arrow[d, "\mathrm{Sq}^j"] \\
H^{r+j-1}(F) \arrow[r, "\partial"] & H^{r+j}(E,F) & H^{r+j}(B,b_0) \arrow[l, "p^*"'] \arrow[r, "j^*"] & H^{r+j}(B)
\end{tikzcd}$$

By this diagram if, say, $\partial(f) = p^*(b)$ then $\partial(\mathrm{Sq}^j(f)) = p^*(\mathrm{Sq}^j(b))$ and $\tau_{r+j}(\mathrm{Sq}^j(f)) = [\mathrm{Sq}^j(j^*(b))]$. □

Theorem 10.12. *Let $n \geq 3$. Then as a graded algebra*

$$H^*(B\,\mathrm{Spin}(n)) \cong \mathbb{F}_2[x] \otimes \mathbb{F}_2[w_2, \ldots, w_n]/(a_0, \ldots, a_{k-1})$$

where $a_0 = w_2$, $a_{l+1} = \mathrm{Sq}^{2^{l-1}}(a_l)$ (hence $\deg a_l = 2^l + 1$), k is the smallest number such that a_k lies in the ideal $(a_0, \ldots, a_{k-1})$, and $\deg x = 2^k$.

In particular $H^*(B\,\mathrm{Spin}(n))$ is polynomial for $n = 3, \ldots, 9$.

Proof. I will use the Serre spectral sequence on the fiber bundle $B\mathbb{Z}_2 \to B\,\mathrm{Spin}(n) \to B\,\mathrm{SO}(n)$, which has trivial LCS since $\pi_1(B\,\mathrm{SO}(n)) = \pi_0(\mathrm{SO}(n)) = 0$. We can use lemma 8.7 which gives that

$$E_2 = H^*(B\mathbb{Z}_2) \otimes H^*(B\,\mathrm{SO}(n)) = \mathbb{F}_2[z] \otimes \mathbb{F}_2[w_2, \ldots, w_n]$$

where $\deg z = (0, 1)$ and $\deg w_i = (i, 0)$. Since $\mathrm{Spin}(n)$ is 1-connected, $B\,\mathrm{Spin}(n)$ is 2-connected, so by the Hurewicz theorem $H^2(B\,\mathrm{Spin}(n) = 0$. Therefore $d_2(z) = w_2$ as w_2 can only be "killed" by z. From this we get

$$E_3 = \mathbb{F}_2[z^2] \otimes \mathbb{F}_2[w_2, \ldots, w_n]/(w_2)$$

Notice that z is transgressive.

Now assume inductively (for $k \geq 1$) that

$$E_{2^{k-1}+2} = \mathbb{F}_2[z^{2^k}] \otimes \mathbb{F}_2[w_2, \ldots, w_n]/(a_0, \ldots, a_{k-1})$$

with $a_{l+1} = \mathrm{Sq}^{2^{l-1}}(a_l)$ and all the a_l's non-zero, that $a_{k-1} = d_{2^{k-1}+1}(z^{2^{k-1}})$, and that $z^{2^{k-1}}$ is transgressive. By the last assumption also $z^{2^k} = \mathrm{Sq}^{2^{k-1}}(z^{2^{k-1}})$ is transgressive (c.f. lemma 10.11) so for all $2^{k-1} + 2 \leq i \leq 2^k$ we have $d_i(z^{2^k}) = 0$ and hence $E_{2^k+1} = E_{2^{k-1}+2}$. Now

$$d_{2^k+1}(z^{2^k}) = d_{2^k+1}(\mathrm{Sq}^{2^{k-1}}(z^{2^{k-1}})) = \mathrm{Sq}^{2^{k-1}}(d_{2^{k-1}+1}(z^{2^{k-1}})) = \mathrm{Sq}^{2^{k-1}}(a_{k-1}) = a_k$$

(here I have used lemma 10.11 in combination with lemma 8.4). If $a_k \neq 0$ the induction step "repeats". Assume $a_k = 0$ (there will be a k for which this is the case, since $\mathbb{F}_2[w_2, \ldots, w_n]$ is Noetherian). Then $E_\infty = E_{2^k+1}$ since for $i > 2^k + 1$ we have $d_i(z^{r2^k}) = r d_i(z^{2^k}) z^{(r-1)2^k} = 0$. Now $E_\infty \cong H^* := H^*(B\,\mathrm{Spin}(n))$ as graded vector spaces; we get such an isomorphism by choosing sections $s^{p,q} \colon E_\infty^{p,q} \cong F^pH/F^{p+1}H^{p+q} \to F^pH^{p+q}$ for all p and q. We can choose all these by just choosing a section $s^{0,2^k}$, putting $x = s^{0,2^k}(z^{2^k})$, and defining $s^{p,r2^k}(z^{r2^k}b) = x^r \smile b$ where $b \in E_\infty^{p,0} \hookrightarrow H^p$ on the right hand side is considered to be an element of H^p. The isomorphism determined by these sections is seen to be an algebra isomorphism by using commutativity of H^*.

That $H^*(B\,\mathrm{Spin}(n))$ is polynomial for $n = 3, \dots, 9$ follows from the following calculations of the Steenrod Squares: By lemma 10.10 $\mathrm{Sq}^j(w_{j+1}) = \sum_{k=0}^{j} \binom{j-k}{j-k} w_{2j+1-k}w_k = \sum_{k=0}^{j} w_{2j+1-k}w_k$ so

$$
\begin{aligned}
\mathrm{Sq}^1(w_2) &= w_3 + w_2w_1 = w_3 && \text{on the } E_3 \text{ page.}\\
\mathrm{Sq}^2(w_3) &= w_5 + w_4w_1 + w_3w_2 = w_5 && \text{on the } E_5 \text{ page.}\\
\mathrm{Sq}^4(w_5) &= w_9 && \text{on the } E_9 \text{ page.}
\end{aligned}
$$

And $\mathrm{Sq}^8(w_9) = 0$ in $H^*(B\,\mathrm{SO}(9))$. □

11. Other calculations

In this section $R = \mathbb{F}_2$

In [2] the following is proven:

Theorem 11.1. *Let $n \geq 0$. Then*

$$H^*(B\,\mathrm{PSp}(2n+1)) = \mathbb{F}_2[x_2, x_3, x_8, x_{12} \dots, x_{8n+4}]$$

Lemma 11.2. *In $H^*(B\,\mathrm{PSp}(m))$ we have $\mathrm{Sq}^1(x_2) = x_3$.*

Proof. In the Serre spectral sequence for $B\mathbb{Z}_2 \to B\,\mathrm{Sp}(m) \to B\,\mathrm{PSp}(m)$ we have $E_2 = \mathbb{F}_2[z] \otimes \mathbb{F}_2[x_2, x_3, x_8, x_{12} \dots, x_{8n+4}]$. Since $H^2(B\,\mathrm{Sp}(m)) = H^3(B\,\mathrm{Sp}(m)) = 0$ we must have $d_2(z) = x_2$ and $d_3(z^2) = x_3$. Then $\mathrm{Sq}^1(x_2) = \mathrm{Sq}^1(d_2(z)) = d_2(\mathrm{Sq}^1(z)) = d_2(z^2) = x_3$. □

Consider the following diagram

$$
\begin{tikzcd}
H^*(X) \arrow[r, hook, "(\_)\otimes 1"] \arrow[d, "Sq^j"] & H^*(X) \otimes H^*(Y) \arrow[r, "\times"] & H^*(X \times Y) \arrow[d, "\mathrm{Sq}^j"] \\
H^*(X) \arrow[r, hook, "(\_)\otimes 1"] & H^*(X) \otimes H^*(Y) \arrow[r, "\times"] & H^*(X \times Y)
\end{tikzcd}
$$

where X and Y are path-connected. We have $x \times 1 = \mathrm{pr}_X^*(x)\,\mathrm{pr}_Y^*(1) = \mathrm{pr}_X^*(x)$, that is, composing the horizontal maps gives pr_X^*. So by naturality of Sq^j the diagram commutes. This gives a way to calculate the Steenrod squares on $H^*(X \times Y)$ in terms of the squares on $H^*(X)$ and $H^*(Y)$ in the cases where the Künneth isomorphism holds.

Theorem 11.3. *Let $A \cong \mathbb{Z}_2$ be a subgroup of the center of $G = \mathrm{Sp}(2n_1+1) \times \mathrm{Sp}(2n_2+1)$. Then*

$$H^*(B(G/A)) \cong H^*(B\,\mathrm{PSp}(2n_1+1)) \otimes H^*(B\,\mathrm{Sp}(2n_2+1))$$

Proof. Let Z be the center of G. Then $Z/A \cong \mathbb{Z}_2$ and $G/Z = \mathrm{PSp}(2n_1+1) \times \mathrm{PSp}(2n_2+1)$. Using the Serre spectral sequence on the fiber bundle $B(Z/A) \to B(G/A) \to B(G/Z))$ we therefore get

$$E_2 = \mathbb{F}_2[z] \otimes \mathbb{F}_2[x_2, y_2, x_3, y_3, \dots]$$

(where I have used the Künneth isomorphism). Since $H^1(B(G/A)) = 0$ we must have $d_2(z) \neq 0$, say $d_2(z) = \epsilon_x x_2 + \epsilon_y y_2$. Then $d_3(z^2) = \mathrm{Sq}^1(d_2(z)) = \epsilon_x x_3 + \epsilon_y y_3$. Next, on $E_5 = E_4$ we get $d_5(z^4) = \epsilon_x \mathrm{Sq}^2(x_3) + \epsilon_y \mathrm{Sq}^2(y_3)$. There are now two possible cases: Either $\mathrm{Sq}^2(x_3) = \mathrm{Sq}^2(y_3) = 0$ or $\mathrm{Sq}^2(x_3) = x_2x_3$ and $\mathrm{Sq}^2(y_3) = y_2y_3$. In all cases $d_5(z^4) = 0$ – also when $\epsilon_x = \epsilon_y = 1$ in the second case, where $d_5(z^4) = x_2x_3 + y_2y_3 = x_2x_3 + x_2x_3 = 0$. Then $E_\infty = E_5$ and the result follows. □

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