Simplicial Spaces and Homotopy Colimits

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Contents

There are many texts that cover the topic of simplicial sets and homotopy colimits and some of the motivation and intuition behind it. However, it is hard to find a source for proofs of the easy statements without using more sophisticated tools from category theory or homotopy theory. This text is an attempt to, in a concise and clear way, give a summary with proofs of the theory from simplicial objects to homotopy colimits and homotopy orbit spaces. This text can be used as complementary notes to the first half of Dwyer's paper Classifying Spaces and Homology Decompositions [DH01], and is aimed to the reader who has seen simplicial complexes and some algebraic topology and category theory but not ventured much further.

1 Preliminary definitions

In this text we will assume some knowledge of category theory. For example, every partially ordered set (poset) P can be viewed as a category with objects being the elements of P and morphisms corresponding to the ordering. Thus for $a \leq b \in P$, we have an arrow $a \to b$ in the corresponding category $\mathbf{P}$.

Definition 1.1. We will denote categories with boldface notation. The category of sets is denoted **Sets**, the category of small categories is denoted **Cat** and the category of topological spaces is denoted **Top**. The category of abelian groups is denoted **AbGrp** and the category of chain complexes of abelian groups is denoted Comp(**AbGrp**).

If $\mathbf{C}$ is a small category, then a functor $\mathbf{C} \to \mathbf{D}$ will be called a diagram (of type $\mathbf{C}$) in $\mathbf{D}$. When $\mathbf{C}$ is a small category, we can form the functor category $\mathbf{D}^{\mathbf{C}}$ with objects being functors $\mathbf{C} \to \mathbf{D}$ and morphisms being natural transformations.

For a category $\mathbf{C}$ we write the opposite category as $\mathbf{C}^{op}$. It is important to note that a functor $\mathbf{C}^{op} \to \mathbf{D}$ is the same thing as a contravariant functor $\mathbf{C} \to \mathbf{D}$.

Definition 1.2. Let $\underline{n}$ denote the ordered set $\{0 < 1 < \ldots < n\}$ for non-negative n. The category corresponding to this poset is denoted $\mathbf{n}$ and thus has one object for every number $0, 1, \ldots, n$ and exactly one morphism from i to j for $i \leq j$.

Definition 1.3. We define the category Δ which has objects $\underline{n}$ and morphisms being weakly order-preserving functions. This is equivalent to the category with objects being the categories $\mathbf{n}$ and morphisms being functors.

As is shown in [ML98, pg. 178], any morphism $\eta \colon \underline{n} \to \underline{m}$ in Δ can be written in terms of morphisms δ^i and σ^i, called coface and codegeneracy maps, respectively. These are defined by

$$\delta^i(j) = \begin{cases} j, & j < i, \\ j+1, & j \geq i, \end{cases}$$

and

$$\sigma^i(j) = \begin{cases} j, & j \leq i, \\ j-1, & j > i\,. \end{cases}$$

These morphisms satisfy a set of relations called the cosimplicial relations:

$$\begin{cases} \delta^j\delta^i = \delta^i\delta^{j-1}, & i<j, \\ \sigma^j\sigma^i = \sigma^i\sigma^{j+1}, & i\le j, \\ \sigma^j\delta^i = \delta^i\sigma^{j-1}, & i<j, \\ \sigma^j\delta^i = \mathrm{Id}, & i=j, i=j+1, \\ \sigma^j\delta^i = \delta^{i-1}\sigma^j, & i>j+1\,. \end{cases}$$

Furthermore, $\eta\colon \underline{n} \to \underline{m}$ can be written *uniquely* on the form

$$\eta = \delta^{i_1}\cdots\delta^{i_s}\sigma^{j_1}\cdots\sigma^{j_t}$$

where $0 \le i_s < \ldots < i_1 \le m$, $0 \le j_1 < \ldots < j_t \le n$ and $n - t + s = m$. Also this is shown in [ML98].

Definition 1.4. Let $\mathbf{D}$ be a category. A simplicial object in $\mathbf{D}$ is a functor $X\colon \Delta^{op} \to \mathbf{D}$. When $\mathbf{D} = \mathbf{Sets}$, we call X a simplicial set. Because of its close connection with topological spaces, we will also call X a *space*.

The category Δ is small, so we form the category with morphisms being natural transformations and we denote this category with $\mathbf{Sp} := \mathbf{Sets}^{\mathbf{\Delta^{op}}}$. A morphism in this category will be called a simplicial map.

In direct analogy, a functor $X\colon \Delta \to \mathbf{D}$ is called a cosimplicial object in $\mathbf{D}$.

By the characterization above, a simplicial object in $\mathbf{D}$ is a sequence of objects $X_n = X(\underline{n})$ together with morphisms $d_i = X(\delta^i)\colon X_n \to X_{n-1}$ and $s_i = X(\sigma^i)\colon X_n \to X_{n+1}$ satisfying the duals of the cosimplicial relations. The dual relations are called the simplicial relations and are thus

$$\begin{cases} d_id_j = d_{j-1}d_i, & i<j, \\ s_is_j = s_{j+1}s_i, & i\le j, \\ d_is_j = s_{j-1}d_i, & i<j, \\ d_is_j = \mathrm{Id}, & i=j, i=j+1, \\ d_is_j = s_jd_{i-1}, & i>j+1\,. \end{cases}$$

Likewise, a simplicial map $\varphi\colon X \to Y$ is a set of morphisms in $\mathbf{D}$ indexed by $\mathbb{N}$, $\varphi_n\colon X_n \to Y_n$, such that φ commutes with all morphisms in Δ^{op}. But since a morphism in Δ^{op} is generated by face maps d_i and degeneracy maps s_i, it is enough to say that φ commutes with all d_i, s_i in each dimension.

Definition 1.5. If $X\colon \Delta^{op} \to \mathbf{D}$ is a simplicial object in $\mathbf{D}$, where $\mathbf{D}$ is a category with objects being sets, then we say that $x_n \in X_n$ is an n-simplex, or just a simplex if we do not want to stress the dimension. It is degenerate if $x_n = s_ix_{n-1}$ for some $x_{n-1} \in X_{n-1}$ and some degeneracy map s_i. A simplex that is not degenerate is said to be non-degenerate.

Definition 1.6. We define the topological n-simplex

$$\Delta_n = \{(t_0,\ldots,t_n) \subseteq \mathbb{R}^{n+1}\ ;\ \sum t_i = 1, 0 \le t_i \le 1\}$$

which has the subspace topology from $\mathbb{R}^{n+1}$.

There is a functor $T\colon \Delta \to \mathbf{Top}$ which on objects is $T(\underline{n}) = \Delta_n$ and for a morphism $\eta\colon \underline{n} \to \underline{m}$, we define $\eta_*\colon \Delta_n \to \Delta_m$ by

$$\eta_*\colon \sum_{i=0}^n t_i\xi_i \mapsto \sum_{i=0}^n t_i\xi_{\eta(i)}$$

where $\{\xi_i\}_{0\leq i\leq n}$ is the standard basis of $\mathbb{R}^{n+1}$.

In particular, for the face and degeneracy maps we have $\delta^i_*(t_0,\dots,t_n) = (t_0,\dots,t_{i-1},0,t_i,\dots,t_n)$ and $\sigma^i_*(t_0,\dots,t_n) = (t_0,\dots,t_i+t_{i+1},\dots,t_n)$ which are the topological face and degeneracy maps.

Note that T is a cosimplicial set by definition.

Remark 1.7. We will often use the notation $\delta^{i^*} := X(\delta^i)$ and $\sigma^{i^*} := X(\sigma^i)$ for the simplicial set X. The upper star does not indicate that it is a precomposition (even though it is for many simplicial sets), but rather to indicate that it changes the contravariation to covariation, so that we know that $d_i = \delta^{i^*}$ decreases the dimension and not the opposite.

This notation is very handy because any combination of face and degeneracy maps in X can then be written as $\eta^* := X(\eta)$ where η is a morphism in the category Δ.

Remark 1.8. In what follows, all proofs can be done either by functorial arguments using only category theory or, because of the characterization above, by doing things "simplicially" with the face and degeneracy maps. It will almost always be the case then that we need to check some kind of commutativity for d_i, s_i in our proofs. Usually the proof for commutativity with s_i is exactly the same as for d_i, mutatis mutandis, and so we only do the verification for d_i.

2 Simplicial sets and the geometric realization

Definition 2.1. For $X \in \mathbf{Sp}$, give each X_n the discrete topology. We define the geometrical realization

$$|X| = \coprod_{n\geq 0} X_n \times \Delta_n / \sim\ .$$

where $(d_i x, p) \sim (x, \delta^i_* p)$ and $(s_i x, p) \sim (x, \sigma^i_* p)$. One can show that $|X|$ is a CW-complex with n-cells corresponding to the non-degenerate n-simplices of X and that $|\cdot|$ is a functor $|\cdot|\colon \mathbf{Sp} \to \mathbf{Top}$ (see [May67, pg. 56]).

Proposition 2.2. *Every diagram $F\colon \mathbf{D} \to \mathbf{Sp}$ has a limit and a colimit. These can be constructed dimensionwise.*

Proof. See the construction in [ML98, pg. 111] for limits and the dual construction for colimits. It is important to note that $\mathbf{D}$ is small in our definition of a diagram. □

Let X, Y be two simplicial sets. From the previous proposition it follows that $(X\times Y)_n = X_n \times Y_n$ and $d_i(x_n, y_n) = (d_i^X x_n, d_i^Y y_n), s_i(x_n, y_n) = (s_i^X x_n, s_i^Y y_n)$ defines $X \times Y$ as the categorical product of X and Y.

Similarly, if we let $(X \coprod Y)_n = X_n \coprod Y_n$ with d_i and s_i defined componentwise, then $X \coprod Y$ is the categorical coproduct of X and Y.

Proposition 2.3. *For $X, Y \in \mathbf{Sp}$, if $|X| \times |Y|$ is a CW-complex, then as topological spaces*

$$|X \times Y| \cong |X| \times |Y|\,.$$

Proof. This is [May67, Thm. 14.3]. □

Remark 2.4. We can think of **Sets** as being a subcategory of **Sp** in the following way. Let $X \in \mathbf{Sets}$ and define $\iota\colon \mathbf{Sets} \to \mathbf{Sp}$ by

$$\iota(X)\colon \Delta^{op} \to \mathbf{Sets}$$

where $\iota(X)$ is the constant functor $\iota(X)_n = X$ for every n. All morphisms are mapped to the identity morphism by the constant functor, so all d_i and s_i are equal to the identity function $1_X : X \to X$.

Since all simplicises except the zeroth simplex $\iota(X)_0 = X$ are degenerate, we see that $|\iota(X)| = X$ with the discrete topology.

3 Simplicial spaces

Definition 3.1. A simplicial object in **Sp**, i.e. a functor $\Delta^{op} \to \mathbf{Sp}$ is called a simplicial space. The category of simplicial spaces (with morphisms being natural transformations) is denoted **SSp**.

Each X_n is a space (i.e. simplicial set) and all the d_i, s_i are maps of simplicial sets. Thus for a simplicial space, we can think of it as a horizontal sequence of simplicial sets X_n. Each of these is a vertical sequences of sets $(X_n)_k$ so we picture X as a grid with $(X_n)_k = X_{n,k}$ and the n-axis is horizontal. We will sometimes call n the outer or external dimension and k the inner or internal dimension.

When taking this viewpoint, we denote the external face and degeneracy maps (i.e. the face and degeneracy maps of the simplicial space X) with d_i^h, s_i^h, where the h is not an index but stands for horizontal. For each space X_n, we denote the inner face and degeneracy maps with d_i^v, s_i^v, where the v stands for vertical.

Another way to view a simplicial space X is as a (bi)functor $\widetilde{X}\colon \Delta^{op} \times \Delta^{op} \to \mathbf{Sets}$ where $\widetilde{X}(\underline{n}, \cdot) = X\colon \Delta^{op} \to \mathbf{Sets}$. We can precompose with the diagonal functor $\mathrm{diag}\colon \Delta^{op} \to \Delta^{op} \times \Delta^{op}$ to obtain a simplicial set $\Delta^{op} \xrightarrow{\mathrm{diag}} \Delta^{op} \times \Delta^{op} \xrightarrow{\widetilde{X}} \mathbf{Sets}$. With a few category-theoretic arguments one can show that this gives us a functor $\mathrm{diag}\colon \mathbf{SSp} \to \mathbf{Sp}$. However, one can also do this simplicially with the notation above:

Definition 3.2. Let X be a simplicial space. Define $\mathrm{diag}(X)_n = X_{n,n}$ and $d_i = d_i^h \circ d_i^v$, $s_i = s_i^h \circ s_i^v$.

Proposition 3.3. *Let X be a simplicial space. Then* $\mathrm{diag}(X)$ *as defined above is a simplicial set and* $\mathrm{diag}\colon \mathbf{SSp} \to \mathbf{Sp}$ *is a functor which takes* $\varphi\colon X \to Y \in \mathbf{SSp}$ *to* $\mathrm{diag}(\varphi)$ *defined by* $\mathrm{diag}(\varphi)(x_{n,n}) = \varphi(x_{n,n})$.

Proof. Since $\mathrm{diag}(X)_n = X_{n,n} = (X_n)_n$ is a set for every $n \in \mathbb{N}$, we only need to check that the simplicial relations hold. Let $i < j$. Each d_j^h is a simplicial map and thus commutes with all d_i^v, hence we find that $d_i d_j = d_i^h d_i^v d_j^h d_j^v = d_i^h d_j^h d_i^v d_j^v$. We can now apply the relations as normal, so that (again by commutativity),

$$d_i d_j = d_i^h d_j^h d_i^v d_j^v = d_{j-1}^h d_i^h d_{j-1}^v d_i^v = d_{j-1}^h d_{j-1}^v d_i^h d_i^v = d_{j-1} d_i\,.$$

The other relations are proved in exactly the same way.

For the second part, the only substantial thing to check is that $\mathrm{diag}(\varphi)$ is a map of simplicial sets. Since φ is a map of simplicial spaces, it commutes with the (external) face and degenerecy maps, so $d_i^h\varphi = \varphi d_i^h$. Furthermore, for every external dimension n, $\varphi_n : X_n \to Y_n$ is a map of simplicial sets, so φ commutes also with d_i^v. Thus

$$d_i\varphi = d_i^h d_i^v \varphi = d_i^v d_i^h \varphi = d_i^v \varphi d_i^h = \varphi d_i^v d_i^h = \varphi d_i^h d_i^v = \varphi d_i \,.$$

The proof that φ commutes with s_i is done mutatis mutandis. $\square$

Definition 3.4. For $\mathbf{n} \in \mathbf{Cat}$, $\mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathbf{n}) : \Delta^{op} \to \mathbf{Sets}$ is a functor. We define the simplicial set $\Delta[n] = \mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathbf{n})$.

Simplicially, for $k, n \in \mathbb{N}$, we have $\Delta[n]_k = \mathrm{Hom}_{\mathbf{Cat}}(\mathbf{k}, \mathbf{n}) = \mathrm{Hom}_\Delta(\underline{k}, \underline{n})$ and $\Delta[n]$ is a simplicial set by precomposition: $d_i = {\delta^i}^*$, $s_i = {\sigma^i}^*$. To be more clear: for $\eta : \underline{k} \to \underline{n}$, we have

$$\begin{cases} d_i(\eta) = \eta \circ \delta^i : \underline{k-1} \to \underline{k} \to \underline{n} \\ s_i(\eta) = \eta \circ \sigma^i : \underline{k+1} \to \underline{k} \to \underline{n} \end{cases}$$

Definition 3.5. Likewise, $\mathrm{Hom}_{\mathbf{Cat}}(\cdot, \cdot) : \Delta^{op} \times \Delta \to \mathbf{Sets}$ is a (bi)functor, so we define the cosimplicial space $\underline{\Delta}$ with $\underline{\Delta} : \mathbf{n} \mapsto \mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathbf{n}) = \Delta[n]$.

This time, the (outer) face and degeneracy maps are defined by postcomposition, so $d_h^i = \delta_*^i$, $s_h^i = \sigma_*^i$. At inner dimension k, for the n-(co)simplex $\eta : \underline{k} \to \underline{n}$, we have

$$\begin{cases} d_h^i(\eta) = \delta^i \circ \eta : \underline{k} \to \underline{n} \to \underline{n+1} \,, \\ s_h^i(\eta) = \sigma^i \circ \eta : \underline{k} \to \underline{n} \to \underline{n-1} \,. \end{cases}$$

When taking this simplicial point of view, we see that $\underline{\Delta}$ is a cosimplicial space precisely because postcomposition commutes with precomposition.

Lemma 3.6. *Let X be a simplicial set. If we have equivalence relations in each X_n such that for all $x_n \sim y_n \in X_n$ it holds that $d_i x_n \sim d_i y_n, s_i x_n \sim s_i y_n$, then $X/\sim$ defined by $(X/\sim)_n = X_n/\sim$ and $d_i[x_n] = [d_i x_n], s_i[x_n] = [s_i x_n]$ is a simplicial set.*

Furthermore, if we have a simplicial map $\varphi : X \to Y$ such that $\varphi(x_n) = \varphi(y_n)$ for all equivalent $x_n \sim y_n \in X_n$, then we have a factorization

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "p"'] & Y \\ X/\sim \arrow[ur, "\widetilde{\varphi}"] & \end{tikzcd}$$

where $\widetilde{\varphi}[x_n] = \varphi(x_n)$ and $p(x_n) = [x_n]$.

Proof. This is an easy example of how to use the definitions. First of all, if everything is well-defined, the diagram certainly commutes.

For the first statement, let us first check that $d_i : (X/\sim)_n \to (X/\sim)_{n-1}$ is well-defined. If $[x_n] = [y_n]$, so that $x_n \sim y_n$, then $d_i x_n \sim d_i y_n$ by assumption, so $d_i[x_n] = [d_i x_n] = [d_i y_n] = d_i[y_n]$ and similarly for s_i. It is obvious that the simplicial relations hold. Thus $X/\sim$ is a simplicial set.

We see that $\widetilde{\varphi}$ is a well-defined function in each dimension since for $[x_n] = [y_n]$ it holds that $\widetilde{\varphi}[x_n] = \varphi(x_n) = \varphi(y_n) = \widetilde{\varphi}[y_n]$ by the assumption given on φ. We need to check that $\widetilde{\varphi}$ commutes with d_i, s_i:

$$d_i\widetilde{\varphi}[x_n] = d_i\varphi(x_n) = \varphi d_i(x_n) = \widetilde{\varphi}[d_i x_n] = \widetilde{\varphi} d_i[x_n]$$

since φ is a simplicial map. As usual, the proof for s_i is exactly the same.

We also see that $d_i p(x_n) = d_i[x_n] = [d_i x_n] = p d_i(x_n)$ and likewise for s_i, so p is a simplicial map. □

In analogy with the geometric realization functor $|-|\colon \mathbf{Sp} \to \mathbf{Top}$ we will now define the realization functor $|-|\colon \mathbf{SSp} \to \mathbf{Sp}$.

Definition 3.7. For a simplicial space X, we define the realization of X to be the space

$$|X| = \coprod_{n\geq 0} X_n \times \Delta[n]/\sim$$

where we in each (internal) dimension k have the identifications

$$(d_i^h x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n-1}) \sim (x_{n,k}, \delta_*^i(\eta)) = (x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n-1} \xrightarrow{\delta^i} \underline{n})$$

and

$$(s_i^h x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n+1}) \sim (x_{n,k}, \sigma_*^i(\eta)) = (x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n+1} \xrightarrow{\sigma^i} \underline{n})$$

Furthermore, for $f\colon X \to Y \in \mathbf{SSp}$, we define

$$|f|\colon |X| \to |Y|, \quad [x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n}] \mapsto [f(x_{n,k}), \underline{k} \xrightarrow{\eta} \underline{n}]\,.$$

Proposition 3.8. *The constructions above makes* $|\cdot|\colon \mathbf{SSp} \to \mathbf{Sp}$ *into a functor.*

Proof. The proof is an easy consequence of Lemma 3.6 and the fact that every relation is generated by the two types of relations $(d_i^h x_{n,k}, \eta) \sim (x_{n,k}, \delta_*^i(\eta))$ and $(s_i^h x_{n,k}, \eta) \sim (x_{n,k}, \sigma_*^i(\eta))$. Let $X \in \mathbf{Sp}$ and $(x_{n,k}, \eta) \in \coprod X_n \times \Delta[n]$. Since $d_j(x_{n,k}, \eta) = (d_j^v x_{n,k}, {\delta^j}^*(\eta))$ and we have commutativity of horizontal and vertical maps, we get

$$\begin{aligned} d_j(d_i^h x_{n,k}, \eta) &= (d_i^h d_j^v x_{n,k}, {\delta^j}^*(\eta)) \\ &\sim (d_j^v x_{n,k}, \delta_*^i {\delta^j}^*(\eta)) \\ &= d_j(x_{n,k}, \delta_*^i(\eta))\,. \end{aligned}$$

There are three more of these to check, but they are near identical to this. The lemma implies that $|X|$ is a well-defined simplicial set.

Let $f\colon X \to Y \in \mathbf{SSp}$. Note that this means that f commutes with all $d_i^h, d_j^v, s_i^h, s_j^v$. We define $\varphi\colon \coprod X_n \times \Delta[n] \to |Y|$ by $\varphi(x_{n,k}, \eta) = [f(x_{n,k}), \eta]$. Then

$$d_i\varphi(x_{n,k}, \eta) = [d_i^v f(x_{n,k}), {\delta^i}^*(\eta)] = [f d_i^v(x_{n,k}), {\delta^i}^*(\eta)] = \varphi d_i(x_{n,k}, \eta)$$

and same for s_i, so φ is a simplicial map.

From the second part of Lemma 3.6, the calculation

$$\varphi(d_i^h x_{n,k}, \eta) = [f d_i^h x_{n,k}, \eta] = [d_i^h f(x_{n,k}), \eta] = [f(x_{n,k}), \delta_*^i] = \varphi(x_{n,k}, \delta_*^i(\eta))$$

(and its counterpart with degeneracy maps) implies that $\widetilde{\varphi}$ is a well-defined simplicial map, and evidently $|f| = \widetilde{\varphi}$. The other axioms for functors are trivial. □

Remark 3.9. It is nice to note that if we have $\mu\colon \underline{m} \to \underline{n}$ so that $\mu_h^*\colon X_{n,k} \to X_{m,k}$ we only need to remember that μ is a combination of face and degeneracy maps to see that we have the easily remembered formula

$$(\mu_h^*(x_{n,k}), \eta) \sim (x_{n,k}, \mu_*(\eta))$$

Indeed, we could have defined the equivalence relation to be generated by all such relations, but it is often more practical to use the fact that every morphism in Δ is generated by face and degeneracy maps so that also the equivalence relation is generated by the two formulas in our definition.

Example 3.10. In particular, we can consider a space X as a simplicial space $\iota(X)$ by letting $\iota(X)_n = X$ for all n in the same way as we did when we considered sets as spaces in Remark 2.4. With this definition, $|\iota(X)| = X$ either by the same reasoning as before or by looking at the diagonal space.

We have so far seen that for a simplicial space X, we can construct two spaces $|X|$ and $\operatorname{diag}(X)$. We will now show in detail that they are naturally isomorphic. For this we will use a preliminary lemma.

Lemma 3.11. *If a simplicial map f is a bijection in each dimension, then f is an isomorphism of simplicial sets.*

Proof. Let $f\colon X \to Y \in \mathbf{Sp}$. In each dimension n we have a two-sided inverse $g_n\colon Y_n \to X_n$. We only need to show that g defined in the obvious way is a simplicial map. We thus want to show that the diagram

$$\begin{tikzcd}
Y_n \arrow[r, "g_n"] \arrow[d, "d_i"] & X_n \arrow[d, "d_i"] \\
Y_{n-1} \arrow[r, "g_{n-1}"] & X_{n-1}
\end{tikzcd}$$

commutes.

Since f is a simplicial map it holds that $d_i f_n = f_{n-1} d_i$. From this it follows that $g_{n-1} d_i f_n = g_{n-1} f_{n-1} d_i = d_i$, hence also $d_i g_n = g_{n-1} d_i f_n g_n = g_{n-1} d_i$. $\square$

Theorem 3.12. *Let X be a simplicial space. Then $|X| \cong \operatorname{diag}(X)$ naturally.*

Proof. A typical k-simplex in $|X|$ is represented by $(x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n}) \in X_{n,k} \times \Delta[n]_k$. Define $\varphi\colon \operatorname{diag}(X) \to |X|$ by

$$\varphi\colon X_{n,n} \to |X|_n, \quad x_{n,n} \mapsto [x_{n,n}, \underline{n} \xrightarrow{\mathrm{Id}} \underline{n}]$$

For $x_{k,k} \in \operatorname{diag}(X)_k$, $d_i(x_{k,k}) = d_i^h d_i^v(x_{k,k})$. In $|X|$, the face map d_i is induced by the product face map $d_i^v \times {\delta^i}^*$, so that

$$d_i[x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n}] = [d_i^v(x_{n,k}), \underline{k-1} \xrightarrow{\delta^i} \underline{k} \xrightarrow{\eta} \underline{n}]$$

We can now check that φ is a simplicial map. On the one hand,

$$d_i(\varphi(x_{k,k})) = d_i[x_{k,k}, \underline{k} \xrightarrow{\mathrm{Id}} \underline{k}] = [d_i^v(x_{k,k}), \underline{k-1} \xrightarrow{\delta^i} \underline{k}]\,.$$

On the other hand,

$$\varphi(d_i(x_{k,k}) = [d_i^h d_i^v(x_{k,k}), \underline{k-1} \xrightarrow{\mathrm{Id}} \underline{k-1}] = [d_i^v(x_{k,k}), \underline{k-1} \xrightarrow{\delta^i} \underline{k}]\,.$$

The case for s_i is identical.

Now define $\psi\colon \coprod_{n\geq 0} X_n \times \Delta[n] \to \mathrm{diag}(X)$ by $\psi(x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n}) \mapsto \eta_h^*(x_{n,k})$. We will show that this is a simplicial map. Since η is a combination of face and degeneracy maps and vertical face and degeneracy maps commute with their horizontal counterparts, we have that

$$d_i\psi(x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n}) = d_i^h d_i^v \eta_h^*(x_{n,k}) = d_i^h \eta_h^* d_i^v(x_{n,k})$$

but also

$$\psi d_i(x_{n,k}, \underline{k} \xrightarrow{\eta} \underline{n}) = \psi(d_i^v(x_{n,k}), \underline{k-1} \xrightarrow{\delta^i} \underline{k} \xrightarrow{\eta} \underline{n}) = d_i^h \eta_h^* d_i^v(x_{n,k})$$

since $(\delta^i)_h^* = d_i^h$ by definition. The case for s_i is done in exactly the same way and so ψ is a simplicial map.

We will now show that ψ respects the relations we use to define $|X|$ from $\coprod_{n\geq 0} X_n \times \Delta[n]$, so that Lemma 3.6 gives us an induced simplicial map

$$\widetilde{\psi}\colon |X| \to \mathrm{diag}(X)\,.$$

Let $x_{n,k} \in X_{n,k}$ and $\underline{k} \xrightarrow{\eta} \underline{n-1}$. Then we have

$$\psi(d_i^h(x_{n,k}), \eta) = \eta_h^* d_i^h(x_{n,k})$$

which equals

$$\psi(x_{n,k}, \delta_*^i(\eta)) = \eta_h^*(\delta^i)_h^*(x_{n,k}) = \eta_h^* d_i^h(x_{n,k})\,.$$

The same holds in the same way for the other relation involving degeneracy maps. Since all relations are sequences of such relations, ψ respects the defining relations of $|X|$.

As the reader probably has guessed, φ and $\widetilde{\psi}$ are inverses of each other, hence simplicial isomorphisms of $|X|$ and $\mathrm{diag}(X)$. We check that for $x_{n,n} \in \mathrm{diag}(X)_n$, $\widetilde{\psi}\psi(x_{n,n}) = \widetilde{\psi}[x_{n,n}, \underline{n} \xrightarrow{\mathrm{Id}} \underline{n}] = x_{n,n}$. For $x_{n,k} \in X_{n,k}$ and $\underline{k} \xrightarrow{\eta} \underline{n}$ it holds that

$$\psi\widetilde{\varphi}[x_{n,k}, \eta] = \varphi(\eta_h^*(x_{n,k})) = [\eta_h^*(x_{n,k}), \underline{k} \xrightarrow{\mathrm{Id}} \underline{k}] = [x_{k,n}, \eta]\,.$$

As for naturality, let $f\colon X \to Y \in \mathbf{SSp}$. We see directly that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{diag}(X) & \xrightarrow{\varphi} & |X| \\ \downarrow{\scriptstyle \mathrm{diag}(f)} & & \downarrow{\scriptstyle |f|} \\ \mathrm{diag}(Y) & \xrightarrow{\varphi} & |Y| \end{array} \qquad \begin{array}{ccc} x_{n,n} & \longmapsto & [x_{n,n}, \underline{n} \xrightarrow{Id} \underline{n}] \\ \downarrow & & \downarrow \\ f(x_{n,n}) & \longmapsto & [f(x_{n,n}), \underline{n} \xrightarrow{Id} \underline{n}] \end{array}$$

This completes the proof. □

4 Nerves and the simplicial replacement

When constructing $\Delta[n] = \mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathbf{n})$ we use that $\mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathbf{D})$ is a functor for $\mathbf{D} \in \mathbf{Cat}$. The generalization is called the nerve construction.

Definition 4.1. For a small category $\mathbf{D}$, we define the simplicial space $N(\mathbf{D})$ called the nerve of $\mathbf{D}$ by

$$N(\mathbf{D}) = \mathrm{Hom}_{\mathbf{Cat}}(\cdot, \mathbf{D}) \colon \Delta^{op} \to \mathbf{Sets}$$

The face and degeneracy maps are again coming from precomposition of the face and degeneracy maps in Δ, so for $\tau \colon \mathbf{n} \to \mathbf{D}$, $d_i \tau = \tau \circ \delta^i$ and $s_i \tau = \tau \circ \sigma^i$

More concretely, an n-simplex $\sigma \in N(\mathbf{D}_n)$ is a sequence of objects and morphisms $\sigma = (\sigma(0) \xrightarrow{\alpha_1} \sigma(1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} \sigma(n))$ in $\mathbf{D}$. The face and degeneracy maps are then defined as follows:

$$d_i(\sigma) = \begin{cases} (\sigma(1) \to \dots \to \sigma(n)), & i = 0 \\ (\sigma(0) \to \dots \to \sigma(i-1) \xrightarrow{\alpha_{i+1}\alpha_i} \sigma(i+1) \to \dots \to \sigma(n)), & 0 < i < n \\ (\sigma(0) \to \dots \to \sigma(n-1)), & i = n \end{cases}$$

and

$$s_i(\sigma) = (\sigma(0) \to \dots \to \sigma(i) \xrightarrow{\mathrm{Id}} \sigma(i) \to \sigma(i+1) \to \dots \to \sigma(n)) \,.$$

In the above, it is understood that the maps not explicitely named are the corresponding ones in σ. In what follows, for $\sigma \in N(\mathbf{D})_n$ the first morphism $\sigma(0) \to \sigma(1)$ will always be denoted α_1. It will play a special role when we define the simplicial replacement.

Proposition 4.2. *The nerve construction gives us a functor $N \colon \mathbf{Cat} \to \mathbf{Sp}$ which on morphisms is defined by postcomposition.*

Proof. This follows immediately from the fact that postcomposition commutes with precomposition. $\square$

Proposition 4.3. *$N \colon \mathbf{Cat} \to \mathbf{Sp}$ respects (finite) products (up to isomorphism).*

Proof. Consider the product category $\mathbf{C} \times \mathbf{D}$. Then $N(\mathbf{C} \times \mathbf{D})_n = \mathrm{Hom}_{\mathbf{Sp}}(\mathbf{n}, \mathbf{C} \times \mathbf{D}) \cong \mathrm{Hom}_{\mathbf{Sp}}(\mathbf{n}, \mathbf{C}) \times \mathrm{Hom}_{\mathbf{Sp}}(\mathbf{n}, \mathbf{D}) = N(\mathbf{C})_n \times N(\mathbf{D})_n$ naturally. Furthermore, this isomorphism clearly commutes with d_i, s_i, so $N(\mathbf{C} \times \mathbf{D}) \cong N(\mathbf{C}) \times N(\mathbf{D})$. $\square$

Definition 4.4. Let $F \colon \mathbf{D} \to \mathbf{Sp}$ be a diagram of spaces. Then we define the simplicial replacement $\amalg_* F \in \mathbf{SSp}$ by

$$(\amalg_* F)_n = \coprod_{\sigma \in N(\mathbf{D})_n} F(\sigma(0))$$

where s_i is sending $F(\sigma(0))$ to $F(s_i\sigma(0))$ by the identity map and d_i is sending $F(\sigma(0))$ to $F(d_i\sigma(0)$ by the identity when $i > 0$ and by the map $F(\alpha_1)$ when $i = 0$.

Remark 4.5. We will work quite a lot with the simplicial replacement $\amalg_* F$ and thus we will need to introduce some notation. The component $F(\sigma(0)) \subseteq \operatorname{hocolim}(F)_n$ indexed by $\sigma \in N(\mathbf{D})_n$ will be denoted with $F(\sigma(0))^{(\sigma)}$. For $x_k \in F(\sigma(0))_k^{(\sigma)}$ we write $x_k^{(\sigma)}$. All the information we need is in this notation: inner dimension, $F(\sigma(0))$ and the index σ.

Proposition 4.6. *If* $\mathbf{D}$ *is a small category, then*

$$\amalg_* \colon \mathbf{Sp}^{\mathbf{D}} \to \mathbf{SSp}$$

is a functor.

Proof. Let $F, G \colon \mathbf{D} \to \mathbf{Sp}$ and let $\tau \colon F \to G$ be a natural transformation so that we have simplicial maps $\tau_d \colon F(d) \to G(d)$ for every $d \in \mathbf{D}$.

For every (outer) dimension n, define

$$(\amalg_* \tau)_n \colon \coprod_{\sigma \in N(\mathbf{D})_n} F(\sigma(0)) \to \coprod_{\sigma \in N(\mathbf{D})_n} G(\sigma(0))$$

by

$$F(\sigma(0))_k^{(\sigma)} \ni x_k^{(\sigma)} \mapsto \tau_{\sigma(0)}(x_k^{(\sigma)}) = \big(\tau_{\sigma(0)}(x_k)\big)^{(\sigma)}$$

on each component. Each of these is a simplicial map, so by Proposition 2.2 we get simplicial maps $(\amalg_* \tau)_n$. We need to prove that $\amalg_* \tau \colon \amalg_* F \to \amalg_* G$ is a map of simplicial spaces. Remember that for $i > 0$, $d_i(x_k^{(\sigma)}) = x_k^{(d_i \sigma)}$ and $d_0(x_k^{(\sigma)}) = (F(\alpha_1)(x_k))^{(d_0 \sigma)}$. Also, $s_i(x_k^{(\sigma)}) = x_k^{(s_i \sigma)}$ for all i.

Thus for $i > 0$,

$$\begin{aligned} \amalg_* \tau(d_i(x_k^{(\sigma)})) &= \tau_{(d_i\sigma)(0)}(x_k^{(d_i \sigma)}) \\ &= (\tau_{\sigma(0)}(x_k))^{(d_i \sigma)} \\ &= d_i((\tau_{\sigma(0)}(x_k))^{(\sigma)}) \\ &= d_i(\amalg_* \tau(x_k^{(\sigma)})) \end{aligned}$$

and correspondingly for all s_i. For the remaining case $i = 0$, we have

$$\amalg_* \tau(d_0(x_k^{(\sigma)})) = \amalg_* \tau((F(\alpha_1)(x_k))^{(d_0\sigma)}) = (\tau_{(d_0\sigma)(0)}(F(\alpha_1)(x_k)))^{(d_0\sigma)}$$

but

$$d_0(\amalg_* \tau(x_k^{(\sigma)})) = d_0((\tau_{\sigma(0)}(x_k))^{(\sigma)}) = (G(\alpha_1)(\tau_{\sigma(0)}(x_k)))^{(d_0\sigma)}.$$

However, τ is a natural isomorphism, so the following diagram commutes:

$$\begin{tikzcd} F(\sigma(0)) \arrow[r, "\tau_{\sigma(0)}"] \arrow[d, "F(\alpha_1)"] & G(\sigma(0)) \arrow[d, "G(\alpha_1)"] \\ F(\sigma(1)) \arrow[r, "\tau_{\sigma(1)}"] & G(\sigma(1)) \end{tikzcd}$$

Because $(d_0\sigma)(0) = \sigma(1)$, this implies that $\amalg_* \tau(d_0(x_k^{(\sigma)})) = d_0(\amalg_* \tau(x_k^{(\sigma)}))$. We conclude that $\amalg_* \tau$ is a map of simplicial spaces.

Finally, if $\tau = \mathrm{Id}_F$, then clearly $\amalg_* \tau$ is also the identity. Let $F \xrightarrow{\tau} G \xrightarrow{\theta} H$ be a sequence of natural transformations. Then for every $d \in \mathbf{D}$,

$$\begin{tikzcd} F(d) \arrow[r, "\tau_d"] \arrow[dr, "(\theta\tau)_d"'] & G(d) \arrow[d, "\theta_d"] \\ & H(d) \end{tikzcd}$$

commutes and hence by the definition of $\amalg_* \tau$, $\amalg_*(\theta\tau) = (\amalg_* \theta) \circ (\amalg_* \tau)$.

$\square$

5 Homotopy colimits

Definition 5.1. If $F\colon \mathbf{D} \to \mathbf{Sp}$ is a diagram in $\mathbf{Sp}$, then we define

$$\mathrm{hocolim}(F) = \mathrm{diag}(\amalg_* F)\,.$$

Example 5.2. It is not hard to verify that if we let $\mathbf{D}$ be the category $\mathbf{1}$ with two objects and a unique morphism $0 \le 1$ and let $F\colon \mathbf{D} \to \mathbf{Sp}$ be defined by $F(0) = F(1) = \Delta[0] = *$, then even though all the spaces in the diagram are sets (discrete spaces), the homotopy colimit is $\Delta[1]$ which is not discrete.

This example is in analogue with the topological example where we have two discrete one-point (topological) spaces and glue them together with an interval $I = |\Delta[1]|$. In both cases we thus see that even though colimits of diagram of discrete spaces are discrete, the homotopy colimit need not be.

Example 5.3. Let $F\colon \mathbf{D} \to \mathbf{Sp}$ be a a constant functor, i.e. $F(d) = X$ for every $d \in \mathbf{D}$ and $F(d \xrightarrow{\alpha} d') = \mathrm{Id}_X$ for every morphism α. Then we prove that $\mathrm{hocolim}(F) \cong X \times N(\mathbf{D})$.

Since $F(d) = X$ for every $d \in \mathbf{D}$, we get that

$$\mathrm{hocolim}(F)_n = \mathrm{diag}(\amalg_* F)_n = \coprod_{\sigma \in N(\mathbf{D})_n} X_n^{(\sigma)}$$

where $X^{(\sigma)} = X$ for every $\sigma \in N(\mathbf{D})_n$. A face map in a diagonal is by definition a composition of the horizontal (outer) face map and the vertical (inner) face map. Thus $d_i(x_n^{(\sigma)}) = (d_i x_n)^{(d_i\sigma)}$ where the first d_i is the inner face map in $X^{(\sigma)}$ and the second one is the face map in $N(\mathbf{D})$. For $i = 0$ we note that $F(\alpha_1)$ is the identity map for any $\alpha_1\colon \sigma(0) \to \sigma(1)$, so also $d_0(x_n^{(\sigma)}) = (d_0 x_n)^{(d_0\sigma)}$ in this particular example. Also, $s_i(x_n^{(\sigma)}) = (s_i x_n)^{(s_i\sigma)}$.

Define $\varphi\colon \mathrm{hocolim}(F) \to X \times N(\mathbf{D})$ at dimension n by $x_n^{(\sigma)} \mapsto (x_n, \sigma)$. This is clearly a bijection dimensionwise, so it suffices to show that φ is a simplicial map. This is easy to verify:

$$\varphi d_i(x_n^{(\sigma)}) = \varphi((d_i x_n)^{(d_i\sigma)}) = (d_i x_n, d_i\sigma) = d_i(x_n, \sigma) = d_i \varphi(x_n^{(\sigma)})$$

and likewise for s_i.

Corollary 5.4. *With the definition above,* $\mathrm{hocolim}\colon \mathbf{D}^{\mathbf{Sp}} \to \mathbf{Sp}$ *is a functor.*

Proof. We compose the functor $\amalg_*\colon \mathbf{Sp}^{\mathbf{D}} \to \mathbf{SSp}$ with $\mathrm{diag}\colon \mathbf{SSp} \to \mathbf{Sp}$. By definition, $\mathrm{hocolim} = \mathrm{diag} \circ \amalg_*$. $\square$

Definition 5.5. Let $F\colon \mathbf{D} \to \mathbf{Sets}$ be a diagram. We will associate a category $\mathrm{Tr}(F)$ called the transport category of F as follows. The objects are pairs (d, x) where $d \in \mathrm{Ob}(\mathbf{D})$ and $x \in F(d)$. A morphism $\widetilde{f}\colon (d, x) \to (d', x')$ is a morphism $f : d \to d' \in \mathbf{D}$ such that $F(f)(x) = x'$. Composition is defined by the composition in $\mathbf{D}$.

Note that the the morphisms from $(d, x) \in \mathrm{Tr}(F)$ are characterized by the morphisms with domain d in $\mathbf{D}$ since x' is determined by x.

Proposition 5.6. *For $F\colon \mathbf{D} \to \mathbf{Sets}$, $N(\mathit{Tr}(F)) \cong \mathrm{hocolim}(F)$.*

Proof. We view each $F(d) \in \mathbf{Sets}$ as discrete spaces. Then

$$(\mathrm{hocolim}(F))_n = \mathrm{diag}(\amalg_* F)_n = \coprod_{\sigma \in N(\mathbf{D})_n} F(\sigma(0))$$

since $F(d)_n = F(d)$ for every $d \in \mathbf{D}$ and $n \geq 0$. We see that n-simplices in $\mathrm{hocolim}(F)$ correspond exactly to pairs (σ, x) where $\sigma \in N(\mathbf{D})_n$ and $x \in F(\sigma(0))$. As noted previously, this also holds for n-simplices in $N(\mathrm{Tr}(F))$ since an element in $N(\mathrm{Tr}(F))_n$ is a sequence

$$(c_0, x) \xrightarrow{\alpha_1} (c_1, F(\alpha_1)(x)) \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} (c_n, F(\alpha_n \cdots \alpha_1)(x))$$

with $x \in F(c_0)$ and $(c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} c_n) \in N(\mathbf{D})_n$. We thus have an obvious isomorphism if this identification respects the face and degeneracy maps in both cases. We first look at $\mathrm{hocolim}(F)$.

Let $\sigma \in N(\mathbf{D})_n$ and $x_n^{(\sigma)} \in F(\sigma(0)) \subseteq (\mathrm{hocolim}(F))_n$. For $i > 0$, $d_i(x_n^{(\sigma)}) = (d_i x_n)^{(d_i\sigma)} = x_n^{(d_i\sigma)}$ since all face and degeneracy maps are identity in the discrete space $F(\sigma(0))$. For $i = 0$, we have $d_0(x_n^{(\sigma)}) = (F(\alpha_1)(x_n))^{(d_0\sigma)}$.

Turning to $N(\mathrm{Tr}(F))$, the n-simplex corresponding to $x_n^{(\sigma)}$ is

$$\widetilde{\sigma} = \Big((\sigma(0), x_n) \xrightarrow{\alpha_1} (\sigma(1), F(\alpha_1)(x_n)) \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_n} (\sigma(n), F(\alpha_n \cdots \alpha_1)(x))\Big) .$$

We will suppress the notation slightly in what follows, but there should be no confusion as to what is meant. For $i > 0$,

$$d_i(\widetilde{\sigma}) = \Big((\sigma(0), x_n) \to \ldots \to (\sigma(i-1), \ldots) \xrightarrow{\alpha_{i+1}\alpha_i} (\sigma(i+1), \ldots) \to \ldots\Big)$$

which corresponds to $x_n^{(d_i\sigma)}$. Finally,

$$d_0(\widetilde{\sigma}) = \Big((\sigma(1), F(\alpha_1)(x_n)) \xrightarrow{\alpha_2} \ldots\Big)$$

which corresponds to $(F(\alpha_1)(x_n))^{(d_0\sigma)}$.

The verification for s_i is of course done in the same way. $\square$

6 Simplicial homotopies

Proposition 6.1. *We call a simplicial map $f\colon X \to Y$ a weak equivalence if $|f|\colon |X| \to |Y|$ is a weak equivalence of topological spaces, meaning that $\pi_n|f|\colon \pi_n(|X|, x_0) \to \pi_n(|Y|, |f|(x_0))$ is an isomorphisms for every $n \in \mathbb{N}$ and every basepoint $x_0 \in |X|$.*

Definition 6.2. We define two simplicial maps $f, g\colon X \to Y$ to be (simplicially) homotopic if there is a simplicial map $h\colon X \times \Delta[1] \to Y$ that restricts to f and g on the subspaces $X \times (0)$ and $X \times (1)$ respectively. We use the notation $f \simeq g$ for homotopic maps.

Corollary 6.3. *If $f \simeq g\colon X \to Y$, then $|f| \simeq |g|\colon |X| \to |Y|$.*

Proof. There is a simplicial map $h\colon X \times \Delta[1] \to Y$ by assumption. We apply the geometric realization functor and obtain $|h|\colon |X| \times I \to |Y|$ by Proposition 2.3 and this is a topological homotopy between $|f|$ and $|g|$. □

Lemma 6.4. *Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be two functors. There is a natural transformation $\tau\colon F \to G$ if and only if there is a functor $\rho\colon \mathbf{C} \times \mathbf{1} \to \mathbf{D}$ with $\rho|_{\mathbf{C}\times 0} = F$, $\rho|_{\mathbf{C}\times 1} = G$.*

Proof. Suppose that we have a functor ρ as stated above. That means that for every $c \in \mathbf{C}$ we have a morphism $\rho(1_c \times (0 \le 1))\colon F(d) \to G(d)$ for every $f\colon c \to c' \in \mathbf{C}$, the following diagram commutes in $\mathbf{C} \times \mathbf{1}$:

$$\begin{tikzcd} c \times 0 \arrow[r, "f\times 1_0"] \arrow[d, "1_c\times(0\le 1)"] & c' \times 0 \arrow[d, "1_{c'}\times(0\le 1)"] \\ c \times 1 \arrow[r, "f\times 1_1"] & c' \times 1 \end{tikzcd}$$

Let $\tau_c = \rho(1_c \times (0 \le 1))$. Then by applying ρ to the diagram we obtain

$$\begin{tikzcd} F(c) \arrow[r, "F(f)"] \arrow[d, "\tau_c"] & F(c') \arrow[d, "\tau_{c'}"] \\ G(c) \arrow[r, "G(f)"] & G(c') \end{tikzcd} .$$

This commutes for every $f\colon c \to c' \in \mathbf{C}$, so τ is a natural transformation.

Conversely, suppose that we have a natural transformation $\tau\colon F \to G$. We define a functor $\rho\colon \mathbf{C} \times \mathbf{1} \to \mathbf{D}$ by $\rho(c, 0) = F(c)$, $\rho(c, 1) = G(c)$ on objects. For morphisms, we define $\rho(1_c, (0 \le 1)) = \rho_d$, $\rho(f, 1_0) = F(f)$ and $\rho(f, 1_1) = G(f)$. A general morphism in $\mathbf{C} \times \mathbf{1}$ is a composition of these and it can easily be shown to be a functor by using that F and G are functors together with the commutativity of τ with F, G. □

Corollary 6.5. *If $\tau\colon F \to G$ is a natural transformation of two functors $F, G\colon \mathbf{C} \to \mathbf{D}$, then $N(F) \simeq N(G)$.*

Proof. By the previous Lemma, there exists a functor $\rho\colon \mathbf{C} \times \mathbf{1} \to \mathbf{D}$ which restricts to F and G on $\mathbf{C} \times 0$ and $\mathbf{C} \times 1$ respectively. We have seen that the nerve functor preserves products, so we get a simplicial map $N(\rho)\colon N(\mathbf{C}) \times N(\mathbf{1}) = N(\mathbf{C}) \times \Delta[1] \to \mathbf{D}$ which restricts to $N(F)$ and $N(G)$. By definition, $N(F) \simeq N(G)$ as simplicial maps. □

Proposition 6.6. *If $F\colon \mathbf{C} \to \mathbf{D}$ is an equivalence of categories, then $N(F)$ is a weak equivalence.*

Proof. The assumption is that there exists a functor $G\colon \mathbf{D} \to \mathbf{C}$ such that $GF \cong 1_{\mathbf{C}}$ and $FG \cong 1_{\mathbf{D}}$ (this only means that they are naturally isomorphic, not necessarily equal). By Corollary 6.5, $N(F)N(G) \simeq 1_{N(\mathbf{C})}$ and $N(G)N(F) \simeq 1_{N(\mathbf{D})}$. From this it follows that $|N(F)|$ is a homotopy equivalences and thus in particular a weak equivalence. □

Definition 6.7. A simplicial space X is said to be weakly contractible if the unique simplicial map $X \to *$ is a weak equivalence. Equivalently, X is weakly contractible if $|X|$ is a contractible topological space.

Proposition 6.8. *If $\mathbf{D}$ is a category with either an initial or a terminal object, then $N(\mathbf{D})$ is weakly contractible.*

Proof. We will only do the proof where $\mathbf{D}$ has an initial object d. The case where $\mathbf{D}$ has a terminal object is very similar.

Let $*$ denote the category with one object and no nonidentity morphisms. We denote the unique functor $\mathbf{D} \to *$ with F and define $G\colon * \to \mathbf{D}$ by $G(*) = d$. Then $FG = 1_*$ of course. We will construct a natural transformation $\tau\colon GF \to 1_{\mathbf{D}}$ and then the statement follows from Corollary 6.5.

Note that $GF(x) = d$ for every $x \in \mathbf{D}$ and for every morphism f we have that $GF(x \xrightarrow{f} x') = (d \xrightarrow{1_d} d)$. Let $\tau\colon GF \to 1_{\mathbf{D}}$ be defined by the unique morphism $\tau_x\colon d \to x$ for every $x \in \mathbf{D}$. Then for the morphism $x \xrightarrow{f} x' \in \mathbf{D}$ we need to show commutativity of the diagram

$$\begin{tikzcd} d \arrow[r, "\tau_x"] \arrow[d, "1_d"] & x \arrow[d, "f"] \\ d \arrow[r, "\tau_{x'}"] & x' \end{tikzcd}$$

but this obviously holds because $f\tau_x$ is a map with domain d, so by the universal property of the initial object d we have $f\tau_x = \tau_{x'}$. By Corollary 6.5, $N(\mathbf{D})$ is weakly equivalent to $N(*) = *$. □

7 G-spaces

Any group G can be considered as a category with one object (usually denoted with $*$) and morphisms corresponding to the elements of the group G. We denote this category with $\mathbf{G}$ in order to distinguish them. Note that a group homomorphism $G \to H$ is the same thing as a functor $\mathbf{G} \to \mathbf{H}$. There are several characterizations of a group action for a group G and a set X. One nice characterization is that a group action ρ is a functor $\rho\colon \mathbf{G} \to \mathbf{Sets}$ with $\rho(*) = X$. Since every morphism in $\mathbf{G}$ is an isomorphism, also $\rho(g)\colon X \to X$ are isomorphism. Furthermore, we define the set of orbits $X/G = \{Gx \,;\, x \in X\}$ of orbits. It is convenient to use the notation $[x] := Gx$ for the equivalence classes in X/G.

This concept is easily generalized to arbitrary categories by simply replacing the category **Sets** with another category of our choice. We will consider G-spaces, that is, functors $\rho\colon \mathbf{G} \to \mathbf{Sp}$. We see that for a given group G, a G-action on the space $X \in \mathbf{Sp}$ concists of simplicial isomorphisms $\rho(g)\colon X \to X$ for $g \in G$, which in turn consists of functions $\rho(g)_n\colon X_n \to X_n$ commuting with the face and degeneracy maps. Because of this fact, we can form the quotient space X/G which is defined by $(X/G)_n = X_n/G$ and $d_i[x_n] = [d_i x_n], s_i[x_n] = [s_i x_n]$.

Example 7.1. In this example we will see that there is no nontrivial $\mathbb{Z}/2$-action on $\Delta[1] \in \mathbf{Sp}$. Denote the 0-simplices with $(0), (1)$ and the non-degenerate 1-simplex with $(0, 1)$.

If g is the generator of $\mathbb{Z}/2$, then the action on $\Delta[1]$ is completely determined by $g\cdot(0, 1)$ since any simplex $x \in \Delta[1]$ can be written as $\eta^*(x)$ for some $\eta \in \Delta$. In other words, $\Delta[1]$ is generated by the simplex $(0, 1)$. For more on this concept, see for example [Fri12]. To get a non-trivial action on $\Delta[1]$, it must thus hold that $g \cdot (0, 1) \neq (0, 1)$. However, then without loss of generality, $g\cdot(0, 1) = (0, 0)$, so $g \cdot (0) = g \cdot (d_1(0, 1)) = d_1(g \cdot (0, 1)) = (0)$ but also $g \cdot (1) = g \cdot (d_0(0, 1)) = d_0(g \cdot (0, 1)) = (0)$, implying that the action on $\Delta[1]_0$ is not a bijection.

If we consider Σ as defined in Example 9.4, then it is not hard to show that $g \cdot (0, 1) = (1, 0)$, $g \cdot (1, 0) = (0, 1)$ defines a group action. We see that Σ/G for $G = \mathbb{Z}/2$ is a space with one 0-simplex v and one degenerate 1-simplex e which is different from $s_0(v)$. Of course, $|\Sigma/G| \cong |\Sigma|/G \cong \mathbb{R}P^1$.

Definition 7.2. For a group G, we define the category $\mathbf{EG}$ with objects corresponding to elements $g \in G$ and exactly one morphism between every object. We also define the simplicial set $EG = N(\mathbf{EG}) \in \mathbf{Sp}$.

Since every morphism $g \to gh = g' \in \mathbf{EG}$ is unique, we will write it simply as $g \xrightarrow{\cdot h} gh$. Note however that with this notation, $\cdot h$ considered as a functor is contravariant.

We define an action on $\mathbf{EG}$, e.g. a functor $\lambda\colon \mathbf{G} \to \mathbf{Cat}$ with $\lambda(*) = \mathbf{EG}$ and $\lambda(* \xrightarrow{g} *) = \lambda_g\colon \mathbf{EG} \to \mathbf{EG}$, which in turn is defined by $\lambda_g(h) = gh$ and $\lambda_g(h_1 \xrightarrow{\cdot h_2} h_1h_2) = (gh_1 \xrightarrow{\cdot h_2} gh_1h_2)$.

By composing this with the nerve functor, we get a a G-action on the space $EG = N(\mathbf{EG})$ by $N \circ \lambda\colon G \to \mathbf{Cat} \to \mathbf{Sp}$. One easily verifies that this action is defined by

$$g \cdot (g_0 \to g_1 \to \ldots \to g_n) = (gg_0 \to gg_1 \to \ldots \to gg_n)$$

for $g \in G$ and $(g_0 \to \ldots \to g_n) \in EG$

Definition 7.3. Let $E\colon \mathbf{G} \to \mathbf{Sp}$ be a weakly contractible G-space. From the theory of covering spaces, we see that $|E| \to |E/G| \cong |E|/G$ is a covering space. Since $|E|$ is contractible and the deck of transformations is G, $|E/G|$ is a classifying space for G in the topological sense. We call E/G a classifying space for G.

For a reference on covering spaces, see [Hat02], in particular Proposition 1.40. For classifying spaces ($K(G, 1)$-spaces), see [Hat02, Ch. 1B].

Proposition 7.4. *Let G be a group. Then $BG := N(\mathbf{G})$ is a classifying space for G.*

Proof. From the discussion above, we see that $EG = N(\mathbf{EG})$ is a weakly contractible space by Proposition 6.8 and we also have a G-action on it which is easily seen to be free. Furthermore, $EG/G \cong N(\mathbf{G})$ by an obvious isomorphism (just think about what the two spaces are!). Hence $N(\mathbf{G})$ is a classifying space for G. $\square$

8 Homotopy orbit spaces

Definition 8.1. For a G-space $X\colon \mathbf{G} \to \mathbf{Sp}$, we define the homotopy orbit space

$$X_{hG} := \operatorname{hocolim}(X)\,.$$

Example 8.2. Let $X\colon \mathbf{G} \to \mathbf{Sp}$ be a constant functor with $X(*) = \widetilde{X}$. Then this is the same as Example 5.3, so $X_{hG} \cong \widetilde{X} \times N(\mathbf{G})$. In particular, for the constant functor with $X(*) = * = \Delta[0]$, $X_{hG} \cong N(\mathbf{G}) = BG$.

Proposition 8.3. *For the G-space $X\colon \mathbf{G} \to \mathbf{Sp}$, it holds that $X_{hG} \cong (X \times EG)/G$ where $X \times EG$ has been given the diagonal action.*

Proof. An n-simplex in $N(\mathbf{G})$, if viewed as EG/G with the notation above, is a sequence $\sigma = (* \xrightarrow{\cdot g_1} * \xrightarrow{\cdot g_2} \dots \xrightarrow{\cdot g_n} *)$ but multiplication from the right is strictly speaking not a group action since it is contravariant. In order to solve this we consider the opposite group G^{op} and the opposite category $\mathbf{G}^{\mathrm{op}}$. Thus when we look at $\operatorname{hocolim}(X)$, we set $X(\cdot g_1)(x_n) := g_1^{-1}(x_n)$.

The proof will be in three steps. First we construct a simplicial map $\varphi\colon X \times EG \to X_{hG}$. After that we verify that φ sends elements in the same orbit of $(X \times EG)/G$ to the same element, so that we get a simplicial map $\widetilde{\varphi}\colon (X \times EG)/G \to X_{hG}$. As a last step, we prove that this is an isomorphism.

One of the hurdles here is the notation. For an element $g_0 \in G$ and an n-simplex $\sigma = (* \xrightarrow{\cdot g_1} * \xrightarrow{\cdot g_2} \dots \xrightarrow{\cdot g_n} *) \in N(G)_n$, we can assign a unique n-simplex σ_{g_0} in $N(\mathbf{EG})_n = EG_n$, namely

$$\sigma_{g_0} = (g_0 \to g_0 g_1 \to \dots \to g_0 \cdots g_n)\,.$$

Using this notation, we define

$$\varphi\colon X \times EG \to X_{hG}, \quad (x_n, \sigma_{g_0}) \mapsto (g_0^{-1} x_n) \in X_n^{(\sigma)}\,.$$

For $i > 0$,

$$d_i \varphi(x_n, \sigma_{g_0}) = d_i((g_0^{-1} x_n)^{(\sigma)}) = (d_i g_0^{-1} x_n)^{(d_i \sigma)} = (g_0^{-1} d_i x_n)^{(d_i \sigma)}$$

since the automorphisms given by the G-action are simplicial maps and so commute with all face and degeneracy maps. On the other hand,

$$\varphi d_i(x_n, \sigma_{g_0}) = \varphi(d_i x_n, (d_i \sigma)_{g_0}) = (g_0^{-1} d_i x_n)^{(d_i \sigma)}$$

so $d_i \varphi = \varphi d_i$. One proves in the same way that $s_i \varphi = \varphi s_i$ for all i. It remains to show that $d_0 \varphi = \varphi d_0$.

Remembering that $X(\cdot g_1) = g_1^{-1}$ by contravariance, we find that

$$d_0 \varphi(x_n, \sigma_{g_0}) = d_0((g_0^{-1} x_n)^{(\sigma)}) = (g_1^{-1} d_0 g_0^{-1} x_n)^{(d_0 \sigma)} = (g_1^{-1} g_0^{-1} d_0 x_n)^{(d_0 \sigma)}\,.$$

Since $d_0(\sigma_{g_0}) = (g_0 g_1 \to \ldots \to g_0 \cdots g_n)$, we get

$$\varphi d_0(x_n, \sigma_{g_0}) = \varphi(d_0 x_n, d_0(\sigma_{g_0})) = ((g_0 g_1)^{-1} d_0 x_n)^{(d_0 \sigma)}$$

an hence we conclude that $\varphi\colon X \times EG \to X_{hG}$ is a simplicial map.

Next, for $g \in G$,

$$\begin{aligned}\varphi(g \cdot (x_n, \sigma_{g_0})) &= \varphi(g x_n, (g g_0 \to \ldots \to g g_0 \cdots g_n)) \\ &= ((g g_0)^{-1} g x_n)^{(\sigma')} = (g_0^{-1} x_n)^{(\sigma')} \\ &= \varphi(x_n, \sigma_{g_0})\end{aligned}$$

since by inspection, $\sigma' := (* \xrightarrow{\cdot g_1} * \xrightarrow{\cdot g_2} \ldots \xrightarrow{\cdot g_n} *) = \sigma$. Thus we have a well-defined map

$$\widetilde{\varphi}\colon (X \times EG)/G \to X_{hG}\,.$$

It remains to show that this is an isomorphism. For this it is sufficient to show bijectivity dimensionwise by Lemma 3.11. For surjectivity, take $x_n^{\sigma} \in X_n^{(\sigma)}$, where $\sigma = (* \xrightarrow{\cdot g_1} * \xrightarrow{\cdot g_2} \ldots \xrightarrow{\cdot g_n} *)$ as before. Then $\widetilde{\varphi}(x_n, \sigma_{1_G}) = x_n^{(\sigma)}$, where 1_G is the identity element in G.

For injectivity, suppose that $\widetilde{\varphi}[x_n, \sigma_{g_0}] = \widetilde{\varphi}[y_n, \tau_{h_0}]$ which, by definition, is $(g_0^{-1} x_n)^{(\sigma)} = (h_0^{-1} y_n)^{(\tau)}$. This can only hold if they lie in the same component, meaning that $\sigma = \tau$. Thus σ_{g_0} and τ_{h_0} differ only by the object in $\mathbf{EG}$ they begin with, say $h_0 = g g_0$, where $g \in G$. But inside this set, we must also have $g_0^{-1} x_n = h_0^{-1} y_n = g_0^{-1} g^{-1} y_n$, so $y_n = g x_n$. Clearly this implies that $g \cdot (x_n, \sigma_{g_0}) = (g x_n, g \sigma_{g_0}) = (y_n, \tau_{h_0})$, so they lie in the same orbit and therefore represent the same element. □

Definition 8.4. Given a group G, let $\mathbf{GSp} := \mathbf{Sp}^{\mathbf{G}}$ be the category of G-spaces. To remind the reader: objects are functors $\mathbf{G} \to \mathbf{Sp}$, and morphisms are natural transformations.

Since hocolim $=$ diag $\circ\, \amalg_*$ is a functor, we have a functor

$$hG = \text{hocolim}\colon \mathbf{Gsp} \to \mathbf{Sp}\,.$$

Thus for a diagram of G-spaces $F\colon \mathbf{D} \to \mathbf{GSp}$, we obtain a diagram of spaces by postcomposing with the functor hG:

$$F_{hG} = hG \circ F\colon \mathbf{D} \to \mathbf{Sp}$$

so we can consider hocolim(F_{hG}).

On the other hand, we can consider the homotopy colimit if we view F simply as a diagram of spaces. Formally, we have the forgetful functor $U\colon \mathbf{GSp} \to \mathbf{Sp}$ sending $\mathbf{G} \xrightarrow{F} \mathbf{Sp}$ to $G(*)$ and postcomposing with this functor is the same thing as considering the diagram F as a diagram of spaces without G-action. With just a slight abuse of notation, we write hocolim$(F) :=$ hocolim$(UF) =$ diag$(\amalg_* UF)$. For $x_n^{(\sigma)} \in UF(\sigma(0))_n$, we define an action $g \cdot x_n^{(\sigma)}$ by the action it has in $F(\sigma(0))_n$. This is an action on hocolim(F) if it commutes with the face and degeneracy maps. In the same way as in previous proofs, this will boil down to commutativity of $UF(\alpha_1)$ with face and degeneracy maps, but this holds true since $F(\alpha_1)$ is a G-map. Thus we can also consider $(\text{hocolim}(F))_{hG}$.

Proposition 8.5. *For a group G, a small category $\mathbf{D}$ and a functor $F\colon \mathbf{D} \to \mathbf{GSp}$, there is a natural isomorphism*

$$(\mathrm{hocolim}(F))_{hG} \cong \mathrm{hocolim}(F_{hG})$$

Proof. An n-simplex in $(\mathrm{hocolim}(F))_{hG}$ consists of a pair $(\tau, y) \in N(\mathbf{G})_n \times \mathrm{hocolim}(F)_n$, where in turn $y = (\sigma, x) \in N(\mathbf{D})_n \times F(\sigma(0))_n$. However, an n-simplex in $\mathrm{hocolim}(F_{hG})$ consists of a pair $(\sigma', z) \in N(\mathbf{D})_n \times F(\sigma'(0))_{hG}$ where $z = (\tau', x') \in N(\mathbf{G})_n \times F(\sigma'(0))_n$.

We define $\varphi\colon (\mathrm{hocolim}(F))_{hG} \to \mathrm{hocolim}(F_{hG})$ by

$$(\tau, (\sigma, x)) \mapsto (\sigma, (\tau, x))\,.$$

The face and degeneracy maps in both cases are given by the diagonal face and degeneracy maps, so

$$\begin{aligned}\varphi d_i(\tau, (\sigma, x)) &= \varphi(d_i\tau, (d_i\sigma, d_i x)) \\ &= (d_i\sigma, (d_i\tau, d_i x)) \\ &= d_i(\sigma, (\tau, x)) \\ &= d_i\varphi(\tau, (\sigma, x))\end{aligned}$$

and likewise for s_i. ◻

9 Homology

For $X \in \mathbf{Sp}$, we will now define the the homology of X with coefficients in an abelian group M.

Definition 9.1. Let $\mathbb{Z}[-]\colon \mathbf{Sets} \to \mathbf{AbGrp}$ be the functor that sends a set to the abelian group generated by the elements of the set.

Let $M \otimes -\colon \mathbf{AbGrp} \to \mathbf{AbGrp}$ be the usual tensor functor.

Furthermore, we define $\mathcal{N}\colon \mathbf{AbGrp} \to \mathrm{Comp}(\mathbf{AbGrp})$ which sends a simplicial abelian group A to the chain complex with $\mathcal{N}(A)_n = A_n/(s_0(A_{n-1} + \dots s_{n-1}(A_{n-1})))$ and differential induced by $\partial = \sum(-1)^i d_i$.

Definition 9.2. Let $X \in \mathbf{Sp}$. We form the simplicial abelian group $M \otimes \mathbb{Z}[X]$ and apply the functor $\mathcal{N}$ to this. The homology of this chain complex is called the homology of X with coefficients in M. We write

$$H_*(X; M) := H(\mathcal{N}(M \otimes \mathbb{Z}[X]))$$

and consider it as a graded abelian group.

Remark 9.3. Since $|X|$ is a CW-complex with cells corresponding to non-degenerate simplices of X, it is easy to see that $H_*(X; M) \cong H_*(|X|; M)$ since $\mathcal{N}(M \otimes \mathbb{Z}[X])$ is exactly the cellular complex of $|X|$.

Example 9.4. Let Σ be the simplicial set generated by $\Sigma_0 = \{(0), (1)\}$, $\Sigma_1 = \{(0,1), (1,0), (0,0), (1,1)\}$ and $d_i(v_0, \dots, v_n) = (v_0, \dots, \widehat{v_i}, \dots, v_n)$ (where as usual $\widehat{v_i}$ means that v_i is left out from the sequence) and $s_i(v_0, \dots, v_n) = (v_0, \dots, v_i, v_i, \dots, v_n)$ for $v_1, \dots, v_n \in \{0, 1\}$.

The non-degenerate simplices are $(0),(1),(0,1)$ and $(1,0)$. There are simplices of every dimension, but the rest are degenerate, for example $s_2s_0(1,0) = (1,1,0,0) \in \Sigma_3$. Using $\mathbb{Z}$-coefficients, we see for example that $\mathbb{Z}[\Sigma]_1 = \mathbb{Z}^4$ and $\mathbb{Z}[\Sigma]_2 = \mathbb{Z}^6$, but after applying the functor $\mathcal{N}$, only the factors coming from non-degenerate simplices remain. Hence $\mathcal{N}(\mathbb{Z}[\Sigma])$ looks like

$$\ldots \to 0 \xrightarrow{\partial_2} \mathbb{Z}^{(0,1)} \oplus \mathbb{Z}^{(1,0)} \xrightarrow{\partial_1} \mathbb{Z}^{(0)} \oplus \mathbb{Z}^{(1)} \to 0\,.$$

To see what the differential ∂_1 is, we first compute

$$\begin{cases} d_0(0,1) = (1), \\ d_1(0,1) = (0), \end{cases} \qquad \begin{cases} d_0(1,0) = (0), \\ d_1(1,0) = (1), \end{cases}$$

so $\partial_1 \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is defined by $\partial_1(x,y) = (y-x, x-y)$. We find that $H_0(\Sigma;\mathbb{Z}) = (\mathbb{Z}\oplus\mathbb{Z})/\langle(1,-1)\rangle \cong \mathbb{Z}$, $H_1(\Sigma;\mathbb{Z}) = \mathrm{Ker}(\partial_1) \cong \mathbb{Z}$ and $H_n(\Sigma;\mathbb{Z}) = 0$ for $n \geq 2$. This is of course the same thing as $H_*(|\Sigma|) \cong H_*(S^1)$.

Remark 9.5. One can generalize the above discussion. Let X be a simplicial space and M an abelian group. We picture X as a sequence

$$\ldots \leftarrow X_{n-1} \xleftarrow{d} X_n \xleftarrow{d} X_{n+1} \leftarrow \ldots$$

with each X_n being a simplicial set vertically. For a fix j, we can apply the functor $H_j(-;M)$ dimensionwise to X, so that $H_j(X;M)_k = H_j(X_k;M)$. This gives us an simplicial abelian group which we picture as

$$\ldots \leftarrow H_j(X_{n-1};M) \xleftarrow{\widehat{d}} H_j(X_n;M) \xleftarrow{\widehat{d}} X_j(X_{n+1};M) \leftarrow \ldots\,.$$

We denote the i:th homology of this with $H_iH_j(X;M)$. This will give the E^2-page of a spectral sequence which is obtained as described below.

Let X be a simplicial space and consider the realization $|X| = \coprod_{i\geq 0} X_i \times \Delta[i]/\sim \in \mathbf{Sp}$. This is associated with a map $q\colon \coprod_{i\geq 0} X_i \times \Delta[i] \to |X|$ and so for the simplicial space

$$X^{(n)} := \coprod_{0\leq i\leq n} X_i \times \Delta[i] \subseteq X$$

we can consider its image under the map q and define:

$$F_n|X| := X^{(n)}/\sim$$

Then we have a filtration

$$F_0|X| \subseteq F_1|X| \subseteq \ldots \subseteq F_n|X| \subseteq \ldots \subseteq |X|\,.$$

To this we apply the procedure described in Definition 9.2 and obtain a filtration of simplicial abelian groups:

$$\mathbb{Z}[F_0|X|] \subseteq \mathbb{Z}[F_1|X|] \subseteq \ldots \subseteq \mathbb{Z}[F_n|X|] \subseteq \ldots \subseteq \mathbb{Z}[|X|]\,.$$

Finally we note that if we then apply the homology functor, we obtain a filtration of the homology $H_*(|X|,\mathbb{Z}) = H(\mathcal{N}(\mathbb{Z}[|X|]))$.

By the general theory of spectral sequences, this filtration gives us a spectral sequence converging to $H_*(\mathrm{diag}(X);\mathbb{Z})$. Bousfield-Kan [BK72] describes the E^2-page as follows:

Theorem 9.6. *For a simplicial space X with the filtration described above, there are natural isomorphisms*

$$E^2_{i,j} \cong H_i H_j(X; M)\,.$$

Proof. See [BK72, XII 5.7]. $\square$

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