Towards the p-completion of classifying spaces of groups

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Homotopy theory is a subfield of algebraic topology that studies spaces and maps up to continuous deformation. It often inherits algebraic structure, and can be studied using algebra's machineries. Among them are the localization and completion of groups, that can also be defined on topological spaces. This internship aims at understanding the p-completion of classifying spaces of groups, and all the preliminary work needed. First, we will introduce the higher homotopy groups and how they behave with two very interesting classes of spaces: cellular spaces, that for us will be CW-complexes, and cocellular spaces, that for us will be Postnikov $\mathbb{Z}$-towers. For the second, we will need more technical constructions that deals with fibrations and cofibrations, necessary to make the toplogical limit and colimit constructions homotopy invariant. We will not have time to introduce the homology and cohomology theories, as well as some category theory, that we will assume are known. If this is not the case, the reader may take a look at [Hat02] and [McL98]. With all these, the definition and construction of localization, and then completion can be introduced. We will discuss some of their properties and characterizations, and present some examples. This report is not intended as a complete presentation of these notions, but more as an overview giving an intuition about how they work for the non specialist. We will however try to define properly every notion we will use, and make a coherent paper, only without the proofs.

Contents

1 Basic Homotopy theory

In this section we recall the definitions and properties of some basic objects we are going to use constantly in our further study. All this matter is exposed in [Hat02] in a more comprehensive way.

1.1 Relative versus absolute objects

Algebraic topology faces a constant trigger between based and unbased topological spaces and maps, or more generally relative versus absolute objects. The two are of course highly linked. A based space can always give an unbased one by forgetting the basepoint, but the other way intuitively requires a choice. This can be solved by adding a basepoint exterior to the space: X_+ is the based space $X \sqcup \{*\}$. Most of the time, the constructions given in one context can be defined similarly in the other.

For example, we can define a based version of the disjoint union (which cannot be based without a choice) by identifying the two basepoints: given two based topological spaces (X, x) and (Y, y), their wedge sum is $X \vee Y = X \sqcup Y/_{x \sim y}$. This is the right notion of a sum in the based context because, if we note $F_*(X, Y)$ the based map between X and Y, then for any based space (Z, z) there is a natural bijection $F_*(X \vee Y, Z) \simeq F_*(X, Z) \times F_*(Y, Z)$, analogous to $F(X \sqcup Y, Z) \simeq F(X, Z) \times F(Y, Z)$ in the unbased case.

The product of two spaces can be based easily, and indeed the usual product still verifies that for any Z, $F_*(Z, X \times Y) \simeq F_*(Z, X) \times F_*(Z, Y)$. However, the product verifies another very useful property in the unbased case. The function spaces $F(X, Y)$ and $F_*(X, Y)$ can be endowed with the compact-open topology, and then the product is adjoint to function space, namely they verify $F(X, F(Y, Z)) \simeq F(X \times Y, Z)$. This bijection is actually a homeomorphism under some weak assumptions on the spaces, namely that they are compactly generated, which we will assume from now on to avoid technicalities, see [May99] for more details. This adjunction is not verified by the usual product in the based context, but by the smash product $X \wedge Y = X \times Y/_{(X \times \{y\} \cup \{x\} \times Y)}$. If we give $F_*(X, Y)$ as basepoint the constant map $X \mapsto y$, then we have the adjunction $F_*(X, F_*(Y, Z)) \simeq F(X \wedge Y, Z)$. This is quite intuitive if we see $F_*(Y, Z)$ as a subspace of $F(Y, Z)$ and use the preceeding adjunction to see that $F_*(X, F_*(Y, Z))$ are just the maps $X \times Y \to Z$ that sends $X \times \{y\}$ to z because each map $Y \to Z$ is based and that also sends $\{x\} \times Y$ to z because x is sent to the basepoint of $F_*(Y, Z)$, namely the constant map at z.

We will soon need many classical topological constructions, among which the ones exposed above. They often come as categorical limits or colimits of some diagram.

Examples : • The limit of a two point diagram $A,\ B$ is the product $A \times B$ and its colimit is the sum $A + B$, that is the disjoint union $A \sqcup B$ in Top, and the wedge sum $A \vee B$ in Top_*.

• For a sequence of inclusions $X_n \subseteq X_{n+1}$, $\mathrm{colim}(X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow \ldots) = \cup_n X_n$ with topology $G \subseteq \cup_n X_n$ is closed if and only if its intersection with every X_n is closed.

• $\mathrm{colim}(A \leftarrow C \rightarrow B) = A \sqcup B/\sim$ with $a \sim b$ if they are the images of a same element $c \in C$. We note this space $A \cup_C B$, called the pushout of A and B along C.

• $\lim(A \rightarrow C \leftarrow B) = \{(a, b) \in A \times B$ that coincide on $C\}$ with subspace topology. We note this space $A \times_C B$, called the pullback of A and B along C.

The idea of relative objects is that in a topological space X, we will work relatively to a subspace A, that we will think as trivial. A maps of pair $(X, A) \to (Y, B)$ is a continuous map $X \to Y$ that kills the trivial part, namely that sends A into B. In particular it induces a map $X/A \to Y/B$. The idea is that relative objects are an analogue of quotients but that behave much better with the initial object. It usually contains much more information than the quotient in the absolute case. Based spaces are the simplest example of relative objects, where A is just a point. Quotienting A do not change the space, but working relatively to A changes quite a bit.

Of course, to study Homotopy theory thoroughly, we need a definition of relative homotopy:

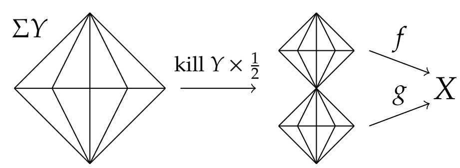
⌜ **Definition 1.1:** Let (X, A) and (Y, B) be topological pairs. Two maps of pairs $f, g : (X, A) \to (Y, B)$ are said to be relative homotopic if there is a map $H : X \times I \to Y$ such that $H(-, 0) = f$, $H(-, 1) = g$ and $H(A, -) \subseteq B$.
In the case of based spaces $(X, \{x\})$ and $(Y, \{y\})$, a relative homotopy is a based homotopy. It is in fact simply a based map $X \wedge I_+ \to Y$ that restricts to f and g on $X \times \{0\}$ and $X \times \{1\}$. ⌟

We note $\langle X, Y\rangle$ the set of equivalence classes of based maps up to based homotopy, and $[X, Y]$ for non based maps up to non based homotopy. We define hTop the category of topological spaces with morphisms $[X, Y]$ and $h\text{Top}_*$ the category of based topological spaces with morphisms $\langle X, Y\rangle$.

1.2 Higher Homotopy groups

The fundamental group is defined as the based homotopy classes of continuous loops on a space X, with product given by concatenation. Another way to say this is that it corresponds to the path-connected components of the loopspace $\Omega X = F_*(S^1, X)$, where a path on ΩX corresponds to a homotopy of loops. Since only this notion will be of interest here, connected will always mean path-connected. The concatenation on ΩX induces a group structure on $\pi_0(\Omega X) = \pi_1(X) = \langle S^1, X\rangle$. A genuine way to define higher homotopy groups is then by $\pi_n(X) = \pi_0(\Omega^n X)$. Below we will use a construction in some way dual to the loopspace: the suspension. Given a based space (X, x), we define $\Sigma X = X \times I/(X \times \partial I \cup \{x\} \times I)$. The loopspace and the suspension are adjoint up to based homotopy, namely they verify $\langle X, \Omega Y\rangle \simeq \langle \Sigma X, Y\rangle$. Hence $\pi_0(\Omega^n X) = \langle S^0, \Omega^n X\rangle \simeq \langle \Sigma^n S^0, X\rangle = \langle S^n, X\rangle$.

The concatenation of two loops $f, g \in \pi_1(X, x) = \langle S^1, X\rangle$ is in fact a manifestation of a more general construction. If we see S^1 as the suspension of S^0, then it is given by $\Sigma S^0 \xrightarrow{\text{kill } S^0\times\frac{1}{2}} \Sigma S^0 \vee \Sigma S^0 \xrightarrow{f\vee g} X$. Analogously we can define a group structure on any $\langle \Sigma Y, X\rangle$ by $\Sigma Y \xrightarrow{\text{kill } Y\times\frac{1}{2}} \Sigma Y \vee \Sigma Y \xrightarrow{f\vee g} X$.



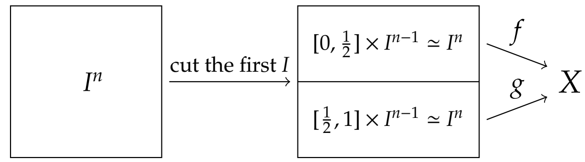
⌜ **Definition 1.2:** The higher homotopy groups of a based space (X, x) are defined for n in $\mathbb{N}$ as $\pi_n(X, x) = \langle S^n, X\rangle$. For $n > 0$ it has a group structure given by the identification $S^n = \Sigma S^{n-1}$. ⌟

Example : The n^{th} homotopy group of the n-dimensional sphere is isomorphic to $\mathbb{Z}$, and is the first non-vanishing one. Namely, all maps $S^k \to S^n$ for $k < n$ are nullhomotopic. This seems fair enough, and it is quite surprising that it doesn't hold for all $k \neq n$, but there is a nontrivial map $S^3 \to S^2$ called the Hopf fibration, generating $\pi_3(S^2) \simeq \mathbb{Z}$, and many others. The first statement can be proved by degree theory, or is just an application of the Hurewicz theorem, that says that the first non-vanishing homology and homotopy groups agree for simply connected spaces.

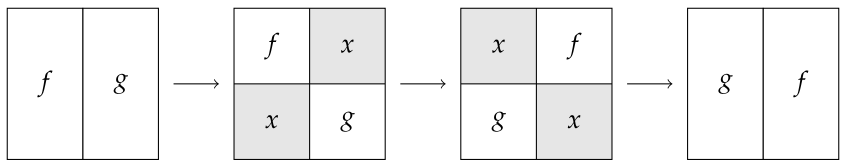
These homotopy groups are functorial in X, namely for a based map $f : X \to Y$ there is an induced map $f_* : \pi_n(X, x) \to \pi_n(Y, y)$. Actually it depends only on the based homotopy class of f. Hence a homotopy equivalence induces an isomorphism on all π_n's. Conversely, a map inducing isomorphisms on all π_n's is called a weak homotopy equivalence. The Hurewicz theorem may be used to show that a weak homotopy equivalence induces an isomorphism on all homology and cohomology groups too.

The homotopy groups distribute on products of spaces : $\pi_n(X \times Y) \simeq \pi_n(X) \times \pi_n(Y)$, as both maps $S^n \to X \times Y$ and homotopies $S^n \wedge I_+ \to X \times Y$ do.

Note that a based map $S^n \to X$ is equivalent to a map of pairs $(I^n, \partial I^n) \to (X, \{x\})$, by $S^n \simeq I^n/\partial I^n$. Then the group structure is simply given by juxtaposition :



When $n \geq 2$, $\pi_n(X, x)$ is actually an abelian group, as one can see from the following figure. This is more generally true for any double suspension $\langle \Sigma^2 Y, X \rangle$:

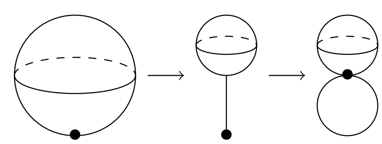


To actually compute higher homotopy groups, it is sometimes very useful to pass by the universal cover, which has trivial fundamental group and is, in general, much simpler to understand.

Proposition 1.3: *If $p : E \twoheadrightarrow B$ is a covering, then the induced maps on higher homotopy groups $p_* : \pi_n(E) \to \pi_n(B)$ is an isomorphism for $n \geq 2$.*

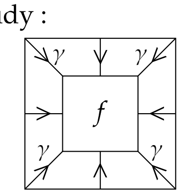
Example : The space $X = S^2 \vee S^1$ is more complicated than it might appear. By the Van-Kampen theorem, since S^2 is simply connected, $\pi_1(X) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. Now all higher homotopy groups of S^1 are trivial because its universal cover is $\mathbb{R}$, which is contractible. Even though, $\pi_2(X)$ is not $\pi_2(S^2) = \mathbb{Z}$, and is actually not even finitely generated. Indeed, the fact the homotopies of the maps $S^2 \to X$ have to be based creates new elements.

For example, the map hereby, that makes a loop around S^1 before "catching" S^2 is not based homotopic to the identity on S^2, even though it is unbased homotopic to it. For the concrete calculation, the universal cover $\tilde{X}$ of X is countably many spheres S^2 successively linked by paths, that are the delooping of S^1. It is then easy to check that $\pi_2(\tilde{X}) \simeq \pi_2(X) \simeq \mathbb{Z}^{(\mathbb{Z})}$.



We will usually drop the base point when there is no ambiguity. A path between basepoints induces an isomorphism between the associated higher homotopy groups. This isomorphism depends on the homotopy class of the given path and is hence not canonical. When considering loops, this construction gives an action of the fundamental group on the higher homotopy groups that will prove of great interest in our study :

Definition 1.4: Let γ be a path on X, denote $x = \gamma(0)$ and $y = \gamma(1)$. Then we can construct an isomorphism $\beta_\gamma : \pi_n(X, y) \to \pi_n(X, x)$ that maps f to the map described here :

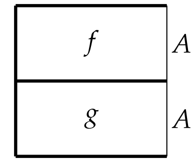


When γ is a loop, this defines an action of ΩX on $\pi_n(X)$ compatible with concatenation that goes down to $\pi_1(X) \curvearrowright \pi_n(X)$.

Example : Again with $X = S^2 \vee S^1$, the π_1 action on $\pi_2(X) = \mathbb{Z}^{(\mathbb{Z})}$ is the shifting up by one in index. Hence, $\pi_2(X)$ is just the group generated by a free action of $\pi_1(S^1)$ on $\pi_2(S^2)$. This may show how essential this action is.

Note that to define the group structure on π_n, we use juxtaposition along one coordinate of maps $(I^n, \partial I^n) \to (X, x)$. For the juxtaposed map to be continuous, we need f and g to coincide on respectively $\{1\} \times I^{n-1}$ and $\{0\} \times I^{n-1}$, which is satisfied since both maps are constant to the basepoint there. For $n \geq 2$, we only need f and g to restrict to the basepoint on these faces, but we don't need them to agree on $I^{n-1} \times \{1\}$ for example.

⌜ **Definition 1.5:** The n^{th} relative homotopy groups $\pi_n(X, A)$ of a based pair (X, A) is the set of maps of triple $(I^n, \partial I^n, J_n) \to (X, A, x)$ up to relative homotopies, where J_n is the subspace of ∂I^n consisting of all faces except $I^{n-1} \times \{1\}$. For $n \geq 2$ it has a group structure given by juxtaposition along the first coordinate and for $n \geq 3$ it is commutative. ⌟



Sending J_n to the basepoint is useful to define the product but one should note that the triple $(I^n, \partial I^n, J_n)$ deformation retracts to the triple $(I^n, \partial I^n, *)$. So an element of $\pi_n(X, A)$ up to homotopy really is like a map $I^n \to X$ with boundary in A. Restricting $f \in \pi_n(X, A)$ to $I^{n-1} \times \{1\}$ gives a map $(I^{n-1}, \partial I^{n-1}) \to (A, x)$ that is well defined in $\pi_{n-1}(A)$. This process defines a boundary homomorphism $\partial : \pi_n(X, A) \to \pi_{n-1}(A)$.

Proposition 1.6: *For any based pair (X, A), the sequence $\ldots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \pi_{n-1}(X) \to \ldots$, where the first map is the one induced by the inclusion $A \hookrightarrow X$ and the second by the map of pair $(X, x) \to (X, A)$, is exact.*

This sequence is functorial in (X, A, x). A path in A between two basepoints also gives an isomorphism between relative homotopy groups, and $\pi_1(A)$ then acts coherently on the whole long exact sequence above.

1.3 CW-complexes and approximations

We usually appreciate working with well behaved topological spaces in algebraic topology. Sometimes the category of compactly generated spaces is not enough. When working with homotopy groups there are well suited spaces, constructed uniquely by glueing disks of different dimensions: the CW-complexes.

⌜ **Definition 1.7:** A CW-complex is a space X of the form $X = \operatorname{colim}(X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow \ldots)$ where X_0 is discrete and X_n is obtained from X_{n-1} as a pushout $X_n = \operatorname{colim}(X_{n-1} \leftarrow \bigsqcup_\alpha S^{n-1} \to \bigsqcup_\alpha D^n)$. The inclusions $D^n \to X_n$ are called the n-dimensional cells of X, and its restriction to $S^{n-1} \to X_{n-1}$ is called the attaching map. ⌟

Hence, X_0 is just a discrete set of points, X_1 is a graph, X_2 is some disks glued on a graph, and so on... All spheres have a CW-complex structure with just one 0-cell and one n-cell, whose boundary is collapsed on the 0-cell.

Note that because S^{n-1} is compact, its image in X_{n-1} hits only finitely many cells. Hence, if there are countably many cells, X can actually be obtained as successive pushouts where we only add one cell at a time. This can prove quite useful to use cell-by-cell arguments.

A sub-CW-complex is subspace $A \subseteq X$ that inherits a CW-complex structure from a subcollection of cells of X. The subspace X_n is a fundamental example, called the n-skeleton of X. A very useful property of CW-complexes is the cellular approximation, which relies heavily on the fact that S^n is $(n-1)$-connected, i.e. $\pi_k(S^n) = 0$ for $k < n$.

Proposition 1.8: *Any based map $f : X \to Y$ between CW-complexes is based homotopic to a cellular map: one that maps X_n into Y_n. Hence $\langle X, Y \rangle$ is in some way the limit of $\langle X_n, Y_n \rangle$.*

There exists a relative version of this: if f is already cellular on a sub-CW-complex A of X then f is homotopic to a cellular map through a homotopy that is constant to $f|_A$ on A.

Moreover, these spaces are extremely well behaved with respect to homotopy groups:

Theorem 1.9 (Whitehead): *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

This is not true in general, but the CW-complex structure enables us to manipulate spaces much more easily, using cell-by-cell arguments for example. A generalization of Whitehead theorem states that a weak homotopy equivalence $f : X \to Y$ induces bijections $\langle Z, X\rangle \to \langle Z, Y\rangle$ for all CW-complexes Z. Hence, when studying homotopy groups, or more generally maps up to homotopy from a CW-complex to a given space, one may want to use an approximation of this space that will be a CW-complex. All we need then is a CW-complex Z along with a weak homotopy equivalence $f : Z \to X$, which is called a CW-approximation. As always we will also want relative versions, that will take the following form:

⌜ **Definition 1.10:** Let (X, A) be a pair where A is a CW-complex. A n-model of this pair is a CW-pair (Z, A) along with a map $f : Z \to X$ that restricts to the identity on A and such that $f_* : \pi_k(Z) \to \pi_k(X)$ is an isomorphism for $k > n$ and an injection for $k = n$, and $i_* : \pi_k(A) \to \pi_k(Z)$ is an isomorphism for $k < n$ and a surjection for $k = n$, and this for any choice of basepoint $x \in A$.
When A is a point in each path component of X and $n = 0$, we recover the definition of a CW-approximation. ⌟

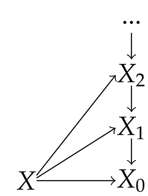
Note that by the long exact sequence, the second condition for a n-model is equivalent to $\pi_k(Z, A) = 0$ for $k < n$ in which case we say that the pair (Z, A) is n-connected.

Proposition 1.11: *Any pair* (X, A)*, where* A *is a CW-complex, admits a* n*-model* (Z, A)*. Moreover we can obtain* Z *simply by adding to* A *cells of dimension strictly greater than* n*. By cellular approximation this ensures that* (Z, A) *is* n*-connected.*

Furthermore, the n-models are natural in n: for $n \geq n'$ and Z, resp Z', a n, resp n', model, there exists a map $g : Z \to Z'$, $g|_A = Id_A$, unique up to relative homotopy, that makes $\begin{array}{ccc} Z & \longrightarrow & X \\ g\downarrow & \nearrow & \\ Z' & & \end{array}$ commute. In particular the n-model of a pair is unique up to homotopy equivalence. The n-models are also natural in X. Naturality enables us to construct bigger coherent structures:

⌜ **Definition 1.12:** A Postnikov tower of a space X is a commutative diagram of CW-complexes as at the right such that $X \to X_n$ induces an isomorphism on π_k for $k \leq n$ and $\pi_k(X_n) = 0$ for $k > n$.

It can be obtained as the coherent n-models of the pair (CX, X), where CX is the cone $X \times I/_{X\times\{1\}}$ and X is seen as the subset $X \times \{0\}$.



⌟

One can also construct it by hand: for each non trivial element $f \in \pi_{n+1}(X)$, add an $n+2$-cell with attaching map f. The element f will be trivial in $X \cup_f e^{n+2}$. We the construct an augmented space $X' = X \cup_\alpha e_\alpha^{n+2}$. By cellular approximation of maps $S^{n+1} \to X'$, every element of $\pi_{n+1}(X')$ can be represented in X, since we only add $(n+2)$-cells. Hence our augmented space X' has trivial π_{n+1}. Adding a $n+2$-cell doesn't change $\pi_k(X)$ for $k \leq n$ by cellular approximation of homotopies $S^k \times I \to X'$. We can reiterate this process for $k \geq n+1$, and obtain spaces $X \subseteq X' \subseteq ... \subseteq X^{(i)} \subseteq X^{(i+1)} \subseteq ...$ with $\pi_k(X) \xrightarrow{\sim} \pi_k(X^{(i)})$ for $k \leq n$ and $\pi_k(X^{(i)}) = 0$ for $n < k \leq n+i$. The n^{th} stage of the Postnikov tower is simply given by the colimit of this sequence.

2 Cocellular spaces

Spheres appear naturally as fundamental spaces through homology theory: they have only one non-trivial (reduced) homology group. Homology behaves nicely with (nice/homotopy) pushouts via Mayer-Vietoris long exact sequences, and quotients via excision. It is no surprise that the good spaces for homology happens be the successive pushouts $X_{n+1} = X_n \cup_\Phi Y$ such that the quotients X_{n+1}/X_n are wedge sums of spheres : the building blocks of CW-complexes.
Homotopy theory has a quite different behavior. Spheres of dimension greater than 1 seem to have infinitely many non trivial homotopy groups. There is a form of excision that treats pushouts but it works only in a range of indices for the π_k's. Actually homotopy theory behaves nicely with the dual notions: (nice/homotopy) pullbacks, for example products and fibrations (a notion we will soon introduce). We will later define a new class of good spaces for homotopy, called cocellular, defined dually to CW-complexes. Their building blocks will be those spaces with only one non trivial homotopy group: Eilenberg-MacLane spaces.

2.1 Eilenberg-MacLane spaces and a model through simplicial sets

⌜ **Definition 2.1:** Let $n \in \mathbb{N}^*$ and G be a group, supposed abelian if $n \geq 2$. We call Eilenberg-MacLane space $K(G, n)$ a CW-complex whose only non trivial homotopy group is in dimension n, and is isomorphic to G. ⌟

It is not very hard to construct a $K(G, n)$: present G as a set of generators and relations $G = \langle g_\alpha \mid r_\beta \rangle$, then let X be a wedge of n-spheres for each generator g_α to which we attach a $n+1$ cell for each relation r_β with attaching map the one realizing r_β in $\pi_n(\vee_{g_\alpha} S^n_\alpha) = \langle g_\alpha \rangle$. One can check that X is $(n-1)$ connected and $\pi_n(X) = G$. Then the n^{th} stage of the Postnikov tower of X is a $K(G, n)$.
Example : The circle S^1 is a $K(\mathbb{Z}, 1)$. It seems that this, and its products, are the only simple Eilenberg-MacLane spaces. A $K(\mathbb{Z}, 2)$ is given by $\mathbb{CP}^\infty = \cup_n \mathbb{CP}^n$, and any $K(\mathbb{Z}/n, 1)$ has to be an infinite dimensional CW-complex.

Proposition 2.2: *If X is a $K(G, n)$ and Y is a $K(H, n)$, then $\langle X, Y \rangle$ is isomorphic, by taking the induced morphism on the n^{th} homotopy group, to Hom(G, H). This proves in particular that a $K(G, n)$ is unique up to homotopy equivalence.*

This proposition suggests that $K(G, n)$'s are very natural with respect to G. Our goal will be to construct them in a functorial way, which we have only up to homotopy for now. The main problem here is the choice of generators and relations, and then of generators for the $\pi_k(X)$ and so on when constructing the Postnikov tower, that may not be functorial. An elegant solution is given by simplicial sets, a very useful tool in algebraic topology. For (much) more information on this subject, see [GJ99].

⌜ **Definition 2.3:** The simplicial category Δ has objects the finite ordered sets $[0, ..., n]$, $n \in \mathbb{N}$, and morphisms the non decreasing maps between them. Among them are face maps $d^k_n : [0, ..., n] \to [0, ..., n+1]$, $i \mapsto i$ if $i < k$ and $i + 1$ otherwise, for $0 \leq k \leq n$, and degeneracy maps $\eta^k_n : [0, ..., n] \to [0, ..., n-1]$, $i \mapsto i$ if $i < k+1$ and $i - 1$ otherwise, for $0 \leq k \leq n-1$. Every morphism can be decomposed as composition of these. ⌟

⌜ **Definition 2.4:** A simplicial set is a contravariant functor $\Delta^{op} \to$ Set. It is thought as a set of n-simplices X_n in each dimension, along with face maps $X_n \to X_{n-1}$, and degenerancy maps the other way. The images of degenerancy maps are called degenerated simplices, they have a combinatorial use but are geometrically trivial.
A simplicial topological space is a contravariant functor $\Delta^{op} \to$ Top. ⌟

Note that the collection of the standard n-simplices (which is the convex hull of $n+1$ point in $\mathbb{R}^n$) do not form a simplicial set or topological space in an obvious way: we actually have a *covariant* functor $\Delta \to \text{Set}$ or Top that maps $[0, ..., n]$ to the n-simplex Δ^n with vertices labeled 0,...,n and mapping a map f to the only linear map $\tilde{f}$ extending f, which is defined on the vertices.

⌜ **Definition 2.5:** The geometric realization of a simplicial set X is $|X| = \sqcup_n X_n \times \Delta^n / \sim$ with $(x, \alpha(t)) \sim (\alpha(x), t)$, $x \in X_n$, $t \in \Delta^n$ and α a map in Δ. In other words X_n is the set of n-simplices, which are glued together according to face maps. The degenerated simplices are glued on their projection to one of their faces, hence degenerated. The geometric realization of a simplicial topological space is the same, with $X_n \times \Delta^n$ endowed with product topology.⌟

The role of degenerated simplices, killed in the geometric realization, becomes apparent when considering products: if X and Y are simplicial sets, then there is a canonical $X \times Y$ with $(X \times Y)_n = X_n \times Y_n$ and face and degenerancy maps the products of the ones in X and in Y. This definition might seem a bit awkward, since a product of two n-simplices should $2n$ dimensional. Moreover, a product of two n-simplices has to be triangulated before giving a bunch of $2n$-simplices. The magic, and purpose, of degenerated simplices is that they precisely compute this triangulation:

Proposition 2.6: *The geometric realization of a product of simplicial sets is homeomorphic to the product of the geometric realizations,* $|X \times Y| \simeq |X| \times |Y|$.

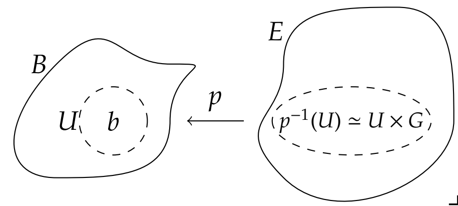
Every small category C has a simplicial set structure, called its nerve $\mathcal{B}C$. First observe that the ordered set $[0, ..., n]$ in Δ is a category, as a poset. We define $\mathcal{B}C_n = \hom_{\text{Cat}}([0, ..., n], C)$, with face and degenerancy maps the obvious contravariant precomposition. Hence a n-simplex of $\mathcal{B}C$ is a string of n composeable arrows, with face maps the composition of two successive arrows, or forgetting the last or first arrow, and degenerancy maps the insertion of an identity. The geometric realization BC of $\mathcal{B}C$ is called the classifying space of C.
A group has two canonical categories associated to it: $\mathcal{E}_G$ with objects the elements of G, and morphisms $\text{Hom}_{\mathcal{E}_G}(g, g') = \{g^{-1}g'\}$, whose classifying space is noted EG, and C_G with only one object, morphisms the elements of G and composition the product in G, whose classifying space is noted BG. These two constructions are of course functorial in G. We can make them more explicit: $(\mathcal{B}\mathcal{E}_G)_n = G^{n+1}$ is the set of n composeable arrows, which is uniquely described by their extremities. The face maps are obtained by removing one element, and degenerancy maps by duplicating one. And $(\mathcal{B}C_G)_n = G^n$, with face maps the multiplication of two successive elements, or forgetting the last or first element, and degenerancy maps the insertion of an identity.
There is a canonical functor $\mathcal{E}_G \to C_G$ sending all objects to the only object of C_G, and arrows to the one with same label. This induces a simplicial morphism (namely a natural transformation between contravariant functors) between their nerves, and hence a map $EG \twoheadrightarrow BG$.

Proposition 2.7: *The classifying space BG of C_G is a $K(G, 1)$, and is functorial in G.*

Proof : BG is a CW-complex as the geometric realisation of a simplicial set (with n-cells the non degenerated n-simplicies). Then we observe that the group G acts on $\mathcal{E}_G$, and hence on EG, by multiplication. The map $EG \twoheadrightarrow BG$ corresponds to the quotient by G. The action is free and we obtain that the preceding map is a covering. The space EG is contractible by deforming any n-simplex $[g_0, ..., g_n] \subseteq [e, g_0, ..., g_n]$ to the first vertex e. Hence EG is the universal cover of BG, and we obtain $\pi_1(BG) \simeq G$ and all of its higher homotopy groups are trivial. □

The last construction can be generalized in a very interesting way. If G is a topological group, then we can turn the simplicial sets $\mathcal{BE}_G$ and $\mathcal{BC}_G$ into simplicial topological spaces just by taking the product topology on $(\mathcal{BE}_G)_n = G^{n+1}$ and $(\mathcal{BC}_G)_n = G^n$. The face and degenerancy maps are still the same. The preceeding construction corresponds to the case where G has discrete topology, so there is no clash of notation if the topology is understood. Actually, we still have a map $p : EG \twoheadrightarrow BG$ that verifies very nice properties: it is a fiber bundle with fiber G.

⌜ **Definition 2.8:** A map $p : E \to B$ is called a fiber bundle if for every point $b \in B$ there exists an open $U \ni b$ and a homeomorphism $\varphi : p^{-1}(U) \to U \times G$ such that $p_1 \circ \varphi = p$, with p_1 the projection on the first coordinate. We call fiber $G = p^{-1}(\{b\})$, so p is in fact a map of pairs $(E, G) \to (B, b)$. ⌟



This of course generalizes coverings when G is discrete. The Proposition 1.3 generalizes as well:

Proposition 2.9: *If $G \hookrightarrow E \twoheadrightarrow B$ is a fibre bundle (or more generally a fibration), then it induces a long exact sequence on homotopy groups: $\ldots \to \pi_n(G) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(G) \to \pi_{n-1}(E) \to \pi_{n-1}(B) \to \ldots$*
This sequence is the one in Proposition 1.6 for the pair (E, G), with $\pi_n(E, G) \simeq \pi_n(B)$ realized by $(E, G) \to (B, b)$.

To construct a $K(G, n)$, for G abelian, first observe that BG is a topological group by Proposition 2.6 and functoriality: $BG \times BG \simeq B(G \times G) \xrightarrow{mult} BG$. Note that we need the hypothesis G abelian for the multiplication $G \times G \to G$ to be a group morphism. Then we can construct a fiber bundle $BG \hookrightarrow EBG \twoheadrightarrow BBG$. If BG is a $K(G, n-1)$ then the long exact sequence gives us, with EBG contractible, that BBG is a $K(G, n)$. Hence a functorial construction of a $K(G, n)$ is just given by $B^n G$.

The Eilenberg-MacLane spaces are of great interest because they represent cohomology with coefficients:

Proposition 2.10: *Let G be an abelian group, and $n \geq 1$. For a CW-complex X, there is a bijection $\langle X, K(G, n)\rangle \to H^n(X; G)$. It is obtained by taking for each map $f \in \langle X, K(G, n)\rangle$ the image of a fundamental class $\alpha \in H^n(K(G, n); G) \xrightarrow{f^*} H^n(X; G)$. It is hence functorial in X.*

This result makes Eilenberg-MacLane spaces of abelian groups even more dual to spheres, that represent the homotopy groups functors. It might also explain why we will later use as building blocks for cocellular spaces only $K(G, n)$'s with G abelian.

2.2 Fibrations and cofibrations

As our last Proposition shows, homotopy groups behave nicely with fiber bundles. However this notion is somewhat too restrictive: the fiber is given in the beginning and cannot vary, even up to homotopy. The good generalization is fibrations, a much more general setting (see Proposition 2.17) for which Proposition 2.9 still holds.

⌜ **Definition 2.11:** A based map $p : E \to B$ is called a fibration if it verifies the Homotopy Lifting Property: suppose given a map $g : X \to B$ with a lift $\tilde{g} : X \to E$, i.e. $p \circ \tilde{g} = g$. Then any homotopy of g, $H : X \times I \to B$ such that $H(-, 0) = g$, can be lifted in E as a homotopy of $\tilde{g}$, namely a map $\tilde{H} : X \times I \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H}(-, 0) = \tilde{g}$. We then call fiber, usually noted F, the preimage of p over the basepoint of B. ⌟

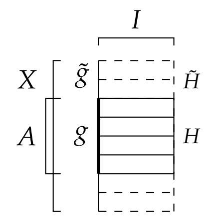
Examples : • The product $B \times F$ with projection on B is a fibration: write $\tilde{g} = (g, f)$ and define $\tilde{H}(x,t) = (H(x,t), f(x))$. It is called the trivial fibration with fiber F and base B. More generally all fiber bundles are fibrations.
• The first non trivial fibration of interest is the path space fibration. For B a connected space, its path space $PB = F_*(I, B)$ with I based at 0. There is a fibration $p : PB \to B$ given by evaluation at 1. Its fiber is precisely the loop space ΩB. Of course, PB is contractible by shortening paths (a "slurping spaghettis" contraction), and Proposition 2.9 gives in another way that $\pi_n(\Omega B) = \pi_{n+1}(B)$.

There is a notion of fiber morphisms: ones that commute with projections, and one of fiber homotopy equivalences: fiber morphisms that have a homotopy inverse such that the homotopies to identity commute with projections. Then we can see how fibrations generalize fiber bundles:

Proposition 2.12: *If B is path connected, the homotopy type of the fiber $F = p^{-1}(\{b\})$ do not depend on the basepoint $b \in B$. Moreover we have local trivializations : around every point $b \in B$ there is an open U such that $p^{-1}(U)$ is fiber homotopy equivalent to the trivial fibration $U \times F$.*

The dual notion is, without surprise, a cofibration. It is somewhat more visual so we will introduce some ideas with it before getting to their duals with fibrations.

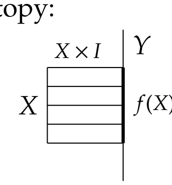
Definition 2.13: A based map $i : A \to X$ is called a cofibration if it verifies the Homotopy Extention Property: suppose given a map $g : A \to Y$ that extends to a map $\tilde{g} : X \to Y$, i.e. $\tilde{g}|_A = g$. Then any homotopy of g, $H : A \times I \to Y$ such that $H(-,0) = g$, can be extended to $X \times I$ as a homotopy of $\tilde{g}$, namely a map $\tilde{H} : X \times I \to Y$ such that $\tilde{H}|_{X \cup A \times I} = \tilde{g} \cup H$.



Example : The fundamental, and only (see [May99]), example is when (X, A) is an NDR-pair, namely A is a deformation retract of a neighborhood in a strong sense: there exists a continuous map $u : X \to I$ such that $u^{-1}(\{0\}) = A$ and a homotopy $H : X \times I \to X$ with $H(a,t) = a$ for all $a \in A$ and $t \in I$, $H(x, 0) = x$ for $x \in X$ and $H(x, 1) \in A$ whenever $u(x) < 1$. This is in particular satisfied by all CW-pairs (X, A). The appropriate notion of cobifration morphism is then simply maps of pairs.

For a cofibration, the collapsing $X \cup_{A \times \{0\}} A \times I \to X$ is a homotopy equivalence. In other words, up to homotopy, there is a copy of $A \times I$ around A in X. This is why we see cofibrations as the *good* kind of inclusions, and more generally of maps. One might ask how often does a map satisfy such idealistic conditions. The great utility of cofibrations comes from the fact that this is virtually always the case when working up to homotopy:

Definition 2.14: Given a continuous map $f : X \to Y$, we define its mapping cylinder M_f as the pushout of $X \times I \overset{\times\{0\}}{\hookleftarrow} X \overset{f}{\to} Y$. It contains a copy $X \times \{1\}$ of X and deformation retracts on Y by collapsing $X \times I$ on $f(X)$.



Proposition 2.15: *Any continuous map $f : X \to Y$ can be written as the composition of a cofibration and a homotopy equivalence: $X \hookrightarrow M_f \twoheadrightarrow Y$.*

We can define a "nondegenerate" analogue of the quotient $Y/f(X)$ by taking the quotient after having turned f into a cofibration. It is called the mapping cone $C_f := M_f/X$, or the homotopy cofiber. If f is a cofibration, we have noted before that this corresponds to the usual quotient. Adding a cell e^{n+1} to the n-skeleton X_n of a CW-complex corresponds precisely to taking the mapping cone of the attaching map $\phi : \partial e^{n+1} \to X_n$.

Happily, similar constructions can be made in the dual case:

⌜ **Definition 2.16:** Given a continuous map $f : X \to Y$, we define its mapping path space N_f as the pullback of $X \xrightarrow{f} Y \xleftarrow{ev_0} F(I, Y)$, namely it is the set of path γ on Y given with a lift in X of $\gamma(0)$. It projects on Y by evaluation at 1. It contains a copy of X by $x \mapsto (x, [f(x)])$, where $[f(x)]$ is the constant path a $f(x)$, and deformation retracts on X by "slurping spaghettis". ⌟

Proposition 2.17: *Any continuous map $f : X \to Y$ can be written as the composition of a homotopy equivalence and a fibration: $X \hookrightarrow N_f \twoheadrightarrow Y$. The fiber of the fibration is called the homotopy fiber of f, and is given by the pullback of $X \xrightarrow{f} Y \xleftarrow{ev_0} PY$, namely the set of elements of X given with a trivialization path to the basepoint in Y.*

2.3 Homotopy limits and colimits

We have to introduce a new notion of homotopy limit and colimit because the usual topological limits and colimits do not behave well with homotopy. Concretely, we cannot define a limit in hTop to be the usual limit in Top, seen in hTop because its homotopy type will depend on the choice of the representants for the homotopy types of its summands.

Example : The pushout of two disks D^k glued on their boundary S^{k-1} is a sphere $S^k = D^k \cup_{S^{k-1}} D^k$. However, disks are contractible, hence homotopy equivalent to points, and the pushout $* \cup_{S^{k-1}} *$ is a point too. Pushouts are not homotopy invariant.

In the above example, it appears that the second pushout is in some way degenerated. The S^{k-1} subspace is killed though it obviously "should" be a subspace of the pushout. The solution is to replace degenerated maps by *good* ones, that is, for pushouts, cofibrations.

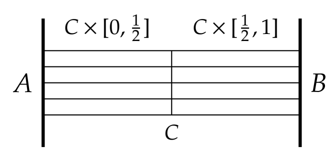
Example (continued): If we replace all maps by cofibrations in the pushout diagram $* \leftarrow S^{k-1} \to *$, we precisely obtain the preceeding diagram $D^k \leftarrow S^{k-1} \to D^k$, because $D^k \simeq S^{k-1} \times I / S^{k-1} \times \{1\}$.

This miracle is not an exception, and just by turning maps into cofibration, we can define the homotopy pushout of a diagram. See [MP12] for more details, among which the following:

Proposition 2.18: *Suppose given two diagrams of cofibrations $A \leftarrow C \to B$ and $A' \leftarrow C' \to B'$ together with a natural transformation between them, namely three maps that makes* $\begin{array}{ccccc} A & \longleftarrow & C & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longleftarrow & C' & \longrightarrow & B' \end{array}$ *commute. If the three vertical maps are homotopy equivalences, then they induce a homotopy equivalence on the pushouts $A \cup_C B \to A' \cup_{C'} B'$.*

The idea is that the homotopies that makes vertical arrows almost isomorphisms can be realized in the $C \times I$ part and give homotopies on the pushout. Actually if only the left horizontal arrows are cofibrations, the result holds still. Roughly, we can still take C to be $C \times \{\frac{1}{2}\}$ in the $C \times I$ copy in A, so that it is a cofibration on both sides.

An explicit construction, or definition, of the homotopy pushout of any diagram $A \leftarrow C \to B$ (not necessarily made of cofibrations) is $A \cup_C^h B := A \cup_{C \times \{0\}} C \times I \cup_{C \times \{1\}} B$. This is the pushout of the two mapping cylinders, i.e. precisely the pushout of the same diagram where all maps have been replaced by cofibrations. We see here why one cofibration is enough. A based version exists too, and is just the same where we have quotiented all basepoints. Namely it is $A \cup_{C \times \{0\}} C \wedge I_+ \cup_{C \times \{1\}} B$.



Sadly, the based homotopy pushout doesn't satisfy the perfect pushout condition in $h\mathrm{Top}_*$ that would be that $\langle A \cup^h_C B, Z\rangle \simeq \langle A, Z\rangle \times_{\langle C,Z\rangle} \langle B, Z\rangle$. A based map from the based homotopy pushout to any space Z is indeed the data of two maps from A and B together with a homotopy between their restrictions on C, as one can see directly on the picture above. However, there needs to be a choice for this homotopy, which makes that the above demanded isomorphism is only a surjection, whose kernel corresponds to different choices of homotopies in C, namely maps $\Sigma C \to Z$. Note that if Z has the homotopy type of a loop space, all sets $\langle -, \Omega Z'\rangle \simeq \langle \Sigma -, Z'\rangle$ inherits a canonical group structure. Then the perfect pushout condition is the exact sequence $0 \to \langle A \cup^h_C B, Z\rangle \to \langle A, Z\rangle \times \langle B, Z\rangle \to \langle C, Z\rangle$. This does not hold in general, but it can be extended to the left in a long exact sequence:

Proposition 2.19: *The homotopy pushout of a diagram $A \leftarrow C \to B$ verifies that for any based space Z,* $... \to \langle \Sigma A \cup^h_{\Sigma C} \Sigma B, Z\rangle \to \langle \Sigma A, Z\rangle \times \langle \Sigma B, Z\rangle \to \langle \Sigma C, Z\rangle \to \langle A \cup^h_C B, Z\rangle \to \langle A, Z\rangle \times \langle B, Z\rangle \to \langle C, Z\rangle$ *is exact, where the end should be replaced by* $\langle A, Z\rangle \times_{\langle C,Z\rangle} \langle B, Z\rangle \to 0$ *if Z doesn't have the homotopy type of a loop space. This is not a problem before because all starting spaces are suspension, with* $\Sigma A \cup^h_{\Sigma C} \Sigma B \simeq \Sigma(A \cup^h_C B)$.

Note that if we take Z to be a $K(G, n)$, that represents cohomology with G coefficients, then the above proposition precisely gives the Mayer-Vietoris sequence for cohomology (using the isomorphism $\tilde{H}^n(\Sigma X) \simeq \tilde{H}^{n-1}(X)$).

Again, it works exactly the same in the dual case (without the drawings though), namely Proposition 2.18 holds with cofibration replaced by fibration, $\bullet \leftarrow \bullet \to \bullet$ replaced by $\bullet \to \bullet \leftarrow \bullet$ and pushout replaced by pullback.
We then can define the homotopy pullback of any diagram $A \xrightarrow{f} C \xleftarrow{g} B$ as $A \times^h_C B := A \times_{ev_0} F(I, C) \times_{ev_1} B$. Its elements are pairs (a, b) together with a path in C joining their images. It corresponds to the pullback of the mapping path spaces along the above described projections $N_f \to C \leftarrow N_g$. An element of this homotopy pullback corresponds to two points in A and B along with a path between their imges in C.
Example : The usual pullback of $* \to X \leftarrow *$ is just a point. However, when we replace both $* \to X$ by the contractible path fibration $PX \to X$, the homotopy pullback is then the quadruples (x, γ, y, δ) such that $x = \gamma(1) = \delta(1) = y$ and $\gamma(0) = \delta(0) = *$ the basepoint of X. By concatenating $\gamma \cdot \delta^{-1}$, we can see that it corresponds to the loops on X. So the homotopy pullback over two points is ΩX.
Again, one fibration is enough as concatenations of paths are just paths. The based version is the same, but it can be written more explicitly as dual of the based homotopy pushout as $A \times_{ev_0} F_*(I_+, C) \times_{ev_1} B$. A map from any Z to the homotopy pullback is the data of two maps to A and B together with a homotopy of their projections on C, that is realized by the paths. There might be multiple choices for these paths, giving rise an error term of maps $Z \to \Omega C$:

Proposition 2.20: *The homotopy pullback of a diagram $A \to C \leftarrow B$ verifies that for any based space Z,* $... \to \langle Z, \Omega A \times^h_{\Omega C} \Omega B\rangle \to \langle Z, \Omega A\rangle \times \langle Z, \Omega B\rangle \to \langle Z, \Omega C\rangle \to \langle Z, A \times^h_C B\rangle \to \langle Z, A\rangle \times \langle Z, B\rangle \to \langle Z, C\rangle$ *is exact, where the end should be replaced by* $\langle Z, A\rangle \times_{\langle Z,C\rangle} \langle Z, B\rangle \to 0$ *if Z doesn't have the homotopy type of a suspension. This is not a problem before because all ending spaces are loop spaces, with* $\Omega A \times^h_{\Omega C} \Omega B \simeq \Omega(A \times^h_C B)$.

Note that if we take Z to be S^n then, using the $\Omega - \Sigma$ adjunction, the above proposition gives a long exact sequence: $\pi_{n+1}(A \times^h_C B) \to \pi_{n+1}(A) \times \pi_{n+1}(B) \to \pi_{n+1}(C) \to \pi_n(A \times^h_C B) \to \pi_n(A) \times \pi_n(B) \to \pi_n(C)$. We recover Proposition 2.9 from the fact that the fiber of a fibration $F \hookrightarrow E \to B$ is the homotopy pullback over a point: $E \to B \leftarrow *$. Actually, the homotopy fiber of a map $f : X \to Y$ is the pullback over the (contractible) path space fibration: $X \to Y \leftarrow PY$.

2.4 Postnikov towers of principal fibrations and k-invariants

Here we (finally) really enter in the world of cocellular spaces, with a first well-known example. Remember that for any path connected space X, we have constructed a Postnikov tower as at the right. Without changing the homotopy type of the spaces X_n, we can turn all vertical maps into fibrations, one by one. We can recover the weak homotopy type of X from this tower, by taking its limit.

$$\begin{array}{ccc} & & \cdots \\ & & \downarrow \\ & \nearrow & X_2 \\ & & \downarrow \\ & \nearrow & X_1 \\ & & \downarrow \\ X & \longrightarrow & X_0 = * \end{array}$$

However, for now, the data of the tower itself remains quite wild: there is no way of constructing X_{n+1} from X_n, as we do construct the $(n+1)$-skeleta of a CW-complex as a pushout of the n-skeleta and some disks along spheres. Moreover, since the inclusion $S^{n-1} \subseteq D^n$ is a cofibration, this is a homotopy pushout and depends only on the homotopy classes of the maps $S^{n-1} \to X_{n-1}$, hence described by elements of $\pi_{n-1}(X_{n-1})$. This shall improve soon: for a very nice class of spaces, namely simple spaces, the fibrations in this tower verify some additional properties.

⌜ **Definition 2.21:** A fibration $F \to E \to B$ is called principal if we can find a commutative diagram as at the right where all vertical maps are weak homotopy equivalences, $E' \to B'$ is a fibration and F' and $\Omega B'$ are obtained by successively taking homotopy fiber.

$$\begin{array}{ccccccc} & & F & \longrightarrow & E & \longrightarrow & B \\ & & \downarrow & & \downarrow & & \downarrow \\ \Omega B' & \to & F' & \longrightarrow & E' & \longrightarrow & B' \end{array}$$

⌟

Example : If a fibration $F \hookrightarrow E \twoheadrightarrow B$ is a product $E = B \times F$ with $F \simeq \Omega B'$ a loop space, then it is principal. Turn the constant map $B \to B'$ into a fibration: that gives a space $E' = \{(b, \gamma) \in B \times F(I, B')/\gamma(1) = b'\} = B \times PB'$ homotopy equivalent to B, with projection on B' the one of PB', and fiber $F' = B \times \Omega B' \simeq E$. We obtain the desired $\Omega B' \hookrightarrow F' = B \times \Omega B' \hookrightarrow E' = B \times PB' \twoheadrightarrow B'$.

Hence any product of Eilenberg-MacLane spaces of abelian groups admits a Postnikov tower of principal fibrations, namely with all vertical maps being principal fibrations. Indeed, it can be written as a product of $K(G, n)$'s with different n's because $\Pi K(G_i, n) \simeq K(\Pi G_i, n)$. Then we can successively add the $K(G, n)$ summand to obtain the n^{th} stage X_n of the Postnikov tower. Since G is abelian, $K(G, n) \simeq \Omega K(G, n+1)$ and the trivial fibration $K(G, n) \to X_n \to X_{n-1}$ is principal by the above example.

⌜ **Definition 2.22:** A connected CW-complex X is called a simple space (or abelian in [Hat02]) if its fundamental group acts trivially on all of its homotopy groups. In particular $\pi_1(X)$ is abelian. ⌟

> **Proposition 2.23:** *A connected space X admits a Postnikov tower of principal fibrations if and only if it is a simple space.*

First note that the fibration $X_{n+1} \to X_n$ induces an isomorphism on all higher homotopy groups except the $(n+1)^{th}$, so by Proposition 2.9 its fiber is a $K(G, n+1)$. If it is a principal fibration, then we have $K(G, n+1) \simeq \Omega B'$, so B' has to be a $K(G, n+2)$. In this case, the $(n+1)^{th}$ stage of the Postnikov tower can be easily constructed from the n^{th} as the homotopy fiber of the map $X_n \to K(G, n+2)$, namely the pullback of $X_n \to K(G, n+2) \leftarrow PK(G, n+2)$. Based homotopy classes of such maps correspond to elements of $H^{n+2}(X_n; G)$, which are, hopefully, not too wild. We call this map the n^{th} k-invariant k_n. They are all trivial when X is a product of abelian $K(G, n)$'s.

Proof : The forward implication comes from the fact that when one builds the $(n+1)^{th}$ stage X_{n+1} as the homotopy fiber $X_{n+1} \to X_n \to K(G, n+2)$, the fundamental group of X_{n+1} acts trivially on all $\pi_i(X_n, X_{n+1}) \simeq \pi_i(K(G, n+2))$ since X_{n+1} maps on the basepoint in $K(G, n+2)$. The backwards implication relies on the relative Hurewicz theorem, drawing a link between homotopy and homology groups, that, here, unables us to use the nice behavior of homology with quotients to understand their homotopy groups. □

We can now see why these spaces should be called cocellular, constructed dually to CW-complexes:

The $(n+1)$-skeleton of a CW-complex is obtained by taking the homotopy cofiber of a disjoint union of spheres S^n mapping to X_n via attaching maps:

$$\begin{array}{ccc} X_{n+1} & \longleftarrow & * \\ \uparrow \cup^h_- & & \uparrow \\ X_n & \longleftarrow & \sqcup S^n \end{array} \qquad \text{or} \qquad \begin{array}{ccc} X_{n+1} & \longleftarrow & \sqcup D^n \\ \uparrow \cup_- & & \uparrow \\ X_n & \longleftarrow & \sqcup S^n \end{array}$$

The $(n+1)$-stage of the Postnikov tower of a simple space is obtained by taking the homotopy fiber of X_n mapping in a $K(G, n+2)$ via k-invariants.

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & * \\ \downarrow \times^h_- & & \downarrow \\ X_n & \longrightarrow & K(G, n+2) \end{array} \qquad \text{or} \qquad \begin{array}{ccc} X_{n+1} & \longrightarrow & PK(G, n+2) \\ \downarrow \times_- & & \downarrow \\ X_n & \longrightarrow & K(G, n+2) \end{array}$$

From this analogy, we also call the map k_n the coattaching map of $K(G, n+2)$, and $PK(G, n+2)$ the $(n+1)$-cocell.

Surprisingly enough, the CW-complexes, made to behave well with homology, are described by their homotopy groups in the sense of Whitehead Theorem. This makes sense in the way that the attaching maps to construct the $(n+1)$-skeleton from the n^{th} are just elements of the higher homotopy groups. In the case of simple spaces, the $(n+1)^{th}$ stage of the Postnikov tower is obtained from the n^{th} with a coattaching map that is just an element of its cohomology with some coefficients. So it is ultimately not so surprising that cohomology describes simple spaces.

Theorem 2.24 (Dual Whitehead): *A map $f : X \to Y$ between simple spaces is a weak homotopy equivalence if and only if it induces an isomorphism on cohomology with all coefficients, and if and only if it induces an isomorphism on homology with $\mathbb{Z}$ coefficients.*

2.5 Postnikov $\mathcal{A}$-towers and nilpotent spaces

The main reason why we construct cellular and cocellular spaces as inductive addition of (co)cells is to use (co)cell-by-(co)cell arguments. In our preceeding constructions, the next (co)cell is attached only on the previous (co)cells of strictly smaller dimension. In the cellular case, cellular approximation guarantees that there is no loss of generality doing so. However, we will win much by allowing n-cocells to attach on previous n-cocells too.

⌜ **Definition 2.25:** A cocellular space, or Postnikov $\mathbb{Z}$-tower in [MP12], is a space X obtained as the limit of a tower $(X_k)_{k\in\mathbb{N}}$ such that X_{k+1} is the homotopy fiber of a coattaching map $X_k \to K(G_k, n_k+1)$, with $X_0 = *$, where G_k's are abelian groups and $(n_k)_k$ is a non stationary and non-decreasing sequence of positive integers. It is called a $\mathbb{Z}$-tower if we do not require the $(n_k)_k$ to be non-decreasing. Note that X is necessarily connected. ⌟

To understand why we still want the n_k's to be non-decreasing, we can study the long exact sequence of homotopy groups associated to the fibration $X_{k+1} \to X_k \to K(G_k, n_k+1)$, namely:

$$0 \to \pi_{n_k+1}(X_{k+1}) \to \pi_{n_k+1}(X_k) \to G_k \to \pi_{n_k}(X_{k+1}) \to \pi_{n_k}(X_k) \to 0$$

Then the n_k's being non-decreasing precisely means that $\pi_i(X_k) = 0$ for $i > n_k$, as one can check by induction. Then we obtain $\pi_i(X_{k+1}) = 0$ for $i > n_k$, $\pi_i(X_{k+1}) = \pi_i(X_k)$ for $i < n_k$ and the only modification is $\pi_{n_k}(X_{k+1})$ that is built from G_k and $\pi_{n_k}(X_k)$ as the extension $0 \to G_k \to \pi_{n_k}(X_{k+1}) \to \pi_{n_k}(X_k) \to 0$. This group G_k is asked to be abelian, and as we have seen in the proof of Proposition 2.23, $\pi_1(X_{k+1})$ has to act trivially on all $\pi_i(K(G_k, n_k+2))$ so in particular on G_k. We will see that these are the only conditions arising from this construction.

On the first steps, for $n_k = 1$, this means we build $\pi_1(X)$ as iterated central extensions $0 \to G_k \to H_{k+1} \to H_k \to 0$ with $X_k = K(H_k, 1)$, so $H_0 = 0$ and $H_n = \pi_1(X)$. Moreover, some group cohomology (see [GS06], chapter 3) gives

$\langle K(H_k,1), K(G_k,2)\rangle \simeq H^2(K(H_k,1); G_k) \simeq H^2(H_k; G_k) \simeq \{\text{central extensions } 0 \to G_k \to ? \to H_k \to 0\}/\text{iso}$. So the data of the first steps of the tower corresponds precisely to the data of these iterated extensions. Hence, cocellular spaces can have as fundamental groups precisely those groups that can be constructed as iterated central extensions.

⌜ **Definition 2.26:** A group G is called nilpotent if it admits a sequence of subgroups $0 = A_n \subseteq ... \subseteq A_1 \subseteq A_0 = G$ such that A_k/A_{k+1} is in the center of G/A_{k+1}.
Equivalently G is nilpotent if it can be built as iterated central extensions, that can be obtained from the preceeding sequence of subgroups with : $0 \to G_k = A_k/A_{k+1} \to H_{k+1} = G/A_{k+1} \to H_k = G/A_k \to 0$. ⌟

Example : The dihedral $D_8 = \langle r, s | r^4, s^2, rsrs\rangle$, namely the group of cyclic order preserving bijections of $\{1, -1, i, -i\}$, is nilpotent. Indeed, take $A_1 = \langle r^2\rangle$, that is actually the center of D_8, then D_8/A_1 is abelian and $0 \subseteq A_1 \subseteq D_8$ fits.

For higher dimensional cocells, the higher homotopy groups of X are necessarily abelian, but it is on the π_1 action that we add a degree of freedom.

⌜ **Definition 2.27:** A group action $\pi \curvearrowright G$ (namely, a homomorphism $\pi \to \mathrm{Aut}(G)$) is nilpotent if there is a sequence of subgroups $0 = A_n \subseteq ... \subseteq A_1 \subseteq A_0 = G$ such that A_k/A_{k+1} is in the center of G/A_{k+1}, A_k is stable by π and π acts trivially on A_k/A_{k+1}. In particular, with $G \curvearrowright G$ by conjugation, it corresponds to the preceeding notion.
A connected space X is called nilpotent if its fundamental group acts nilpotently on all of its homotopy groups. ⌟

Proposition 2.28: *A space X is weak homotopy equivalent to a cocellular space if and only if it is nilpotent.*

Such a weak homotopy equivalence $X \to Z$, with Z a cocellular space, is called a cocellular approximation. It can be made functorial in X up to homotopy.

We will later need some variants of these cocellular spaces. Let $\mathcal{A}$ be a collection of abelian groups, for example $\mathrm{Mod}(R)$, the collection of modules over a commutative ring R. Then a $\mathcal{A}$-cocellular space or a Postnikov $\mathcal{A}$-tower, resp an $\mathcal{A}$-tower, is a cocellular space, resp $\mathbb{Z}$-tower, where all G_k's are in $\mathcal{A}$. An action $\pi \curvearrowright G$ is $\mathcal{A}$-nilpotent if it is nilpotent with A_k/A_{k+1} in $\mathcal{A}$. A space is $\mathcal{A}$-nilpotent if its fundamental group acts $\mathcal{A}$-nilpotently on all of its homotopy groups, who are hence in $\mathcal{A}$. The preceeding definitions correspond to the case where $\mathcal{A}$ is all abelian groups, namely $\mathrm{Mod}(\mathbb{Z})$. Again a space is weak homotopy equivalent to a Postnikov $\mathcal{A}$-tower if and only if it is $\mathcal{A}$-nilpotent. Now we can verify that these spaces generalize simple spaces not too wildly:

Theorem 2.29 ($\mathcal{A}$-Dual Whitehead): *A map $f : X \to Y$ between $\mathcal{A}$-nilpotent spaces is a weak homotopy equivalence if and only if it is a $\mathcal{A}$-cohomology isomorphism, namely that it induces an isomorphism on cohomology with coefficients in any group in $\mathcal{A}$. If $\mathcal{A}$ = Mod(R), then the above holds if and only if f is a R-homology isomorphism, namely that it induces an isomorphism on homology with R coefficients.*

The notions of $\mathrm{Mod}(R)$-cohomology isomorphism and of R-homology isomorphism are not equivalent in general (when the spaces are not R-nilpotent) but they are as soon as R is a PID by universal coefficient. This theorem states that Postnikov $\mathcal{A}$-towers see those *quasi*-isomorphisms as actual isomorphisms in $h\mathrm{Top}_*$. This is true in an even stronger sense, and actually holds for all $\mathcal{A}$-towers.

Theorem 2.30 ($\mathcal{A}$-Dual Whitehead, second form): *A map $f : X \to Y$ between connected spaces is a $\mathcal{A}$-cohomology isomorphism if and only if it induces a bijection $\langle Y, Z\rangle \to \langle X, Z\rangle$ for all $\mathcal{A}$-tower Z.*

3 Localization and completion

Localization is a very useful tool, well known in commutative algebra. It essentially consists in formally inverting some elements of a given ring. We will see how this idea can be generalized to any category, and completion will then be only a particular case of localization. This generalization actually describes some notions we have already used, and will provide us a very useful language. Our goal will then be to study how these general definitions behave on spaces, or at least some of them. These notions are presented in [MP12] with a very topological approach and in [GH16] with a more categorical and simplicial point of view.

3.1 Categorical point of view

Sometimes, in a category C, there are some special morphisms, say $\mathcal{W}$, that we would like to think as isomorphisms, but that are not really. For example, $\mathcal{W}$ could be the homotopy equivalences between spaces in Top. This is why we have formed the category hTop, that indeed will see homotopy equivalences as isomorphisms, and them only. But can we do the same with weak homotopy equivalences? Or just with any collection of morphisms?

⌜ **Definition 3.1:** A localization context is a pair $(C, \mathcal{W})$ where C is a category and $\mathcal{W}$ is a collection of morphisms in C, or, to make things clearer, a subcategory containing all objects and all isomorphisms. Morphisms in $\mathcal{W}$ are called weak equivalences. The subcategory $\mathcal{W}$ is usually required to satisfy the 2-out-of-3 property, namely that if any two of f, g and $g \circ f$ are weak equivalences, so is the third. We then define the localization of C at $\mathcal{W}$ to be the category where we formally add inverses of weak equivalences: $C[\mathcal{W}^{-1}]$ has objects the ones of C and morphisms the zigzags $\bullet \to \bullet \leftarrow \ldots \to \bullet$ where we only authorize weak equivalences to go backwards, and where we identify the obvious compositions. ⌟

Most often, and in all cases we shall consider, the category $C[\mathcal{W}^{-1}]$ satisfies the universal property of a localization: any functor $C \to C'$ sending every weak equivalence to an isomorphism factors uniquely through $C[\mathcal{W}^{-1}]$.

Examples : • The localization of Top at homotopy equivalences is equivalent to hTop. Of course, homotopy equivalences are isomorphisms in hTop. Now we have to understand why Top$[h.\mathrm{eq}^{-1}]$ do not see the difference between homotopic maps. Consider a homotopy $H : X \times I \to Y$. The projection $p : X \times I \to X$ is a homotopy equivalence, and is a right inverse of both inclusion i_0 and $i_1 : X \to X \times I$. Then in Top$[h.\mathrm{eq}^{-1}]$, $p \circ i_0 = p \circ i_1$ but p has an inverse so $i_0 = i_1$. Hence the two homotopic maps $H \circ i_0$ and $H \circ i_1$ are equal too.

• We will also consider HoTop, the localization of Top$_*$ at weak homotopy equivalences. Here, some miracle happens. We have seen in 1.3 that every topological space is weak homotopy equivalent to a CW-complex in a functorial way, and a weak homotopy equivalence between CW-complexes is a homotopy equivalence. Hence there is a "section" HoTop $\to hCW_*$, namely the category of based CW-complexes localized at homotopy equivalences. This section is in fact an equivalence of categories. So if we localize in two steps and see HoTop as hTop$_*$ localized at weak homotopy equivalences, HoTop simply corresponds to the full subcategory hCW_* of hTop$_*$ whose objects verify that amazing property of seeing weak equivalences as isomorphisms. Moreover, there is a retraction $r : h\mathrm{Top}_* \to hCW_*$ that corresponds to CW-approximation. This retraction is adjoint to the inclusion $i : hCW_* \to h\mathrm{Top}_*$, namely $\mathrm{Hom}_{hCW_*}(X, r(Y)) \simeq \mathrm{Hom}_{h\mathrm{Top}_*}(i(X), Y)$. This comes from the fact that the weak homotopy equivalence of the CW-approximation $r(Y) \to Y$ induces a bijection $\langle X, r(Y)\rangle \to \langle X, Y\rangle$ for all CW-complexes X. Replacing hCW_* by the equivalent HoTop, r corresponds to the localization at weak homotopy equivalences and we have exhibited an adjoint, which is quite useful to describe morphisms in HoTop: they are simply the morphisms in hCW_* between the CW-approximations of the two spaces.

⌜ **Definition 3.2:** In a localization context $(C, \mathcal{W})$, an object Z is called local if every weak equivalence $f : X \to Y$ induces a bijection $\mathrm{Hom}_C(Y, Z) \to \mathrm{Hom}_C(X, Z)$. It is called colocal if every weak equivalence induces a bijection $\mathrm{Hom}_C(Z, X) \to \mathrm{Hom}_C(Z, Y)$. In other words we cannot distinguish weak equivalent objects by mapping them into Z or dually by mapping Z into them, as explained in [Dwy04].
Given any object A, a weak equivalence $A \to Z$ to a local object is called a localization. By definition of local objects we can check that it is unique up to canonical isomorphism when it exists. A weak equivalence $Z \to A$ from a colocal object is called a colocalization and is seamingly unique.
The localization context $(C, \mathcal{W})$ is said to have good localizations if every object has a localization. We then have a localization functor $L_{\mathcal{W}} : C \to Loc(C)$, namely the subcategory of local objects. All weak equivalences between local objects are isomorphisms, so this functors factors $C[\mathcal{W}^{-1}] \to Loc(C)$. This is an equivalence of categories whose inverse is the restriction of the at $\mathcal{W}$ on $Loc(C)$.
It is said to have good colocalizations if every object has a colocalization. Then there is a colocalization functor $C_{\mathcal{W}} : C \to Coloc(C)$ that factors as an equivalence of categories $C[\mathcal{W}^{-1}] \to Coloc(C)$.
Weak equivalences are then characterized by the fact that they induce isomorphisms between (co)localizations. ⌟

We earlier observed that CW-complexes are colocal in $h\mathrm{Top}_*$ with weak homotopy equivalences, namely actually the generalization of Whitehead Theorem. CW-approximation guarantees that they are the only ones up to homotopy equivalence (i.e. isomorphism here), and that $h\mathrm{Top}_*$ has good colocalizations.
We actually had another interesting result stating the locality of some spaces. With $\mathcal{W}$ the $\mathcal{A}$-cohomology isomorphisms in $h\mathrm{Top}_*$, the second form of the $\mathcal{A}$-Dual Whitehead Theorem precisely states that Postnikov $\mathcal{A}$-towers are local. The $\mathcal{A}$-version of Proposition 2.28 means that they are the only ones in the category of $\mathcal{A}$-nilpotent spaces, which has good localizations at $\mathcal{A}$-cohomology isomorphisms.

3.2 Localization of groups

The localization of modules is well known. If S is a multiplicative subset of a commutative ring R, and M is a R-module, then $S^{-1}M := \{\frac{m}{s},\ m \in M,\ s \in S\}/\frac{m}{s} \sim \frac{m'}{s'}$ if $\exists t \in S/\ t(s'm - sm') = 0 \simeq S^{-1}R \otimes M$. It comes with a map $\phi_M : M \to S^{-1}M$ sending m to $\frac{m}{1}$, whose kernel is the S-torsion of M. Note that the localization of M at S is uniquely S-divisible: every element $s \in S$, the multiplication by s is an isomorphism. If M is already uniquely S-divisible, then $S^{-1}M = M$.

We would like to see this construction as a localization in the categorical sense. First, what we really want is multiplication by $s \in S$ to be isomorphisms. The simplest way to do so is to define $\mathcal{W}_1$ as the possible composite of isomorphisms and multiplication by $s \in S$ on a given module. A module is then $\mathcal{W}_1$-local if and only if it is uniquely S-divisible. However, the localization $M \to S^{-1}M$ is rarely a multiplication by s, unless M is s-torison, or uniquely S-divisible. There are too few weak equivalences, even though they already define the good local objects. Let's work the other way around: we define $\mathcal{W}_2$ to be the morphisms $f : M \to N$ that induce bijections $\mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(M, L)$ for all uniquely S-divisible L.

> **Proposition 3.3:** *A morphism is in $\mathcal{W}_2$ if and only if its kernel and its cokernel are S-torison.*

Proof : For $L = S^{-1}M$, by surjectivity ϕ_M factors through f, so $\mathrm{Ker}\ f \subseteq \mathrm{Ker}\ \phi_M$ which is the S-torsion of M, so is S-torison. For $L = S^{-1}(N/\mathrm{im}\ f)$, note that $\phi_{/M} : N \to N/\mathrm{im}\ f \to S^{-1}(N/\mathrm{im}\ f)$ goes to the zero map $M \to N \to N/\mathrm{im}\ f \to S^{-1}N/\mathrm{im}\ f$, so by injectivity $\phi_{/M} = 0$ so $\phi_{N/\mathrm{im}\ f} = 0$ and the cokernel of f is S-torison.

For the converse: if $\phi : N \to L$ maps to $0 : M \to N$ then for all $n \in N$, $\exists s \in S/\ sn \in \text{im } f$ and $\phi(sn) = 0$ but 0 is *uniquely* s-divisible so $\phi(n) = 0$. Now take any $\psi : M \to L$, then any element $x \in \text{Ker } f$ is killed by some s and $\psi(sx) = 0$ so $\psi(x) = 0$ and $\text{Ker } f \subseteq \text{Ker } \psi$. Then for $n \in N$, $sn = f(m) \in \text{im } f$, just define $\phi(n)$ to be $\frac{\psi(m)}{s}$. □

It so appears that the right localization context is with maps whose kernel and cokernel are S-torison, which are killed when mapped in a uniquely S-divisible module. It is then easy to check that the localization $\phi_M \to S^{-1}M$ is such a map, and hence $\text{Mod}(R)$ has good localization at S.

As announced in the title, what we are really interested in is to define localization of groups. This is done for abelian groups, that are $\mathbb{Z}$-modules. Multiplicative subsets $S \subseteq \mathbb{Z}$ are then described by a bunch of generators $s_i = \Pi p_{i,k}^{\alpha_{i,k}}$. Being of $\langle s_i \rangle$-torsion is the same as being of $\langle p_{i,k} \rangle$-torsion so we will assume S is generated by a bunch of primes T. We define $\neg T$ as the set of primes not in T. Quite often in commutative algebra we consider $S = R \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, and we write $R_\mathfrak{p}$ for $S^{-1}R$. Hence, $\mathbb{Z}_{(p)}$ is $\langle \neg\{p\} \rangle^{-1}\mathbb{Z}$, and we define similarly $\mathbb{Z}_T = \langle \neg T \rangle^{-1}\mathbb{Z}$, so corresponding to $S = \langle \neg T \rangle$, the set of products of (or monoid generated by) elements in $\neg T$. This might be a little confusing, but the idea is that the elements of S are trivialized (turned into units) by the localization process, so $\mathbb{Z}_T$ only "sees" the elements of T. The group $\mathbb{Z}_T$ is naturally a subgroup of $\mathbb{Q}$, and inherits a ring structure that makes it a Principal Ideal Domain. An abelian group A is T-local if and only if it is a $\mathbb{Z}_T$-module. When it's not, we note A_T its localization at T. A group morphism $G \to H$ between abelian groups is a T-equivalence if and only if its kernel and cokernel are of $\langle \neg T \rangle$-torsion.

This last definition can be generalized on groups. We again have two possible definitions of T-equivalences: $\mathcal{W}_1$ that are just composition of isomorphisms and power by an element of $\neg T$, and $\mathcal{W}_2$ that are the maps whose kernel and cokernel are of $\langle \neg T \rangle$-torsion, containing $\mathcal{W}_1$. There is a little abuse here because cokernels aren't always defined. We mean that each element of the ending group lands in the image of our morphism when raised at some power in $\langle \neg T \rangle$. This is usually called a T-epimorphism. They are indeed precisely the morphisms that will land on an epimorphism when localized, at least when the morphism is between nilpotent groups. The same holds for T-monomorphisms: morphisms whose kernel are of $\langle \neg T \rangle$-torsion.
The p^{th} power is not a group morphism in general so these definitions behave pretty bad with non abelian groups. They behave well enough with short exact sequence though, so we can define them on successive central extensions of abelian groups: nilpotent groups. This is exposed with great details in [MP12], II.5.4-5, among which:

> **Proposition 3.4:** *In the category of nilpotent groups, the weak equivalences $\mathcal{W}_1$ and $\mathcal{W}_2$ define the same local objects, that are the groups on which the p^{th} power is a bijection for all $p \in \neg T$. They correspond to the $\mathbb{Z}_T$-nilpotent groups.*
> *This category has good localization for $\mathcal{W}_2$, and localization is exact (namely it preserves short exact sequences).*

3.3 Localization of spaces

We have noted before that when some localization context has good localizations, the weak equivalences are characterized by the fact that they induce isomorphisms between localizations. In the preceeding section, weak equivalences are those maps that become isomorphisms when tested on groups built out of $\mathbb{Z}_T$-modules.

⌜ **Definition 3.5:** A continuous map $f : X \to Y$ is called a T-equivalence if it induces an isomorphism $f_* : H_*(X; \mathbb{Z}_T) \to H_*(Y; \mathbb{Z}_T)$. We call localization of a space at T the localization at these T-equivalences.
These are what we have earlier called $\mathbb{Z}_T$-homology isomorphisms. We write in general $\mathcal{W}_R$ for the collection of R-homology isomorphisms, where R is a commutative ring, and we call localization at R the localization at $\mathcal{W}_R$. ⌟

We shall also be interested in another class of weak equivalances. For any collection of abelian groups $\mathcal{A}$, we write $\mathcal{W}^*_{\mathcal{A}}$ the collection of $\mathcal{A}$-cohomology isomorphisms. Remember that when R is a PID, $\mathbb{Z}_T$ for example, $\mathcal{W}_R = \mathcal{W}^*_{\mathrm{Mod}(R)}$. This will be of great use because cohomology is representable and hence way easier to deal with.

Note that the second form of the R-Dual Whitehead Theorem precisely states that Postnikov R-towers are R-local spaces. On the other hand, for the particular case $R = \mathbb{Z}_T$ which is our interest now, we can easily see why the homotopy groups $\pi_n(X)$ of a T-local space X should be T-local themselves. For simply connected spaces, this implies that X has to be $\mathbb{Z}_T$-nilpotent. Indeed, the degree p map $S^n \to S^n$ is a T-equivalence for $p \in \neg T$, for it induces multiplication by p on $\tilde{H}_*(S^n;\mathbb{Z}_T) \simeq \mathbb{Z}_T \to \mathbb{Z}_T$. Locality implies that the multiplication by $p : \langle S^n, x\rangle = \pi_n(X) \to \pi_n(X)$ is a bijection.

We already have a big class of T-local spaces, that seems quite exhaustive, since it is for simply connected spaces. Now, we would like to know how to localize spaces. A good place to start is Eilenberg-MacLane spaces of abelian groups: it is of great interest to know how to localize them because they are the building blocks of cocellular spaces, and they are defined by a single group, so their localization might be understandable by localization of groups. This is indeed true:

Proposition 3.6: *Let $n \in \mathbb{N}^*$. An abelian group A is T-local if and only if its Eilenberg-MacLane space $K(A, n)$ is T-local. In general, the T-localization is given by the map $K(A, n) \to K(A_T, n)$ obtained by functoriality from $A \to A_T$.*

Proof : (first line) We already saw that $\pi_n(K(A, n)) = A$ should be T-local if $K(A, n)$ is. For the other direction, $K(A, n)$ is local if for any T-equivalence $f : X \to Y$, the map $\langle Y, K(A, n)\rangle \to \langle X, K(A, n)\rangle$ is an isomorphism. By representability of cohomology, this is the map $f^* : H^n(Y;A) \to H^n(X;A)$. Since A is a $\mathbb{Z}_T$-module, it is sufficient that f be a $\mathrm{Mod}(\mathbb{Z}_T)$-cohomology isomorphism, which holds because $\mathcal{W}_{\mathbb{Z}_T} = \mathcal{W}^*_{\mathrm{Mod}(\mathbb{Z}_T)}$. □

Theorem 3.7: *Every nilpotent space X admits a T-localization $X \to X_T$ to a Postnikov $\mathbb{Z}_T$-tower.*
On nilpotent groups it gives: G is T-local if and only if $K(G, 1)$ is, and a map $G \to H$ is a T-localization if and only if $K(G, 1) \to K(H, 1)$ is.

Proof : We can construct this localization cocell-by-cocell. We can suppose that X is a Postnikov $\mathbb{Z}$-tower $\lim X_i$, because we know by Proposition 2.28 that it is weak homotopy equivalent, so T-equivalent, to one. Then $\langle \lim X_i, X_T\rangle \simeq \langle X, X_T\rangle$ and X_T is canonically a localization of X.
Suppose we have a localization $X_i \overset{\phi_i}{\to} (X_i)_T$ to a Postnikov $\mathbb{Z}_T$-tower with cocells of dimension at most n, and a cocell $K(G, n) \to X_{i+1} \to X_i \to K(G, n+1)$. We already know how to localize 3 out of the 4 spaces here. Functoriality of localization on the coattaching map provides the following diagram, commutative up to homotopy, where X_{i+1} is defined as the homotopy fiber of $(k_i)_T$.

$$\begin{array}{ccccccc} K(G,n) & \longrightarrow & X_{i+1} & \longrightarrow & X_i & \xrightarrow{k_i} & K(G,n+1) \\ \downarrow & & \vdots & & \downarrow{\scriptstyle \phi_i} & & \downarrow \\ K(G_T,n) & \longrightarrow & (X_{i+1})_T & \longrightarrow & (X_i)_T & \xrightarrow{(k_i)_T} & K(G_T,n+1) \end{array}$$

The Lemma 1.2.3 of [MP12] states that there exists a fill-in vertical map $X_{i+1} \overset{\phi_i}{\to} (X_{i+1})_T$ such that the middle square commutes. Then, we can define $X_T = \lim (X_i)_T$. The space X_{i+1} is T-local because it is a Postnikov $\mathbb{Z}_T$-tower, and we can check that ϕ_{i+1} is indeed a T-localization. It is easy to deduce that $X \to X_T$ is a T-localization, because an element in homology is always defined in a compact, hence in one step of the tower.

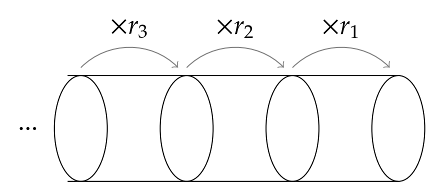
When X is a $K(G, 1)$ with G T-local, by Proposition 3.4 we can find a cocellular construction of X with cocells T-local abelian $K(G_k, 1)$'s, so all vertical maps are isomorphisms in the above diagram, and $X \simeq X_T$ is T-local. □

Knowing how to build localization, it becomes much easier to describe T-local spaces, and T-localization. We already know that simply connected T-local spaces are the $\mathbb{Z}_T$-nilpotents, because its homotopy groups must be T-local. By uniqueness of localization, last theorem implies that this is still true for nilpotent spaces:

Theorem 3.8: *A nilpotent space X is T-local if and only if it is $\mathbb{Z}_T$-nilpotent, if and only if all of its homotopy groups are T-local and if and only if all of its reduced homology groups are T-local.*
A map $X \to X_T$ from a nilpotent space to a T-local space is a T-localization if and only if it induces a T-localization on all homotopy groups and if and only if it induces a T-localization on all reduced homology groups.

Corollary 3.9: *For X and Y two nilpotent spaces, $\Sigma(X_T) \simeq (\Sigma X)_T$ and $X_T \times Y_T \simeq (X \times Y)_T$, because homotopy groups of the suspension and of the product are functorially by the ones of the base spaces.*
Moreover, if X is simply connected to avoid non-connectivity issues on ΩX, and if $X \vee Y$ is nilpotent (note that nilpotency is not stable by wedge sum, $S^1 \vee S^1$ is not nilpotent for example) then $\Omega(X_T) \simeq (\Omega X)_T$ and $X_T \vee Y_T \simeq (X \vee Y)_T$, because homology groups of the loop space and of the wedge sum are functorially by the ones of the base spaces.

Generally, localization commutes with homotopy limits or colimits in hTop, provided that the resulting space is still nilpotent, because homotopy groups behave well with the first, and homology groups with the second.
Example : We can construct a $K(\mathbb{Z}_T, 1) = S^1_T$ in a way that illustrates very well how localization works. We start with S^1, that is $\mathbb{Z}$, and to construct $\mathbb{Z}_T$ we need to introduce some $\frac{1}{p}$ for all $p \in \neg T$. We can do that by taking the mapping cylinder of the degree p map $S^1 \xrightarrow{z^p} S^1$. But then we need the $\frac{1}{p^2}$ and the $\frac{1}{pq}$ and so on... Write $\neg T = \{p_1, ..., p_n, ...\}$ and set $r_n = p_1 \times ... \times p_n$. Then we just need to introduce a $\frac{1}{r_1}$, $\frac{1}{r_1 \times r_2}$ and so on... This can be made by successively attaching the mapping cylinder of the r_n^{th} power to the last circle of the preceeding mapping cylinder. We then take the colimit and obtain an infinite mapping telescope as below:
This is the homotopy colimit of $... \to S^1 \xrightarrow{r_{n+1}} S^1 \xrightarrow{r_n} S^1 \to ... \to S^1$, where we first turn all maps into cofibration, and then take the usual colimit. Similarly, $\mathbb{Z}_T$ can be obtained as the colimit of $... \to \mathbb{Z} \xrightarrow{r_{n+1}} \mathbb{Z} \xrightarrow{r_n} \mathbb{Z} \to ... \to \mathbb{Z}$, which just π_1 applied on the first. Homology behaves nicely with homotopy colimits and we can easily check that this is indeed a $K(\mathbb{Z}_T, 1)$, the T-localization of the circle.



This construction also gives the localization of all spheres by taking suspensions, and many more by taking homotopy colimits of some spheres. It is possible to do a cellular construction of localization for simply connected CW-complexes, by attaching cells in the very same way, but localized. Namely, a cell $S^n \xrightarrow{\phi} X_{n-1}$ becomes a cell $S^n_T = \Sigma^{n-1} S^1_T \xrightarrow{\phi_T} (X_{n-1})_T$ by functoriality. Actually we can do the same for all CW-complexes that admit a cell-by-cell construction that is nilpotent at every step.

3.4 Completion of abelian groups

Completion has several different definitions in the literature, though they all agree on finitely generated abelian groups. We will follow [MP12] here. The basic principle is that we consider a prime p as small, so that "infinite series" in p will be convergent. The most fundamental example is the p-completion of $\mathbb{Z}$ that will be the p-adic integers $\mathbb{Z}_p$, which can be described as formal series $\sum_{i=0}^{\infty} b_i p^i$ with $b_i \in \{0, 1, ..., p-1\}$.

⌜ **Definition 3.10:** The p-adic completion of an abelian group A is the limit $\hat{A}_p = \varprojlim A/p^r A$ with $A/p^{r+1}A \twoheadrightarrow A/p^r A$ the quotient. The coherent projections $A \twoheadrightarrow A/p^r A$ induce a morphism $\hat{\phi}_p : A \to \hat{A}_p$ called the p-adic completion. The group A is called p-adic complete if it is an isomorphism. ⌟

This construction gives a functor $A \mapsto \hat{A}_p$, which is exact when restricted to finitely generated abelian groups. Moreover, it is idempotent: the group $\hat{A}_p$ is always p-adic complete.

Examples : • The p-adic completion of $\mathbb{Z}$ is the p-adic integers $\mathbb{Z}_p$.

• The prime p is already small in $\mathbb{Z}/p$, so it is p-adic complete. However it is invertible in $\mathbb{Z}/n$ with n prime to p, and $(\widehat{\mathbb{Z}/n})_p = 0$. Any p-divisible abelian group has trivial p-adic completion, since all $A/p^r A$ are trivial. Hence for $T \not\ni p$, the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}_T \to \mathbb{Z}_T/\mathbb{Z} \to 0$ gives after p-adic completion $0 \to \mathbb{Z}_p \to 0 \to 0 \to 0$ which is of course not exact.

• A countably generated free abelian group $F = \bigoplus_{i\in\mathbb{N}} f_i\mathbb{Z}$ has p-adic completion $\hat{F}_p = \varprojlim \bigoplus_{i\in\mathbb{N}} f_i\mathbb{Z}/p^r$ which corresponds to the subset of the product $\prod_{r\in\mathbb{N}} \bigoplus_{i\in\mathbb{N}} f_i\mathbb{Z}/p^r$ where the projection of the $(r+1)^{th}$ term of the product is the r^{th}. Hence an element of $\hat{F}_p$ is a sequence $(\sum_i a_i^r f_i)_{r\in\mathbb{N}}$ with $0 \leq a_i^r < p^r$, $a_i^{r+1} \equiv a_i^r[p^r]$ and each sum finite. We can see $\hat{F}_p$ as the subset of the product $\prod_{i\in\mathbb{N}} \mathbb{Z}_p$ where the p-valuation of the terms diverges to infinity when i does.

When $T = \neg\{p\}$, we define $\mathbb{Z}/p^\infty = \mathbb{Z}_T/\mathbb{Z} = \mathbb{Z}[p^{-1}]/\mathbb{Z}$. This is actually the colimit of $\mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p^2 \xrightarrow{\times p} \mathbb{Z}/p^3 \xrightarrow{\times p} ...$, hence its name. Visually, we successively add a pth of every element, starting with $\mathbb{Z}/p$. If an abelian group A is p-divisible but not uniquely, then there are nontrivial group morphisms $\mathbb{Z}/p^\infty \to A$, sending $1 \in \mathbb{Z}/p$ to an element $a \in A \setminus \{0\}$ such that $pa = 0$ and then choosing successive pth of a.

When the prime p is fixed, we will drop it in the notations. The above definition of completion is too bad behaved, with respect to short exact sequences most importantly. To make it more regular, we will use its left derived functor. This is a general categorical construction, that needs not be introduced here. We will use the good behavior of p-adic completion on finitely generated abelian groups and their sums, mostly free abelian groups, to recover what completion *should* be. We will use the following definition:

⌜ **Definition 3.11:** Consider any abelian group A. We can find a free abelian group F that surjects onto A, for example by taking the free abelian group generated by all elements of A. Then set F' to be the kernel of this surjection. As a subgroup of a free abelian group, it is one itself. Hence we have a short exact sequence $0 \to F' \xrightarrow{\iota} F \to A \to 0$.
Applying the p-adic completion, we get a sequence $\hat{F'}_p \xrightarrow{\hat{\iota}_p} \hat{F}_p \to \hat{A}_p$ that has no reasons to be exact. Only the $\hat{F'}_p \to \hat{F}_p$ are the correct completions here, and if completion were to be exact, the one of A would have to be the cokernel of this map.
We define $L_0(A) = \operatorname{coker} \hat{\iota}_p$ and $L_1(A) = \ker \hat{\iota}_p$, so that $0 \to L_1(A) \to \hat{F'}_p \xrightarrow{\hat{\iota}_p} \hat{F}_p \to L_0(A) \to 0$ is exact. The group A is called completable if $L_1(A) = 0$. There is a canonical map $\phi : A \to L_0(A)$, which is called a completion when A is completable. A is called complete if this map is an isomorphism. This actually implies that A is completable, as will show the following. We call $\mathcal{A}_p$ the class of completable abelian groups, and $\mathcal{B}_p$ the class of complete ones. ⌟

Proposition 3.12: *For an abelian group A, the groups $\hat{A}_p$, $L_0(A)$ and $L_1(A)$ are completable and complete. An exact sequence of abelian groups $0 \to A \to B \to C \to 0$, gives an exact $0 \to L_1(A) \to L_1(B) \to L_1(C) \to L_0(A) \to L_0(B) \to L_0(C) \to 0$, which means that completion is indeed exact on completable abelian groups.*

These groups $L_0(A)$ and $L_1(A)$ can be reinterpreted in terms of the very useful Ext groups. There are canonical isomorphisms $L_0(A) \simeq \mathrm{Ext}(\mathbb{Z}/p^\infty, A)$ and $L_1(A) \simeq \mathrm{Hom}(\mathbb{Z}/p^\infty, A)$. Hence an abelian group is not completable if and only if there are non trivial group morphisms $\mathbb{Z}/p^\infty \to A$, for example non-uniquely p-divisible groups. The exact sequence above is the 6-term sequence for Ext groups with $\mathbb{Z}/p^\infty$. See [Hat02] 3.F and [MP12] 10.1.3.

Example : The group $A = \mathbb{Z}/p^\infty$ is a fundamental example of non-completable abelian group. It is generated as $\mathbb{Z}$-module by the projections of the $\frac{1}{p^i} \in \mathbb{Z}[p^{-1}]$. We get a free resolution $0 \to F' \to F \to A \to 0$ by setting F the free abelian group with basis $(f_i)_{i\in\mathbb{N}^*}$ and F' its subgroup with basis $(f'_i = f_i - pf_{i+1})_{i\in\mathbb{N}}$, where $f'_0 = -pf_1$. Its p-adic completion gives $0 \to L_1(A) \to \hat{F'}_p \to \hat{F}_p \to L_0(A) \to 0$. Take an element $(\sum_i a_i^r f_i)_{r\in\mathbb{N}}$ of $\hat{F}_p$ as described in the last example. Each $a_i^r f_i$ can be obtained as the image of an element in $\hat{F'}_p$: $a_i^r f_i = a_i^r(f_i - pf_{i+1}) + pa_i^r(f_{i+1} - pf_{i+2}) + p^2 \ldots$. Now since the p-valuation of the a_i^r diverges, the sum of all the preceeding element converges in $\hat{F'}_p$, and has image our chosen element in $\hat{F}_p$. We just proved that $\hat{F'}_p \to \hat{F}_p$ is surjective so $L_0(\mathbb{Z}/p^\infty) = 0$.

To compute $L_1(\mathbb{Z}/p^\infty)$, we will use the above identification with $\mathrm{Hom}(\mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty)$. Such an endomorphism f is determined by the images of each $\frac{1}{p^i}$, with only condition that $pf(\frac{1}{p^{i+1}}) = f(\frac{1}{p^i})$. Hence, $f(\frac{1}{p^{i+1}})$ is determined by $f(\frac{1}{p^i})$ up to a p-torsion element, namely one of the subgroup $\mathbb{Z}/p \subseteq \mathbb{Z}/p^\infty$. An element of $L_1(\mathbb{Z}/p^\infty)$ is just the data of a countable sequence $(a_n)_{n\in\mathbb{N}^*}$ of elements of $\{0, ..., p-1\}$, with $f(\frac{1}{p^i}) = \sum_{n=1}^{i} a_{i-n+1}\frac{1}{p^n}$. The group structure is given by doing the usual addition with carrying, that is precisely the sum in $\mathbb{Z}_p$. Hence $L_1(\mathbb{Z}/p^\infty) = \mathbb{Z}_p$.

3.5 Completion of spaces

To define completion on spaces we use the same idea than with localization. The localization at T of a space is the categorical localization where the weak equivalences are the ones inducing cohomology isomorphisms at all T-local abelian groups (the $\mathbb{Z}_T$-modules). The completion will simply be the categorical localization where the weak equivalences are the ones inducing cohomology isomorphisms at all complete abelian groups, namely $\mathcal{W}^*_{\mathcal{B}_p}$.

Proposition 3.13: *The $\mathcal{B}_p$-cohomology isomorphisms $\mathcal{W}^*_{\mathcal{B}_p}$ are precisely the $\mathbb{Z}/p$-homology equivalences. Moreover $\mathcal{B}_p$ is the biggest class of abelian groups $\mathcal{B}$ such that $\mathcal{W}^*_{\mathcal{B}} = \mathcal{W}_{\mathbb{Z}/p}$.*

Again, the $\mathcal{A}$-Dual Whitehead Theorem with $\mathcal{A} = \mathcal{B}_p$ says that all Postnikov $\mathcal{B}_p$-towers are complete spaces. As before, we will first try to understand how this completion, or $\mathbb{Z}/p$-localization, behaves on Eilenberg-MacLane Spaces. The complete Eilenberg-MacLane spaces are the expected ones:

Proposition 3.14: *Let $n \in \mathbb{N}^*$. An abelian group A is complete if and only if its Eilenberg-MacLane space $K(A, n)$ is.*

However, the completion of non-complete Eilenberg-MacLane spaces is more complicated. On the completable groups $\mathcal{A}_p$, completion is exact and the completion of the Eilenberg-MacLane space of a completable abelian group A is the expected $K(A, n) \to K(L_0(A), n)$. But in general, the fact that completion on abelian groups is not exact will give rise to significant difficulties.

Proposition 3.15: *The completion of an abelian Eilenberg-MacLane space $K(A,n)$ is a Postnikov $\mathcal{B}_p$-tower $K(A,n)^\wedge_p$ whose only non-vanishing homotopy groups are $\pi_{n+1}(K(A,n)^\wedge_p) = L_1(A)$ and $\pi_n(K(A,n)^\wedge_p) = L_0(A)$. The induced map on π_n is the completion of abelian groups.*

Proof : To construct the completion in the general case we will use the free case, as for abelian groups. Take a free resolution $0 \to F' \to F \to A \to 0$. Check by Proposition 2.9 that the homotopy fiber of $K(F', n+1) \to K(F, n+1)$ is indeed a $K(A,n)$. We have a great candidate for the completion of $K(A,n)$: the homotopy fiber of the completion of $K(F', n+1) \to K(F, n+1)$, noted $K(A,n)^\wedge_p$. This all fits in a homotopy commutative diagram as below. Again with Lemma 1.2.3 of [MP12] there is a fill-in map ϕ:

$$\begin{array}{ccccccc} K(F,n) & \longrightarrow & K(A,n) & \longrightarrow & K(F', n+1) & \longrightarrow & K(F, n+1) \\ \downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\ K(\hat{F}_p, n) & \longrightarrow & K(A,n)^\wedge_p & \longrightarrow & K(\hat{F'}_p, n+1) & \longrightarrow & K(\hat{F}_p, n+1) \end{array}$$

The constructed space $K(A,n)^\wedge_p$ is complete because it is a $\mathcal{B}_p$-tower, and we can check that the fill-in map ϕ is a $\mathbb{Z}/p$-homology equivalence, hence the completion of $K(A,n)$. Now this space is a homotopy fiber, so Proposition 2.9 gives the exact $0 \to \pi_{n+1}(K(A,n)^\wedge_p) \to \hat{F'}_p \to \hat{F}_p \to \pi_n(K(A,n)^\wedge_p) \to 0$, which is the definition of $L_0(A)$ and $L_1(A)$.□

Example : There can be a full degree shift: the p-completion of $K(\mathbb{Z}/p^\infty, n)$ is a $K(\mathbb{Z}_p, n+1)$. We already computed in a previous example that $L_0(\mathbb{Z}/p^\infty) = 0$ and $L_1(\mathbb{Z}/p^\infty) = \mathbb{Z}_p$.
By induction on the cocells, exactly like for localization, we get:

Theorem 3.16: *Every nilpotent space X admits a p-completion $X \to X^\wedge_p$ to a Postnikov $\mathcal{B}_p$-tower.*

⌜ **Definition 3.17:** For a nilpotent group G, the space $K(G,1)$ is nilpotent too, and admits a p-completion $K(G,1)^\wedge_p$. We define, analogously to abelian groups, $L_0(G) = \pi_1(K(G,1)^\wedge_p)$ and $L_1(G) = \pi_2(K(G,1)^\wedge_p)$. We then can define completable and complete nilpotent groups, as well as completion of nilpotent groups. ⌟

Theorem 3.18: *A nilpotent space X is p-complete if and only if it is $\mathcal{B}_p$-nilpotent, if and only if all of its homotopy groups are p-complete. If X is an $\mathcal{A}_p$-nilpotent space, so all of its homotopy groups are completable, then a map $X \to X^\wedge_p$ to a p-complete space is a p-completion if and only if it induces a p-completion on all homotopy groups.*

Corollary 3.19: *For X and Y two nilpotent spaces, $X^\wedge_p \times Y^\wedge_p \simeq (X \times Y)^\wedge_p$, and if we further require that X is simply connected to avoid non-connectivity issues on ΩX, then $\Omega(X^\wedge_p) \simeq (\Omega X)^\wedge_p$.*

As for modules, the localized and completed versions are much easier to deal with in some contexts, and they describe very well the initial object. A fundamental result is the arithmetic square for spaces, that states that a nilpotent space can be re-obtained as a pullback of its rationalization (localization at $T = \varnothing$) and the product its localization at all primes along the rationalization of this product. The same holds for completion with localization at a prime p replaced by p-completion.

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