University of Copenhagen

Master's Thesis

Topological Types and Group Cohomology of Reductive Linear Algebraic Groups

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Abstract

We construct the étale topological type of a simplicial scheme $(X_\bullet)_{\acute{e}t}$, a pro-object generalising the Čech nerve with the intent of applying the homotopy theory techniques of pro-objects to give a way of calculating some group cohomologies of reductive linear algebraic groups. The Lang-Steinberg theorem is proved and used to exhibit a principal fibration of simplicial schemes to which the étale type is applied along with the homotopy theory developed to obtain a 'cohomological pullback square'. We then restrict attention to the case of general linear groups over finite fields to give a calculation of the K-theory of finite fields.

Acknowledgements

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Contents

0 Introduction

An object of interest in algebraic topology is the classifying space of a group G, this space BG, in conjunction with a certain map $EG \to BG$ has the property that it classifies principal G-bundles (or G-fibrations) over a paracompact space X: given a map $X \to BG$ we obtain a principal G-bundle over X by means of a pullback and moreover principal G-bundles over X are in bijection with $[X, BG]$ - the homotopy classes of maps from X to BG. The space BG also provides a way of extending the definition of group (co)homology from discrete groups to all topological groups. In the case that G is discrete we have that $K(G, 1)$, the Eilenberg-Maclane space of G, is a model for BG and that there is a natural isomorphism $H_*(K(G,1); R) \cong H_*(G; R)$, where the right hand side is the group (co)homology of G. So we can take this as a definition for group (co)homology, i.e. we set $H_*(G; R) := H_*(BG; R)$. A more indepth discussion of the above can be found in [Ben91].

We note that for a topological group G it is not necessarily the case that $K(G, 1)$ is a model for BG. A construction of BG goes as follows: find a space EG which is contractible with free G action and define the map and classifying space by $EG \to BG := EG/G$ to be the quotient which is a principal G-bundle. As an example S^1 has the structure of a topological group by viewing it as a subset of $\mathbb{C}$ where the multiplication and inverse maps are continuous. S^1 acts freely on S^∞ so that by the long exact sequence of a fibration 8.2.2 we have $\pi_2(BS^1) = \mathbb{Z}$ and all other homotopy groups are trivial, thus $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$ is a model for BS^1.

The major result we work towards in this paper is a method for calculating the cohomology of certain finite groups. Specifically those that are the fixed points of a surjective endomorphism of a reductive linear algebraic group over some algebraically closed field of characteristic p. We will specify to the case of the Forbenius morphism acting on $GL_n(k)$ with fixed points $GL_n(\mathbb{F}_p)$. In [Qui72] Quillen provided a calculation for these (co)homologies with coefficients $\mathbb{F}_l$ for l coprime to p and in this paper we set out to extend these results to a more general setting, albeit by somewhat different means.

The interest in this specific group is the relation to the K-theory of rings as defined by Quillen which extends previously constructed groups.

Unlike S^1 or $GL_n(\mathbb{C})$ the groups we shall be interested in will be subgroups of $GL_n(k)$ and so do not come with a useful Hausdorff topology, however we do have a Zariski topology on them by viewing them as linear algebraic groups, i.e. a variety equipped with a group structure. This topology does not yield anything too interesting however, it is sufficient if we want to examine vector bundles over a space but not prinicpal G-bundles, a finer topology is needed to investigate these. A counterexample for instance would be the case of the punctured affine complex line wound n times round itself. As such we will redefine out notion of topology to that of a Grothendieck topology (or site) which can be though of as replacing the open sets with maps from some other variety satisfying reasonable conditions such as composites should also be considered as 'opens' as should pullbacks. If we leave this definition as is we are left with what is esentially the Zariski topology so we impose an extra condition for our maps to be considered as 'opens', that of being an étale morphism - intuitively a morphism which is locally a homeomorphism. On $\mathbb{C}$ an étale map would be one which has non-vanishing derivative at all points, or on $\mathbb{C}^n$ non-vanishing Jacobian.

We provide a couple of pieces of motivation for the above, firstly a theorem concerning sheaves on the étale site and secondly some motivation for the redefining of 'topology'.

Given a variety X of finite type over $\mathbb{C}$ we can form it's analytification $X(\mathbb{C})$ by transporting the structure of $\mathbb{C}^n$ in the Euclidean topology across to X, associated to this construction there is a comparison theorem:

Theorem. *Let X be a variety and $X(\mathbb{C})$ its analytification, let M be an abelian sheaf on the étale site of X which is locally constant and torsion and denote by the same the corresponding abelian local coefficient system on $X(\mathbb{C})$. Then there is a natural isomorphism:*

$$H_*(X(\mathbb{C}); M) \cong H_*(X_{\acute{e}t}; M)$$

□

As motivation for replacing 'topology' with 'Grothendieck topology' we state the Čech nerve theorem:

Theorem (Čech Nerve Theorem)**.** *Let $\{U_i \to X\}$ be a covering of a paracompact space X with the U_i and their intersections contractible and open, and let $C(\mathcal{U})$ be the Čech nerve of the covering:*

$$C(\mathcal{U}) := \left(\dots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \coprod_{i,j} U_i \cap U_j \substack{\longrightarrow\\ \longrightarrow} \coprod_i U_i \right)$$

then after applying the connected components functor and taking geometric realization we obtain a space homotopy equivalent to X. □

The hope is that by forming an analogous object to the Čech nerve we can produce a simplicial set homotopy equivalent to BG thereby passing to some more manageable combinatorial data. This is however too optimistic, but we can impose further conditions on what a covering should be and then consider all such coverings, this gives us a functor indexed by such coverings to the category of simplicial sets a so called pro-object:

$$(X_\bullet)_{\acute{e}t} : \text{Rigid Hypercoverings} \to \text{Simplicial Sets}$$

We call this the étale topological type of the variety X and it has the property that it determines (co)homology on the étale site for certain sheaves. Having defined this object it will be necessary to consider the homotopy theory of pro-objects, we shall define analogues of the homotopy and (co)homology groups as well as Hurewicz and Whitehead theorems and recall some p-completions and limits in the homotopy category.

Using the above construction we can then show the major result, existence of a cohomological pullback square relating the fixed points H of G, for G a reductive linear algebraic group, to the Bousfield-Kan l-completion of BG where l is coprime to the characteristic of the field k:

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & (BG)\hat{}_l \arrow[d] \\ (\mathbb{Z}/l)_\infty(Sing(BG)) \arrow[r] & (\mathbb{Z}/l)_\infty(Sing(BG)) \times (\mathbb{Z}/l)_\infty(Sing(BG)) \end{tikzcd}$$

By a cohomological pullback square we mean that the induced map from BH to the actual pullback is an isomorphism on cohomology with $\mathbb{Z}/l$ coefficients. In nice cases this will make the (co)homology groups with $\mathbb{F}_l$ coefficients of $BGL_n(k)$ susceptible to calculation via means of the Eilenberg-Moore spectral sequence or similar. If we have any hope of this working we need knowledge of the (co)homology of $(\mathbb{Z}/l)_\infty(Sing((BG))$, so long as we make extra assumptions about G, i.e. that G is reductive, we can relate the linear algebraic group G_k to $G_\mathbb{C}$.

Theorem. *Let $G_{\mathbb{Z}}$ be a reductive linear algebraic group. For l coprime to $p = char(k)$ there is an isomorphism:*

$$H^*_{\acute{e}t}(G_k; \mathbb{Z}/l) \cong H^*_{\acute{e}t}(G_{\mathbb{C}}; \mathbb{Z}/l)$$

We give a breakdown of the chapter structure of this thesis.

Chapter 1 will give some of the prerequisites we require from algebraic geometry and provides a small discussion on étale morphisms. Some results about simplicial objects will be recalled, specifically the construction of the coskeleta which will be vital for our definition of the 'nice' class of coverings. Finally we give the definition of a 'Grothendieck Topology/site' and introduce the étale site.

Chapter 2 will define the notion of (pre)sheaves on a site, mention that these form an abelian category and give conditions on when a sequence of sheaves is exact. The inverse and direct image functors of sheaves will be introduced as will the right derived functors of the latter. Also introduced are two cohomology groups associated to these and finally we provide some cohomology results - that of the Comparison theorem stated earlier as well as the proper base change theorem to do with the higher direct image sheaves.

Chapter 3 introduces the étale topological type and the category of rigid hypercoverings used to define it.

Chapter 4 discusses some of the work of Artin and Mazur in [AM69] regarding pro-objects and provides most of the homotopy theoretic background necessary for studying the étale type.

Chapter 5 will introduce the corresponding notion of a principal G-fibration (when G is discrete) for (simplicial) schemes with the intention of concluding that just as maps $\pi_1(X) \to G$ classify principal G-fibrations for *topological spaces* so too then for *(simplicial) schemes* so long as the fundamental group is replaced by the fundamental group of the étale topological type. We can then deduce that the étale topological type is sufficient for cohomology calculations on the étale site for locally constant abelian sheaves.

Chapter 6 will fill in some missing details regarding firstly some equivalences between various topological types (after some form of pro-completion) and secondly some comparisons of fibre types - since we are passing from (simplicial) schemes to pro-objects in the homotopy category we could either take first the geometric fibre in the realm of schemes and then consider its topological types or first take the topological type and then perform a fibrant replacement and look at the pro-fibres. A result stated for instance will say that the cohomology groups of the two cases are isomorphic under certain conditions.

Chapter 7 states the Lang-Steinberg theorem and is used to exhibit a pullback square as above and the Eilenberg-Moore spectral sequence is then applied to calculate the group cohomology of H with coefficients in $\mathbb{F}_l$ for the case of $GL_n(\mathbb{F}_q)$.

Chapter 8 introduces the K-theory as defined by Quillen and uses the above calculation to find the K-theory of finite fields.

Some proofs of a few results have been relegated to the appendices, namely existence and construction of the coskleton functor and a condition on a map being a homology isomorphism in integral coefficients.

We make some last remarks for what is to come. Various constructions we have eluded to on varieties can in fact be given on schemes as well so this is how they will be stated. In later sections when we apply this to calculate (co)homology we will be working with linear algebraic groups so it will be more convinient to refer to varieties then. We will, for the most part, ignore any set theoretic issues when they arise and refer elsewhere for these considerations. Our passing from the category of topological space to the category of schemes/varieties will require the construction of classifying spaces and principal G-fibrations for schemes, these definitions are near identical to their topological analogues.

The majority of the results concerning the étale type within as well as the method for showing the cohomological pullback square can be found in the book [Fri82] of Friedlander. Also very helpful for understanding the material was the paper [Fri76] of the same author.

Some Notations and Conventions

When we say homotopy group we implicitly mean set in degree 0, group in degree 1 and abelian group in all higher degrees. The term ring shall mean commutative unital ring. Colimits and limits shall be written as colim and lim without an arrow underset to distinguish them. Subscripts (resp. superscripts) on functors shall mostly be used to denote covariance (resp. contravariance). When we say a map we shall mean a continuous function. Topology shall refer to the notion of a Grothendieck topology (or site) introduced in 1.4.1.

We will ignore set theoretic size issues when they appear however these are accounted for by Friedlander in [Fri82, Chp 4] using Grothendieck universes.

Many of the constructions of schemes are more precisely schemes over a base scheme, most commonly Spec of a field, but this shall for the most part be left implicit for brevity. For instance the construction of the coskeleton functor works equally well over a base scheme.

1 Preliminaries and Definitions

1.1 Algebraic Geometry

We recall here some scheme terminology and lemmas.

Definition 1.1.1. *A **scheme** (X, O_X) is a locally ringed space such that every $x \in X$ has an open neighbourhood U with $(U, O_X|_U) \cong (Spec(A), O_{Spec(A)})$ as locally ringed spaces.*

Definition 1.1.2. *The **stalk** at a point x in a scheme X is*

$$O_{X,x} := \operatorname{colim}_{U \ni x} O_X(U)$$

Definition 1.1.3. *A morphism of schemes is **surjective** if it is surjective on the underlying set*

Definition 1.1.4. *A morphism of schemes, $f\colon X \to Y$, is **étale** if it is:*

1. ***locally of finite presentation** - i.e for all $x \in X$ there are affine opens $x \in Spec(A) \subset X$ and $Spec(B) \subset Y$ with $f(Spec(A) \subset Spec(B)$ such that the corresponding ring morphism $B \to A$ is of finite presentation, i.e. $A \cong B[x_1, x_2, \ldots, x_n]/(g_1, g_2, \ldots g_m)$*
2. ***flat** - i.e. all induced maps on stalks $f_{\mathfrak{p}}\colon O_{Y,f(\mathfrak{p})} \to O_{X,\mathfrak{p}}$ are flat*
3. ***unramified** - i.e. it is locally of finite type and for each $\mathfrak{p} \in X$ the morphism of local rings $f_{\mathfrak{p}}\colon O_{Y,f(\mathfrak{p})} \to O_{X,\mathfrak{p}}$ sends the unique maximal ideal surjectively onto the unique maximal ideal and is also such that $\kappa(\mathfrak{p})$ is a finite separable extension of $\kappa(f(\mathfrak{p}))$ - where $\kappa(\mathfrak{p})$ is the residue field of the local ring at $\mathfrak{p}$*

These are very much the 'nice' morphisms we will work with in schemes, there are many alternate definition but these will be the easiest to work with without defining sheaves of relative differentials. However with this one can realise the étale morphims as being the 'smooth morphisms of relative dimension 0', i.e. the scheme analogue to local diffeomorphism in manifolds. Reinforcing this idea is the special case of varieties where the conditions are equivalent to invertibility of the Jacobian matrix at each point.

Lemma 1.1.5. *The composition of étale (surjective) scheme morphisms is étale (surjective).*

Proof. Apparent from the definition since composition of flat (resp. locally of finite presentation, resp. unramified, resp. surjective) morphisms are flat (resp. locally of finite presentation, resp. unramified, resp. surjective). □

Lemma 1.1.6. *The category of Schemes $\mathcal{Sch}$ admits all fibre products. In particular all finite limts exist as $Spec(\mathbb{Z})$ is terminal.*

Proof. [Har77, Thm 3.3] □

Lemma 1.1.7. *The pullback of étale (resp. surjective) scheme morphisms is étale (resp. surjective).*

Proof. We show the pullback of an étale morphism is étale. For locally of finite presentation we consider the diagram:

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, dashed] & Spec(B) \arrow[d, "f"] \\ Spec(C) \arrow[r] & Spec(A) \end{tikzcd}$$

where f is of finite presentation so that we may write $B \cong A[x_1, x_2, \ldots, x_n]/(g_1, g_2, \ldots, g_m)$. Finding this pullback is then equivalent to finding the pushout of:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, dashed] \\ C \arrow[r, dashed] & R \end{tikzcd}$$

We abuse notation by writing f for both the scheme map and ring map. The pushout R is just the tensor product $C \otimes_A B$ and so the pullback is:

$$X = Spec(C \otimes_A B) = Spec(C \otimes_A A[x_1, x_2, \ldots, x_n]/(g_1, g_2, \ldots, g_m))$$

so that $C \to C \otimes_A B$ is of finite presentation.

For flatness consider again the pushout square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, dashed] \\ C \arrow[r, dashed] & C \otimes_A B \end{tikzcd}$$

Since flatness of the ring morphisms $A_{f^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$ for all primes $\mathfrak{p} \in B$ is equivalent to flatness of the morphism $A \to B$ we will assume that f is flat and shows it's pushout is also flat. Take $0 \to M \to N$ to be an exact sequence of C modules, tensoring with $B \otimes_A C$ over C we obtain:

$$0 \to B \otimes_A C \otimes_C M \to B \otimes_A C \otimes_C N$$

But this is just the sequence:

$$0 \to B \otimes_A M \to B \otimes_A N$$

with A acting on M and N via the $A \to C$ algebra structure, but the above is injective as $A \to B$ is flat, hence $C \to C \otimes_A B$ is flat.

It remains then to show pullbacks of unramified morphisms are unramified. Again we check this affine locally. We are given that for $\mathfrak{p} \subset B$ we have $\mathfrak{p} \cdot B_{\mathfrak{p}} = (f(f^{-1}(\mathfrak{p}))) \cdot B_{\mathfrak{p}}$ or equivalently for each $p/1 \in \mathfrak{p} \cdot B_{\mathfrak{p}}$ there is a $p_1 \in f^{-1}(\mathfrak{p})$ and $b_1/b_2 \in B_{\mathfrak{p}}$ with $f(p_1) \cdot b_1/b_2 = p/1$. We are required firstly to show that for any prime ideal $\mathfrak{q} \subset C \otimes_B A$ that $\mathfrak{q} \cdot C \otimes_A B = (g(g^{-1}(\mathfrak{q}))) \cdot C \otimes_A B$ and secondly that there is the finite separable extension condition on the residue fields.

The following psuhout diagram can be localized at the prime $\mathfrak{q}$ (and its preimages).

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "f"] & C \arrow[d, "g"] \\ B \arrow[r, "h"] & C \otimes_A B \end{tikzcd}$$

We write $\mathfrak{n}$ for the prime ideal $\mathfrak{n} := f^{-1}(h^{-1}(\mathfrak{q})) = j^{-1}(g^{-1}(\mathfrak{q}))$ and so we have:

$$\begin{tikzcd} A_{\mathfrak{n}} \arrow[r, "j"] \arrow[d, "f"] & C_{g^{-1}(\mathfrak{q})} \arrow[d, "g"] \\ B_{h^{-1}(\mathfrak{q})} \arrow[r, "h"] & (C \otimes_A B)_{\mathfrak{q}} \end{tikzcd}$$

We choose then some element $c \otimes b \in q \subset C \otimes_A B$ which is also $(c \otimes 1) \times (1 \otimes b)$. The first of these elements is in the image of g when restricted to $g^{-1}(q)$ which contains c. For the second note $b \mapsto 1 \otimes b$ and that $b \in h^{-1}(q)$ and we can then write $b/1 = f(n) \cdot b_1/b_2$ for some $n \in \mathfrak{n}$ and $b_1/b_2 \in B_{h^{-1}(\mathfrak{q})}$ by f being unramified. Observe then the equalities:

$$1 \otimes b = 1 \otimes h(f(n))b_1/b_2 = g(j(n)) \otimes b_1/b_2 = (g(j(n)) \otimes 1) \times (1 \otimes b_1/b_2)$$

and $j(n) \in g^{-1}(\mathfrak{q})$. We omit the remaining check on the residue fields.

$\square$

Definition 1.1.8. *A geometric point $\overline{x}$ of a scheme X is a morphism $\overline{x}\colon Spec(K) \to X$ with K algebraically closed.*

1.2 Simplicial Objects

We write Δ for the category with objects $[n] = \{0, 1, \ldots, n\}$ and morphisms the weakly increasing functions $f\colon [n] \to [m]$. A simplicial object in a category $\mathcal{C}$ is a functor $X\colon \Delta^{op} \to \mathcal{C}$. The category of simplicial objects in $\mathcal{C}$ will be denoted $s\mathcal{C}$ which has morphisms the natural transformations of functors. X evaluated on $[n]$ will be written as X_n .

We can restrict the simplex category Δ to the full subcategory with objects $[k]$ for $k \leq n$ and we denote this category $\Delta_{\leq n}$. A functor $X\colon (\Delta_{\leq n})^{op} \to \mathcal{C}$ will be called an n-truncated simplicial object in $\mathcal{C}$, we also obtain the full category $s_n\mathcal{C}$ of $s\mathcal{C}$ consisting of the n-truncated simplicial objects.

Note 1.2.1. *Any simplicial object X gives rise to an n-truncated simplicial object via the inclusion $i\colon \Delta_{\leq n} \hookrightarrow \Delta$ and so we have an n-truncation functor $\tau_n\colon s\mathcal{C} \to s_n\mathcal{C}$.*

The following two definitions are used in the proof of existence of a right adjoint to the truncation functor.

Definition 1.2.2. *For a category $\mathcal{C}$ and $x \in ob(\mathcal{C})$ we can form the overcategory, denoted $\mathcal{C}_{/x}$, where:*

- *objects are morphisms $y \to x$ in $\mathcal{C}$*
- *morphisms are commuting triangles* $\begin{tikzcd} y \arrow[rr] \arrow[dr] & & z \arrow[dl] \\ & x & \end{tikzcd}$.

The under category can be defined in an analogous way.

Definition 1.2.3. *Given a k-truncated simplicial object X in $\mathcal{C}$ we define a functor $X^n \colon \Delta^{op}_{/[n]} \to \mathcal{C}$ sending:*

- *objects $[l] \to [n]$ to X_l*
- *morphisms* $\begin{tikzcd} {[l]} \arrow[rr, "\varphi"] \arrow[dr] & & {[m]} \arrow[dl] \\ & {[n]} & \end{tikzcd}$ *to $\varphi^* \colon X_m \to X_l$.*

Theorem 1.2.4. *Let $\mathcal{C}$ admit all finite fibre products and have a terminal object then the m-truncation functor admits a right adjoint $\mathcal{cosk}_m$. (This is temporary notation).*

Proof. Proof detailed in Appendix A. □

Corollary 1.2.5. *In particular since $\mathcal{Sch}$ or $\mathcal{Sch}_{/X}$ have finite fibre products and terminal objects the right adjoint $\mathcal{cosk}_m$ is defined on the respective simplicial categories.* □

Definition 1.2.6. *For a category $\mathcal{C}$ with all finite limits the **m-coskeleton** functor is the composition*

$$\operatorname{cosk}_n := \mathcal{cosk}_m \circ \tau_m \colon s\mathcal{C} \to s\mathcal{C}$$

By adjointness we have a natural isomorphism $\operatorname{Hom}_{s_n\mathcal{C}}(\tau_n X, \tau_n Y) \cong \operatorname{Hom}_{s\mathcal{C}}(X, \operatorname{cosk}_n Y)$
We will make most use of this for simplicial schemes and the coskeleton will feature in the definition of a hypercovering of (simplicial) schemes with the étale site, although it can be defined on other topologies.

Example 1.2.7. *As an example suppose we view an object $Y \in \mathcal{C}$ as a 0-truncated simplicial set, i.e. in $s_0\mathcal{C}$. We give a calculation for $\mathcal{cosk}_0 Y$. This will be useful later when we give an example of a hypercovering for any étale surjection $U \to X$. Suppose we have a morphism $\tau_0 X \to Y$ on the left side of the adjunction $\operatorname{Hom}_{s_0\mathcal{C}}(\tau_0 X, Y) \cong \operatorname{Hom}(X, \mathcal{cosk}_0 Y)$. When defined we must have commutativity in the following:*

$$\begin{tikzcd} X_k \arrow[r, "\varphi_k"] \arrow[d, "d"] & \mathcal{cosk}_k Y \arrow[d, "d"] \\ X_0 \arrow[r, "\varphi_0"] & \mathcal{cosk}_0 Y \end{tikzcd}$$

whenever $d \colon [0] \to [k]$ is a morphism in the ordinal category. This gives $(k+1)$ morphisms and an obvious candidate for $\mathcal{cosk}_0 Y$ is given in dimension k by the $(k+1)$ fold fibre product of Y since there will be a unique induced morphism $X_k \to Y \times \ldots \times Y$ by the universal property of pullbacks. It can be checked that there are face and degeneracy maps on these commuting with the induced map φ and so we have given a construction of $\mathcal{cosk}_0 Y$.

$$\mathcal{cosk}_0 Y := \left(\begin{tikzcd} Y \arrow[r, shift left=2] \arrow[r, shift right=2] & Y \times Y \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & Y \times Y \times Y \ldots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd} \right)$$

As a briefer second example we consider the simplicial object $\operatorname{cosk}_{t-1}(\mathcal{cosk}_0 U)$ which will feature in the definition of hypercoverings. Specifically its t-degree part.

Example 1.2.8. *From the proof in Appendix A we have:*

$$(\operatorname{cosk}_{t-1}(\mathcal{cosk}_0 U))_t := \lim_{(\Delta_{/[t]})_{\leq t-1}} (\mathcal{cosk}_0 U)^t$$

using also the notation X^n of 1.2.3. We conclude from this that $(\operatorname{cosk}_{t-1}(\mathcal{cosk}_0 U))_t$ is some finite fibre product of $U \times \ldots \times U$ of lengths less than $(t-1)$ over other such objects. In particular the map $((\mathcal{cosk}_0 U)_t = U \times \ldots \times U \to (\operatorname{cosk}_{t-1}(\mathcal{cosk}_0 U))_t$ is étale surjective where there are $(t+1)$ copies of U.

Definition 1.2.9. *The **Geometric Realization** of a simplicial set X is*

$$|X| := \left(\coprod_{i \geq 0} X \times \Delta^i \right) / \sim$$

where $\left|\Delta^i\right|$ is the standard topological i-simplex and the equivalence relation $\sim$ identifies $(\varphi^*(x), t) \sim (x, \varphi_*(t))$ for all morphisms φ in the category Δ.

Remark 1.2.10. *When we say that a simplicial object $X_\bullet$ in some category has the property 'P' we shall usually mean that each X_n has the property P. E.g. for a simplicial scheme to be locally noetherian means that X_n is locally noetherian for each n. For the most part we will be working with simplicial schemes, however simplicial spaces and simplicial simplicial sets (or bisimplicial sets) will make an appearance.*

The following lemma will be useful when we compare various topological types.

Lemma 1.2.11. *If $T_{\bullet\bullet}$ is a bisimplicial space, i.e. functor $\Delta^{op} \times \Delta^{op} \to \mathcal{T}op$ we can consider the geometric realization of the $T_{n,\bullet}$ and $T_{\bullet,n}$ which assemble to form simplicial spaces $[n] \to |T_{n,\bullet}|$ and $[n] \to |T_{\bullet,n}|$ so we can take their geometric realizations also. We have homeomorphisms between these and the geometric realization of the diagonal of $T_{\bullet\bullet}$.*

$$|\{[n] \to |T_{\bullet,n}|\}| \cong |\Delta T_{\bullet\bullet}| \cong |\{[n] \to |T_{n,\bullet}|\}|$$

Proof. [Qui10, §1] □

By a group scheme G we mean a scheme equipped with scheme morphisms $e\colon Spec(\mathbb{Z}) \to G$ and $\mu\colon G \times G \to G$ such that these morphisms satisfy the usual conditions for forming a group. The definition of the classifying simplicial scheme will resemble the definition of $\mathcal{cosk}_0 G$ in that in each degree it is a product of G, albeit offset by a shift of 1 degree, however the face and degeneracy maps are induced by the morphisms e and μ. We write $*$ for the unique scheme morphism $G \to Spec(\mathbb{Z})$. More generally one can consider group schemes over a base scheme S.

Definition 1.2.12. *The **Classifying Simplicial Scheme** of a group scheme G or **Bar Construction**, denoted by $BG_\bullet$, has BG_n an n-fold product of G and face and degeneracy maps from $BG_n = G^n$ given by:*

$$s_i := \begin{cases} e \times id_{G^n}, & i = 0 \\ id_{G^i} \times e \times id_{G^{n-i}}, & 1 \leq i \leq n \end{cases}$$

$$d_i := \begin{cases} * \times id_{G^{n-1}}, & i = 0 \\ id_{G^{i-1}} \times \mu \times id_{G^{n-3-i}}, & 1 \leq i \leq n-1 \\ id_{G^{n-1}} \times *, & i = n \end{cases}$$

Note the construction is similar to that of the nerve, $NG_\bullet$, of the category obtained from the discrete group G, when viewed as a category with one object and morphisms are the elements of G. The geometric realization of $NG_\bullet$ provides a model for $K(G,1)$. We will later remark that the simplicial scheme $BG_\bullet$ will carry enough information to study the topological $BG = K(G,1)$ in the cases we are interested in.

More generally we can form a bar construction involving schemes X and Y with right and left G actions respectively. If we write $l\colon G \times Y \to Y$ and $r\colon X \times G \to X$ then we make the defintion:

Definition 1.2.13. *The **Double Bar Construction** of X, G and Y is the simplicial scheme, denoted $B(X,G,Y)_\bullet$, which has $B(X,G,Y)_n := X \times G^n \times Y$ and face and degeneracy maps are as above (with identity maps for X and Y), except for d_0 and d_n which replace the maps $*$ with r and l respectively.*

Restricting our attention temporarily to varieties we have that if $H < G$ is a closed subgroup of the variety G then G/H is a homogenous G-space and can be given the structure of a variety (see discussion following [MT11, Thm 5.5]). We can then consider the double bar constructions of the form $B(G/H, G, *)_\bullet$. In a case to be considered later this will turn out to model the topological space BH.

1.3 Local Coefficient Systems

Recall by Yoneda $\mathrm{Hom}_{\mathcal{S}ets^{\Delta^{op}}}(\Delta(-,[n]), S) \cong S([n])$ for a simplicial set $S\colon \Delta^{op} \to \mathcal{S}ets$ so that we may talk about a simplex $s_n \in S([n])$.

We define a **local coefficient system** $\mathcal{M}$ on a simplicial set $S_\bullet$ to be for each simplex $s_n \in S_n := S([n])$ an assignment of a set (abelian group) $\mathcal{M}(s_n)$ and for any map in the ordinal category $\varphi\colon [m] \to [n]$ there is a map $\varphi^*\colon \mathcal{M}(s_n) \to \mathcal{M}(s_m)$ which is an isomorphism of sets (abelian groups). Further we want $\mathcal{M}(\varphi \circ \chi) = \mathcal{M}(\chi) \circ \mathcal{M}(\varphi)$ so that $\mathcal{M}$ is a functor. If $\mathcal{M}(s_n) = A$ for some simplex we say that $\mathcal{M}$ has fibres isomorphic to A.

Equivalently $\mathcal{M}$ can also be viewed as a group homomorphism from the fundamental group to the group of automorphisms of A.

$$M\colon \pi(S_\bullet, s_0) \to Aut(A)$$

Cohomology of a simplicial set can be defined with coefficients in a local coefficient system similar to its usual definition where we define p-cochains by:

$$C^p(S_\bullet, \mathcal{M}) := \{\sigma\colon x_p \to \mathcal{A}(x_p) | x_p \in S_p\}$$

and where differentials take into account the isomorphisms between groups in the local coefficient system.

1.4 Topologies

Definition 1.4.1. *A **Topology** (or **Site**) T on a category $\mathcal{C}$ is a collection of families of morphisms $\{U_i \to U\}$ for each U, called coverings, satisfying:*

1. *Any isomorphism is a covering*
2. *If $\{U_i \to U\}$ and $\{V_{ij} \to U_i\}$ are coverings for each i then the composition $\{V_{ij} \to U\}$ is a covering*

3. *If $\{U_i \to U\}$ is a covering and $V \to U \in Mor(\mathcal{C})$ then $U_i \underset{U}{\times} V$ exists and $\{U_i \underset{U}{\times} V \to V\}$ is a covering*

We will write $Cat(T)$ for the underlying category and $Cov(T)$ for the collection of coverings.

Definition 1.4.2. *For topologies T, T' we define a **morphism of topologies** $F\colon T \to T'$ to be a functor $F\colon Cat(T) \to Cat(T')$ satisfying:*

1. *If $\{U_i \overset{\alpha_i}{\to} U\} \in Cov(T)$ then $\{FU_i \overset{F\alpha_i}{\to} FU\} \in Cov(T')$*
2. *If $\{U_i \to U\} \in Cov(T)$ and $V \to U$ is a morpism in $Cat(T)$ then the induced morphism on pullback squares is an isomorphism, i.e.*

$$F(U_i \underset{U}{\times} V) \overset{\cong}{\to} FU_i \underset{FU}{\times} FV$$

Note some definitions of morphisms of topologies modify this to be a map in the reverse direction so that it agrees with the case that when we have a map of topological spaces $X \to Y$ and cover of Y then the preimage of the cover is a cover for X so that the map of sites goes $Y \to X$. We do not make take this convention.

Definition 1.4.3. *A **presheaf** on a topology T with values in $\mathcal{C}$ is a contravariant functor $P\colon Cat(T) \to \mathcal{C}$. If in addition $\mathcal{C}$ has all products and the following is an equalizer diagram:*

$$PU \longrightarrow \prod_i PU_i \rightrightarrows \prod_{i,j} P(U_i \underset{U}{\times} U_j)$$

*then we say P is a **sheaf** on T.*

Definition 1.4.4. *A **representable presheaf** on a topology T is a presheaf of the form $U \to \mathrm{Hom}(U, X)$ for some $X \in Cat(T)$.*

We also say that the presheaf $U \to \mathrm{Hom}(U, X)$ is represented by the object X.

Representable presheaves are of interest due to the Yoneda lemma asserting there is a natural bijection between morphisms from a representable presheaf to a presheaf and elements of the section of the presheaf over the representing object:

$$\mathrm{Hom}(\mathrm{Hom}(-, X), P) \cong P(X)$$

Definition 1.4.5. *The **Étale Site**, $Et(X_\bullet)$ of a simplicial scheme $X_\bullet$ has the underlying category with:*

- *objects the étale morphisms $U \to X_n$ for some U and n*
- *morphisms the commutative squares where $\varphi\colon [n] \to [m]$*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ X_m \arrow[r, "\varphi^*"] & X_n \end{tikzcd}$$

The coverings of an object $U \to X_n$ will be collections of morphisms with the maps $U_i \to U$ étale and with the U_i jointly surjective onto U.

$$\left\{ \begin{array}{ccc} U_i & \longrightarrow & U \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{id} & X_n \end{array} \right\}_{i \in I}$$

We shall supress the X_n and abbreviate this by $\{U_i \to U\}_{i\in I}$.

Proof. We verify the definition of the étale site does indeed define a site. Isomorphisms are clearly etale and surjective. If $\{U_i \to U\}$ and $\{V_{ij} \to U_i\}$ are coverings then they are étale and jointly surjective and compositions of étale are étale and compositions of jointly surjective morphisms are jointly surjective 1.1.5. The third condition of a site follows from 1.1.7 as pullbacks preserve the étale (resp. surjective) property. □

The étale site can also be defined without the simplicial structure and in this setting is sometimes referred to as the **small étale site**. In contrast the **big étale site** relaxes the condition that objects $U \to X$ be étale although we shall never make use of this definition. Also introduced later is the classical site (or local homeomorphism site) on the analytification of some scheme of finite type over $Spec(\mathbb{C})$ (or more generally some space). It will turn out these are closely related, in fact 6.1.2 asserts there is an equivalence of categories between the classical site of a scheme and the (small) étale site. We will for the most part work on the étale site of a simplicial scheme much for this reason and the étale topological type defined later will capture the cohomological information of sheaves on the étale site.

Returning to representable presheaves we have the following proposition, useful for the definition of constant sheaves:

Proposition 1.4.6. *For a scheme X the representable presheaves on $\acute{E}t(X)$ are in fact sheaves.* □

Proof. [Tam94, Thm 3.1.2] □

2 Cohomology Groups

We restrict attention now to (pre)-sheaves in abelian groups on a topology T which we call abelian (pre)-sheaves. Given an abelian sheaf $\mathcal{F}$ on a topology T we can define two cohomology theories the first on abelian sheaves and the second on abelian presheaves. We will briefly recall the main cohomology groups associated to an abelian sheaf without any simplicial structure. First we note the following so that we can use the techniques of homological algebra:

Proposition 2.0.1. *The category of abelian (pre-)sheaves on a topology T is abelian.*

Proof. [Tam94, Props 2.1.1 & 3.2.1] □

Proposition 2.0.2. *The category of abelian (pre-)sheaves on a topology T has enough injectives.*

Proof. [Tam94, Cors 2.1.2 & 3.2.2] □

For the first we recall the section functor, $\Gamma_U(-)$, associated to an object $U \in Cat(T)$ evaluates an abelian (pre-)sheaf $\mathcal{F}$ at U, i.e. $\Gamma_U(\mathcal{F}) = \mathcal{F}(U)$. This functor is exact on the category of presheaves but only left exact on the category of abelian sheaves and so we can form the p^{th} right derived functors.

Definition 2.0.3. *The p^{th}* ***Cohomology group*** *functor of U is defined to be the p^{th} right derived functor of the section functor $\Gamma_U(-)$ on the category of abelian sheaves*

$$H^p(U; -) := R^p\Gamma_U(-)$$

For the second suppose we have a covering $\{U_i \to U\}$ of U in a topology. We can form a functor from abelian presheaves to abelian groups:

$$H^0(\{U_i \to U\}; F) := \ker\left(\prod_i (F(U_i) \rightrightarrows \prod_{i,j} F(U_i \underset{U}{\times} U_j)\right)$$

When we restrict to abelian sheaves this gives the sections over U, $F(U)$, by the equalizer property of sheaves. But again the sections functor is left exact so we can right derive to form the **Čech Cohomology group of a cover**:

$$H^p(\{U_i \to U\}; F) := R^p H^0(\{U_i \to U\}; F)$$

which agree with the previous cohomology groups of U when F is an abelian sheaf. However the extra structure of the covering allows us to take colimits over all coverings once we define what a map of coverings should be. A **refinement of a covering** $\{V_j \to U\} \to \{U_i \to U\}$ consists of a function $\alpha\colon I \to J$ and maps $f_j\colon V_j \to U_{\alpha(j)}$ over U. It can be checked that these induced maps on all the right derived functors [Tam94] and so we have:

Definition 2.0.4. *The p^{th}* ***Čech Cohomology group of*** *U on abelian sheaves is given by:*

$$\check{H}^p(U; -) := \operatorname{colim}_{\{U_i \to U\}} H^p(\{U_i \to U\}; -)$$

Theorem 2.0.5. *The Čech cohomology group of a cover $H^*(\{U_i \to U\}, \mathcal{F})$ can be computed using the complex of Čech cochains:*

$$\prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_{i_1} \underset{U}{\times} U_{i_2}) \rightrightarrows \cdots$$

Proof. [Tam94, Thm 2.2.3] □

These two cohomology groups need not agree. We consider sheaves on the topology induced by the open sets, i.e. the Zariski site. In [Gro57] Grothendieck constructs a counterexample, the affine plane with sheaf supported on two circles intersecting at two points in the affine plane. In fact there are counterexamples even if we impose Hausdorffness on our space, in [Sch13], Schröer constructs a space formed by a countable wedge of 2-discs which is homeomorphic to the CW structure away from the wedge point but which is weakened at the wedge point, an open contatining the wedge necessarily contains all but perhaps finitely many of the discs (without the antipodes of the wedge point). The sheaf is given by the locally constant $\mathbb{Z}$-valued sheaf on the complement of the 1-skeleton and extended by 0 on the whole space.

2.1 Cohomology Groups on the Étale Site

In this section we introduce the corresponding cohomology group functors for simplicial schemes on the étale site. First however we note that the functor $(-)_n$ restricting an abelian sheaf on $\acute{E}t(X_\bullet)$ to an abelian sheaf on $\acute{E}t(X_n)$ admits both a left and right adjoint.

Definition 2.1.1. *Let A be an abelian sheaf on $\acute{E}t(X_n)$, we define $L_n(A)$ and $R_n(A)$ in degree m by:*

$$(L_n(A))_m := \bigoplus_{\varphi \in \Delta[n,m]} \varphi^* A$$

$$(R_n(A))_m := \prod_{\varphi \in \Delta[m,n]} \varphi_* A$$

Since the functor φ^* is exact so is $L_n(-)$. $R_n(-)$ is merely left exact.

Lemma 2.1.2. *$L_n(-)$ $\big($ resp. $R_n(-)\big)$ is a left (resp. right) adjoint of the restriction functor $(-)_n$.* □

Definition 2.1.3. *The p^{th} **Cohomology group** functor on the étale site $Et(X_\bullet)$ assigns to a sheaf $\mathcal{F}$ the p^{th} right derived functor of the functor sending $\mathcal{F}$ to* $\ker\left(d_0^* - d_1^* \colon \mathcal{F}(X_0) \to \mathcal{F}(X_1)\right)$

$$H^p(X_\bullet; \mathcal{F}) := R^p\left(\ker(d_0^* - d_1^*)\right)$$

If we take $X_\bullet$ to be the same scheme X in each degree and all induced maps $\theta^* \colon X_n \to X_m$ being the identity then the above just becomes the right derived functor of the section functor on $X_0 = X$. We could also define the cohomology group functor by using representability of the section functor. Writing $\mathbb{Z}$ for the constant sheaf (defined later) we could have made the definition:

$$H^*(X_\bullet; -) := Ext^*_{\mathcal{A}b\mathcal{S}h}(\mathbb{Z}, -)$$

We have a spectral sequence which computes the sheaf cohomology on simplicial schemes in terms of the sheaves in each degree. This will turn out to be a very useful decomposition of the cohomology as we will state many cohomological results on the étale site of a non-simplicial scheme so we will need a method to push these results onto cohomology on simplicial schemes.

Lemma 2.1.4. *Let $\mathcal{F}$ be an abelian sheaf on the étale site of a simplicial scheme $X_\bullet$, then there is a first quadrant spectral sequence:*

$$E_1^{s,t} := H^t(X_s; \mathcal{F}_s) \Longrightarrow H^{t+s}(X_\bullet; \mathcal{F})$$

Proof. [Fri82, Prop 2.4] □

Similarly if $\acute{E}t(X_{\bullet\bullet})$ is the étale site for a bisimplicial scheme sheaf cohomology can be defined on $X_{\bullet\bullet}$, either by:

$$H^*(X_{\bullet\bullet}; -) := R^p \ker(d_0^* \times d_0^* - d_1^* \times d_1^*))$$

or using representability of the section functor by:

$$H^*(X_{\bullet\bullet}; -) := Ext^*_{\mathcal{A}b\mathcal{S}h}(\mathbb{Z}, -)$$

The diagonal functor $\Delta\colon s(s\mathcal{S}ets) \to s\mathcal{S}ets$ restricts the functor $X_{\bullet\bullet}\colon \Delta \times \Delta \to \mathcal{S}et$ to the subcategory with objects $([n],[n])$ and morphisms $\theta \times \theta$, i.e. the same object (morphism) in both components. Then sheaf cohomology of bisimplicial schemes can be caluclated on the level of simplicial schemes by restriction:

Proposition 2.1.5. *There is a natural isomorphism of δ-functors:*

$$H^*(X_{\bullet\bullet}; -) \cong H^*(\Delta(X_{\bullet\bullet}), (-)^\Delta)$$

Proof. The proof is Proposition 2.5 of [Fri82] and is shown by observing both are δ-functors, both are 0 on the higher cohomology groups when computed on an injective abelian sheaf and that there is a natural isomorphism in degree 0. The proposition then follows from standard results of cohomological δ-functors. □

In analogy of 2.0.5 we define the Čech Cohomology groups on $Et(X_\bullet)$ using the Čech nerve.

Definition 2.1.6. *We define the p^{th} **Čech Cohomology group** functor by:*

$$\check{H}^p(X_\bullet; \mathcal{F}) := \operatorname{colim}_{U_\bullet \to X_\bullet} H^p(\mathcal{F}(N_{X_\bullet}(U_\bullet)))$$

Replacing sheaf cohomology by Čech cohomology in 2.1.4 we get a similar spectral sequence.

Henceforth when we write $H^*(X_\bullet; -)$ or refer to sheaf cohomology on a (simplicial) scheme we shall mean cohomology of an abelian sheaf on the étale site of $X_\bullet$.

2.2 Constructions on (Pre)sheaves

We have observed that the categories of abelian (pre)sheaves denoted $\mathcal{P}$ and $\mathcal{S}$ on a topology T are abelian categories.

Theorem 2.2.1. *The inclusion functor $i\colon \mathcal{S} \to \mathcal{P}$ admits a left adjoint $(-)^\#$ which we call the sheafification functor.*

$$\mathrm{Hom}_{\mathcal{P}}(i(\mathcal{F}), \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{S}}(\mathcal{F}, \mathcal{G}^\#)$$

Proof. [Tam94, Thm 3.1.1] □

As an immediate application we define constant and locally constant abelian sheaves. Let A be an abelian group with the discrete topology, then we can define a sheaf, also denoted A, on the étale site $\acute{E}t(X)$. Define the sheaf by the sheafification of the constant abelian presheaf $U \to A$.

$$A(-) := ((-) \to A)^{\#}$$

An explicit construction of the sheaf associated to a presheaf is given in [Tam94, Chp. 1 §3.1] where a functor $(-)^{\dagger}$ is defined on an abelian presheaf by:

$$\mathcal{F}^{\dagger}(U) := \check{H}^{0}(U; \mathcal{F})$$

If the abelian presheaf $\mathcal{F}$ is separated, i.e. for any covering $\{U_i \to U\}$ the morphism $\mathcal{F}(U) \to \prod \mathcal{F}(U_i)$ is injective then $\mathcal{F}^{\dagger}$ is a sheaf and is the sheafification of $\mathcal{F}$. Even when $\mathcal{F}$ is not separated we have $\mathcal{F}^{\dagger}$ is separated so that $\mathcal{F}^{\dagger\dagger}$ is a sheaf and in fact the sheafification of $\mathcal{F}$. This is the result [Tam94, Chp. 1 Prop 3.1.3].

The constant presheaf A above is easily seen to be separated so that we need only apply the construction $(-)^{\dagger}$ once. If given two (étale) morphisms $U \to X$ and $V \to X$ the scheme theoretic intersection is non empty $U \underset{X}{\times} V \to X$ then the restriction maps in the presheaf A are the identity, otherwise they are 0. Consider now the connected components of the scheme X. They are closed but not necessarily open (but indeed open if for instance we have finitely many such components), we can take a covering $\{U_i \to X\}$ where each U_i consists of all but finitely many of the connected components - we can restrict to such coverings as the restriction morphisms are either the identity or zero morphism depending on intersection. The colimit over the cohomology of the complex of Čech cochains can now be seen to depend only on the connected components and thus we have the sheafification of the constant presheaf A is:

$$A^{\#}(U) := \prod_{\pi(U)} A$$

where we write $\pi(U)$ for the set of connected components of U. This can be seen to be equivalent to the representable sheaf:

$$\mathrm{Hom}_{\acute{E}t(X)}(-, \coprod_{A} X)$$

which really is a sheaf also since representable presheaves on the étale site are sheaves.

We call such an abelian sheaf a **constant sheaf**. More generally for a sheaf $\mathcal{F}$ on $\acute{E}t(X)$ if we have for any object U of $\acute{E}t(X)$ a covering $\{U_i \to U\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf then we say that $\mathcal{F}$ is **locally constant**.

We will use locally constant sheaves mostly in the case where the groups are torsion in which case we also consider a system of local coefficients determined by the sheaf.

Definition 2.2.2. *The* ***Stalk of a (pre)sheaf*** $\mathcal{F}$ *on* $X_{\acute{e}t}$ *at a geometric point* $\overline{x}$ *is given by the colimit over the étale neighbourhoods of* $\overline{x}$*:*

$$\mathcal{F}_{\overline{x}} := \mathrm{colim}_{(U,u)}\, \mathcal{F}(U)$$

Proposition 2.2.3. *For a sequence of abelian sheaves on* $X_{\acute{e}t}$

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

the following are equivalent:

1. *The sequence of abelian sheaves is exact*
2. *For all geometric points $\overline{x}\colon Spec(\Omega) \to X$ the sequence on stalks at $\overline{x}$ is exact*

$$0 \to \mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \mathcal{H}_{\overline{x}} \to 0$$

Proof. [Mil, Prop 7.6] □

Let $f\colon X \to Y$ be a morphism of schemes, $\mathcal{F}$ an abelian sheaf on $X_{ét}$ and $V \to Y$ and étale morphism.

Definition 2.2.4. *The **Direct Image Sheaf** of $\mathcal{F}$ on $V \to Y$ is*

$$f_*\mathcal{F}(V) := \mathcal{F}(V \underset{Y}{\times} X)$$

Keeping the above notation we have:

Lemma 2.2.5. *$f_*\mathcal{F}$ is a sheaf*

Proof. Take $\{V_i \to V\}_I$ to be a covering of V for $V \to Y$ in the étale site over Y. Pulling back along $X \to Y$ we get a covering $\{V_i \underset{Y}{\times} X \to V \underset{Y}{\times} X\}_I$ in the étale site on X, hence since $\mathcal{F}$ is a sheaf we have an equalizer sequence:

$$\mathcal{F}(V \underset{Y}{\times} X) \longrightarrow \prod_I \mathcal{F}(V_i \underset{Y}{\times} X) \rightrightarrows \prod_{I\times I} \mathcal{F}(V_i \underset{Y}{\times} X \underset{Y}{\times} V_j \underset{Y}{\times} X)$$

which is the condition than $f_*\mathcal{F}$ needs to satisfy to be a sheaf. □

Suppose now $\mathcal{G}$ is an abelian sheaf on $Y_{ét}$ and $U \to X$ is étale.

Definition 2.2.6. *The **Inverse Image Sheaf** of $\mathcal{G}$ on $U \to X$ is defined to be the sheafification of the presheaf given by*

$$U \to \operatorname{colim} \mathcal{G}(V)$$

where the colimit is indexed over morphisms $U \to V$ commuting over $X \to Y$. We denote this by f^*.

We have the following adjunction of direct and inverse image sheaves:

Proposition 2.2.7. *f^* is left adjoint to f_**

Proof. We have the following natural isomorphisms:

$$\begin{aligned}\operatorname{Hom}(f^*\mathcal{G}, \mathcal{F}) &= \operatorname{Hom}((\operatorname{colim} \mathcal{G}(-))^{\#}, \mathcal{F}) \\ &\cong \operatorname{Hom}(\operatorname{colim} \mathcal{G}(-), \mathcal{F}) \\ &\cong \operatorname{Hom}(\mathcal{G}, f_*\mathcal{F})\end{aligned}$$

□

In fact f^* also admits a left adjoint $f_!$ a construction of which can be found in [Mil].

Corollary 2.2.8. *The functor f^* is exact and f_* is left exact.* □

Thus given the following proposition we can right derive the direct image functor.

Proposition 2.2.9. *The category of abelian sheaves on $X_{ét}$ has enough injectives.* □

Keeping the above notation we call the p^{th}-right derived functor $R^p f_* \mathcal{F}$ the p^{th}**-higher direct image**. We have the p^{th}-higher direct image is the sheafification of the cohomology group of the pullback of an open V on $\mathcal{F}$.

Proposition 2.2.10. *$R^p f_* \mathcal{F}$ is the sheafification of $V \mapsto H^p(V \underset{Y}{\times} X; \mathcal{F})$ on $X_{ét}$.*

Proof. [Mil, Prop 12.1] □

2.3 Cohomological Results on the Étale Site

Suppose X is a scheme locally of finite type over $Spec(\mathbb{C})$, i.e. every $x \in X$ has an affine open $x \in Spec(A) \subset X$ where the induced map $\mathbb{C} \to A$ is of finite type so that A is a finitely generated $\mathbb{C}$ algebra. We equip the set of $\mathbb{C}$-rational points of X (also just called rational points) denoted $X(\mathbb{C})$ with the 'classical' or 'analytic' topology (here we use the usual notion of topology, not that of 1.4.1). The term ananlytic topology implies a sheaf of functions on the topology which we shall not use so we use the former terminology.

We define this topology as follows: on the above open $Spec(A)$ of X we write $A = \mathbb{C}[x_1, \ldots, x_n]/I$ and so a $\mathbb{C}$-rational point with image in A corresponds to a ring map $\mathbb{C}[x_1, \ldots, x_n]/I \to \mathbb{C}$, i.e. a choice $(x_i) \in \mathbb{C}^n$ satisfying the equations in I. Transporting the Euclidean topology of $\mathbb{C}^n$ across to $X(\mathbb{C})$ gives the set $X(\mathbb{C})$ the classical topology.

Using this we now define the classical or analytic site of a scheme, with the understanding $X(\mathbb{C})$ comes equipped with the topology described above.

Definition 2.3.1. *The* ***Classical Site*** *of a scheme X denoted X_{cl} has underlying topology with:*

- *objects being local homeomorphisms $U \to X(\mathbb{C})$*
- *morphisms commuting triangles over $X(\mathbb{C})$ of local homeomorphisms*

We say a collection $\{U_i \to U\}$ of morphisms over $X(\mathbb{C})$ is a covering if $\cup U_i = U$

We can also define the classical site of any topological space in a similar way.

Proof. Isomorphisms are clearly local homeomorphisms. Compositions of local homeomorphisms are local homeomorphisms as are pullbacks. So the classical site is indeed a site. □

Before we state the comparison theorem we need to define a morphism of sites from the étale site to the classical site. Assume also that Y is locally of finite type over $Spec(\mathbb{C})$. If $f\colon X \to Y$ is a morphism of schemes we have an induced function $f_*\colon X(\mathbb{C}) \to Y(\mathbb{C})$ by post composing a rational point of X with f. We would like this function to be not only continuous but also a local homeomorphism so that when $U \to X(\mathbb{C})$ is a local homeomorphism so is $U \to X(\mathbb{C}) \to Y(\mathbb{C})$. This is true if we further assume that f is étale.

Lemma 2.3.2. *If $f\colon X \to Y$ is étale then $f_*\colon X(\mathbb{C}) \to Y(\mathbb{C})$ is a local homeomorphism.*

Proof. The étale property is a local one and on locally finite type is equivalent to invertibility of the Jacobian at a point. f_* being invertible now follows from the implicit function theorem. □

The above lemma further reinforces the idea that étale morphisms really are the scheme analogue to local homeomorphisms of topological spaces/manifolds. What this gives us is a morphism of topologies from the étale site to the classical one:

$$\epsilon\colon X_{\acute{e}t} \to X_{cl} \tag{2.3.3}$$

For a definition of a constructible sheaf see [Tam94, Chp. 2, §9.3], but note we shall use this theorem in the case that the sheaf is locally constant on some covering which are in addition finite on such a covering and these are examples of constructible sheaves.

Definition 2.3.4. *A morphism $X \to S$ of schemes is proper if it universally closed, i.e. the pullback of a morphism $Y \to S$ is closed.*

For the following theorems a notion of a *smooth morphism of schemes* is also needed. Most generally this says that 'the sheaf of relative differentials is locally free of dimension the same as the relative dimension of X over S' along with a flatness condition. Again we will restrict attention later to linear algebraic groups defined as varieties so we suffice to think about smoothness, over $Spec(k)$, in terms of varieties where the condition is equivalent to having a cover by affine opens $Speck[x_1, x_2, \ldots, x_n]/(f_1, f_2, \ldots, f_r)$ such that the jacobian of all these affines has the same corank d. Either way one of the cases we will later consider is the projection from a fibre product so equppied also with the knowledge that smoothness is preserved by base change we needn't check explicitly these conditions.

With the above notation the Comparison Theorem states:

Theorem 2.3.5. *Suppose X is a locally finite scheme smooth over $\mathbb{C}$ and let $\mathcal{F}$ be an abelian torsion sheaf on the étale site. Then if either $\mathcal{F}$ is constructible or the map $X \to Spec(\mathbb{C})$ is proper then there is a natural isomorphism for each p:*

$$H^p(X_{\acute{e}t}; \mathcal{F}) \cong H^p(X_{cl}; \mathcal{F})$$

Proof. [Art73, Exposé XVI Thm 4.1] □

Finally we mention the Smooth base change theorem on the étale site. A consequence of the smooth base change used later will be that the higher direct images of constant sheaves are themselves constant.

Suppose we have the pullback square:

$$\begin{tikzcd} X \underset{S}{\times} S' \arrow[r, "b'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "b"] & S \end{tikzcd}$$

We define the **base change morphism**:

$$b^*(R^p f_*)(\mathcal{F}) \to (R^p f'_*)b'^*(\mathcal{F})$$

Note since both sides are δ functors it suffices to define the morphism in degree 0 where we need a morphism $b^* f_* \mathcal{F} \to f'_* b'^* \mathcal{F}$. Equivalently by the adjunction $b^* \vdash b_*$:

$$f_* \mathcal{F} \to b_* f'_* b'^* \mathcal{F}$$

which is then equivalent by commutativity of the pullback square to:

$$f_*\mathcal{F} \to f_*b'_*b'^*\mathcal{F}$$

We have a natural choice of such a morphism now by taking f_* applied to the unit of the adjunction:

$$id \to b'_*b'^*$$

and so this defines for us a morphism in degree 0. With the base change morphism defined and using the notation of the above we can state the smooth base change theorem. Recall a quasi-compact morphism is one where the inverse of every affine open is quasi-compact.

Theorem 2.3.6 (Smooth Base Change Theorem)**.** *If f is smooth, b quasi-compact and $\mathcal{F}$ an abelian torsion sheaf with torsion prime to the characteristic of the scheme S then the base change morphism is an isomorphism.* □

Proof. [Mil16, Chp VI Thm 4.1] □

3 Topological Types

In the introduction we stated the Čech nerve theorem which to an open cover of a paracompact topological space X, satisfying contractibility conditions on all intersections, associated a simplicial set which is weakly equivalent to X. Thus we have replaced the rather unwieldy cover with the combinatorial data of a simplicial set. We would like to carry out a similar construct for (simplicial) schemes but of course the contractible intersection criterion fails rather badly in schemes but this failure can be repaired by considering all such covers and forming these into a formal limit object, thus we obtain an object in pro$-$*sSets* instead of *sSet*.

We will define a Čech Topological Type and an Étale Topological Type, the latter having the nice property that it determines sheaf cohomology on locally noetherian simplicial schemes when we restrict to locally constant abelian sheaves.

3.1 Pro-Objects

Definition 3.1.1. *We say a non-empty category I is **cofiltered** if the following are satisified:*

- *For any $j, k \in \mathrm{ob}(I)$ there exists $i \in \mathrm{ob}(I)$ and morphisms $i \to j$ and $i \to k$*
- *For any two morphisms $f, g\colon j \to k$ there exists a third $h\colon i \to j$ with $fh = gh$*

Definition 3.1.2. *For a category $\mathcal{C}$ the category of **pro-objects** in $\mathcal{C}$ denoted* pro$-\mathcal{C}$ *has*

- *objects the functors $X\colon I \to \mathcal{C}$ with I small and cofiltering - which we also denote $\{X_i\}_{i\in I}$*
- *morphisms $\mathbf{Hom}(X, Y) = \lim_J \operatorname{colim}_I \operatorname{Hom}(X_i, Y_j)$*

In general a morphism of pro-objects $f\colon \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$ need not have the same indexing category, however we have the following useful result which allows us to reindex by the same category.

Lemma 3.1.3. *A morphism $f\colon \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J}$ can be represented by a system of morphisms $\{f_i\colon X_i \to Y_i\}_{i\in I}$.*

Proof. [AM69, Appendix Cor 3.2] □

3.2 The Čech Nerve

Given an étale covering of a simplicial scheme $U_\bullet \to X_\bullet$ we can define, in an analogous way to the topological Čech nerve, a Čech nerve for this simplicial cover. The difference being in that intersection is replaced fibre product (which of course agrees in $\mathcal{T}op$).

Definition 3.2.1. *The **Čech nerve** of an étale covering $U_\bullet \to X_\bullet$ is:*

$$N_{X_\bullet}(U_\bullet)_{s,t} := N_{X_s}(U_s)_t$$

where $N_{X_s}(U_s)_t$ is the $(t+1)$ fold fibre product of X_s over U_s. This is a bisimplicial scheme.

We introduce the notion of a rigid covering which will have the effect of restricting our coverings so that they form a cofiltered category. We can also think of them as behaving in an analogous way to covering spaces in that maps of covering spaces are determined by a point which is the assertion of the following proposition.

Proposition 3.2.2. *Let $U \to X$ be an étale morphism of schemes with U connected and $V \to X$ separated and étale. Then two morphisms $f, g\colon U \to V$ over X are equal iff they agree on a geometric point of U.*

Proof. [Fri82, Prop 4.1] □

With this lemma in mind we now define a rigid covering.

Definition 3.2.3. *Let X be a locally noetherian scheme, we say a morphism of schemes $\varphi\colon U \to X$ is a **Rigid Covering** if φ is a disjoint union of étale separated morphisms:*

$$\varphi_{\overline{x}}\colon (U_{\overline{x}}, u_{\overline{x}}) \to (X, \overline{x})$$

with $U_{\overline{x}}$ connected and $\varphi_{\overline{x}}$ takes the geometric point $u_{\overline{x}}$ to $\overline{x}$

We now show the **category of rigid coverings** over X, $RC(X)$, is cofiltered. Morphisms of rigid coverings over a scheme morphism $f\colon X \to Y$ are those morphisms taking a geometric point over $\overline{x}$ to a geometric point over $f(\overline{x})$, restricting f to be the identity 3.2.2 gives uniqueness of morphisms. Given two rigid coverings of X a subobject of the pullback can be formed with the property that it is also a rigid covering of X, one takes a disjoint union indexed of connected components of $U_{\overline{x}} \times V_{\overline{x}}$ containing the geometric point $u_{\overline{x}} \times v_{\overline{x}}$ showing that any two rigid coverings have a third mapping to both so that $RC(X)$ is cofiltered.

We can now define our first topological type based on these coverings by taking connected components of the diagonal:

Definition 3.2.4. *The **Čech Topological Type** of a simplicial scheme $X_\bullet$ is the pro simplicial set given by:*

$$\pi \circ \Delta \circ N_{X_\bullet}(-)\colon RC(X_\bullet) \to \mathcal{sSets}$$

Each component of this pro-object is then providing an approximation for the simplicial scheme and the hope of the inclusion of the pro-object of the definition is to circumvent the problem that n-fold intersections need not be contractible. However we have already commented previously that the systems of Čech complexes computes the Čech cohomology groups and not the sheaf cohomology and we have commented also that these do not necessarily agree. Since we are specifically interested in the sheaf cohomology of the étale site due to the theorems alluded to in the introduction we are forced to consider other pro-objects.

3.3 Hypercoverings and Rigid Hypercoverings

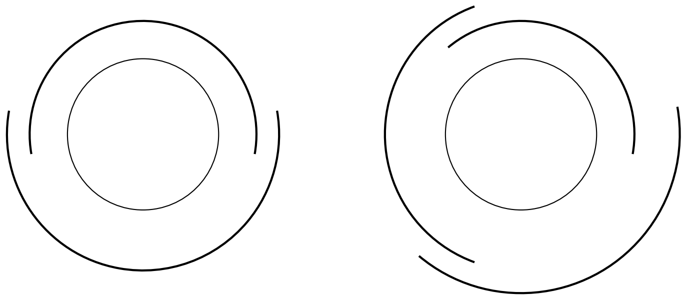
In order to obtain elements of pro$-\mathcal{sSets}$ so we can employ the techniques of étale homotopy theory we need our indexing categories to be cofiltering. The previous construction of Čech nerves is in general not cofiltering. We introduce an additional requirement on our covers, that of being a hypercovering.

Definition 3.3.1. *A **Hypercovering of a scheme** is a simplicial scheme over X, $U_\bullet \to X$, such that there is an étale surjection of U_t onto its $(t-1)$-skeleton:*

$$U_t \to (\operatorname{cosk}^X_{t-1}(U_\bullet))_t$$

where we take the $t = 0$ case to mean $U_0 \to X$ is étale surjective.

In a topological setting where the Čech nerve may have failed due to intersections being non-contractible we have replaced this condition with the intersections are covered by a covering. Consider for example the following topological situation as coverings of S^1 by copies of $(0,1)$:



Applying the Čech nerve theorem of the introduction to the first diagram fails to model S^1. In fact below we show that the Čech nerves are a suubset of the hypercoverings so that we have included more general objects in an attempt to counter the problem of sheaf and Čech cohomology disagreeing.

In some sense the n^{th} stage of the simplicial scheme $U_\bullet$ provides a covering for the $(n-1)^{th}$ stage. In degree 0 we have the simple statement that there is a covering:

$$U_0 \to X$$

In degree 1 the condition of a hypercovering amounts to there being an étale surjection:

$$U_1 \to U_0 \underset{X}{\times} U_0$$

and in higher degrees some generalisation of these. We could alternatively view this construction by writing the maps $U_t \to \mathrm{cosk}_{t-1}^X U_\bullet$ as $U_t \to (cosk_{t-1} \circ \tau_{t-1}(U_\bullet))_t$. Then once we have an étale surjection $U_0 \to X$ and we have constructed some covering of the 1^{st} degree of the Čech complex if we can form U_0 and U_1 into a 1-truncated simplicial scheme then the hypercovering condition says we need an étale surjection of U_2 onto $cosk_{t-1} \circ (U_\bullet)_{\leq 1}$ so this covering condition is in some sense accounting for all lower levels of intersections.

Another observation to be made about the definition of Hypercoverings is the following sequence of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{s_{t-1}C}(\partial(\Delta[-,t]), (U_\bullet)_{\leq t-1}) &\cong \mathrm{Hom}_{s_{t-1}C}((\Delta[-,t])_{\leq t-1}, (U_\bullet)_{\leq t-1}) \\ &\cong \mathrm{Hom}_{sC}(\Delta[-,t], \mathrm{cosk}_{t-1}U_\bullet) \\ &\cong (\mathrm{cosk}_{t-1}U_\bullet)_t \end{aligned}$$

The first isomorphism is just the definition of the boundary of the standard n simplex, the second is the truncation-coskeleton adjunction and the third follows by an application of the Yoneda lemma. Notice also by Yoneda that $U_t \cong \mathrm{Hom}_{sC}(\Delta[-,t], U_\bullet)$. Now supposing we had defined a hypercovering for simplicial sets instead, so that $U_\bullet$ is now a simplicial set rather than a simplicial scheme and the étale surjection condition has been replaced with some appropriate condition which also includes surjective. And for further simplicity suppose $X = *$. Then the condition of a hypercovering amounts to there being a surjection:

$$\mathrm{Hom}_{sSets}(\Delta[-,t], U_\bullet) \cong \mathrm{Hom}_{s_{t-1}Sets}(\partial(\Delta[-,t])_{\leq t-1}, (U_\bullet)_{\leq t-1})$$

I.e. that the simplicial set $U_\bullet$ is a Kan fibration.

As an example of a hypercovering we show the Čech nerves satisfy the conditions:

Example 3.3.2. *This is a continuation of example 1.2.8. As an example of a hypercovering, which will be useful later, suppose $U \to X$ is an étale surjective scheme morphism. Then we claim there is a hypercovering of X:*

$$(\operatorname{cosk}_0^X U)_\bullet \to X$$

recalling that $(\operatorname{cosk}_0^X)_t$ is the $(t+1)$-fold fibre product $U \underset{X}{\times} \dots \underset{X}{\times} U$.

Proof. We already have that the morphism in degree 0 is étale surjective so we need only show we have an étale surjection $(\operatorname{cosk}_0^X U)_t \to (\operatorname{cosk}_{t-1}^X(\operatorname{cosk}_0^X U))_t$. This is the conclusion of 1.2.8. $\square$

We extend the definition of hypercoverings to simplicial schemes:

Definition 3.3.3. *A **Hypercovering of a simplicial scheme** $X_\bullet$ is a covering $U_{\bullet\bullet} \to X_\bullet$ such that the restriction to degree s is a hypercovering of schemes, i.e. for all s we have étale surjections:*

$$U_{s,t} \to (\operatorname{cosk}_{t-1}^{X_s}(U_{s,\bullet}))_t \tag{3.3.4}$$

We define the homotopy category of such coverings and show it does indeed form a cofiltered category.

Definition 3.3.5. *The **Category of Hypercoverings** $HR(X_\bullet)$ has objects the hypercoverings $U_{\bullet\bullet} \to X_\bullet$ and morphisms are equivalence classes of maps of hypercovering over $X_\bullet$ where two maps are equivalent if they are restrictions of a simplicial homotopy $U_{\bullet\bullet} \otimes (\Delta[0] \times \Delta[1]) \to V_{\bullet\bullet}$*

Proposition 3.3.6. *The category $HR(X_\bullet)$ is cofiltered.*

We only sketch the proof as we will not make much use of this category in preference of defining rigid hypercoverings afterward.

Proof. For full details see [Fri82, Prop 3.4]. The construction of the equalizer is the tricky part. For this the construction roughly goes: Construct the simplicial scheme $W_{s,\bullet} := \operatorname{Hom}(X_s \otimes \Delta[-,1], V_{s,\bullet})$. By the tensor-internal hom adjunction we have:

$$\operatorname{Hom}_{\mathcal{S}ch_{/X_s}}(-, \operatorname{Hom}(X_s \otimes \Delta[1]), V_{s,\bullet}) \cong \operatorname{Hom}_{\mathcal{S}ch_{/X_s}}((-) \otimes \Delta[-,1], V_{s,\bullet})$$

There are two maps from $W_{s,\bullet}$ to $V_{s,\bullet}$ over X_s given by sending an element of $W_{s,\bullet}$ to its image at either end of the simplicial homotopy. The required equalizer is then given as a pullback of the diagram:

$$\begin{tikzcd} Z_{\bullet\bullet} \arrow[r] \arrow[d] & U_{\bullet\bullet} \arrow[d] \\ W_{\bullet\bullet} \arrow[r] & V_{\bullet\bullet} \underset{X_\bullet}{\times} V_{\bullet\bullet} \end{tikzcd}$$

$\square$

The use of hypercoverings becomes apparent with the following proposition:

Proposition 3.3.7. *Let $U_{\bullet\bullet} \to X_\bullet$ be a hypercovering of a simplicial scheme. Then there is a natural isomorphism of δ functors:*

$$H^*(X_\bullet, -) \cong H^*(U_{\bullet\bullet}, -)$$

Proof. The natural isomorphism is induced by the diagonal $\Delta U_{\bullet\bullet} \to X_\bullet$ and can be shown using the standard machinery of δ functors to demonstrate the isomorphism by showing there is a natural isomorphism in degree 0 and that they agree on injectives. □

In fact there is another isomorphism:

Proposition 3.3.8. *There is a natural isomorphism of δ functors:*

$$H^*(X_\bullet; -) \cong \operatorname{colim}_{HR(X_\bullet)} H^*(U_{\bullet\bullet}; -)$$

Proof. [Fri82, Thm 3.8] □

One disadvantage of the category $HR(X_\bullet)$ is that we are required to take homotopy classes of morphisms in order to make it cofiltered. As an improvement on the above we can further simplify the category $HR(X_\bullet)$ so that there can be at most 1 map between any two hypercoverings as above by enforcing a rigidity condition on morphisms.

Definition 3.3.9. *A **Rigid Hypercovering** of a locally noetherian simplicial scheme $X_\bullet$ is a hypercovering $U_{\bullet\bullet} \to X_\bullet$ where the morphisms of 3.3.4 are rigid coverings for all s and t and such that $\varphi\colon [s'] \to [s]$ induces a morphism of rigid coverings for all s, t.*

We obtain a **category of rigid hypercoverings** over $X_\bullet$ denoted $HRR(X_\bullet)$ by taking morphisms of rigid coverings $U_{\bullet\bullet} \to X_\bullet$ and $V_{\bullet\bullet} \to X_\bullet$ over $X_\bullet$ to be a morphism of bisimplicial schemes $U_{\bullet\bullet} \to V_{\bullet\bullet}$ over $X_\bullet$ which is compatible with the maps of 3.3.4, i.e. the following commutes:

$$\begin{array}{ccc} U_{s,t} & \longrightarrow & V_{s,t} \\ \downarrow & & \downarrow \\ (\operatorname{cosk}^{X_s}_{t-1} U_{s,\bullet})_t & \longrightarrow & (\operatorname{cosk}^{X_s}_{t-1} V_{s,\bullet})_t \end{array}$$

Without the need of taking homotopy classes of maps this category is cofiltered. In fact in the presence of 3.2.2 we have the stronger statement that it is left directed as any two morphisms of rigid hypercoverings $U_{\bullet\bullet} \to V_{\bullet\bullet}$ over $X_\bullet$ must agree.

Lemma 3.3.10. *The category of rigid hypercoverings $HRR(X_\bullet)$ is cofiltered.*

Proof. It remains to show the property that for any two rigid hypercoverings there is a third having the property that there are maps from the third to the first two. For this there is a rigid product which keeps track of geometric points: take $U_{\bullet\bullet}$ and $V_{\bullet\bullet}$ in $HRR(X_\bullet)$ and suppose there is a morphism $U_{\bullet\bullet} \to V_{\bullet\bullet}$, suppose further the morphisms of 3.3.4 are given by the disjoint union of étale separated morphisms

$$((U_{s,t})_{\overline{a}}, (u_{s,t})_{\overline{a}}) \to ((\operatorname{cosk}^{X_s}_{t-1} U_{s,\bullet})_t, \overline{a})$$

$$((V_{s,t})_{\overline{b}}, (v_{s,t})_{\overline{b}}) \to ((\operatorname{cosk}^{X_s}_{t-1} V_{s,\bullet})_t, \overline{b})$$

Then we define $Z_{s,t}$ to be the disjoint union of étale separated maps each of the form:

$$(U_{s,t} \times V_{s,t})_0 \to (\operatorname{cosk}^{X_s}_{t-1} U_{s,\bullet})_t \times (\operatorname{cosk}^{X_s}_{t-1} V_{s,\bullet})_t \supset (\operatorname{cosk}^{X_s}_{t-1}(U_{s,\bullet} \times V_{s,\bullet}))_t \tag{3.3.11}$$

where the disjoint union is over the product set of geometric points $((u_{s,t})_{\bar{a}}) \times ((v_{s,t})_{\bar{b}})$, the connected component functor $(-)_0$ is taking the component containing the image of $((u_{s,t})_{\bar{a}}) \times ((v_{s,t})_{\bar{b}})$ and we restrict to those that have image in coskeleton of the product sitting inside the product of the coskeleton. Together the $Z_{\bullet\bullet}$ define a hypercovering of $X_\bullet$ that is rigid: hypercovering following from the fact that the first map in 3.3.11 will be étale surjective when we take the disjoin union, as $U_{\bullet\bullet}$ and $V_{\bullet\bullet}$ are hypercoverings, and rigidity from the definition of $Z_{s,t}$. Furthermore it is clear there are maps from $Z_{\bullet\bullet}$ to $U_{\bullet\bullet}$ and $V_{\bullet\bullet}$. □

The advantage of working with the category of rigid hypercoverings becomes even more apparent when we make the following definition of the étale topological type which is a pro-object of $s\mathcal{S}ets$ rather than in the homotopy category $\mathcal{H}$.

Definition 3.3.12. *The* ***Étale Topological Type*** *is the object in* pro$-s\mathcal{S}ets$ *given by the composition of the diagonal functor and the connected components functor:*

$$(X_\bullet)_{ét} := \pi \circ \Delta \colon HRR(X_\bullet) \to s\mathcal{S}ets$$

This is indeed a pro-object since $HRR(X_\bullet)$ is cofiltered by the above lemma. Given a scheme morphism $f \colon X_\bullet \to Y_\bullet$ we have an induced map $f_{ét} \colon (X_\bullet)_{ét} \to (Y_\bullet)_{ét}$ and hence a functor:

Definition 3.3.13. *The* ***Étale Topological Type Functor*** *from the category of locally Noetherian simplicial schemes, $s\mathcal{S}ch_{ln}$ to* pro$-s\mathcal{S}ets$ *is:*

$$(-)_{ét} \colon s\mathcal{S}ch_{ln} \to \text{pro}-s\mathcal{S}ets$$

as described above.

One can also define the **(étale) homotopy type** by replacing the domain of the functor with $HR(X_\bullet)$ and the codomain by the homotopy category. Denote this $(-)_{ht}$.

This concludes the transfer of our object that lies in the realm of algebraic geometry to something we can conceivably do homotopy theory on. The study of the étale topological site will continue in Chapter 5 but the next chapter will first provide some homotopy results on pro objects to convince ourselves that they are well behaved homotopically speaking in that many results carry over from the homotopy categroy.

4 Homotopy Theory

4.1 Homotopy Theory of Pro-Objects

We denote by $\mathcal{H}_{*c}$ the homotopy category of connected pointed simplicial sets and we consider in this section the pro-category pro$-\mathcal{H}_{*c}$ and various constructions on it. Recall we denote a pro-object $X\colon I \to \mathcal{C}$ of $\mathcal{C}$ where I is cofiltered by $\{X_i\}_{i\in I}$ or simply $\{X_i\}$.

We start by stating an analogue of the Whitehead Theorem for the pro-homotopy category the proof of which is found as [AM69, Thm 4.3].

Theorem 4.1.1 (Artin, Mazur). *The following are equivalent for a map $f\colon \{(X_\bullet)_i\} \to \{(Y_\bullet)_j\}$ in* pro$-\mathcal{H}_{*c}$ *:*

1. *f is a weak equivalence*
2. *We have isomorphisms on pro-homotopy groups for all $k > 0$:*
$$f_*\colon \pi_k(\{(X_\bullet)_i\}) \stackrel{\simeq}{\to} \pi_k(\{(Y_\bullet)_j\})$$
3. *We have an isomorphism on the first pro-homotopy group and isomorphisms on cohomology for all local abelian coefficient systems M :*
$$f^*\colon H^*(\{(Y_\bullet)_j\}, M) \stackrel{\simeq}{\to} H^*(\{(X_\bullet)_i\}, M)$$

We of course need to define the pro-homotopy (cohomology) groups and what a weak equivalence is.

Definition 4.1.2. *For an object $\{(X_\bullet)_i\} \in$ pro$-\mathcal{H}_{*c}$ and corresponding abelian local coefficient system $\{A_i\}$ we define:*

- *The k^{th} **Homotopy Group** is the pro-group object $\pi_k \circ X\colon I \to \mathcal{G}rp$*
- *The k^{th} **Homology Group** with coefficients in $\{A_i\}$ is $\{H_k((X_\bullet)_i; A_i)\}$*
- *The k^{th} **Cohomology Group** with coefficients in $\{A_i\}$ is $\operatorname{colim} H_k((X_\bullet)_i; A_i)$*

For definitions of (co)homology in an abelian local coefficient system we refer to [Hal83, Chps 12, 14]. We note we take the colimit in the case of cohomology but not the limit for homology as colimit is an exact functor, the limit is merely left exact.

The weak equivalences are not quite as simple as a morphism $f\colon \{(X_\bullet)_i\} \to \{(Y_\bullet)_j\}$ inducing isomorphism on the homotopy groups, they need encompass a slightly larger collection so that we obtain a Whitehead theorem. As a counterexample [AM69] provide the following.

Counterexample 4.1.3. *Let $X_\bullet$ be a simplicial set considered as a pro-object indexed by the trivial category, and form another pro-object $\{\operatorname{cosk}_n(X_\bullet)\}$ indexed by $\mathbb{N}$ with maps $n \to m$ whenever $n \geq m$. The map $X \to \{\operatorname{cosk}_n(X_\bullet)\}$ of pro-objects induces an isomorphism on pro-homotopy groups: $\pi_i X_\bullet \to \pi_i(\operatorname{cosk}_n X_\bullet)$ is an isomorphism for $i < n$ and is 0 otherwise, and inverse map $\pi_i(\operatorname{cosk}_n X_\bullet) \to \pi_i X_\bullet$ also an isomorphism for $i < n$ and 0 otherwise. Composing these (in either order) in the pro-category gives the identity morphism.*

However:

$$\operatorname{Hom}_{\text{pro}-\mathcal{H}_{*c}}(\{\operatorname{cosk}_n X_\bullet\}, X_\bullet) = \operatorname{colim}_{\mathbb{N}} \operatorname{Hom}_{\mathcal{H}_{*c}}(\operatorname{cosk}_n X_\bullet, X_\bullet)$$

and if the homotopy groups of $X_\bullet$ don't vanish above some degree then we cannot find morphisms $\operatorname{cosk}_n X_\bullet \to X_\bullet$ which provide an inverse when we apply homotopy group functors.

We use this as motivation to define weak equivalence and for a more general object in pro$-\mathcal{H}_{*c}$, $\{(X_\bullet)_i\}$, we form a pro-object by resolving each $X_\bullet(i)$ into its Postnikov tower.

Definition 4.1.4. *We denote this pro-object by $\{(X_\bullet)_i\}^\natural := \{(\mathrm{cosk}_n X_\bullet)_i\}_{I\times\mathbb{N}}$ where there is a unique morphism $n \to m$ in $\mathbb{N}$ whenever $n \geq m$.*

We say a morphism $f\colon \{(X_\bullet)_i\} \to \{(Y_\bullet)_j\}$ is a weak equivalence if the induced morphism $\natural f\colon \{(X_\bullet)_i\}^\natural \to \{(Y_\bullet)_j\}^\natural$ is an isomorphism in pro$-\mathcal{H}_{*c}$.

The notational use of $\natural$ is that of [AM69], $\sharp$ is used in [Fri82] for the same construction.

We also have analogous results to the Hurewicz isomorphism, uniqueness of Eilenberg-Maclane spaces and uniqueness of spheres although we will not need these.

Theorem 4.1.5 (Hurewicz in pro$-\mathcal{H}_{*c}$). *Let $\{(X_\bullet)_i\} \in$ pro$-\mathcal{H}_{*c}$ and suppose the first $n-1$ pro-homotopy groups are 0, then the canonical map $\pi_n(\{(X_\bullet)_i\}) \to H_n(\{(X_\bullet)_i\})$ is an isomorphism in* pro$-\mathcal{Ab}$.

Proof. [AM69, Cor 4.5] □

Theorem 4.1.6 (Uniqueness of Eilenberg-Maclane Spaces). *Let $\{(X_\bullet)_i\} \in$ pro$-\mathcal{H}_{*c}$ have $\pi_n(\{(X_\bullet)_i\}) = \{G_i\}$ and all other pro-homotopy groups being 0. Then there is a weak equivalence $\{(X_\bullet)_i\} \to K(\{G_i\}, n)$*

Proof. [AM69, Cor 4.14] □

For uniqueness of sphere we write pro$-C\mathcal{H}_{*c}$ for the subcategory with pro-finite pro-homotopy groups and write $\widehat{\mathbb{Z}}$ and $\widehat{S^n}$ for there pro-finite completions (explained in the next section):

Theorem 4.1.7. *Let $\{(X_\bullet)_i\} \in$ pro$-C\mathcal{H}_{*c}$ with $H_n(\{(X_\bullet)_i\}, \mathbb{Z}) = \widehat{\mathbb{Z}}$ and all other pro-homology groups 0, then there is a weak equivalence $\widehat{S^n} \to \{(X_\bullet)_i\}$.*

Proof. [AM69, Cor 14.5] □

Another remark to make is that we can also take fibrant replacements so that there is, for a map of pro objects $\{(E_\bullet)_i\} \to \{(B_\bullet)_j\}$, a long exact sequence on homotopy groups:

$$\ldots \pi_{n+1}(\{(B_\bullet)_j\}) \to \pi_n(\{(F_\bullet)_k\}) \to \pi_n(\{(E_\bullet)_i\}) \to \pi_n(\{(B_\bullet)_j\}) \to \pi_{n-1}(\{(F_\bullet)_k\}) \to \ldots$$

We now introduce various completions in pro$-\mathcal{H}_{*c}$. For a subset L of all primes P we denote by $L\mathcal{H}_{*c}$ the subcategory consisting of those objects whose homotopy groups are finite L-torsion groups, and similarly we define pro$-L\mathcal{H}_{*c}$.

We will make use of the following:

$(-)^{\widehat{L}}$	pro$-\mathcal{H}_{*c} \to$ pro$-L\mathcal{H}_{*c}$	the pro$-L$ completion functor
$\mathrm{holim}^{Su}(-)$	pro$-P\mathcal{H}_{*c} \to \mathcal{H}_{*c}$	the Sullivan homotopy limit functor
$\mathrm{holim}^{Su}((-)^{\widehat{L}})$	pro$-\mathcal{H}_{*c} \to \mathcal{H}_{*c}$	the composite of the above two is the Sullivan pro$-L$ completion functor
$(\mathbb{Z}/l)_\infty(-)$	$s\mathcal{S}ets \to s\mathcal{S}ets$	the Bousfield-Kan $\mathbb{Z}/l$ completion functor
$\mathrm{holim}(-)$	$s\mathcal{S}ets^I \to s\mathcal{S}ets$	the Bousfield-Kan homotopy limit functor

We will construct here the first two and refer to [Bou72] for construction and theory of the latter two.

4.2 The pro$-L$ Completion

Definition 4.2.1. *A* ***Class*** *of groups $\mathcal{C}$ is a collection of groups such that:*

- $0 \in \mathcal{C}$
- *If* $0 \to A \to B \to C \to 0$ *is exact then* $A, C \in \mathcal{C}$ *iff* $B \in \mathcal{C}$
- *If* $A, B \in \mathcal{C}$ *then so is* $\prod_B A$

In particular note that the classes of finite L-torsion groups are classes. The following can be done more generally in the case of classes but we suffice to state if for these particular cases.

Given an object $\{G_i\}$ in the pro-category of groups, pro$-\mathcal{G}rps$, there is an associated pro-object lying in the full subcategory pro$-L$. We construct this by considering the under category $(\text{pro}-L)_{G/}$ where objects are morphisms from $\{G_i\}$ to an object in pro$-L$ and morphisms are commutative triangles, see 1.2.2.

$$\begin{tikzcd} & \{G_i\} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \{A_i\} \arrow[rr, "f"] & & \{B_i\} \end{tikzcd}$$

This category is cofiltered and there is a small cofinal subcategory, choosing such a subcategory and forgetting the structure of an under category, i.e. we only retain the pro$-L$ objects and maps between them, we obtain an object of pro$-L$. We'll denote this pro-object $(\{G_i\})^{\widehat{L}}$ and it comes with a map $\{G_i\} \to (G)^{\widehat{L}}$. We would like a similar construction for objects in pro$-\mathcal{H}_{*c}$ and this is given by [AM69, Thm 3.4].

Theorem 4.2.2. *The inclusion* pro$-L\mathcal{H}_{*c} \to$ pro$-\mathcal{H}_{*c}$ *admits a left adjoint* $(-)^{\widehat{L}}\colon$ pro$-\mathcal{H}_{*c} \to$ pro$-L\mathcal{H}_{*c}$. □

The construction is similar to that of the groups case in that one considers the category $(\text{pro}-L\mathcal{H}_{*c})_{X_\bullet/}$, (note $X_\bullet$ is not necessarily in pro$-L\mathcal{H}_{*c}$ and shows it is cofiltering and and that there is a small final subcategory.

$$\begin{tikzcd} & X_\bullet \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ Y_{\bullet 1} \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Y_{\bullet 2} \end{tikzcd}$$

With the notation of the above diagram where $Y_{\bullet i} \in$ pro$-L\mathcal{H}_{*c}$ existence of a third object mapping to the $Y_{\bullet i}$ follows by taking for instance their product. To show there is a $(X_\bullet \to Y_\bullet) \in (\text{pro}-L\mathcal{H}_{*c})_{X_\bullet/}$ and map $j\colon Y_\bullet \to Y_{\bullet 1}$ with $f \circ j = g \circ j$.

Equivalently to the above theorem we could say that there is a map $\{(X_\bullet)_i\} \to (\{(X_\bullet)_i\})^{\widehat{L}}$ initial with respect to maps from $\{(X_\bullet)_i\}$ into pro$-L\mathcal{H}_{*c}$.

There is also a variant of 4.1.1 for the case of pro$-L$ completion.

Theorem 4.2.3. *Let* $f\colon \{(S_\bullet)_i\} \to \{(T_\bullet)_j\}$ *be a map in* pro$-\mathcal{H}_{*c}$ *then the* pro$-L$ *completion* $(f)^{\widehat{L}}$ *is a weak equivalence if and only if the following two conditions hold:*

1. $(f_*)^{\widehat{L}}$ *is an isomorphism between fundamental groups* $(\pi_1(\{(S_\bullet)_i\}))^{\widehat{L}} \overset{\cong}{\to} (\pi_1(\{(T_\bullet)_j\}))^{\widehat{L}}$
2. *whenever* $\mathcal{A}$ *is a collection of compatible coefficient systems on* $\{(T_\bullet)_j\}$ *with all groups over a point being finite L-groups and whose representing homomorphism* $\pi_1(\{(T_\bullet)_j\}) \to Aut(A_t)$ *, for some point t, factors through the* pro$-L$ *completion of the fundamental group then there is a cohomology isomorphism:*

$$f^*\colon H^*(\{(T_\bullet)_j\};\mathcal{A}) \overset{\cong}{\to} H^*(\{(S_\bullet)_i\};f^*\mathcal{A})$$

Proof. This follows from the previous pro-Whitehead theorem along with the fact that for a local coefficient system $\mathcal{A}$ on $\{(T_\bullet)_j\}$ whose groups at a point are finite L-groups the cohomology on $\{(T_\bullet)_i\}$ with $\mathcal{A}$ coeffcients is detected on the the pro$-L$ completion of $\{(T_\bullet)_j\}$ with the same coeffcients. □

4.3 The Sullivan Homotopy Limit

We sketch the construction found in [Sul74]. Suppose $F_\bullet \in P\mathcal{H}_{*c}$ and consider the contravariant functor $[-,F_\bullet]\colon \mathcal{H}_{*c} \to \mathcal{S}et$ sending an $X_\bullet \in P\mathcal{H}_{*c}$ to the set of classes of pointed maps up to homotopy. This is a so called compact Brownian functor. Suppose now $\{(F_\bullet)_i\} \in$ pro$-P\mathcal{H}_{*c}$, i.e. an inverse system in $\mathcal{H}_{*c}$ with finite homotopy groups. We can now consider the functor $\lim[-,(F_\bullet)_i]$ sending an $X_\bullet$ to $\lim[X_\bullet,(F_\bullet)_i]$, this can also be shown to be a compact Brownian functor and so by Brown reprsentability is isomorphic to $[-,\widehat{F}]$ for some $\widehat{F}$. We thus make the following definition:

Definition 4.3.1. *The Sullivan Homotopy Limit Functor is given by:*

$$\begin{aligned} \mathrm{holim}^{Su}(-)\colon \mathrm{pro}-P\mathcal{H}_{*c} &\to \mathcal{H}_{*c} \\ \{(F_\bullet)_i\} &\mapsto \widehat{F}_\bullet \end{aligned}$$

In particular we have isomorphisms:

$$\mathrm{Hom}(X_\bullet, \mathrm{holim}^{Su}(\{(F_\bullet)_i\})) \cong \lim \mathrm{Hom}(X_\bullet, \{(F_\bullet)_i\})$$

A result we use later is the following:

Proposition 4.3.2. *Suppose* $f\colon \{(S_\bullet)_i\} \to \{(T_\bullet)_j\}$ *is a map in* pro$-\mathcal{H}_{*c}$ *which becomes an isomorphism after taking a Sullivan* pro$-L$ *completion. Then there is an isomorphism:*

$$\mathrm{holim}^{Su}(\{(S_\bullet)_i\}^{\widehat{L}}) \to \mathrm{holim}^{Su}(\{(T_\bullet)_j\}^{\widehat{L}})$$

□

I.e. isomorphic objects in pro$-L\mathcal{H}_{*c}$ are represented by the same object of $\mathcal{H}_{*c}$ in the above compact Brownian functor construction.

4.4 The Bousfield Kan $\mathbb{Z}/l$-completion

The Bousfield-Kan $\mathbb{Z}/l$ completion functor for a prime l is a functor defined from *sSets* to *sSets* which is determined by the fact that for any map $S_\bullet \to T_\bullet$ there is a weak equivalence

$$(\mathbb{Z}/l)_\infty(S_\bullet) \to (\mathbb{Z}/l)_\infty(T_\bullet)$$

if and only if there is a homology (or cohomology) isomorphism:

$$H^*(S_\bullet;\mathbb{Z}/l) \xrightarrow{\cong} H^*(T_\bullet;\mathbb{Z}/l)$$

This construction is defined on the whole of *sSets* but it particularly ncie when restriced to nilpotent spaces, those whose fundamental group acts nilpotently on all homotopy groups. Take $S_\bullet$ to be a nilpotent space, then there is an isomorphism of (co)homology groups:

$$H^*(S_\bullet;\mathbb{Z}/l) \xrightarrow{\cong} H^*((\mathbb{Z}/l)_\infty(S_\bullet);\mathbb{Z}/l)$$

Further we have the result [Bou72, Thm 5.1] stating:

Theorem. *If $f\colon E \to B$ is a fibration of pointed connected spaces such that the action of $\pi_1(B)$ on the homology $H_*(F;\mathbb{Z}/l)$ is nilpotent then $(\mathbb{Z}/l)_\infty(f)$ is also a fibration and there is a homotopy equivalence between the $\mathbb{Z}/l$-completion of the fibre and the fiber of the $\mathbb{Z}/l$-completion.* □

4.5 The Bousfield-Kan Homotopy Limit

The homotopy limit of Bousfield-Kan, holim, provides a homotopy invariant replacement for lim on the level of *sSets* instead of the homotopy category. More precisely we have the homotopy limit is a functor:

$$\operatorname{holim}\colon sSets^I \to sSets$$

such that if $\alpha\colon F \to G$ is a natural transformation of functors of the form $I \to sSets$ such that each $\alpha_i\colon F(i) \to G(i)$ is a homotopy equivalence then and both are fibrant simplicial sets then the induced map is a homotopy equivalence:

$$\operatorname{holim}\alpha\colon\ \operatorname{holim} F \xrightarrow{\simeq} G$$

Another useful property of the homotopy limit is the following lemma:

Lemma 4.5.1 (Fibration Lemma of Bousfield-Kan)**.** *Let $\alpha\colon F \to G$ be a natural transformation of functors of the form $I \to sSets$ and such that for all $i \in I$ we have $\alpha_i\colon F(i) \to G(i)$ is a fibration. Then the induced map is a fibration:*

$$\operatorname{holim}\alpha\colon\ \operatorname{holim} F \to \operatorname{holim} G$$

Proof. [Bou72, Ch XI Lem 5.5] □

4.6 Comparisons of Completions and Homotopy Limits

Under certain nice conditions we can relate the previous defined constructions to each other up to homotopy equivalence. Its use is quite apparent for the defined completions: the Sullivan completion has a very simple definition so that the object is quite simple to work with however the Bousfield-Kan $\mathbb{Z}/l$ completion satisfies the very useful condition of inducing weak equivalence if and only if there is an isomorphism on cohomology with $\mathbb{Z}/l$ coefficients.

The following are proved as proposition 6.10 in [Fri82].

Proposition 4.6.1. *Let $S_\bullet \in \mathcal{H}_{*c}$ have finite homotopy groups, then there is an isomorphism in* pro$-l\mathcal{H}_{*c}$*:*

$$(\mathbb{Z}/l)_\infty(S_\bullet) \stackrel{\cong}{\to} (S_\bullet)^{\widehat{l}}$$

□

Proposition 4.6.2. *Suppose $\{(S_\bullet)_i\} \in$ pro$-\mathcal{sSets}$ is such that each component has finite $\mathbb{Z}/l$ cohomology groups (or that it is weakly equivalent to such a pro-space), i.e. $H^n((S_\bullet)_i; \mathbb{Z}/L)$ if finite for each i and n. Then there is an isomorphism:*

$$\operatorname{holim}(\mathbb{Z}/l)_\infty(S_\bullet) \stackrel{\cong}{\to} \operatorname{holim}^{Su}(\{(S_\bullet)^{\widehat{l}}\}_i)$$

□

In particular we note the following case where the second is applicable: Suppose $(X_\bullet, x)$ is a pointed connected noetherian simplicial scheme, in addition to defining hypercoverings type we could also restrict our hypercoverings to those $U_{\bullet\bullet} \to X_\bullet$ with each $U_{s,t}$ noetherian. It is a fact that the inclusion of the noetherian hypercoverings into the hypercoverings is final. Since in addition the inclusion of rigid hypercoverings into hypercoverings is also final we can conclude that the noetherian homotopy type:

$$(X_\bullet, x)_{nht} := \pi \circ \Delta \colon nHR(X_\bullet, x) \to \mathcal{sSets}$$

is weakly equivalent to the étale topological type $(X_\bullet, x)_{ét}$. Since by definition the components of the noetherian homotopy type are finite in each dimension so is the cohomology with $\mathbb{Z}/l$ coefficients so that the conditions of 4.6.2 are satisifed in the case that we consider the étale topological type of a connected pointed noetherian simplicial scheme. For an explicit proof of finality of $nHR(-)$ in $HR(-)$ see [Fri82, Prop 7.1].

4.7 Homotopy Theory of Topological Types

Proposition 4.7.1. *If morphisms $f, g \colon X_\bullet \to Y_\bullet$ of locally noetherian simplicial schemes are related by a simplicial homotopy then $f_{ét} = g_{ét}$.*

Proof. [Fri82, Cor 4.8] □

Definition 4.7.2. *The **pro-homotopy groups** of a locally Noetherian pointed simplicial scheme $(X_\bullet, x)$ are given by composing the étale topological type functor with π_i.*

$$\pi_i((X_\bullet, x)_{ét}) := \pi_i \circ (X_\bullet, x)_{ét}$$

To define the cohomology groups of $(X_\bullet)_{\acute{e}t}$ we first fix an object $U_{\bullet\bullet} \to X_\bullet$ of $HRR(X_\bullet)$ and consider the overcategory $(HRR(X_\bullet))_{/U_{\bullet\bullet}}$. To an abelian local coefficient system M on $\pi \circ \Delta(U_{\bullet\bullet})$ and a map $f\colon V_{\bullet\bullet} \to U_{\bullet\bullet}$ in the overcategory we have the pullback f^*M on $V_{\bullet\bullet}$ and for any commuting square:

$$\begin{tikzcd} V_{\bullet\bullet} \arrow[rr, "j"] \arrow[dr, "f"'] & & W_{\bullet\bullet} \arrow[dl, "g"] \\ & U_{\bullet\bullet} & \end{tikzcd}$$

we get a morphism $j^*\colon H^*(\pi \circ \Delta(W_{\bullet\bullet}); g^*M) \to H^*(\pi \circ \Delta(V_{\bullet\bullet}); f^*M)$ and so we make the following definition:

Definition 4.7.3. *The cohomology groups of the étale topological type are given by:*

$$H^*((X_\bullet)_{\acute{e}t}; M) := \operatorname{colim} H^*(V_{\bullet\bullet}; f^*M)$$

Once we have developed the theory of prinicpal G-fibrations in the setting of simplicial schemes we will be able to show that the étale topological type is a suitable object in pro$-$*sSets* for computing the sheaf cohomology of locally constant abelain sheaves in the setting of locally noetherian connected simplicial schemes, i.e. we have:

Proposition. *Let $X_\bullet$ as above, A a locally constant abelian sheaf on $\acute{E}t(X_\bullet)$. There exists some corresponding local coefficients system $\mathcal{A}$ on $(X_\bullet)_{\acute{e}t}$ and we have:*

$$H^*(X_\bullet; A) \cong H^*((X_\bullet)_{\acute{e}t}; \mathcal{A})$$

5 Principal G-fibrations of Simplicial Schemes

In this chapter we develop a notion of principal G-fibration for a scheme and extend it to simplicial schemes. It will be shown that we can reduce principal G-fibrations of simplicial schemes to the fibration over the X_0 component so long as we keep track of some additional data, known as descent data, which provides all the required information to reform the full fibration over $X_\bullet$. Just as in the topological case we will then have principal G-fibrations are then classified by the maps from fundamental groups into G.

This chapter concludes by transferring the calculation of sheaf cohomology to the calculation of the cohomology of a local coefficient system on the étale topological type.

5.1 Definitions and Descent Data

We take G to be a discrete group.

Definition 5.1.1. *A **Principal G-fibration over a scheme** X is a map of schemes $Y \to X$ commuting with a right G action which acts trivially on X and furthermore there is an étale surjective morphism $U \to X$ and an isomorphism of schemes $U \underset{X}{\times} Y \xrightarrow{\cong} U \otimes G$ over U commuting with the G action.*

In the definition $U \otimes G$ should be read as the disjoint union of copies of U indexed by G with free G-right action permuting the copies of U, thus we should view the isomorphism as locally trivialising the fibration. A map of principal G-fibrations over X, $(p_1\colon Y_1 \to X) \to (p_2\colon Y_2 \to X)$, will be a map $\rho\colon Y_1 \to Y_2$ commuting with the G action and with $p_2 \circ \rho = p_1$. We will denote the category defined with objects the principal G-fibrations over X and maps as described by $\Pi(X, G)$.

Remark 5.1.2. *It can be shown that the map $Y \to X$ is **fppf**, i.e. faithfully flat and of finite presentation and it is a fact that a scheme morphism over some base scheme is an isomorphism if and only if its base change along an fppf morphism is an isomorphism. Using this the morphisms in $\Pi(X, G)$ can be shown to be isomorphisms.*

This definition can be easily extended to simplicial schemes.

Definition 5.1.3. *A **Principal G-fibration over a simplicial scheme** $X_\bullet$ is a map $p\colon Y_\bullet \to X_\bullet$ of simplicial schemes commuting with a right G action which acts trivially on X satisfying:*

1. *In degree 0, p is a principal G-fibration over X_0*
2. *For any $\varphi\colon [n] \to [m]$ we have a pullback square*

$$\begin{array}{ccc} Y_m & \xrightarrow{\theta^*} & Y_n \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p} \\ X_m & \xrightarrow{\theta^*} & X_n \end{array}$$

A morphism of principal G-fibrations over a simplicial scheme is defined similarly and again it can be shown such a morphism is necessarily an isomorphism of simplicial schemes. The category consisting of such objects and morphisms will be denoted $\Pi(X_\bullet, G)$.

Note 5.1.4. *We observe that the restriction of a $(p\colon Y_\bullet \to X_\bullet) \in \Pi(X_\bullet, G)$ to degree n is an element of $\Pi(X_n, G)$. Indeed pullback of an étale surjection is an étale surjection by 1.1.7 so we can show pullbacks preserve the property of being a principal G-fibrations. Let $p\colon Y \to X$ and U be as above, and $Z \to X$ a morphism. $U \underset{X}{\times} Z \to Z$ is étale and surjective and the isomorphism follows by commuting pullbacks:*

$$(Y \underset{X}{\times} Z) \underset{Z}{\times} (U \underset{X}{\times} Z) \cong Y \underset{X}{\times} (Z \underset{Z}{\times} U) \underset{X}{\times} Z \cong (Y \underset{X}{\times} U) \underset{X}{\times} Z \cong (U \otimes G) \underset{X}{\times} Z \cong (U \underset{X}{\times} Z) \otimes G$$

It should not come as too much of a surprise in light of the second condition that $\Pi(X_\bullet, G)$ can be interpreted as $\Pi(X_0, G)$ with some 'descent data'. Recall the standard face maps on simplicial objects are denoted d_i and that we can form the pullback of a principal G-fibration p by these maps which we denote $d_i^*(p)$. We define a new category $\Pi_{dd}(X_0, G)$ to have objects consisting of a principal G-fibration over X_0, $p_0\colon Y_0 \to X_0$, with an isomorphism $\varphi\colon d_0^*(p_0) \to d_1^*(p_0)$ of principal G-fibrations over X_1 satisfying $d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi$ as morphisms of principal G-fibrations over X_2 (where $d_i^*\varphi$ is the morphism induced by pullback).

For the definition of morphisms in $\Pi_{dd}(X_0, G)$ let $p_0\colon Y_0 \to X_0$ and $q_0\colon Z_0 \to X_0$ be principal G-fibrations with isomorphisms φ and χ respectively. A morphism f of these objects is then a map $Y_0 \to Z_0$ over X_0 satisfying $\chi \circ d_0^* f = d_1^* f \circ \varphi$.

Many of the propositions to come involve the technique of reducing principal G-fibrations over some object to principal G-fibrations over an object of simplicial freedom one fewer with some descent data. We will then show once how this comes about in 5.1.5 and omit the proofs in subsequent propositions involving this reduction.

Proposition 5.1.5. *There is an equivalence of categories $\Pi(X_\bullet, G) \cong \Pi_{dd}(X_0, G)$.*

The map in one direction is clear: to a principal G-fibration of simplicial schemes $p\colon Y_\bullet \to X_\bullet$ we can associate the object of $\Pi_{dd}(X_0, G)$ consisting of p_0 and take φ to be the composite identifying the principal G-fibrations over X_1, $d_0^*(p_0) \cong p_1 \cong d_1^*(p_0)$ - the isomorphisms following from 5.1.2. The maps inducing these morphisms are given by universal property of the pullback. The first of the two isomorphisms for instance comes from the following diagram:

$$\begin{tikzcd}
Y_1 \arrow[dr, dashed, "\cong"] \arrow[drr, bend left, "d_0"] \arrow[ddr, bend right, "p_1"'] & & \\
& X_1 \underset{X_0}{\times} Y_0 \arrow[r] \arrow[d, "d_0^*(p_0)"] & Y_0 \arrow[d, "p_0"] \\
& X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}$$

The outer square commutes by the property of a principal G-fibration being a simplicial map. We now check the conditions on objects of $\Pi_{dd}(X_0, G)$ and show the reverse map but first we recall breifly the simplicial identities:

$$
\begin{array}{rcll}
d_i \circ d_j & = & d_{j-1} \circ d_i, & i < j \\
s_j \circ s_i & = & s_i \circ s_{j-1}, & i < j \\
d_i \circ s_j & = & \begin{cases} s_{j-1} \circ d_i, & i < j \\ id, & i = j, j+1 \\ s_j \circ d_{i-1}, & j < i \end{cases}
\end{array}
$$

Proof. We need to verify $d_1^*\varphi = d_2^*\varphi \circ d_1^*\varphi$. The first of the simplicial identities, for $i \leq j$, gives $d_i \circ d_j = d_{j-1} \circ d_i$ and hence the commuting hexagon:

$$
\begin{tikzcd}
 & X_1 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\
X_0 & & X_0 \\
 & X_2 \arrow[uu, "d_2"] \arrow[dl, "d_0"] \arrow[dr, "d_1"'] & \\
X_1 \arrow[uu, "d_1"] \arrow[dr, "d_0"'] & & X_1 \arrow[uu, "d_1"'] \arrow[dl, "d_0"] \\
 & X_0 &
\end{tikzcd}
\tag{5.1.6}
$$

We expand either side of the relation to be verified using the formula for φ.

$$d_1^*\varphi = d_1^*(d_0^*p_0 \to d_1^*p_0) = d_1^*(d_0^*p_0 \overset{\cong}{\to} p_1 \overset{\cong}{\to} d_1^*p_0) = d_1^*d_0^*p_0 \overset{\cong}{\to} d_1^*p_1 \overset{\cong}{\to} d_1^*d_1^*p_0$$

$$d_2^*\varphi = d_2^*(d_0^*p_0 \to d_1^*p_0) = d_2^*(d_0^*p_0 \overset{\cong}{\to} p_1 \overset{\cong}{\to} d_1^*p_0) = d_2^*d_0^*p_0 \overset{\cong}{\to} d_2^*p_1 \overset{\cong}{\to} d_2^*d_1^*p_0$$

$$d_0^*\varphi = d_0^*(d_0^*p_0 \to d_1^*p_0) = d_0^*(d_0^*p_0 \overset{\cong}{\to} p_1 \overset{\cong}{\to} d_1^*p_0) = d_0^*d_0^*p_0 \overset{\cong}{\to} d_0^*p_1 \overset{\cong}{\to} d_0^*d_1^*p_0$$

Using commutativity of 5.1.6 we can identify pullbacks of pullbacks with the commuting squares and the final equality in the following follows from commutativity of the entire hexagon.

$$
\begin{aligned}
d_2^*\varphi \circ d_0^*\varphi &= (d_1^*d_0^*p_0 = d_0^*d_0^*p_0 \overset{\cong}{\to} d_0^*p_1 \overset{\cong}{\to} d_0^*d_1^*p_0 = d_2^*d_0^*p_0 \overset{\cong}{\to} d_2^*p_1 \overset{\cong}{\to} d_2^*d_1^*p_0 = d_1^*d_1^*p_0) \\
&= d_1^*\varphi
\end{aligned}
$$

We now need to construct a principal G-fibration of simplicial schemes from an object of $\Pi_{dd}(X, G)$.

By the second condition of principal G-fibrations we are forced to define Y_n as the pullback along some $\varphi : [n] \to [0]$. Using $\varphi = (d_0)^n$, the n-fold composition of d_0 makes the caluclations simple and automatically gives us the maps $\mathbf{d_0} \colon Y_{n+1} \to Y_n$.

$$
\begin{tikzcd}
Y_n \arrow[rr, dashed] \arrow[dr] \arrow[dd, "p_n"'] & & Y_m \arrow[dl] \arrow[dd, "p_m"] \\
& Y_0 \arrow[dd, "p_0"] & \\
X_n \arrow[rr, "\theta^*", crossing over] \arrow[dr, "(d_0)^n"'] & & X_m \arrow[dl, "(d_0)^m"] \\
& X_0 &
\end{tikzcd}
$$

It suffices to define the induced maps for $\theta^* = s_i$ and d_i. By the simplicial identities for each i we have $(d_0)^{n+1} \circ s_i = (d_0)^n \colon X_n \to X_0$, for $i \le n$ we have $(d_0)^n \circ d_i = (d_o)^{n+1}$, and for $i = n+1$, $(d_0)^n \circ d_{n+1} = d_1 \circ (d_0)^n$. So we define the simplicial maps on $Y_{n+1} := X_{n+1} \underset{X_0}{\times} Y_0$ by:

$$
\begin{array}{lll}
\mathbf{s_i} := s_i \times id_{Y_0} & & \forall i \\
\mathbf{d_i} := \begin{cases} d_i \times id_{Y_0} \\ (d_{n+1} \times id_{Y_0}) \circ ((d_0)^n)^* \varphi \end{cases} & & \begin{array}{l} i \le n \\ i = n+1 \end{array}
\end{array}
$$

The definition for $\mathbf{d_{n+1}}$ coming from the following diagram noting that all squares and the triangle involving solid arrows commute by definition and so the dashed arrow exists and is unique by the pullback property of Y_n. Note $\mathbf{d_{n+1}}$ is also the pullback $((\mathbf{d_0})^n)^*((d_1^* p_0 \to Y_0) \circ \varphi)$.

$$
\begin{tikzcd}
& Y_{n+1} \arrow[rrr, dashed, "\mathbf{d_{n+1}}"] \arrow[dl, "(\mathbf{d_0})^n"'] \arrow[ddr, "p_{n+1}"] & & & Y_n \arrow[dl, "(\mathbf{d_0})^n"'] \arrow[dd, "p_n"] \\
Y_1 \arrow[r, "\varphi", "\cong"'] \arrow[ddr, "p_1"'] & d_1^*(p_0) \arrow[rr, crossing over] \arrow[dd] & & Y_0 \arrow[dd, "p_0"] & \\
& & X_{n+1} \arrow[rr, "d_{n+1}", crossing over] \arrow[dl, "(d_0)^n"'] & & X_n \arrow[dl, "(d_0)^n"] \\
& X_1 \arrow[rr, "d_1"] & & X_0 &
\end{tikzcd}
$$

The simplicial identities clearly hold for the relations not involving $\mathbf{d_{n+1}}$ so we check $\mathbf{d_i} \circ \mathbf{d_{n+1}} = \mathbf{d_n} \circ \mathbf{d_i}$ in the case $i \le n-1$. The following gives us equality after we post compose by $(\mathbf{d_0})^{n-1}$ as we know each of the inner diagrams commute (but not yet the outer diagram).

$$\begin{tikzcd}
Y_n \arrow[dddd, "\mathbf{d_n}"'] \arrow[dr, "(\mathbf{d_0})^{n-1}"] & & & Y_{n+1} \arrow[lll, "\mathbf{d_i}"'] \arrow[dl, "\mathbf{d_0}"] \arrow[dddd, "\mathbf{d_{n+1}}"] \\
& Y_1 \arrow[d, "\varphi", "\cong"'] & Y_n \arrow[l, "(\mathbf{d_0})^{n-1}"] & \\
& d_1^* p_0 \arrow[d] & & \\
& Y_0 & Y_{n-1} \arrow[l, "(\mathbf{d_0})^{n}"] & \\
Y_{n-1} \arrow[ur, "(\mathbf{d_0})^{n-1}"] & & & Y_n \arrow[lll, "\mathbf{d_i}"] \arrow[ul, "d_0"']
\end{tikzcd}$$

Commutativity of all squares but the top in the following then gives equality after postcomposing by p_{n-1}.

$$\begin{tikzcd}
& Y_{n+1} \arrow[rr, "\mathbf{d_{n+1}}"] \arrow[dl, "\mathbf{d_i}"'] \arrow[dd] & & Y_n \arrow[dl, "\mathbf{d_i}"] \arrow[dd] \\
Y_n \arrow[rr, "\mathbf{d_n}"] \arrow[dd] & & Y_{n-1} \arrow[dd] & \\
& X_{n+1} \arrow[rr, "d_{n+1}"] \arrow[dl, "d_i"] & & X_n \arrow[dl, "d_i"] \\
X_n \arrow[rr, "d_n"] & & X_{n-1} &
\end{tikzcd}$$

It can also be checked that mapping then to X_0 makes the maps equal, i.e. we have compatible maps in the following so that by uniqueness of maps into the pullback we have the desired equality $\mathbf{d_i} \circ \mathbf{d_{n+1}} = \mathbf{d_n} \circ \mathbf{d_i}$.

$$\begin{tikzcd}
Y_{n+1} \arrow[dr, "\mathbf{d_i} \circ \mathbf{d_{n+1}}"] \arrow[dr, "\mathbf{d_n} \circ \mathbf{d_i}"'] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y_{n-1} \arrow[r] \arrow[d] & Y_0 \arrow[d] \\
& X_{n-1} \arrow[r] & X_0
\end{tikzcd}$$

For the case of $\mathbf{d_n} \circ \mathbf{d_{n+1}} = \mathbf{d_n} \circ \mathbf{d_n}$ with $n = 1$ we have the following diagram which is 5.1.6 after applying pullbacks of maps to $Y_0 \to X_0$ and using the given relation $d_1^* \varphi =$

$d_2^*\varphi \circ d_0^*\varphi$.

$$
\begin{tikzcd}
 & & & & d_0^*p_0 \arrow[rr, "\varphi", "\cong"'] \arrow[dll] & & d_1^*p_0 \arrow[drr] & & & & \\
 & & Y_0 & & d_2^*d_0^*p_0 \arrow[u] \arrow[rr, "d_2^*\varphi", "\cong"'] \arrow[dl, equal] & & d_2^*p_0 \arrow[u] \arrow[dr, equal] & & Y_0 & & \\
 & & & d_0^*d_1^*p_0 \arrow[dlll] & & & & d_1^*d_1^*p_0 \arrow[drrr] & & & \\
d_1^*p_0 \arrow[uurr] & & & & d_0^*d_0^*p_0 \arrow[ul, "d_0^*\varphi", "\cong"'] \arrow[rr, equal] \arrow[dlll] & & d_1^*d_0^*p_0 \arrow[ur, "d_1^*\varphi"', "\cong"] \arrow[drrr] & & & & d_1^*p_0 \arrow[uull] \\
 & d_0^*p_0 \arrow[ul, "\varphi", "\cong"'] \arrow[rrrr] & & & & Y_0 & & & & d_0^*p_0 \arrow[ur, "\varphi"', "\cong"] \arrow[llll] &
\end{tikzcd}
\tag{5.1.7}
$$

The map $\mathbf{d_1} \circ \mathbf{d_2}$ is then the composition $Y_2 := d_0^*d_0^*p_0 \cong d_0^*d_1^*p_0 = d_2^*d_0^*p_0 \to d_0^*p_0 \cong d_1^*p_0 \to Y_0$ and the map $\mathbf{d_1} \circ \mathbf{d_1}$ is the composite $Y_2 := d_0^*d_0^*p_0 = d_1^*d_0^*p_0 \to d_0^*p_0 \cong d_1^*p_0 \to Y_0$ which are equal by commutativity of 5.1.7. The relation for higher n can now be seen by induction on the pullback cubes along $\mathbf{d_0}$:

$$
\begin{tikzcd}
 & Y_n \arrow[dl, "\mathbf{d_0}"'] \arrow[dd, "\mathbf{d_n}" near start] & & Y_{n+1} \arrow[ll, "\mathbf{d_n}"] \arrow[dl, "\mathbf{d_0}"] \arrow[dd, "\mathbf{d_{n+1}}"] \\
Y_{n-1} \arrow[dd, "\mathbf{d_{n-1}}"] & & Y_n \arrow[ll, "\mathbf{d_{n-1}}" near start] \arrow[dd, "\mathbf{d_n}" near end] & \\
 & Y_{n-1} \arrow[dl, "\mathbf{d_0}"] & & Y_n \arrow[ll, "\mathbf{d_n}"' near start] \arrow[dl, "\mathbf{d_0}"] \\
Y_{n-2} & & Y_{n-1} \arrow[ll, "\mathbf{d_{n-1}}"] &
\end{tikzcd}
$$

We omit the verification of the morphisms and remaining simplicial identities - it amounts to similar diagram chasing. So we conclude that restricting from principal G-fibrations over a simplicial scheme to the degree 0 part (with the added descent data) and lifting a principal G-fibration with descent data to the whole simplicial scheme gives the equivalence of categories as claimed. □

We have a similar proposition involving the diagonal of a bisimplicial scheme $\Delta X_{\bullet\bullet}$ where we descend to simplicial principal G-fibrations. The category $\Pi_{dd}(X_{0,\bullet}, G)$ has objects principal G-fibrations of simplicial schemes $p\colon Y_\bullet \to X_{0,\bullet}$ with isomorphism $\varphi\colon d_0^*(p) \to d_1^*(p)$ in $\Pi(X_{1,\bullet}, G)$ satisfying $d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi$.

Proposition 5.1.8. *There is an equivalence of categories* $\Pi(\Delta X_{\bullet\bullet}, G) \cong \Pi_{dd}(X_{0,\bullet}, G)$. □

Recall, for the topological case, from the introduction that principal G-fibrations over some space X (which we assume to be path connected) are in bijection with homotopy classes of maps $[X, BG]$ where BG is the classifying space of the group G. If G is finite, or more generally viewed with the discrete topology, then a choice of a principal G-fibration over X amounts to a choice of group homomorphism $\Pi(X) \to G$ as $K(G,1)$ is a model for BG. A natural question now given our definition of principal G-fibrations for (simplicial) schemes is do we have an analogue to this? If we use instead the étale homotopy group of $(X_\bullet, x)$ we have a correspondence between these fibrations and pro-group homomorphisms for sufficiently nice schemes.

For the remainder of this chapter we shall take $X_\bullet$ to be locally Noetherian and connected. We will make use of two lemmas. The restriction functor $(-)_n\colon s\mathcal{S}ch_{X_\bullet} \to \mathcal{S}ch_{X_n}$ restricting a simplicial scheme over $X_\bullet$ to degree n admits a right adjoint $\Gamma_n^{X_\bullet}(-)$ that extends étale (resp. surjective) morphisms in degree n to a simplicial scheme morphism which is étale (resp. surjective) in all degrees. This is the content of [Fri82, Prop 1.5].

Lemma 5.1.9. *The functor $\Gamma_n^{X_\bullet}(-)$ of taking a covering $U_{\bullet n} \to X_n$ to a covering $W_{\bullet\bullet} \to X_\bullet$ restricts to a functor $\Gamma_n^{X_\bullet}(-)\colon HRR(X_n) \to HRR(X_\bullet)$* □

Lemma 5.1.10. *For a hypercovering $U_\bullet \to X$ the category of principal G-fibrations over $\pi(U_\bullet)$ is equivalent to the category of principal G-fibrations trivial over U_0.*

Proof. [AM69, Cor 10.7] □

Proposition 5.1.11. *Let $X_\bullet$ be pointed by x. There is a natural bijection:*

$$\mathrm{Hom}(\pi_1((X_\bullet, x)_{\acute{e}t}, G) \cong \{\textit{objects of } \Pi(X_\bullet, G) \textit{ up to iso}\}$$

For a pointed hypercovering $U_{\bullet\bullet} \to X_\bullet$ there is a natural bijection:

$$\mathrm{Hom}(\pi_1(\pi \circ \Delta(U_{\bullet\bullet}), u), G) \cong \{\textit{objects of } \Pi(X_\bullet, G) \textit{ trivial over } U_{0,0} \textit{ up to iso}\}$$

Proof. We show how the first statement follows from the second.

$$\mathrm{Hom}(\pi_1((X_\bullet, x)_{\acute{e}t}), G) := \operatorname{colim} \mathrm{Hom}(\pi_1(\pi(\Delta U_{\bullet\bullet}), u), G)$$

From a principal G-fibration $Y_\bullet \to X_\bullet$ trivialized by an étale surjection $U_{0,0} \to X_0$, then there is a hypercovering $(\mathrm{cosk}_0^{X_0} U_{0,0})_\bullet \to X_0$ by 3.3.2 and applying $\Gamma_0^{X_\bullet}(-)$ we obtain a hypercovering of $X_\bullet$ which we can rigidify, this gives an element in $HRR(X_\bullet)$. By the second bijection this gives an element of $\mathrm{Hom}(\pi_1(\pi(\Delta U_{\bullet\bullet}, u)), G)$ and hence an element of the colimit $\operatorname{colim} \mathrm{Hom}(\pi_1(\pi(\Delta U_{\bullet\bullet}, u)), G)$ since $HRR(X_\bullet)$ is cofiltering.

Conversely given an element of $\mathrm{Hom}(\pi_1((X_\bullet, x)_{\acute{e}t}), G)$ choose a representative of it for some $U_\bullet$ in the colimit so that by the second bijection we have a principal G-fibration, which happens to be trivial over $U_{0,0}$.

For the second statement since G is discrete so that $BG \simeq K(G,1)$, homomorphisms $\pi_1(\pi(\Delta U_{\bullet\bullet}, u)) \to G$ are in bijection with principal G-fibrations of $\pi(\Delta U_{\bullet\bullet}, u)$ which are equivalent to principal G-fibrations of $(\pi(U_{\bullet,0}), u)$ with descent data by the simplicial set version of 5.1.8 which is equivalent, by 5.1.10, to principal G-fibrations trivial over $U_{0,0}$. □

5.2 Cohomology of the Étale Topological Type

We now aim to show that for such simplicial schemes $X_\bullet$ that are connected and locally noetherian the étale cohomology of a locally constant abelian sheaf on $\acute{E}t(X_\bullet)$ is determined by the étale topological type of $X_\bullet$ with some appropriate local coefficient system. We recall that the Čech nerve theorem expressed a space X with the combinatorial data of a simplicial set which has the same homotopy type, so then we replace our scheme with some pro-combinatorial data that is good enough for calculating cohomology when we restrict to locally constant abelian sheaves (corresponding to abelian local coefficient systems).

Lemma 5.2.1. *Let $U_{\bullet\bullet} \to X_\bullet$ be a hypercovering, then there is a natural bijection between locally constant sheaves on $\acute{E}t(X_\bullet)$ with all stalks isomorphic to a set S and local coefficient systems on $(X_\bullet)_{\acute{e}t}$ with fibres isomorphic to S.*

Proof. On $\pi(\Delta U_{\bullet\bullet})$ a local coefficient system $\mathcal{A}$ with stalks isomorphic to S is equivalent to a group homomorphism $\pi_1(\pi(\Delta(U_{\bullet\bullet})) \to G$ and so also to a principal G-fibration over $X_{\bullet\bullet}$ with fibres isomorphic to S which is trivialized by $U_{0,0}$. □

Theorem 5.2.2. *Let A be a locally constant abelian sheaf on $\acute{E}t(X_\bullet)$ and $\mathcal{A}$ the corresponding local coefficient system on $(X_\bullet)_{\acute{e}t}$, then there is a natural isomorphism:*

$$H^*(X_\bullet; A) \cong H^*((X_\bullet)_{\acute{e}t}; \mathcal{A})$$

Proof. Let $U_{\bullet\bullet} \to X_\bullet$ be a hypercovering and recall that we have the natural isomorphisms of δ functors:

$$H^*(X_\bullet; -) \cong \operatorname{colim}_{HR(X_\bullet)} H^*(U_{\bullet\bullet}; -)$$

$$H^*(\Delta U_{\bullet\bullet}; (-)^\Delta) \cong H^*(U_{\bullet\bullet}; -)$$

of 3.3.8 and 2.1.5. If we choose $U_{\bullet\bullet}$ such that the locally constant sheaf A on $X_\bullet$ is constant when restricted to $U_{0,0}$ (i.e. we apply the inverse image functor on $U_{0,0} \to X_0$) then the cohomology groups $H^*(\pi(\Delta U_{\bullet\bullet}); \mathcal{A})$ and $H^*(\Delta U_{\bullet\bullet}; (A)^\Delta)$ are naturally isomorphic.

We can indeed make such a choice of hypercovering $U_{\bullet\bullet}$, for instance such a construction could go: choose opens of X_0 making A constant and covering X_0, take their disjoint union and set this as $U_{0,0}$. The natural map $U_{0,0} \to X_0$ is an étale, as a disjoint union of open immersions, and surjective morphism hence a covering in $\acute{E}t(X_0)$ and so we have that the map $(\operatorname{cosk}_0^{X_0} U_{0,0})_\bullet \to X_0$ is a hypercovering. Next recall the functor $\Gamma_0^{X_\bullet}(-)$ takes a hypercovering of X_0 to a hypercovering of $X_\bullet$ and so we have $(\Gamma_0^{X_\bullet}(\operatorname{cosk}_0^{X_0}(U_{0,0}))_{\bullet\bullet} \to X_\bullet$ is a hypercovering with the desired property.

Thus we need to show the isomorphism labelled ? in the following and then the theorem will follow from the above isomorphisms of δ functors.

$$H^*((X_\bullet)_{\acute{e}t}; \mathcal{A}) := \operatorname{colim}_{(HRR(X_\bullet))_{/U_{\bullet\bullet}}} H^*(\pi(\Delta V_{\bullet\bullet}); f^*A) \overset{?}{\cong} \operatorname{colim}_{\substack{V_{\bullet\bullet} \in HRR(X_\bullet) \\ A_{V_{0,0}} constant}} H^*(\pi(\Delta V_{\bullet\bullet}; A)$$

It will suffice to show that the subcategory of rigid hypercoverings restricting the sheaf to be constant are cofinal in the category of rigid hypercoverings which can be seen by considering the pullback of some hypercovering along the above hypercovering $U_{\bullet\bullet} \to X_\bullet$. □

6 Comparisons of Topological Types and Fibres

In this chapter we relate various topologial types in the category pro$-\mathcal{H}_{*c}$ as well as various types of fibres.

We keep the conditions connected, locally Noetherian on a pointed simplicial scheme $(X_\bullet, x)$.

6.1 Some Comparisons of Topological Types

Lemma 6.1.1. *Let $f\colon U_{\bullet\bullet} \to X_\bullet$ be a pointed hypercovering, then there is a weak equivalence in* pro$-\mathcal{H}_{*c}$

$$(g)_{ét}\colon (\Delta U_{\bullet\bullet}, u)_{ét} \to (X_\bullet, x)_{ét}$$

Proof. 4.1.1 says it is sufficient to check there are isomorphisms on the first homotopy group and all cohomology groups induced by $(g)_{ét}$. On π_1 this follows from the descent theory of Chapter 5 and for cohomology from 3.3.8 and 5.2.2. □

We now further impose that X is finite type over $Spec(\mathbb{C})$ so that we may take the analytification of X. Recall also from 2.3.1 the classical site of a topological space T_{cl}. In the case that $T_\bullet$ is a paracompact simplicial topological space we have the following as a lemma ([Fri82, Lem 8.3]), which relates sheaf cohomology on the classical site to cohomology of a simplicial set via a natural isomorphism.

$$H^*((Y_\bullet)_{cl}; A) \cong H^*(\mathcal{Sing}(|T_\bullet|); \mathcal{A})$$

The following theorem of Grauert and Remmert [Art73, Exposés XI Thm 4.3] relates the classical site to the étale site via the functor of 2.3.3.

Theorem 6.1.2. *The functor $\epsilon\colon X_{ét} \to X_{cl}$ is an equivalence of categories between the category of finite coverings of $X(\mathbb{C})$ that are local homeomorphisms and the category of étale coverings of X.* □

Before stating the comparison theorem we introduce another topological type denoted $(-)_{s.ét}$. Like the étale topological type it is also defined on the hypercoverings of $X_\bullet$, but unlike the étale topological type it features the analytification of X. Temporarily we denote $X(\mathbb{C})$ by X^{top} for clarity. $(-)_{s.ét}$ is then given by the composite functor:

$$\Delta \circ \mathcal{Sing} \circ (-)^{top} \circ \Delta\colon HRR(-) \to \mathcal{sSets}$$

With this we have natural maps

$$\rho\colon (-)_{s.ét} \to (-)_{ét}$$

Let $U_{\bullet\bullet} \to X_\bullet$ be a hypercovering, then the map ρ on this hypercovering sends an element $\Delta^n \to (U^{top})_{n,n}$ to the connected component of the image of this map, which is well defined as Δ^n is connected and defined on the diagonal, so does define a map to the element of $(X_\bullet)_{ét}$ corresponding to $U_{\bullet\bullet}$.

$$\tau\colon (-)_{s.ét} \to \mathcal{Sing}(|(-)^{top}|)$$

Theorem 6.1.3. *τ is an isomorphism and $(\rho)^{\widehat{P}}$ a weak equivalence in* pro$-\mathcal{H}_{*c}$.

This is [Fri82, Thm 8.4].

Proof. Explicitly τ is the composite, on a hypercovering $U_{\bullet\bullet} \to X_\bullet$ of a simplicial scheme, of maps:

$$\tau := \Delta(\mathcal{S}ing((\Delta U_{\bullet\bullet})^{top})) \to \Delta(\mathcal{S}ing(X_\bullet^{top})) \to \mathcal{S}ing(|X_\bullet^{top}|)$$

The first map is just the induced map coming from the hypercovering and the second is the natural map sending a singular map $\alpha\colon \Delta^k \to X_k$ to the image in the geometric realization of $\alpha \times id\colon \Delta^k \to X_k \times \Delta^k$. τ being an isomorphism in pro$-\mathcal{H}_{*c}$ follows since the second map is a homotopy equivalence (using 1.2.11) and the first can be shown similarly to 6.1.1 by descent data for isomorphisms on π_1 and by analogous results for cohomology on the local homeomorphism site.

Recall the comparison theorem 2.3.5 which identifies sheaf cohomology on the étale and classical sites on a finite type scheme over $\mathbb{C}$ for constructible sheaves, we shall use this to show that $(\rho)^{\widehat{P}}$ is a weak equivalence. Friedlander proves this by expressing ρ as a composite. Hypercoverings can also be defined on the classical site in the same way so we have the category $HRR((X_\bullet)^{top})$ and can define an analogue of $(-)_{s.\acute{e}t}$ (just omit the $(-)^{top}$ functor).

$$(-)_{s.cl}\colon HRR((X_\bullet)^{top}) \to \mathit{sSets}$$

Just as in the case with τ there is an isomorphism $(-)_{s.cl} \cong \mathcal{S}ing(|-|)$ so that we have $\beta\colon (-)_{s.cl} \cong (-)_{s.\acute{e}t}$ induced by forgetting the structure from the classical site. Define γ similar to ρ, again without the $(-)^{top}$ functor and define a map $\delta\colon (X_\bullet)_{cl} \to (X_\bullet)_{\acute{e}t}$ to be induced by the morphism of sites $\epsilon\colon (X_\bullet)_{\acute{e}t} \to (X_\bullet)_{cl}$. The following now commutes:

$$\begin{tikzcd}
(X_\bullet)_{s.cl} \arrow[r, "\beta"] \arrow[d, "\gamma"] & (X_\bullet)_{s.\acute{e}t} \arrow[d, "\rho"] \\
(X_\bullet)_{cl} \arrow[r, "\delta"] & (X_\bullet)_{\acute{e}t}
\end{tikzcd}$$

since forgetting the structure of the site and taking the natural map to connected components is the same as taking connected components and forgetting the site structure. The limitation on ρ being a weak equivalence only after a P-completion comes from δ. By this theorem then we have that δ induces cohomology isomorphisms with such coefficients. By Whitehead it then suffices to prove isomorphisms on the pro-fundamental group which follows from descent and 6.1.2. Thus $(\rho)^{\widehat{P}}$ is a weak equivalence. □

6.2 On Fibres

There are multiple choices of definition of fibres we can take for a simplicial scheme morphism $f\colon X_\bullet \to Y_\bullet$, either we can take the (étale) topological type of a scheme theoretic fibre or we can define a fibre of the induced map $f_{\acute{e}t}$.

We first define fibres for the types $(-)_{ht}$ and $(-)_{\acute{e}t}$. For a pointed simplicial scheme morphism $f\colon X_\bullet \to Y_\bullet$ we consider morphisms of hypercoverings over f, i.e. for hypercoverings

$U_{\bullet\bullet} \to X_\bullet$ and $V_{\bullet\bullet} \to Y_\bullet$ we consider the morphisms φ in:

$$\begin{array}{ccc} U_{\bullet\bullet} & \xrightarrow{\varphi} & V_{\bullet\bullet} \\ \downarrow & & \downarrow \\ X_\bullet & \xrightarrow{f} & Y_\bullet \end{array}$$

Since we defined the category of hypercoverins $HR(X_\bullet)$ up to equivalence by simplicial homotopy we consider the maps φ up to simplicial homotopy. We say φ and ψ are equivalent over the pointed map f if there is a pointed simplicial map $\varphi \otimes \Delta[-,1] \to \psi$.

Definition 6.2.1. *We denote the category of maps of hypercoverings over f up to pointed simplicial homotopy by $HR(f)$.*

Similarly if we impose rigidity conditions as in $HRR(X_\bullet)$ we can define:

Definition 6.2.2. *We denote by $HRR(f)$ the category of of maps of rigid hypercoverings over the rigid morphism f up to pointed simplicial homotopy.*

Let $\overline{y}\colon Spec(\Omega) \to Y_\bullet$ be the pointed data for the pointed map f so that y has image in Y_0 and recall that for a map of simplicial sets $\alpha\colon S_\bullet \to T_\bullet$ we have a functorial replacement of α converting it to a fibration, this will replace the domain $S_\bullet$ by another simplicial set which is weakly equivalent to it. Denote this new map $\alpha^\sim\colon (S_\bullet)^\sim \to T_\bullet$ following the notation of [Fri82].

Definition 6.2.3. *The* ***homotopy fibre of the homotopy topological type****, denoted* $\mathrm{fib}(f_{ht})$*, is defined to be the pro-object sending a map of hypercoverings φ as above to the equivalence class $(((\pi \circ \Delta)(\varphi))^\sim)^{-1}(y)$ in the homotopy category $\mathcal{H}_*$.*

$$\mathrm{fib}(f_{ht}) := (((\pi \circ \Delta)(-))^\sim)^{-1}(y)\colon HR(f) \to \mathcal{H}_*$$

Definition 6.2.4. *The* ***homotopy fibre of the étale topological type****, denoted* $\mathrm{fib}(f_{\acute{e}t})$*, is defined to be the pro-object sending a map of rigid hypercoverings φ as above to the pointed simplicial set $((\pi\Delta)^\sim)^{-1}(\varphi)$*

$$\mathrm{fib}(f_{\acute{e}t}) := (((\pi \circ \Delta)(-))^\sim)^{-1}(y)\colon HRR(f) \to sSets_*$$

These then are the defintion of the fibres if we first pass to the map of topological types and then take appropriate fibres, alternately we can first take scheme theoretic fibres:

Definition 6.2.5. *The* ***topological type of the scheme theoretic fibre*** *is denoted:*

$$(X_\bullet \underset{Y_\bullet}{\times} y)_{ht}\colon HR(X_\bullet \underset{Y_\bullet}{\times} y) \to \mathcal{H}_*$$

Definition 6.2.6. *The* ***étale topological type of the scheme theoretic fibre*** *is denoted:*

$$(X_\bullet \underset{Y_\bullet}{\times} y)_{\acute{e}t}\colon HRR(X_\bullet \underset{Y_\bullet}{\times} y) \to sSets_*$$

There is nothing new in these definitions.

We have the inclusion map of the scheme theoretic fibre into $X_\bullet$ induced from the pullback $i\colon X_\bullet \underset{Y_\bullet}{\times} y \to X_\bullet$. We also have a projection onto the first vertical map in the square:

$$\begin{array}{ccc} U_{\bullet\bullet} & \xrightarrow{\varphi} & V_{\bullet\bullet} \\ \downarrow & & \downarrow \\ X_\bullet & \xrightarrow{f} & Y_\bullet \end{array}$$

Their compositions induce maps from the topological types of the scheme theoretic fibres to the fibres of the topological type:

$$(X_\bullet \underset{Y_\bullet}{\times} y)_{ht} \to \mathrm{fib}(f_{ht})$$

$$(X_\bullet \underset{Y_\bullet}{\times} y)_{\acute{e}t} \to \mathrm{fib}(f_{\acute{e}t})$$

We can also compare fibres of different topological types. We have maps:

$$(X_\bullet \underset{Y_\bullet}{\times} y)_{ht} \to (X_\bullet \underset{Y_\bullet}{\times} y)_{\acute{e}t}$$

$$fib(f_{ht}) \to fib(f_{\acute{e}t})$$

All 4 of these maps assemble to give a square that is homotopy commutative, the third map is a weak equivalence and the fourth a weak equivalence if there is a bijection on π_0 on the first map as can be seen by comparing long exact sequence of fibrations of the étale and homotopy types and applying a 5-lemma argument. [Fri82, Prop 10.2] provides the proof of these results. What is harder to prove and aruguably more useful is a comparison on the first or second maps. The following will be of great importance later.

Theorem 6.2.7. *Suppose the X_n and Y_n are connected and noetherian, that $\pi_1((Y_n)_{ht})$ if pro-finite and $\pi_0(X_\bullet \underset{Y_\bullet}{\times} y)$ finite. Let $\mathcal{A}$ be a locally constant abelian sheaf on the étale site of $X_\bullet$ with all right derived functors of the push forward $R^q(f_n)_*\mathcal{A}_n$ locally constant with base change map an isomorphism $(R^q(f_n)_*\mathcal{A}_n)_y \to H^q(X_n \underset{Y_n}{\times} y; i^*\mathcal{A}_n)$. Then there is an isomorphism on cohomology:*

$$H^*(fib(f_{\acute{e}t}); \mathcal{A}) \to H^*(X_\bullet \underset{Y_\bullet}{\times} y; i^*\mathcal{A})$$

The proof of this is Proposition 10.7 of [Fri82] for which we sketch a proof. The map i is the second comparison map given above and note that we will apply this in conjunction with the smooth base change theorem so that the isomorphism condition is satisfied.

Sketch proof. A third type of fibre is introduced defined similarly over those maps which are in addition special as well as hypercoverings. The category of special maps is cofiltered and we have that the fibres of the homotopy type and special type are isomorphic (although this does not arise from a finality argument):

$$fib(f_{ht}) \overset{\cong}{\to} fib(f_{sp})$$

The use of special maps is apparent from the following results. Defining $f_{sp}^{-1}(y)$ by:

$$f_{sp}^{-1}(y) := \{\pi(g)^{-1}(y) | g \in Sp(f)\}$$

where the g are maps of hypercoverings over $f\colon X \to Y$ then there is an isomorphism:

$$H^*(f_{sp}^{-1}(y);\mathcal{A}) \xrightarrow{\cong} (R^*f_*\mathcal{A})_y$$

which moreover fits in a homotopy commutative diagram:

$$\begin{array}{ccc} H^*(f_{sp}^{-1}(y);\mathcal{A}) & \xrightarrow{\quad\cong\quad} & (R^*f_*\mathcal{A})_y \\ \downarrow & & \downarrow \\ H^*((X \underset{Y}{\times} y)_{ht}; i^*\mathcal{A}) & \xrightarrow{\cong} & H^*(X \underset{Y}{\times} y; i^*\mathcal{A}) \end{array}$$

the bottom map being an isomorphism since the étale cohomology of a scheme is the same as the cohomology of the étale type under our assumptions. The weak equivalence relating the homotopy and etale types of the geometric fibres means we can replace the bottom left with the étale type so that if there is an isomorphism for the right vertical map (part of the condition of this theorem) so that there is an isomorphism on the left vertical arrow and if we relate the top left to the fibre of the étale type we are done (after a bootstrap to the simplicial setting). The proof of these results are presented as Lemmas 10.3, 10.4 and 10.5 of [Fri82]. □

7 A Principal Fibration of Linear Algebraic Groups

This chapter creates a principal fibration from which we can derive the cohomological pullback square which we then use for the case of general linear groups over finite fields.

7.1 The Lang-Steinberg Theorem

We first set up some theory of linear algebraic groups.

Definition 7.1.1. *A **Linear Algebraic Group** is an affine algebraic variety equipped with a group structure where the multiplication and inverse functions are morphisms of varities.*

We write $R(G)$ for the radical of G, i.e. the maximal closed connected normal subgroup of G. We say an element u of $GL(V)$, for V finite dimensional, is unipotent if all it's eigenvalues are 1, or equivalently $u - 1$ is nilpotent. The additive Jordan decomposition of g into a diagonalisable (semisimple) plus nilpotent part (which commute) can be turned into a multiplicative decomposition into a diagonalisable times unipotent part (which again commute), [MT11, Prop 2.2].

Further there exists a closed embedding ρ of any linear algebraic group G into $GL_n(k)$ where k is the base field of G as a variety, [MT11, Corollary to Thm 5.5]. Then $g \in G$ can be expressed uniquely as $g = g_s g_u$ with $\rho(g_s)$ diagonalisable and $\rho(g_u)$ unipotent, moreover this expression is independent of embedding, [MT11, Thm 2.5], and we say g_u is unipotent.

With the notation of the above and writing H_u for the unipotent elements of a subgroup $H \leq G$ we make the following definition:

Definition 7.1.2. *A linear algebraic group G is said to be **Reductive** if the unipotent part of its radical, $R(G)_u$, is trivial.*

The major result will be stated in terms of reductive linear algebraic groups instead of anything more general as in this instance we will have a means, 7.3.1, of relating étale cohomology of the group over the fields $\mathbb{C}$ and k by means of base change. It also provides for finding a $\mathbb{Z}$ model for the group variety from which we can base change to obtain the linear algebraic groups of interest. More we have the following theorem attributed to Chevalley (in the case we base change from $\mathbb{Q}$ instead) and Demazure.

Proposition 7.1.3. *Let k be a field, then for all reductive k-groups G_k there exists a reductive $\mathbb{Z}$-group $G_{\mathbb{Z}}$ such that G_k is obtained from G by the base change:*

$$\begin{tikzcd} G_k \arrow[r] \arrow[d] & G \arrow[d] \\ Spec(k) \arrow[r] & Spec(\mathbb{Z}) \end{tikzcd}$$

Proof. Either [Mil13, Exposé XXV Cor 1.3] □

Theorem 7.1.4 (Lang-Steinberg)**.** *Let $\varphi : G \to G$ be a surjective endomorphism of a connected linear algebraic group over algebraically closed field k. If the fixed point group $H := G^{\varphi}$ is finite then the Lang map is surjective:*

$$\begin{aligned} (1/\varphi) \colon G &\to G \\ g &\mapsto g \cdot (\varphi(g))^{-1} \end{aligned}$$

Lang's original theorem was for $k = \overline{\mathbb{F}}_q$ and Steinberg gave this generalisation in [Ste68] and it's this proof we present. Before however we note how we'll apply this, our ultimate aim is to calculate the group cohomology of $GL_n(\mathbb{F}_q)$ and since the frobenius morphism $F_p \colon GL_n(\overline{\mathbb{F}}_q) \to GL_n(\overline{\mathbb{F}}_q)$ is surjective, where $q = p^d$, 7.1.4 says we have a a principal H-fibration $(1/\varphi) \colon GL_n(\overline{\mathbb{F}}_q) \to GL_n(\overline{\mathbb{F}}_q)$. Where the fixed points $H = (GL_n(\overline{\mathbb{F}}_q))^{F_p} = GL_n(\mathbb{F}_q)$ are the group we are interested in.

We will give a proof due to Steinberg: first we prove the result for G solvable then using existence of a Borel subgroup fixed by φ we generalise to the full case.

Definition 7.1.5. *A group is called solvable if the derived series of a group (the series of iterated commutator subgroups):*

$$G \geq G_1 := [G, G] \geq G_2 := [G_1, G_1] \geq G_3 := [G_2, G_2] \geq \ldots \geq \{e\}$$

terminates at $\{e\}$ after finitely many steps.

Definition 7.1.6. *A Borel subgroup of a linear algebraic group G is a maximal subgroup with respect to being closed, connected and solvable.*

Lemma 7.1.7. *Let G be a linear algebraic group, then the following hold:*

1. *All Borel subgroups are conjugate*
2. *If G is connected it is the union over all conjugates of a Borel subgroup*
3. *A surjective endomorphism of G fixes some Borel subgroup*
4. *If G is connected and φ a surjective endomorphism of G and B a Borel subgroup fixed by φ then the following is also surjective:*

$$\begin{aligned} \alpha \colon G \times B &\to G \\ (g, b) &\mapsto gb(\varphi(g))^{-1} \end{aligned}$$

Proof. These are proved in the following:

1. [MT11, Thm 6.4(a)]
2. [MT11, Thm 6.10]
3. [Ste68, Thm 6.2]
4. [Ste68, Lem 7.3]

□

Proof of Lang-Steinberg. G^φ is finite iff $(1/\varphi)G$ contains a dense open of G - follows from a dimension argument. Using this we can prove for $H \lhd G$, if G^φ is finite so is $(G/H)^{\overline{\varphi}}$.

The solvable case — We proceed by induction on length of the derived series. Let $g \in G$, we show $g \in \operatorname{im}(1/\varphi)$. Consider G/H where H is the abelian entry in the derived series, i.e. the last non-identity group. G/H has derived series of length 1 fewer than that of G so by induction we have a $k \in G$ with $g + H = (k + H)((\varphi(k))^{-1} + H)$ in G/H and so $k^{-1} g \varphi(k) \in H$. As H is abelian $(1/\varphi)$ is a group homomorphism so $(1/\varphi)H$ is closed, [MT11, Prop 1.5], and as H^φ is finite we have $(1/\varphi)H = H$ and so $k^{-1}g\varphi(k) = h(\varphi(h))^{-1}$ for some $h \in H$. This gives $g = (kh)(\varphi(kh))^{-1}$, i.e. $(1/\varphi)$ is surjective on G.

The general case — Again we show $g \in \mathrm{im}(1/\varphi)$. Choose a Borel subgroup $B \leq G$ fixed by φ by (3) of 7.1.7 and $k \in G, b \in B$ with $g = kb(\varphi(k))^{-1}$ by (4) of 7.1.7. Since B is solvable by the solvable case there exists $c \in B$ with $b = c(\varphi(c))^{-1}$. Combining these we get $g = kc(\varphi(kc))^{-1}$ showing surjectivity of $(1/\varphi)$. □

Remark 7.1.8. *In the case of $GL_n(k)$ with $\varphi = F_q$ the Frobenius morphism, which acts by raising each component of the matrix to the power q, we can take the Borel subgroup B to be the subgroup with zeroes below the diagonal and it is clear this is fixed by F_q.*

More generally for any linear algebraic group over k we take the Frobenius morphism (as a ring homomorphism) to raise the variables X_i to the q-th power.

$$\begin{aligned}\varphi\colon k[X_1,\ldots,X_n]/I &\to k[X_1,\ldots,X_n]/I\\ k \ni a &\mapsto a\\ X_i &\mapsto X_i^q\end{aligned}$$

The effect on points of $GL_{n,k}$ for instance is as described above.

7.2 A Principal Fibration of Linear Algebraic Groups

Denoting by H the fixed points, $H := G^\varphi$ we now have a principal H-fibration $(1/\varphi)\colon G \to G$ when H is finite. This is called the **Lang Isogeny**. Checking the definition we need an étale surjective morphism to G which we take to be $(1/\varphi)$ and we need to verify the pullback along two copies of $(1/\varphi)$ is isomorphic to $G \otimes H$. Note $(1/\varphi)$ is indeed étale: we are working in varities so we can check the étale property by computing the map on tangent spaces and verifying it is an isomorphism for all points. It can be shown that $\mathrm{d}\varphi = 0$ (the ring homomorphism sends $X_i \mapsto X_i^q$ so that the derivative is 0 since we are in characteristic p) and then $\mathrm{d}(1/\varphi) = \mathrm{d}(id) - \mathrm{d}(\varphi) = id$, so that the tangent map is an isomorphism.

Suppose we have a scheme X and maps f and g to G making the diagram commute.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\theta"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& G \otimes H \arrow[r, "\mu"'] \arrow[d, "pr_1"] & G \arrow[d, "(1/\varphi)"] \\
& G \arrow[r, "(1/\varphi)"'] & G
\end{tikzcd}$$

We define θ by $\theta(x) = f(x) \otimes f(x)^{-1} \cdot g(x)$, this is a scheme morphism as f and g are, as are multiplication and inverse in G. The morphism does indeed have codomain $G \otimes H$ as $(1/\varphi)(f(x)^{-1} \cdot g(x)) = 1$ and makes the diagram commute, moreover θ is unique by definition: the first component is forced on us by commutativity in $f = pr_1 \circ \theta$ and the second by commutativity in $g = \mu \circ \theta$ and knowledge of the first component. Thus we have shown the conditions of a principal H-fibration.

7.3 Some $\mathbb{Z}/l$ Cohomology Isomorphisms

In the above we have a commutative diagram involving a linear algebraic group G over the algebraically closed field k, of charateristic p and it's subfield $\mathbb{F}_q$ with $q = p^d$. The commutative diagram will not be of any use in the finite field case unless we know more about the algebraically closed case.

As it turns out when we take l coprime to p we have isomorphisms in étale cohomology between the cohomologies of G over k, the Witt vectors of k denoted R, the field complex numbers $\mathbb{C}$ and a fourth algebraically closed field K containing R and $\mathbb{C}$.

We do not discuss the Witt vectors here in depth as doing do would distract from the exposition. For now we suffice to say they are a functorial construction from commutative rings to commutative rings. We will later mention some more properties for the specific case mentioned above.

Theorem 7.3.1. *Let $G_{\mathbb{Z}}$ be a reductive linear algebraic group. With k, R and K as above and l coprime to $char(k)$ the base change maps $G_k \to G_R \leftarrow G_K \to G_{\mathbb{C}}$ induce isomorphisms on étale cohomology*

$$H^*(G_k, \mathbb{Z}/l) \xleftarrow{\simeq} H^*(G_R, \mathbb{Z}/l) \xrightarrow{\simeq} H^*(G_K, \mathbb{Z}/l) \xleftarrow{\simeq} H^*(G_{\mathbb{C}}, \mathbb{Z}/l)$$

Proof. [FP81] □

By means of the Spectral Sequence 2.1.4 we also deduce isomorphisms on classifying simplicial schemes.

Corollary 7.3.2. *There are isomorphisms of sheaf cohomology groups on the étale site:*

$$H^*(BG_k, \mathbb{Z}/l) \xleftarrow{\simeq} H^*(BG_R, \mathbb{Z}/l) \xrightarrow{\simeq} H^*(BG_K, \mathbb{Z}/l) \xleftarrow{\simeq} H^*(BG_{\mathbb{C}}, \mathbb{Z}/l)$$

□

Theorem 7.3.3. *There is a weak equivalence:*

$$\operatorname{holim} \circ (\mathbb{Z}/l)_\infty((BG_k)_{\acute{e}t}) \simeq (\mathbb{Z}/l)_\infty((BG(\mathbb{C}))$$

Proof. Recall from 6.1.3 we have for $X_\bullet$ a connected, pointed simplicial scheme of finite type over $\mathbb{C}$:

$$(X_\bullet, x)_{\acute{e}t} \xleftarrow{\rho} (X_\bullet, x)_{s.\acute{e}t} \xrightarrow{\tau} \mathcal{S}ing(|X_\bullet^{top}, x|)$$

with ρ a weak equivalence after taking the pro-P completion $(-)^{\widehat{P}}$ and τ an isomorphism. The conclusion of 6.1.3 also holds if we instead take a pro-l completion. Substituting $BG_{\mathbb{C}}$ for $X_\bullet$ we obtain weak equivalences:

$$((BG_{\mathbb{C}})_{\acute{e}t})^{\widehat{l}} \xleftarrow{\simeq} ((BG_{\mathbb{C}})_{s.\acute{e}t})^{\widehat{l}} \xrightarrow{\simeq} (\mathcal{S}ing(BG(\mathbb{C})))^{\widehat{l}}$$

from which we now apply the Sullivan homotopy limit from which we obtain, by 4.3.2, weak equivalences:

$$\operatorname{holim}^{Su}((BG_{\mathbb{C}})_{\acute{e}t})^{\widehat{l}} \xleftarrow{\simeq} \operatorname{holim}^{Su}((BG_{\mathbb{C}})_{s.\acute{e}t})^{\widehat{l}} \xrightarrow{\simeq} (\mathcal{S}ing(BG(\mathbb{C})))^{\widehat{l}}$$

Since the simplicial scheme $BG_{\mathbb{C}}$ is Noetherian we have that the homology with $\mathbb{Z}/l$ coefficients of each component of $(BG_{\mathbb{C}})_{\acute{e}t}$ is finite and under these conditions, by 4.6.2, we have that there is a natural isomorphism between the Sullivan homotopy limit of the pro l-completion and the Bousfield-Kan homotopy limit of the Bousfield-Kan $\mathbb{Z}/l$-completion, i.e.

$$\operatorname{holim} \circ (\mathbb{Z}/l)_\infty(\{S_\bullet\}_i) \simeq \operatorname{holim}^{Su}(\{S_\bullet\}_i)^{\widehat{l}}$$

Combining this with the above sequence of weak equivalence we obtain the weak equivalence:

$$\operatorname{holim} \circ (\mathbb{Z}/l)_\infty((BG_\mathbb{C})_{\acute{e}t}) \simeq (\mathbb{Z}/l)_\infty \mathcal{S}ing(BG(\mathbb{C}))$$

We are then left with showing that there is a weak equivalence:

$$\operatorname{holim} \circ (\mathbb{Z}/l)_\infty((BG_k)_{\acute{e}t}) \simeq \operatorname{holim} \circ (\mathbb{Z}/l)_\infty((BG_\mathbb{C})_{\acute{e}t})$$

which we do by using 7.3.1, showing there are no principal G-fibrations so that the fundamental groups of the étale topological types of BG_A (for $A \in \{k, R, K, \mathbb{C}\}$) are trivial, and so by 4.2.3 we have that the base change maps after pro$-l$ completion give weak equivalence (since the fundamental groups are trivial there are no non-trivial local coefficient systems so it suffices to check isomorphisms on $\mathbb{Z}/l$ cohomology which is the result of 7.3.1).

We now prove there are no non-trivial principal G-fibrations (this G bears no relation to the G_A). By descent we can reduce to consider $\Pi((BG_A)_0, G)$ and an isomorphism in $\Pi((BG_A)_1, G)$. But $(BG_A)_0$ is just $Spec(A)$ and observe that for the cases $A \in \{k, K, \mathbb{C}\}$ an étale morphism $U \to Spec(A)$, for U connected, is necessarily induced by $A \to B$ for B a finite separable field extension of A but since A is algebraically closed $A = B$, so in this case principal G-fibrations are trivial. For the case R see the following remark. Further the conditions $d_1^*\varphi = d_2^*\varphi \circ d_0^*\varphi$ enforce no non-trivial isomorphisms for the descent data.

We then have, by 5.1.5, that since the category of principal G-fibrations with descent data is trivial so is $\Pi((BG_A)_\bullet, G)$ and then by 5.1.11 that the fundamental groups are trivial. This completes the proof. □

Remark 7.3.4. *In the above case the Witt vectors are a complete local ring and so form a Hensel local ring [Tam94, Ch II Def 6.1.2]. Moreover since the finite field $\mathbb{F}_p$ is perfect so that its algebraic closure F coincides with its separable closure we have that R is a strict Hensel local ring [Tam94, Ch II Def 6.1.8] (its residue field is F). Equivalently [Tam94, Ch II Prop 6.1.7] this says that finite étale R-algebras are trivial, i.e. isomorphic to a finite product of copies of R, and so we have that $\Pi((BG_R)_0, G)$ is trivial.*

7.4 A Cohomological Pullback Square of Linear Algebraic Groups

As before we take k to be an algebraically closed field of charateristic p, l coprime to p and $\varphi\colon G_k \to G_k$ a surjective endomorphism of linear algebraic groups with finite fixed points $H := G_k^\varphi$. The major result is the following:

Theorem 7.4.1. *Let $G(\mathbb{C})$ be a reductive Lie Group and $G_\mathbb{Z}$ the integral algebraic group associated to it. Then for the purposes of cohomology calculations with coefficients in $\mathbb{Z}/l$ we may assume that 7.4.2 is a pullback.*

$$\begin{array}{ccc} BH & \longrightarrow & (\mathbb{Z}/l)_\infty(Sing(BG(\mathbb{C}))) \\ \downarrow & & \downarrow \\ (\mathbb{Z}/l)_\infty(Sing(BG(\mathbb{C}))) & \longrightarrow & (\mathbb{Z}/l)_\infty(Sing(BG(\mathbb{C})^2)) \end{array} \tag{7.4.2}$$

In particular we can use the Eilenberg-Moore spectral sequence to commute $H^*(BH;\mathbb{Z}/l)$. 7.4.1 is the result [Fri82, Thm 12.2].

Consider the following square:

$$\begin{tikzcd}
U_\bullet \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& B(G_k/H, G_k, *) \arrow[r, "\Psi"] \arrow[d] & B((G_k \times G_k)/\Delta G_k, G_k \times G_k, *) \arrow[d] \\
& BG_k \arrow[r, "id\times\varphi"] & B(G_k \times BG_k)
\end{tikzcd} \tag{7.4.3}$$

We will show the square is a pullback. Both vertical maps project away from the left most variety and are principal G_k/H and $(G_k \times G_k)/\Delta G_k)$ fibrations (verification of the conditions is straightforward), the bottom horizontal map is the identity cross the surjective endomorphism and the top morphism sends rational points (or elements in the groups/homogenous spaces) by the formula:

$$\Psi_m(\overline{n}, g_1, g_2, \ldots, g_m) := ((\overline{n}, \varphi(\overline{n})), (g_1, \varphi(g_1)), \ldots, (g_m, \varphi(g_m)))$$

The simplicial scheme morphism γ_m can be defined degree wise uniquely using α_m and β_m making 7.4.3 commute. The image on the G_k components is determined by the map β and the image on the homogenous space G_k/H is determined by α, projecting on to $(G_k \times G_k)/\Delta G_k$ and the isomorphisms : $G_k \to (G_k \times G_k)/\Delta$ and $G_k/H \cong (G_k \times G_k)/\Delta G_k$ given by $\overline{n} \to \overline{(\overline{n}, \varphi(\overline{n}))}$. It can be checked the γ_m assemble to a simplicial map and so we have 7.4.3 is indeed a pullback.

The next lemma shows the pullback is a model for BH.

Lemma 7.4.4. *There is a weak equivalence between $B(G_k/H, G_k, *)_{ét}$ and BH.*

Proof. First the pro object $B(G_k, G_k, *)_{ét}$ is shown to be contractible by exhibiting a homotopy from the identity on $B(G_k, G_k, *)$ to the map factoring as:

$$B(G_k, G_k, *) \to Spec(k) \to B(G_k, G_k, *)$$

Explicitly there are maps

Observe the claim now holds as k is a Hensel local ring and so admits no non-trivial étale covers. It then suffices to show either there is a weak equivalence on the fibres of the étale type of the diagram:

$$\begin{tikzcd}
B(H, H, *) \arrow[r] \arrow[d] & B(G_k, G_k, *) \arrow[d] \\
BH \arrow[r] & B(G_k/H, G_k, *)
\end{tikzcd}$$

Note the BH in this diagram is the classifying space of schemes not of topological spaces! The weak equivalence on fibres follows from [Fri82, Lem 10.6]. Essentially those hypercoverings of $B(G_k/H, G_k, *)$ that factor through $B(G_k, G_k, *)$ equip the étale type with a free H action whose quotient is the étale type of the base. Now note that H is just the disjoint union of copies of $Spec(k)$, being finite and having a discrete topology, equipped with the group action so that it's etale type agrees with the discrete group H. Then by the contractibility of $B(G_k, G_k, *)_{ét}$ we have a weak equivalence:

$$BH \simeq B(G_k/H, G_k, *)_{ét}$$

where the BH is the topological classifying space of the discrete group H. □

Proof of 7.4.1. By the above we have the following commuting diagram of simplicial schemes with an isomorphism of varieties on the fibres:

$$\begin{tikzcd}
G_k/H \arrow[r, "\simeq"] \arrow[d] & (G_k \times G_k)/\Delta G_k \arrow[d] \\
B(G_k/H, G_k, *) \arrow[r] \arrow[d, "p_1"] & B((G_k \times G_k)/\Delta G_k, G_k, *) \arrow[d, "p_2"] \\
BG_k \arrow[r, "id \times \varphi"] & BG_k \times BG_k
\end{tikzcd} \tag{7.4.5}$$

Applying the étale topological type functor $(-)_{ét}$ followed by $\operatorname{holim} \circ (\mathbb{Z}/l)_\infty$ to the bottom square we obtain:

$$\begin{tikzcd}
\operatorname{holim} \circ (\mathbb{Z}/l)_\infty((B(G_k/H, G_k, *))_{ét}) \arrow[rr] \arrow[dd] \arrow[dr] & & \operatorname{holim} \circ (\mathbb{Z}/l)_\infty((B((G_k \times G_k)/\Delta G_k, G_k, *))_{ét}) \arrow[dd] \\
& P \arrow[ur] \arrow[dl, "\zeta"'] & \\
\operatorname{holim} \circ (\mathbb{Z}/l)_\infty((BG_k)_{ét}) \arrow[rr] & & \operatorname{holim} \circ (\mathbb{Z}/l)_\infty((BG_k \times BG_k)_{ét})
\end{tikzcd}$$

This doesn't necessarily preserve pullbacks so we write P for the new pullback. We will however show that the induced map $\operatorname{holim} \circ (\mathbb{Z}/l)_\infty((B(G_k/H, G_k, *))_{ét}) \to P$ is an isomorphism in cohomology with $\mathbb{Z}/l$ coefficients i.e. after applying $(\mathbb{Z}/l)_\infty$ we obtain a weak equivalence $(\mathbb{Z}/l)_\infty((B(G_k/H, G_k, *))_{ét}) \to (\mathbb{Z}/l)_\infty(P)$.

We consider then the map on fibres if we only apply $(-)_{ét}$ to 7.4.5.

$$\begin{tikzcd}
fib((p_1)_{ét}) \arrow[r] \arrow[d] & fib((p_2)_{ét}) \arrow[d] \\
(B(G_k/H, G_k, *))_{ét} \arrow[r] \arrow[d, "(p_1)_{ét}"'] & (B((G_k \times G_k)/\Delta G_k, G_k, *))_{ét} \arrow[d, "(p_2)_{ét}"] \\
(BG_k)_{ét} \arrow[r] & (BG_k \times BG_k)_{ét}
\end{tikzcd} \tag{7.4.6}$$

Since the lower vertical maps p_1 p_2 and in 7.4.5 are smooth being merely projections away from the homogenous spaces we can apply 6.2.7 with the constant sheaf $\mathbb{Z}/l$ to obtain isomorphisms on cohomology:

$$H^*(fib((p_1)_{ét});\mathbb{Z}/l) \cong H^*((G_k/H_k)_{ét};i^*(\mathbb{Z}/l))$$

$$H^*(fib((p_2)_{ét});\mathbb{Z}/l) \cong H^*(((G_k \times G_k)/\Delta G_k)_{ét};i^*(\mathbb{Z}/l))$$

However the geometric fibres are known to be isomorphic as varities by the Lang map so we have an isomorphism between the fibres of the étale topological types of p_1 and p_2. Now applying $holim \circ (\mathbb{Z}/l)_\infty$ we would like to show a $\mathbb{Z}/l$ cohomology isomorphism between the fibres of the maps $\operatorname{holim} \circ(\mathbb{Z}/l)_\infty(p_{ét})$ and ζ.

$$\begin{tikzcd}
\operatorname{holim} \circ(\mathbb{Z}/l)_\infty fib((p_1)_{ét}) \arrow[rr] \arrow[d] & & \operatorname{holim} \circ(\mathbb{Z}/l)_\infty fib((p_2)_{ét}) \arrow[d] \\
\operatorname{holim} \circ(\mathbb{Z}/l)_\infty((B(G_k/H,G_k,*))_{ét}) \arrow[rr] \arrow[dd, "\operatorname{holim} \circ(\mathbb{Z}/l)_\infty(p_1)_{ét}"'] \arrow[dr] & & \operatorname{holim} \circ(\mathbb{Z}/l)_\infty((B((G_k \times G_k)/\Delta G_k,G_k,*))_{ét}) \arrow[dd, "\operatorname{holim} \circ(\mathbb{Z}/l)_\infty(p_2)_{ét}"] \\
 & P \arrow[ur] \arrow[dl, "\zeta"] & \\
\operatorname{holim} \circ(\mathbb{Z}/l)_\infty((BG_k)_{ét}) \arrow[rr] & & \operatorname{holim} \circ(\mathbb{Z}/l)_\infty((BG_k \times BG_k)_{ét})
\end{tikzcd} \tag{7.4.7}$$

The fibre of ζ is the same as that of $\operatorname{holim} \circ(\mathbb{Z}/l)_\infty(p_2)_{ét}$ as it is a pullback so we are left with showing a $\mathbb{Z}/l$ cohomology isomorphism between $\operatorname{holim} \circ(\mathbb{Z}/l)_\infty fib((p_1)_{ét})$ and $\operatorname{holim} \circ(\mathbb{Z}/l)_\infty fib((p_2)_{ét})$. We then consider the two fibrations with the intent of showing isomorphism on the $\mathbb{Z}/l$ cohomology groups of the fibres.

$$\begin{tikzcd}
fib((p_1)_{ét}) \arrow[d] & \operatorname{holim} \circ(\mathbb{Z}/l)_\infty fib((p_1)_{ét}) \arrow[d] \\
B(G_k/H,G_k,*)_{ét} \arrow[d, "(p_1)_{ét}"] & \operatorname{holim} \circ(\mathbb{Z}/l)_\infty B(G_k/H,G_k,*)_{ét} \arrow[d, "\operatorname{holim} \circ(\mathbb{Z}/l)_\infty((p_1)_{ét})"] \\
(BG_k)_{ét} & \operatorname{holim} \circ(\mathbb{Z}/l)_\infty(BG_k)_{ét}
\end{tikzcd}$$

Now by the previous lemma the pro-space $B(G_k/H,G_k,*)_{ét}$ is weakly equivalent to BH and so, since H is finite so that $BH \simeq K(H,1)$ and therefore nilpotent, we have that there is a $\mathbb{Z}/l$ cohomology isomorphism between the total spaces. We claim that there is a $\mathbb{Z}/l$ cohomology isomorphism of the base spaces as well. This isomorphism can be seen by recalling the base change maps are weak equivalences after pro$-l$ completion and the results in 7.3.3 so that we have a series of maps all inducing $\mathbb{Z}/l$ cohomology isomorphisms. A $\mathbb{Z}/l$ cohomology isomorphism on the fibres now follows by the mod-$\mathcal{C}$ class theory of Serre and the Serre spectral sequence. A similar argument applies with p_2 in place of p_1. So we've deduced:

$$H^*(fib((p_i)_{\acute{e}t});\mathbb{Z}/l) \cong H^*(\text{holim}\circ(\mathbb{Z}/l)_\infty(fib((p_i)_{\acute{e}t}));\mathbb{Z}/l)$$

for $i = 1,2$. Combining these with the previously deduced isomorphism $H^*(fib((p_1)_{\acute{e}t});\mathbb{Z}/l) \cong H^*(fib((p_2)_{\acute{e}t});\mathbb{Z}/l)$ we have that there is a $\mathbb{Z}/l$ cohomology isomorphism on the fibres in 7.4.7 so that a $\mathbb{Z}/l$ cohomology isomorphism on the induced map $\text{holim}\circ(\mathbb{Z}/l)_\infty B(G_k/H, G_k, *)_{\acute{e}t} \to P$ follows by the next lemma.

We finish by restating this result in terms of topological space. Applying the previous lemma to replace $(B(G_k/H, G_k, *))_{\acute{e}t}$ with $(BH)_{\acute{e}t}$ and the result of 7.3.3 to replace $\text{holim}\circ(\mathbb{Z}/l)_\infty((BG_k)_{\acute{e}t})$ with $(\mathbb{Z}/l)_\infty \mathcal{S}ing(BG(\mathbb{C}))$ we now have that there is a homotopy commutative square which suffices to calculate $\mathbb{Z}/l$ cohomology on:

$$\begin{tikzcd}
BH \arrow[r] \arrow[d] & (\mathbb{Z}/l)_\infty \mathcal{S}ing(BG(\mathbb{C})) \arrow[d, "\Delta"] \\
(\mathbb{Z}/l)_\infty \mathcal{S}ing(BG(\mathbb{C})) \arrow[r, "id\times\varphi^q"] & (\mathbb{Z}/l)_\infty \mathcal{S}ing(BG(\mathbb{C}) \times BG(\mathbb{C}))
\end{tikzcd}$$

Note the right hand vertical map is indeed the diagonal map as p_2 was just the projection. We prove later what the map $1 \times \varphi^q$ is on cohomology. □

Lemma 7.4.8. *Suppose there are two nilpotent fibrations and maps between them such that the map on fibres is an isomorphism on $\mathbb{Z}/l$ (co)homology as is the induced map on the bases:*

$$\begin{tikzcd}
F_1 \arrow[r] \arrow[d, "f"] & E_1 \arrow[r] \arrow[d, "e"] & B_1 \arrow[d, "b"] \\
F_2 \arrow[r] & E_2 \arrow[r] & B_2
\end{tikzcd}$$

then so is the induced map on the $\mathbb{Z}/l$ homology of the total spaces.

Proof. Since the fibrations are nilpotent we can apply the Bousfield-Kan $\mathbb{Z}/l$-completion functor and have it preserve the fibration property.

$$\begin{tikzcd}
(\mathbb{Z}/l)_\infty(F_1) \arrow[r] \arrow[d, "(\mathbb{Z}/l)_\infty(f)"] & (\mathbb{Z}/l)_\infty(E_1) \arrow[r] \arrow[d, "(\mathbb{Z}/l)_\infty(e)"] & (\mathbb{Z}/l)_\infty(B_1) \arrow[d, "(\mathbb{Z}/l)_\infty(b)"] \\
(\mathbb{Z}/l)_\infty(F_2) \arrow[r] & (\mathbb{Z}/l)_\infty(E_2) \arrow[r] & (\mathbb{Z}/l)_\infty(B_2)
\end{tikzcd}$$

Since we have cohomology isomorphisms with $\mathbb{Z}/l$ coefficients for the maps f and b the maps are $(\mathbb{Z}/l)_\infty(f)$ and $(\mathbb{Z}/l)_\infty(b)$ are weak equivalences, by the property of Bousfield-Kan completions, and so, by the five lemma, is $(\mathbb{Z}/l)_\infty(e)$ hence e induces (co)homology isomorphisms again with $\mathbb{Z}/l$ coefficients. □

7.5 The Cohomology of $GL_n(\mathbb{F}_q)$

In this section we calculate, by means of the Eilenberg-Moore spectral sequence and a simplification thereof, the cohomology of $GL_n(\mathbb{F}_q)$ away from the characteristic. This simplification will come in the form of the 'Big Collapse Theorem' which gives an explicit form for the E_2 page and the knowledge that this agrees with the E_∞ page. We first describe the form of this spectral sequence using as a reference [McC01, §7]. Suppose that we have a fibration $p\colon E \to B$ with fibre F and a map $f\colon X \to B$, then we consider the pullback square:

$$\begin{tikzcd} F \arrow[r, equal] \arrow[d] & F \arrow[d] \\ E_f \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"] & B \end{tikzcd} \tag{7.5.1}$$

The statement of the Eilenberg-Moore spectral sequence is then the following:

Theorem 7.5.2 (Eilenberg-Moore Spectral Sequence)**.** *Suppose the fibre F is connected and that the system of local coefficients on B determined by π is simple. Then there is a second quadrant spectral sequence converging to the cohomology $H^*(E_f;R)$ with E_2 page given by:*

$$E_2^{*,*} := Tor_{H^*(B;R)}(H^*(X;R), H^*(E;R))$$

□

Recall the conclusion of the previous section was that for the puposes of cohomology calculations with coefficients in $\mathbb{Z}/l$, i.e. away from the characteristic, we may say there is a pullback square:

$$\begin{tikzcd} BGL_n(\mathbb{F}_q) \arrow[r] \arrow[d] & (\mathbb{Z}/l)_\infty \mathcal{S}ing(BGL_n(\mathbb{C})) \arrow[d, "\Delta"] \\ (\mathbb{Z}/l)_\infty \mathcal{S}ing(BGL_n(\mathbb{C})) \arrow[r, "id\times\varphi"] & (\mathbb{Z}/l)_\infty \mathcal{S}ing(BGL_n(\mathbb{C}) \times BGL_n(\mathbb{C})) \end{tikzcd} \tag{7.5.3}$$

Here we have taken our reductive linear algebraic group $G_\mathbb{Z}$ to be $GL_{n,\mathbb{Z}}$ and φ to be the Frobenius so that when we consider the group $GL_{n,k}$, where as before k is algebraically closed of characteristic p and the Frobenius raises coefficients to the $q = p^d$ power, we have that the fixed points is the group $GL_{n,\mathbb{F}_q}$.

Let us take 7.5.3 as our pullback square in 7.5.1. To use this spectral sequence we must then know firstly the cohomology of $BG(\mathbb{C})$ with coefficients in $\mathbb{Z}/l$ and secondly the algebra structures of $H^*(E;\mathbb{Z}/l)$ and $H^*(X;\mathbb{Z}/l)$ as $H^*(B;\mathbb{Z}/l)$ algebras. In other words we need to know the effect of the maps Δ and $id \times \varphi$ on cohomology. Note the cohomology of the product space of B is determined by the Kunneth theorem as we are taking coefficients in a field.

We denote by $R[a_1, a_2, \ldots, a_n]$ the polynomial algebra over the ring R with generators $a_1, \ldots, a_n$.

Proposition 7.5.4. *The cohomology of $BGL_n(\mathbb{C})$ with integer coefficients is given by the polynomial algebra over $\mathbb{Z}$ with generators the c_{2i} in degree $2i$ for $i \in \{1, 2, \ldots, n\}$.*

$$H^*(GL_n(\mathbb{C});\mathbb{Z}) = \mathbb{Z}[c_2, c_4, \ldots c_{2n}]$$

Proof. By Gram-Schmidt orthogonalisation we have a deformation retract of $GL_n(\mathbb{C})$ onto $U(n)$ so we can instead calculate the cohomology of $U(n)$ which is known to be a polynomial algebra generated by the Chern classes c_{2i} in degree $2i$ for $1 \leq i \leq n$. □

Replacing $\mathbb{Z}$ by $\mathbb{Z}/l$ we also obtain:

$$H^*(GL_n(\mathbb{C}); \mathbb{Z}/l) = \mathbb{Z}/l[c_2, c_4, \ldots, c_{2n}]$$

So the cohomology groups in question are given by:

$$H^*(E; \mathbb{Z}/l) \cong \mathbb{Z}/l[e_2, e_4, \ldots, e_{2n}]$$

$$H^*(X; \mathbb{Z}/l) \cong \mathbb{Z}/l[x_2, x_4, \ldots, x_{2n}]$$

$$H^*(B; \mathbb{Z}/l) \cong \mathbb{Z}/l[b_2, b_2', b_4, b_4', \ldots, b_{2n}, b_{2n}']$$

where the letters e, b, x serve to indicate which algebra they belong to but they should be thought of as coming from the Chern classes so that the b_{2i}' also reside in degree $2i$. For determining the algebra structure note the map Δ is the diagonal map so that $b_{2i} \mapsto e_{2i}$ and $b_{2i}' \mapsto e_{2i}$. It is also clear that under $(id \times \varphi)^*$ we have $b_{2i} \mapsto x_{2i}$ so we are left with determining the map φ^*. For this we use a proposition.

Proposition. *Let $i\colon \mathbb{T} \to GL_n(\mathbb{C})$ be the inclusion of a maximal torus of $GL_n(\mathbb{C})$. Then the induced map $i^*\colon H^*(GL_n(\mathbb{C}); \mathbb{Z}/l) \to H^*(\mathbb{T}; \mathbb{Z}/l)$ is an injection.*

Proof. Either [Ben91, Lem 2.9.2] or [Qui72, Lems 12 & 13]. □

Taking S^1 as a torus we have $\mathbb{T} = \prod_1^n S^1$. Recall from the introduction we have a covering space $S^\infty \to \mathbb{CP}^\infty$ with S^1 acting on S^∞ so that $\mathbb{CP}^\infty$ is a model for BS^1, hence we have the cohomology is the polynomial algebra:

$$H^*\left(\prod_1^n BS^1; \mathbb{Z}/l\right) = \mathbb{Z}/l[s_2, s_2', s_2'', \ldots s_2^{(n)}]$$

where $s_2^{(i)}$ is a generator in degree 2. A consequence of the proposition is that the effect of φ on this polynomial algebra determines the effect on $H^*(BGL_n(\mathbb{C}); \mathbb{Z}/l)$ by means of the commuting diagram:

$$\begin{tikzcd} H^*(BGL_n(\mathbb{C}); \mathbb{Z}/l) \arrow[r, "\varphi^*"] \arrow[d, hook] & H^*(BGL_n(\mathbb{C}); \mathbb{Z}/l) \arrow[d, hook] \\ H^*(\mathbb{T}; \mathbb{Z}/l) \arrow[r, "\varphi^*"] & H^*(\mathbb{T}; \mathbb{Z}/l) \end{tikzcd} \tag{7.5.5}$$

It then suffices to determine the effect of φ^* on $H^*(BS^1; \mathbb{Z}) = \mathbb{Z}$. By considering the sequence of pro$-L$ weak equivalences of homotopy types that were established:

$$(BG_k)_{\acute{e}t} \to (BG_R)_{\acute{e}t} \leftarrow (BG_{\mathbb{C}})_{\acute{e}t} \leftarrow (BG_{\mathbb{C}})_{s.\acute{e}t} \to \mathcal{S}ing(BG(\mathbb{C}))$$

we see that since $G = S^1$ and on the left we have the Frobenius the induced map on the right is going to be homotopic to the map $x \mapsto x^q$ since this map is defined on $GL_1(\mathbb{C})$ (the map raising elements of $GL_n(\mathbb{C})$ to the power q is not well defined for $n \geq 2$) and so induces multiplication by q on degree 2 cohomology. By the cup product structure we have $\varphi(s_{2k}) = \varphi(s_2^k) = \varphi(s_2)^k = (q.s_2)^k = q^k.s_{2k}$, more generally if we consider the maximal torus of $GL_n(\mathbb{C})$, taking cohomology coefficients to be $\mathbb{Z}/l$ then φ acts on a homogenous element of degree $2k$ by multiplication by q^k.

Replacing the cohomology by their polynomial rings the square 7.5.5 becomes:

$$\begin{array}{ccc} \mathbb{Z}/l[c_2, c_4, \ldots, c_{2n}] & \xrightarrow{\varphi^*} & \mathbb{Z}/l[c_2, c_4, \ldots, c_{2n}] \\ \big\downarrow & & \big\downarrow \\ \mathbb{Z}/l[s_2, s_2', \ldots, s_2^{(n)}] & \xrightarrow{\varphi^*} & \mathbb{Z}/l[s_2, s_2', \ldots, s_2^{(n)}] \end{array}$$

from which we determine that the upper map φ^* must also act in degree $2k$ by multiplication by q^k. This concludes the determination of the algebra structures of the cohomology algebras. Calculations involving the Eilenberg-Moore Spectral Sequence can become rather involved so we follow Kleinerman's calculation, [Kle82], and simplify using a theorem of Smith.

Definition 7.5.6. *A **Borel Ideal** of a ring R is an ideal generated by a finite sequence of elements $(r_1, r_2, \ldots, r_n)$ such that r_1 is not a zero divisor in R, r_2 is not a zero divisor in $R/ < r_1 >$ and more generally r_k is not a zero divisor in $R/ < r_1, r_2, \ldots, r_{k-1} >$ for all k.*

We state the Big Collapse Thoerem of Smith keeping the notation defined for the Eilenberg-Moore spectral sequence and writing $\Lambda_R[e_1, e_2, \ldots, e_n]$ for the exterior algebra over R on the generators $e_1, e_2, \ldots, e_n$. In terms of the calculation this is a large simplification as it avoids any need for finding a projective resolution of the $H^*(B; \mathbb{Z}/l)$ algebra $H^*(E; \mathbb{Z}/l)$. In [McC01] a general construction is detailed where a projective resolution of the graded algebra is given in degree n by the bar construction:

$$\mathcal{B}^{-n}(H^*(B;\mathbb{Z}/l), H^*(E;\mathbb{Z}/l)) := H^*(B;\mathbb{Z}/l) \otimes \overline{H^*(B;\mathbb{Z}/l)} \otimes \ldots \otimes \overline{H^*(B;\mathbb{Z}/l)} \otimes H^*(E;\mathbb{Z}/l)$$

where there are n copies of $\overline{H^*(B;\mathbb{Z}/l)}$ the algebra without unit. This quickly becomes unmanageable when the algebras are polynomial rings with multiple generators.

Theorem (Big Collapse Theorem). *Let $\mathbb{F}$ be a field and suppose the following are satisfied:*

1. *$p^* \colon H^*(B; \mathbb{F}) \to H^*(E; \mathbb{F})$ is surjective*
2. *$\ker(p^*) \subset H^*(B; \mathbb{F})$ is a Borel ideal*
3. *$f^*(\ker(p^*))$ generates a Borel ideal of $H^*(X; \mathbb{F})$ which we denote J*

then we have the following conclusions:

1. *The spectral sequence collapses at the E_2 page, i.e. $E_2^{*,*} \cong E_\infty^{*,*}$*
2. *$E_2 = H^*(X; \mathbb{F})/J \otimes \Lambda_{\mathbb{F}}[e_1, e_2, e_3, \ldots]$ with all e_i in degree -1.*

Proof. [Smi67, Part II, Thm 3.1] □

To be more explicit for a minimal set of generators for $H^*(X;\mathbb{F})/J$ we have e_i for each, the an element $x \in H^*(X;\mathbb{F})/J$ has bidegree $(0, \deg(x))$ and the e_i have bidegree $(-1, \deg(g_i))$ for the corresponding element g_i of the minimal generators.

We state again the notation that k is an algebraically closed field of characteristic p and $q = p^d$. Let r be the least positive integer such that $q^r \equiv 1(\mathrm{mod}\ l)$. We verify the conditions of the Big Collapse Theorem. p^* is surjective since $b_{2i} \mapsto e_{2i}$, $\ker(p^*)$ is generated by the elements $b_{2i} - b'_{2i}$ and the sequence $(b_2 - b'_2, b_4 - b'_4, \ldots, b_{2n} - b'_{2n})$ clearly defines a Borel ideal which shows the first two conditions. For the third the image of the generators $b_{2i} - b'_{2i}$ under $1 \times \varphi^*$ is given by $(1 - q^i)x_{2i}$ as φ acts by multiplication by q^i. If r divides i this image is 0. Since $\mathbb{Z}/l$ is a field we have the image is generated by the x_{2i} where r does not divide i. And these give a Borel ideal.

We restrict to $l \neq 2$, the reason being that by [Smi67, Part II, Cor 3.3] there is an algebra isomorphism $E_2 = E_\infty$. The preceding discussion immediately gives the corollary:

Corollary 7.5.7. *The cohomology of $GL_n(\mathbb{F}_q)$ with coefficients in $\mathbb{Z}/l$, $l \neq 2$, is given by:*

$$H^*(GL_n(\mathbb{F}_q);\mathbb{Z}/l) = \mathbb{Z}/l[\{x_{2jr}\}] \otimes \Lambda_{\mathbb{Z}/l}[\{y_{2jr-1}\}]$$

where as before r is the least positive integer with $q^r \equiv 1(mod\ l)$ and $1 \leq jr \leq n$ with subscripts denoting the degree. □

We would only have this result at most as an associated grading for $l = 2$ by this method. Quillen gives relations for the algebra structure in [Qui72] where it splits into two cases: a typical case which behaves as above and an exceptional case where generators of the exterior algebra no longer square to 0.

We also mention here the two other cases we shall need firstly note that since the groups $GL_n(\mathbb{F}_q)$ are finite the cohomology groups are torsion and annhilated by the order of $GL_n(\mathbb{F}_q)$ so that rational cohomology is trivial. The case where $l = p$ (divides the order of the characteristic) is not known in general however we do have a vanishing range, [Qui72, Thm 6].

Theorem 7.5.8. *There is a vanishing range, for $0 < i < d(p-1)$ we have:*

$$H^i(GL_n(\mathbb{F}_q);\mathbb{Z}/p) = 0$$

8 The K-Theory of Finite Fields

8.1 A Brief Discussion of K-Theory

For the initial discussion in the following we refer to [Hat] for topological K-theory and [Ros94] for algebraic K-theory. The results stated in this section can be found in either of the two cited works.

Recall that topological K-theory is a cohomology theory on compact Hausdorff spaces represented by the classifying space $BU \times \mathbb{Z}$, or BU for the reduced case $\widetilde{K}$. The former is the Grothendieck group completion of the monoid of vector bundles on X where $E_1 - E_1' = E_2 - E_2'$ iff $E_1 \oplus E_2' \approx_s E_2 \oplus E_1'$ i.e. isomorphic after adding a trivial bundle of the same dimension to both sides. The latter is the group with elements the vector bundles over X subject to $E_1 \sim E_2$ whenever $E_1 \oplus \epsilon^n \cong E_2 \oplus \epsilon^m$. This cohomology theory can be shown to be 2-periodic with $K^1(X) = K^0(S^1 \wedge X)$.

Similar to above given a ring R we can consider the semigroup $Proj(R)$ of finitely generated projective R-modules and define $K_0(R)$ to be the Grothendieck group completion of $Proj(R)$. With this notation we have the following theorem and corollary of Swan.

Theorem 8.1.1 (Swan). *Let X be compact Hausdorff and $C(X) := \{f\colon X \to \mathbb{R} \mid f \text{ continuous}\}$, then there is a 1-1 correspondence between vector bundles over X and finitely generated projective $C(X)$-modules given by sending a vector bundle to the module of its sections $\Gamma(X)$.* □

Corollary 8.1.2. *There is an isomorphism $K^0(X) \xrightarrow{\cong} K_0(\Gamma(X))$.* □

There is then a natural question of given that topological K-theory extends to a cohomology theory on spaces can something similar be done with the algebraic K_0? As further motivation a relative form of K_0 can be defined such that there is an exact sequence resembling the exact sequences of a pair of topological spaces. We take $I \lhd R$ to be an ideal and then there is an exact sequence of the form:

$$K_0(R, I) \to K_0(R) \to K_0(R/I)$$

Further this is natural in (R, I) and the relative K_0 group is isomorphic to $K_0(I)$ (this is the Excision theorem). These algebraic analogues were found in degrees 1 and 2 and defined algebraically. We give constructions here.

We define the **infinite general linear group** over a commutative, unital ring R to be the colimit of $GL_n(R)$ with maps the inclusion of $GL_n(R)$ into the first $n \times n$ entries of $GL_{n+1}(R)$.

$$GL(R) := \operatorname{colim}\left(GL_1(R) \hookrightarrow GL_2(R) \hookrightarrow GL_3(R) \hookrightarrow GL_4(R) \hookrightarrow \ldots\right)$$

Recall that a group G is said to be **perfect** if its abelianization $G_{ab} := G/[G, G]$ is trivial. We can also define the **elementary subgroup** of $GL_n(R)$ generated by those matrices of the form $I_n + r\delta_{ij}$ for $i \neq j$. The elementary subgroup is a perfect normal subgroup of $GL_n(R)$. Similarly to the above we define the **infinite elementary subgroup** $E(R)$ and the following theorem says it is also perfect and normal in $GL(R)$.

Proposition 8.1.3 (Whitehead). $[GL(R), GL(R)] = E(R) = [E(R), E(R)]$ □

Bass introduced a group now called $K_1(R)$ defined to be the quotient $K_1(R) := GL(R)/E(R)$. The above result of Whitehead says that K_1 is the abelianization of the infinite general linear group over the ring R.

Example 8.1.4. *Take the field $\mathbb{F}_4 := \mathbb{F}_2[X]/(X^2 + X + 1)$. We denote the elements of $\mathbb{F}_4$ also by $\{0, 1, X, X^2\}$. The determinant homomorphism takes all elementary matrices to 1 so in $K_1(\mathbb{F}_4)$ the element of $GL(\mathbb{F}_4)$ represented by (X) cannot be in $E(\mathbb{F}_4)$ as it has determinant X. Similarly for X^2, thus there must be at least 3 cosets so that $|K_1(\mathbb{F}_4)| \geq 3$. In fact it is 3 as will be shown later and so isomorphic to $\mathbb{Z}/3$. Similarly for $\mathbb{F}_q$ there are $q-1$ invertible elements so $|K_1(\mathbb{F}_q)| \geq q-1$, which again is an equality and a cyclic group of order $q-1$.*

More generally it is a fact that if R is a field the first K group is the group of invertible elements of the field, $K_1(R) = R^\times$. The K_1 groups extended the natural exact sequence on K_0 above to a natural exact sequence:

$$K_1(R, I) \to K_1(R) \to K_1(R/I) \to K_0(R, I) \to K_0(R) \to K_0(R/I)$$

Milnor's K_2 is defined to be the kernel of a ring map $\varphi\colon St(R) \to E(R)$ where $St(R)$, the Steinberg group of R, is the group freely generated by elements $x_{ij}(r)$ whenever $i \neq j$ and $r \in R$ quotiented by relations corresponding to those appearing in $E(R)$. The map φ is then just $x_{ij}(r) \mapsto I_n + r\delta_{ij}$ (we abuse notation here slightly to give a map into $GL_n(R)$ and therefore into $GL(R)$). So $K_2(R)$ is detecting extra relations amongst the infinite elementary group.

Quillen's work sought to find a definition of the higher algebraic K groups rooted in homotopy theory, for instance arising as homotopy groups of some space, and in [Qui71] he gave the following definition for $i \geq 1$.

$$K_i(R) := \pi_i(BGL(R)^+) \tag{8.1.5}$$

This definition agrees in degrees 1 and 2 with those given by Bass and Milnor but not in degree 0, this can be repaired by instead defining $K_i(R) := K_0(R) \times \pi_i(BGL(R)^+)$ although we will later consider the case $R = \mathbb{F}_q$ where K_0 is of little interest (modules over a field are just vector spaces, and so all are projective).

With this definition in terms of homotopy groups a long exact sequence of these K-groups should then arise as the long exact sequence of a fibration, and indeed relative forms of these are defined to be homotopy fibres of appropriate maps so that there is such a long exact sequence which extends those of the above, [Ros94, Def 5.2.14].

The following section is a sketch of the calculation Quillen gave in [Qui72] for the K-theory of finite fields.

8.2 Quillen's Higher K Groups and Calculations for Finite Fields

Theorem 8.2.1. *Let X be a connected topological space and $N \trianglelefteq \pi_1(X)$ a perfect, normal subgroup. Then there exists a topological space X^+ and map $\rho\colon X \to X^+$ such that $\pi_1(X^+) = \pi_1(X)/N$ and ρ_* is a homology isomorphism. Moreover this map is initial amongst all maps $X \to Y$ sending N to 0.*

Proof. [Hat03, Prop 4.40] □

In fact as defined $(-)^+$ is a functor. Recall that when G is discrete $K(G, 1)$ is a model for B so that $\pi_1(BG) = G$.

We have the Adams operations Ψ^q which are ring endomorphisms $\Psi^q\colon K^*(X) \to K^*(X)$ and which can be represented as a homotopy class of maps $[BU, BU]$ which we pick a representative of and call σ^q.

The claim is that there is a space $F\Psi^q$ and map $BGL(\mathbb{F}_q)^+ \xrightarrow{\sim} F\Psi^q$ which is a homotopy equivalence where $F\Psi^q$ is a homotopy theoretic fixed point set of the Adams operation which can be shown to be homotopy equivalent to a fibre and so its homotopy groups are easily calculated by means of the long exact sequence on homotopy groups of a fibration:

$$\ldots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \pi_{n-1}(B) \to \ldots \tag{8.2.2}$$

The two interpretations of $F\Psi^q$ as homotopy fixed points and as a fibre are given by the two diagrams, the first expressing it as homotopy fixed points.

$$\begin{tikzcd} F\Psi^q \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BU^I \arrow[d, "\Delta"] \\ BU \arrow[r, "(id{,}\sigma)"'] & BU \times BU \end{tikzcd} \tag{8.2.3}$$

This is a pullback diagram with Δ sending a path in BU^I to its endpoints. For the second diagram the difference operation $(a, b) \mapsto a - b$ can also be represented by a map $d\colon BU^2 \to BU$ and it can be chosen such that some point $b \in BU$ is an identity element.

$$\begin{tikzcd} F\Psi^q \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BU^I \arrow[r, "d^I"] \arrow[d, "\Delta"] & PBU \arrow[d, "p"] \\ BU \arrow[r, "(id{,}\sigma)"'] & BU \times BU \arrow[r, "d"'] & BU \end{tikzcd}$$

We recall that pullbacks of fibrations are fibrations so that since Δ is a fibration so is φ, the right hand map is the path space fibration. Choosing the element b in the lower left BU it's images along the lower horizontal maps are (b, b) and b by the choices of d and σ outlined above and the fibres over each of these are clearly homotopic to ΩBU.

$$\begin{tikzcd} \Omega BU \arrow[r, equal] \arrow[d] & \Omega BU \arrow[r, equal] \arrow[d] & \Omega BU \arrow[d] \\ F\Psi^q \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BU^I \arrow[r, "d^I"] \arrow[d, "\Delta"] & PBU \arrow[d, "p"] \\ BU \arrow[r, "(id{,}\sigma)"'] & BU \times BU \arrow[r, "d"'] & BU \end{tikzcd}$$

We claim that $F\Psi^q$ is the pullback of p along $d \circ (id, \sigma)$. Suppose we have maps $f\colon D \to BU$ and $g\colon D \to PBU$ with $d \circ (id, \sigma) \circ f = p \circ g$. Then the two dashed arrow can clearly be found by first using the property of pullback in the right hand square and then in the second, which gives us existence of a map $D \to F\Psi^q$ making 8.2.4 commute.

$$
\begin{tikzcd}
D \arrow[dr, dashed] \arrow[drr, dashed] \arrow[drrr, bend left=10, "g"] \arrow[ddr, bend right, "f"'] & & P_{a,b}BU \arrow[r, equal] \arrow[d] & P_{a,b}BU \arrow[d] \\
& F\Psi^q \arrow[r, "\gamma"] \arrow[d, "\varphi"'] & BU^I \arrow[r, "d^I"] \arrow[d, "\Delta"] & PBU \arrow[d, "p"] \\
& BU \arrow[r, "(id,\sigma)"'] & BU \times BU \arrow[r, "d"'] & BU
\end{tikzcd}
\tag{8.2.4}
$$

There is an issue of uniqueness however, suppose we are given two maps $j, k \colon D \to F\Psi^q$ making 8.2.4 commute and such that $\gamma \circ j = \gamma \circ k$ then it is necessarily the case that $j = k$ as the left hand square is a pullback so it will suffice to show there is only one map after composing with γ. The fibre of the map Δ over the point $(a, b) \in BU \times BU$ is the space of paths $\xi \colon a \rightsquigarrow b$ in BU and similarly for the fibre of p over a, we write this fibre as $P_{a,b}BU$. If we let $\xi := \gamma(j(d))$ and $\xi' := \gamma(k(d))$ be paths in BU^I they are certainly equal after applying d^I as this composite is just the map g. But then since the map on the fibres over (a, b) and a is an isomorphism and the paths are equal after applying d^I they must also be equal in BU^I so $\gamma \circ j = \gamma \circ k$ which shows uniqueness. So we have shown $F\Psi^q$ is the pullback of the composite of the two squares.

Thus we have that $F\Psi^q$ is the homotopy fibre of the composite $d \circ (id, \sigma)$ so we can now consider the long exact sequence of 8.2.2 applied to the fibration $F\Psi^q \to BU \overset{1-\Psi^q}{\to} BU$. The homotopy groups of BU are the K-theory groups of the spheres:

$$\pi_n(BU) = [S^n, BU] = \widetilde{K}(S^n) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Further we have the Adams operation Ψ^q induces multiplication by q on $\widetilde{K}(S^2) = \pi_2(BU)$ and therefore by q^n on $\pi_{2n}(BU)$. The long exact sequence then gives short exact sequences:

$$0 \longrightarrow \pi_{2n}(F\Psi^q) \longrightarrow \mathbb{Z} \xrightarrow{1-q^n} \mathbb{Z} \longrightarrow \pi_{2n-1}(F\Psi^q) \longrightarrow 0$$

The middle map is injective for $n \geq 1$ and so we obtain:

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/(q^n - 1) & i \text{ odd} \\ 0 & i > 0 \text{ even} \end{cases}$$

We still have to explain the homotopy equivalence $BGL(\mathbb{F}_q)^+ \xrightarrow{\sim} F\Psi^q$. It follows from the long exact sequence 8.2.2 applied to $F\Psi^q \to BU \xrightarrow{1-\Psi^q} BU$ that $F\Psi^q$ is a simple space and $BGL(\mathbb{F}_q)^+$ is simple since it is a H-space, see [Ada78, §3.2] or [Ros94, Thm 5.2.12], and so it suffices by the Hurewicz and Whitehead theorems to show the map is a homology isomorphism in integral coefficients. We can split this into showing homology isomorphisms with coefficients in $\mathbb{Q}, \mathbb{F}_p$, and $\mathbb{F}_l$ for all l which are prime and coprime to p where p is the characteristic of the field. An explanation of this can be found in the appendix.

8.3 The Homology Isomorphism of Quillen

This section details the construction of the map $BGL(\mathbb{F}_q) \to F\Psi^q$. Via a series of constructions we will pass from virtual $\mathbb{F}_q$ representations of a finite group G to virtual $\mathbb{C}$ representations from which there is a map to the elements of reduced K-theory of BG fixed by the Adams operations, the first few results show this to be equivalent to homotopy classes of maps $BG \to F\Psi^q$. We do not show this map is a homology isomorphism here but we will show in the next section a cohomology isomorphism by the techniques of Friedlander.

Theorem 8.3.1. *There is a homology isomorphism $BGL(\mathbb{F}_q)^+ \to F\Psi^q$* □

We explain how this map arises. Recall that the representation ring of a group G over the field $\mathbb{F}$ is defined to be the group completion of the character ring which is denoted $R_{\mathbb{F}}(G)$. This map will then appear from elements of the representation rings of $GL_n(\mathbb{F}_q)$ over the field $\mathbb{F}_q$. Elements of the representation ring will be referred to as virtual representations. Since the space $F\Psi^q$ is simple finding the above map is equivalent to giving a map:

$$BGL(\mathbb{F}_q) \to F\Psi^q$$

This shall be built from compatible maps on the finite subgroups $GL_n(\mathbb{F}_q)$:

$$BGL_n(\mathbb{F}_q) \to F\Psi^q$$

Quillen shows homotopy classes of such maps are isomorphic to fixed points under the Adams operation of the reduced K-theory of $BGL_n(\mathbb{F}_q)$. More precisely we have the following lemma (Lemma 1 (ii) in [Qui72]):

Lemma 8.3.2. *Let $\varphi\colon F\Psi^q \to BU$ be the map of the preceding section and X a space with $[X, \Omega BU] = 0$ then we have:*

$$\varphi_*\colon [X, F\Psi^q] \xrightarrow{\cong} [X, BU]^{\Psi^q}$$

Proof. Surjectivity follows since $F\Psi^q$ is a pullback so a map in $[X, F\Psi^q]$ is equivalent to maps $g\colon X \to BU$ and $X \to BU^I$ with a homotopy relating g to $\sigma \circ g$, but σ represents Ψ^q so this lands in the fixed points of the Adams operation on reduced K-theory, i.e. $[X, BU]^{\Psi^q}$.

For injectivity note since we have a fibration:

$$F\Psi^q \xrightarrow{\varphi} BU \xrightarrow{d\circ(id,\sigma)} BU$$

we obtain a long exact sequence:

$$\ldots \to [X, \Omega BU] \to [X, \Omega BU] \to [X, F\Psi^q] \xrightarrow{\varphi_*} [X, BU] \to \ldots$$

so if φ_* weren't injective then $[X, \Omega BU] \neq 0$ but we assumed the contrary, so φ_* is injective. □

To apply this to the space $BGL_n(\mathbb{F}_q)$ one needs to know that $[BGL_n(\mathbb{F}_q), \Omega BU] = 0$ which follows from the Atiyah-Segal completion theorem of equivariant K-theory, [AS69]. If G-equivariant topological K-theory is denoted K_G^* then this results says that for a G-space X there is an isomorphism:

$$K_G^*(X)\widehat{\ }_I \xrightarrow{\cong} K^*(X \times_G EG)$$

between the completion of G-equivariant K-theory with respect to the augmentation ideal I and the standard topological K-theory of $X \times_G EG$. Taking X as a point this result relates the K-theory of BG and a completion of the complex character ring $R_{\mathbb{C}}G$. The result $[BG, \Omega BU] = 0$ now follows by the loop space adjunction and the equalities (we are taking reduced K-theory):

$$K^1(BG) = K^1_G(*)\widehat{\ }_I = K^0_G(S^1)\widehat{\ }_I$$

where the first equality is the completion theorem and the second the suspension isomorphism and now observe that:

$$K^0_G(S^1) = K^0(S^1) \otimes R_{\mathbb{C}}(G) = 0$$

since the action of G on S^1 is trivial and $K^0(S^1) = 0$.

So we are now left with describing a map of the right hand side in:

$$[BGL_n(\mathbb{F}_q), F\Psi^q] \to [BGL_n(\mathbb{F}_q), BU]^{\Psi^q}$$

Such a map will be found by lifting (virtual) representations over $\mathbb{F}_q$ to (virtual) representations over $\mathbb{C}$. Let's note first how a $\mathbb{C}$ representation will give a class in K-theory. If $G \to GL_n(V)$ is an actual representation over $\mathbb{C}$ then we can form an element in unreduced K-theory which we can then map to reduced K-theory. The element is given by the class of the vector bundle:

$$EG \underset{G}{\times} V \to BG$$

So this gives a map from the representation ring of G over $\mathbb{C}$ to the reduced K-theory of BG.

$$R_{\mathbb{C}}(G) \to [BG, BU]$$

We are left with extending this to a map from $R_{\mathbb{F}_q}(G)$ to $[BG, BU]$ and showing it lands in the fixed points of the Adams operations. The extension is given by the Brauer lift.

Fix an embedding $\tau \colon \overline{\mathbb{F}_q}^{\times} \to \mathbb{C}^{\times}$ of multiplicative groups. From an $\overline{\mathbb{F}_q}$ representation $G \to GL_n(V, \overline{\mathbb{F}_q})$ of a finite group G one can define the Brauer character. Denote by $\{\lambda_i\}$ the set of eigenvalues for the representation (with repeats accounting for multiplicity) the Brauer character is the class function given by:

$$\chi(g) := \sum_i \tau(\lambda_i)$$

Green shows that this character is in fact a character of a unique complex virtual representation, [Gre55, Thm 1] , and this element of $R_{\mathbb{C}}(G)$ is called the Brauer lift. Given then instead a $\mathbb{F}_q$ representation we can extend this to its algebraic closure by tensoring $\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} -$ and then take the Brauer lift, this gives a map from $\mathbb{F}_q$ representations to $\mathbb{C}$ representation which then extends by group completion to a map $R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)$.

Denote the Brauer lift on V by $BR(V)$. The Adams operations are also defined on the representation rings and the effect on characters of a representation ρ with character χ_ρ is given by $\chi_{\Psi^q \rho}(g) = \chi(g^q)$. Moreover they are compatible with the Adams operations on K-theory in the sense they commute with the map $R_{\mathbb{C}}(G) \to [BG, BU]$. Since the eigenvalues of a $\mathbb{F}_q$ representation are fixed under the Frobenius morphism we have that $\Psi^q BR(\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} V) = BR(\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} V)$. So in fact we have a map:

$$R_{\mathbb{F}_q}(G) \to R_{\mathbb{C}}(G)^{\Psi^q}$$

Combining this with the previous maps and isomorphisms we obtain a map:

$$R_{\mathbb{F}_q}(G) \to [BG, F\Psi^q]$$

which to summarise is formed by the following composites:

$$\begin{aligned}
\overline{\mathbb{F}_q} \otimes_{\mathbb{F}_q} (G)- \colon R_{\mathbb{F}_q} &\to R_{\overline{\mathbb{F}_q}}(G) \\
BR(-) \colon R_{\overline{\mathbb{F}_q}}(G) &\to R_{\mathbb{C}}(G)^{\Psi^q} \\
B \colon R_{\mathbb{C}}(G)^{\Psi^q} &\to [BG, BU]^{\Psi^q} \\
(\psi_*)^{-1} \colon [BG, BU]^{\Psi^q} &\to [BG, F\Psi^q]
\end{aligned}$$

Then taking $G = GL_n(\mathbb{F}_q)$ and applying the composite to the standard representation we obtain a homotopy class of maps

$$BGL_n(\mathbb{F}_q) \to F\Psi^q$$

Moreover these maps are compatible so that we can form the colimit over them to obtain finally a map:

$$BGL(\mathbb{F}_q) \to F\Psi^q$$

Equipped with these last two maps and knowledge of the (co)homologies $H^*(F\Psi^q; \mathbb{Z}/l)$ Quillen sets about calculating the (co)homologies of the $BGL_n(\mathbb{F}_q)$ and $BGL(\mathbb{F}_p)$.

8.4 The Cohomology Isomorphism of Friedlander

The preceding discussion set up the framework Quillen used to show a homology isomorphism. In the following we show how the cohomology isomorphism $BGL(\mathbb{F}_q)^+ \xrightarrow{\simeq} F\Psi^q$ follows from the cohomological pullback squares 7.4.1, in a somewhat more simple fashion, for all n. We will still have need for the vanishing range described above and the vanishing of cohomology with $\mathbb{Q}$ coefficients. This theorem features as [Fri82, Thm 12.7].

Theorem 8.4.1. *There is a cohomology isomorphism $BGL(\mathbb{F}_q)^+ \to F\Psi^q$.*

Proof. We are left with showing the case for $\mathbb{Z}/l$ coefficients, for l coprime to p. We can construct the following commutative cube:

$$\begin{tikzcd}
BGL(\mathbb{F}_q) \arrow[rr] \arrow[dr, dashed, "\theta"] \arrow[dd] & & (\mathbb{Z}/l)_\infty(\mathcal{S}ing(GL(\mathbb{C}))) \arrow[dr] \arrow[dd] & \\
& (\mathbb{Z}/l)_\infty(F\Psi^q) \arrow[rr] \arrow[dd] & & (\mathbb{Z}/l)_\infty(\mathcal{S}ing(GL(\mathbb{C}))) \arrow[dd] \\
(\mathbb{Z}/l)_\infty(\mathcal{S}ing(GL(\mathbb{C}))) \arrow[rr] \arrow[dr] & & (\mathbb{Z}/l)_\infty(\mathcal{S}ing(GL(\mathbb{C}) \times GL(\mathbb{C}))) \arrow[dr] & \\
& (\mathbb{Z}/l)_\infty(\mathcal{S}ing(GL(\mathbb{C}))) \arrow[rr] & & (\mathbb{Z}/l)_\infty(\mathcal{S}ing(GL(\mathbb{C}) \times GL(\mathbb{C})))
\end{tikzcd}$$

The back square is the colimit over n of 7.4.1 and the front square is $(\mathbb{Z}/l)_\infty$ applied to the pullback square 8.2.3. Since all spaces in 8.2.3 were simply connected on application of $(\mathbb{Z}/l)_\infty$ this gives another pullback square hence the dashed arrow θ exists, where we have taken the other maps between the squares to be identities. This commutative cube gives us a map between the Eilenberg Moore Spectral Sequences for the two squares with $\mathbb{Z}/l$ coefficients and since the three non-dashed horizontal arrows are identities we obtain an isomorphism on the spectral sequences, hence that θ is a $\mathbb{Z}/l$-cohomology isomorphism. □

8.5 Some Brief Comparisons

This section serves to draw some brief comparisons between the étale homotopy method devloped within leading up to the cohomology and K-theory calculations and the approach of Quillen in determining the same.

An obvious first remark to make is that both methods at some point make a transition to the more familiar characteristic 0 algebraically closed scenario. Recall this manifested itself for Friedlander's method in the form of 7.3.1, which stated for the purposes of étale cohomology calculations with coefficients in $\mathbb{Z}/l$ we can replace the classifying scheme BG with the easily understood $BG_{\mathbb{C}}$. More specifically it appeared as the base change map $R \to F$ where R was the Witt vectors of $F := \overline{\mathbb{F}_q}$. The corresponding transition to characteristic 0 in [Qui72] occurs by introducing the Brauer lift.

The method of constructing the cohomological pullback square also has the immediate advantage of being much more applicable. Many more group cohomologies can be obtained from the cohomological pullback square, e.g. [Kle82] and further more if we replace GL with SO or Sp, the special orthogonal and symplectic groups, we can define real oriented and symplectic K-theories and calculate these for finite fields with a little more work, see [Fri82, Chp 12] and [Fri76]. Much of the input in Quillen's calculation has also been employed in Friedlander's - Eilenberg-Moore spectral sequence and the restriction to maximal tori for instance - and many more powerful results from algebraic geometry have been incorporated to achieve this generality.

A Coskeleta

Theorem. *Let $\mathcal{C}$ admit all fibre products and have a terminal object then the m-truncation functor admits a right adjoint $cosk_m$.*

When we write $((\Delta_{/[n]})_{\leq m}$ for the m-truncated under category we do allow $n > m$, only the domain $[k]$ of objects need have $k \leq m$.

Proof. We will first define what each stage should be and then show these assemble to give a simplicial object, finally we'll demonstrate the adjunction. Let $X_\bullet \in s_n\mathcal{C}$.

Step 1 – Consider the category $(\Delta_{/[n]})_{\leq m}$, (note $n \geq m$ is possible). For $k \leq m$ and an object $(\varphi\colon [k] \to [n]) \in (\Delta_{/[n]})_{\leq m}$ we have $X^n(\varphi) = X_k$.

To a morphism α in this category we can associate a map $\alpha^{n,*}\colon X^n(\varphi) \to X^n(\varphi')$ to just be $\alpha^*\colon X_k \to X_{k'}$. We then define the n^{th} stage by $(cosk_m X)_n := \lim_{(\Delta_{/[n]})_{\leq m}} X^n$.

$$\begin{tikzcd} {[k']} \arrow[rr, "\alpha"] \arrow[dr, "\varphi'"'] & & {[k]} \arrow[dl, "\varphi"] \\ & {[n]} & \end{tikzcd}$$

Step 2 – We now assemble these into a simplicial object, given a map $\theta\colon [n] \to [l]$ we must show there is an induced map $(cosk_m X)_l \to (cosk_m X)_n$.

Giving this map is equivalent to giving maps from $(cosk_m X)_l$ to the $X^n(\varphi)$ commuting with the maps $X^n(\varphi) \to X^n(\varphi')$. We define the structure maps by

$$\gamma_\varphi\colon (cosk_m X)_l = \lim_{(\Delta/[l])_{\leq m}} X^l \to X^l(\theta \circ \varphi) = X_k = X^n(\varphi)$$

to be the projection onto $X^l(\theta \circ \varphi)$ followed by identifying X_k with $X^l(\theta \circ \varphi)$ and $X^n(\varphi)$.

We have commutativity in the form $\alpha^{n,*} \circ \gamma_\varphi = \gamma_{\varphi'}$. The left triangle commutes by the defining property of the limit and the right square by examination: the horizontal maps are the identity and the vertical maps are equal. So we have an induced map $(cosk_m X)_l \to (cosk_m X)_n$.

$$\begin{tikzcd} (cosk_m X)_l \arrow[r] \arrow[dr] & X^l(\theta \circ \varphi) \arrow[r] \arrow[d] & X^n(\varphi) \arrow[d] \\ & X^l(\theta \circ \varphi') \arrow[r] & X^n(\varphi') \end{tikzcd}$$

Step 3 – It remains to show the adjunction $\tau_m \dashv cosk_m$, i.e. a natural isomorphism

$$\mathrm{Hom}_{s_m\mathcal{C}}(\tau_m Y, X) \cong \mathrm{Hom}_{s\mathcal{C}}(Y, cosk_m X) \tag{A.0.1}$$

First we note that $\tau_m(cosk_m X) \cong X$ as objects in $s_m\mathcal{C}$: for $n \leq m$ the degree n part $(cosk_m X)_n$ is certainly isomorphic to X_n as the index category $(\Delta_{/[n]})_{\leq m}$ has terminal object $([n] \xrightarrow{id} [n])$ so the limit is identified with $X^n([n] \to [n]) = X_n$. With this identification the induced map of *Step 2* is exactly θ^* (assuming $n, l \leq m$) : we can ignore all objects in the index category bar the terminal object where the structure map $\gamma_{id_{[n]}}$ corresponds to the map $X^l(id_{[l]}) \to X^l(\theta \circ id_{[l]})$ and so the induced map is exactly θ^*. So to a morphism $X \to cosk_m T$ we obtain a morphism $\tau_m X \to \tau_m cosk_m Y \cong Y$, this is one half of the bijection.

Suppose now we have a morphism $f\colon \tau_m X \to Y$, to define a morphism $X \to cosk_m Y$ is equivalent to giving maps $\zeta_\varphi\colon X_n \to Y^n(\varphi)$ for each $\varphi \in (\Delta_{/[n]})_{\leq m}$ which also satisfy commutativity conditions with the induced maps from $\theta\colon [n] \to [l]$.

$$\begin{tikzcd}
 & X_l \arrow[rr, "\theta^*"] \arrow[ddl, "\zeta^l_{\theta\circ\varphi'}"'] \arrow[dr, "\zeta^l_{\theta\circ\varphi}"] & & X_n \arrow[dr, "\zeta^n_\varphi"] \arrow[dd, "\zeta^n_{\varphi'}"'] & \\
 & & Y^l(\theta\circ\varphi) \arrow[rr, equal] \arrow[dll, "\alpha^{l,*}"'] & & Y^n(\varphi) \arrow[dl, "\alpha^{n,*}"] \\
Y^l(\theta\circ\varphi') \arrow[rrr, equal] & & & Y^n(\varphi') &
\end{tikzcd} \tag{A.0.2}$$

A priori we should expect the morphism $X \to cosk_m Y$ to be extending the morphism $f\colon \tau_m X \to Y$ and so the projection $\zeta^n_{id_{[n]}}$ (for $n \leq m$) to the terminal object $Y^n(id_{[n]})$ is f_n. With this in mind the back square of A.0.2 becomes $\zeta^l_\theta = f_n \circ \theta^*$ and we take this as our definition when the domain $[n]$ of θ has $n \leq m$. We in fact need only define ζ^l_θ with this condition on θ. Commutativity of the subdiagrams involving the X_i now follows easily: e.g. the left triangle —

$$\alpha^{l,*} \circ \zeta^l_{\theta\circ\varphi} = \alpha^* f_k(\theta\circ\varphi)^* = f_{k'}\alpha^*(\theta\circ\varphi)^* = f_{k'}\alpha^*\varphi^*\theta^* = f_{k'}(\varphi\circ\alpha)^*\theta^* = f_{k'}\varphi'^*\theta^* = \zeta^l_{\theta\circ\varphi'}$$

Where we use that the m-truncated simplicial structure on both X and Y for the second equality as $k, k' \leq m$. Thus we have the two maps in the adjunction A.0.1 which are evidently inverse: one is given by extending and the other by truncating, and we note that the extension is unique by construction. Note we have not shown naturality of the adjunction. $\square$

B Homology Isomorphisms

We prove here that a map $f\colon X \to Y$ inducing an isomorphism in homology with integral coefficients is equivalent to inducing isomorphism in homology with $\mathbb{Q}$ coefficients and $\mathbb{Z}/p$ coefficients for all primes p.

Proposition B.0.1. *Let $f\colon X \to Y$ be a map of simple spaces. Then f_* is an isomorphism in homology with $\mathbb{Z}$ coefficients iff it is an isomorphism in homology with $\mathbb{Q}$ and $\mathbb{Z}/p$ coefficients for all primes p.*

We write $C(f)$ for the cofibre of the map f and M_f for the mapping cylinder of f so that we have a cofibration:

$$X \to M_f \to C(f)$$

Associated to this is a long exact sequence on homology groups for any ring R:

$$\ldots \to H_n(X;R) \to H_n(M_f;R) \to H_n(C(f);R) \to H_{n-1}(X;R) \to H_{n-1}(M_f;R) \to \ldots$$

There is also for any space W a Bockstein long exact sequence for the prime p determined by the short exact sequence on coefficients:

$$0 \to \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

The Bockstein long exact sequence is then:

$$\ldots \to H_n(W;\mathbb{Z}) \xrightarrow{\cdot p} H_n(W;\mathbb{Z}) \to H_n(W;\mathbb{Z}/p) \xrightarrow{\beta} H_{n-1}(W;\mathbb{Z}) \xrightarrow{\cdot p} H_{n-1}(W;\mathbb{Z}) \to \ldots$$

Lastly we have the universal coefficient theorem stating:

Theorem B.0.2 (UCT). *For A an abelian group and W a space we have:*

$$0 \to H_i(W;\mathbb{Z}) \otimes A \to H_i(W;A) \to Tor_1(H_{i-1}(W;\mathbb{Z}),A) \to 0$$

□

We use these to prove the proposition.

Proof. First note the forward direction follows simply from UCT since $\mathbb{Z}/p$ and $\mathbb{Q}$ are both fields so that the Tor term vanishes and so the result follows from the five lemma:

$$\begin{tikzcd}
0 \arrow[r] & H_i(X;\mathbb{Z}) \otimes A \arrow[r] \arrow[d, "f_*"', "\cong"] & H_i(X;A) \arrow[r] \arrow[d] & Tor_1(H_{i-1}(W;\mathbb{Z}),A) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_i(Y;\mathbb{Z}) \otimes A \arrow[r] & H_i(Y;A) \arrow[r] & Tor_1(H_{i-1}(Y;\mathbb{Z}),A) \arrow[r] & 0
\end{tikzcd}$$

For the reverse direction consider the long exact sequence for the cofibration $X \to M_f \to C(f)$:

$$\begin{tikzcd} \ldots \arrow[r] & H_n(X;R) \arrow[r, "\cong"] & H_n(M_f;R) \arrow[r] \arrow[d, equal] & H_n(C(f);R) \arrow[r] & H_{n-1}(X;R) \arrow[r, "\cong"] & H_{n-1}(M_f;R) \arrow[r] \arrow[d, equal] & \ldots \\ & & H_n(Y;R) & & & H_{n-1}(Y;R) & \end{tikzcd}$$

which shows the $\mathbb{Z}/p$ and $\mathbb{Q}$ homology of $C(f)$ is trivial since f_* is inducing isomorphisms with these coefficients for all p. Consider next the Bockstein of the cofibre $C(f)$ with multiplication by p:

$$\begin{tikzcd} \ldots \arrow[r] & H_{n+1}(C(f);\mathbb{Z}/p) \arrow[r, "\beta"] \arrow[d, equal] & H_n(C(f);\mathbb{Z}) \arrow[r, "\cdot p", "\cong"'] & H_n(C(f);\mathbb{Z}) \arrow[r] & H_n(C(f);\mathbb{Z}/p) \arrow[r] \arrow[d, equal] & \ldots \\ & 0 & & & 0 & \end{tikzcd}$$

So that we have multiplication by p is an isomorphism on the homology of the cofibre for all p, i.e. the homology groups with $\mathbb{Z}$ coefficients are $\mathbb{Q}$ vector spaces. But as shown above $H_*(C(f);\mathbb{Q})$ is trivial and therefore so is $H_*(C(f);\mathbb{Z})$, i.e. f_* induces an isomorphism on $\mathbb{Z}$ homology. $\square$

Corollary B.0.3. *If $f\colon X \to Y$ is a map of simply connected spaces inducing isomorphisms on homology with $\mathbb{Z}/p$ and $\mathbb{Q}$ coefficients then it is a weak equivalence.* $\square$

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