# Group Cohomology

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### Abstract

This report briefly sketches some parts of my internship on group cohomology. Even though the (co)homology of a group appears in a multitude of situations, the actual computations are far from obvious and make use of techniques found in related fields. Indeed, knowledge of homological algebra, topology or even algebraic geometry proves to be useful in a variety of ways. To keep this document short and concise, we focus more precisely on the interaction between group cohomology and topology.

### Resumé

Denne rapport kort præsenterer nogle dele af min praktik om gruppekohomologi. Selvom (ko)homologi af en gruppe vises i mange situationer, de faktiske beregninger er virkelig hårdt i almindelighe og gøre brug af teknikker findes i relaterede områder. Kendskab til homologisk algebra, topologi eller algebraisk geometri viser sig at være nyttigt på forskellige måder. For at holde dette dokument kort og kortfattet, vi fokuserer mere præcist på samspillet mellem gruppekohomologi og topologi.

Jeg vil gerne takke min vejleder Jesper Grodal for hans tid og forklaringer. Jeg vil også gerne takke Institutet for Matematiske Fag for at lade mig blive i Københavns Universitetet i løbet af min praktik.

# Contents

The cohomology theory of groups arose from both algebraic an topological sources. The idea behind this report is to provide a tiny walk-through for anyone wanting to take a peek at some basic results in this field, with a special emphasize on the relation between two worlds: algebra and topology. We first start with an example where they meet in an unusual fashion.

Let $G$ be a group. Since their appearance in a 1936 paper by Hurewicz [3], topologists have used and constructed aspherical spaces (connected topological spaces with vanishing higher homotopy groups, ie. such that $\pi_n = 0$ for $n \neq 1$). It is a result that we can construct a classifying space of $G$: an aspherical space $BG$ with $G$ as fundamental group. Following the standard notation, we will say that $BG$ is an Eilenberg-MacLane space $K(G, 1)$. This idea is central in topology, indeed the ordinary cohomology theory can defined with the Eilenberg-MacLane spectrum [7]. In his paper, Hurewicz proved that these spaces are determined up to homotopy by their fundamental group. Therefore, it seems that the (ordinary) cohomology of $BG$ is a good candidate for our definition of the cohomology of $G$.

Before jumping to formal definitions, let us put our algebraist's hat on. A first way of studying our group is to consider a representation of it (briefly, a functor $\{\bullet\} \longrightarrow$ **Vect** where the morphisms of $\{\bullet\}$ are the group elements). By slightly modifying this approach (ie. by replacing **Vect** by $\mathbb{Z}G$**-Mod**), we can consider $G$-modules, that is to say abelian groups with a $G$ action (or modules over the group ring $\mathbb{Z}G$). In particular we are interested in the submodule of $G$-invariants (the fixed elements under the $G$ action). Let us be more specific: we take $M$ to be a $G$-module and we write $-^G$ for the functor associated with taking the $G$-invariants. Basic computations show that this functor is left exact and our algebraic description of group cohomology relies on its right derived functors $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, -)$.

In the end, $H^n(G, M)$ (with our topological defition) is naturally isomorphic to $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$ (our algebraic construction).

From this situation, we see that the topological and the algebraic points of view are closely intertwined for the strong reason that they describe the same object in two different languages. This relation will sometimes be straightforward, for example when constructing the standard resolution of a group $G$, we will essentially build a $K(G, 1)$, but will require more work in other situations.

Starting with basic definitions to introduce the subject, we will quickly delve into the real world with computations related to topology. We finish by briefly introducing the ring structure and spectral sequences, both essential tools to continue studying the subject. The appendix is miles away from standard introductions and can be skipped at first reading.

# 1 Some Homological Algebra and Group Cohomology

The study of group (co)homology makes use of objects and constructions found in homological algebra. We provide here a short summary of the basic definitions and properties we will use in this report. For more in-depth introductions, we refer to [10] or [2].

## 1.1 Chain Complexes and Homology

Let $R$ be a ring. A **chain complex** $(C_\bullet, d_\bullet)$ is a sequence of $R$-modules $C_i$ together with homomorphisms, called **boundary operators** or **differentials**, $d_i : C_i \longrightarrow C_{i-1}$ such that $d^2 := d \circ d = 0$ (we will often drop the indexing subscript when the domain and codomain are understood). Likewise, a **cochain complex** is a sequence of $R$-modules $C_i$ and differentials $d_i : C_i \longrightarrow C_{i+1}$ such that $d_{i+1} \circ d_i = 0$. Because $d^2 = 0$, we see that $\operatorname{Im} d \subset \operatorname{Ker} d$ and we can consider the quotient $H_i(C_\bullet) := \operatorname{Ker} d_i / \operatorname{Im} d_{i+1}$, called the **homology** of the chain complex (and the same reasoning applies to a cochain complex). As in ordinary (co)homology, we will call **(co)boundaries** elements of $\operatorname{Im} d$ and **(co)cycles** elements of $\operatorname{Ker} d$. If $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$ are chain complexes, we will say that $f_\bullet : C_\bullet \longrightarrow C'_\bullet$ is a **chain map** if it commutes with the differentials. That is to say, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r, "d'_{i+2}"'] & C'_{i+1} \arrow[r, "d'_{i+1}"'] & C'_i \arrow[r, "d'_i"'] & C'_{i-1} \arrow[r, "d'_{i-1}"] & \cdots
\end{tikzcd}$$

Two chain maps $f, g : C \longrightarrow C'$ are **homotopic**, denoted $f \simeq g$ if there is a chain map $h : C \longrightarrow C'$ of degree 1 (a chain map $C \longrightarrow C'$ has **degree** $r$ if the maps on the modules are of the form $C_i \longrightarrow C'_{i+r}$) such that $d'h + hd = f - g$. Pictorially, the situation is as follows:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+2}"] & C_{i+1} \arrow[r, "d_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[dl, "h_i"'] \arrow[d, dashed, "f_i - g_i"] & C_{i-1} \arrow[r, "d_{i-1}"] \arrow[dl, "h_{i-1}"] & \cdots \\
\cdots \arrow[r, "d'_{i+2}"'] & C'_{i+1} \arrow[r, "d'_{i+1}"'] & C'_i \arrow[r, "d'_i"'] & C'_{i-1} \arrow[r, "d'_{i-1}"] & \cdots
\end{tikzcd}$$

Writing down the commutativity of squares in the definition of a chain map $f$, we see that it sends cycles to cycles and boundaries to boundaries. Therefore, it induces a map on the homology of the complex, often denoted $H(f)$ (this makes $H(-)$ into functor $\operatorname{Ch}(\mathcal{A}) \longrightarrow \mathbf{Ab}$ for $\mathcal{A}$ an abelian category and $\operatorname{Ch}(\mathcal{A})$ the category of chain complexes with elements in $\mathcal{A}$). This allows us to call a chain map a **weak equivalence** if the induced map on homology is an isomorphism.

Given three chain complexes $A$, $B$ and $C$, we call $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ a short exact sequence if every row of the following diagram is exact, and every square is commutative:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{i+1} \arrow[r] \arrow[d] & B_{i+1} \arrow[r] \arrow[d] & C_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_i \arrow[r] \arrow[d] & B_i \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r] \arrow[d] & B_{i-1} \arrow[r] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Given such a short exact sequence, a diagram chase shows that we have a long exact sequence:

$$\cdots \longrightarrow H_{n+1}(C) \xrightarrow{\delta} H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \cdots$$

where $\delta$ is called the **connecting homomorphism**.

## 1.2 Special Modules and Resolutions

In order to compute the (co)homology of a group, we will need to consider specific algebraic constructions. We here provide the minimal background to define them.

We will say that a module $Q$ is **injective**, and that a module $P$ is **projective** if they satisfy the following properties (not universal, we do not require uniqueness):

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\forall f"] \arrow[d, "\forall g"'] & Y \arrow[dl, dashed, "\exists h"] \\
& Q &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& P \arrow[dl, dashed, "\exists h"'] \arrow[d, "\forall g"] & \\
X \arrow[r, "\forall f"'] & Y \arrow[r] & 0
\end{tikzcd}$$

Let $F : R\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$ be a covariant additive (ie., it is a group homomorphism on the each hom-set) functor and $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ a short exact sequence of modules. We say that $F$ is **right exact** if $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$ is exact. We call it **left exact** when $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ is exact. It is **exact** if $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$ is exact.

Before giving the definition of the (co)homology of a group, we need a final algebraic construction. Let $A$ be a $R$-module. We say that an exact sequence

$$P_\bullet \longrightarrow A := (\cdots \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0)$$

is a **projective resolution** of $A$ over $R$ if all the $P_i$'s are projective modules. Likewise, we have an **injective resolution** of $A$ over $R$:

$$A \longrightarrow I_\bullet := (0 \longrightarrow A \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow \cdots)$$

when the $I_i$'s are injective modules and the sequence is exact. We can show that the category of modules has enough injectives and projectives, which implies that we can always find an injective or a projective resolution for a given module. In the case of projective resolution, we can even construct a free resolution (defined as before with free modules) in the following way: recall that every module is the quotient of a free module and construct the resolution inductively (where every projection is a quotient map found by the previous remark):

$$\begin{tikzcd}
& & \mathrm{Ker}\, f_2 \arrow[dr, hook] & & & & \mathrm{Ker}\,\epsilon \arrow[dr, hook] & & & \\
& F_3 \arrow[ur, two heads] \arrow[rr, dashed, "f_3"'] & & F_2 \arrow[rr, dashed, "f_2"] \arrow[dr, two heads] & & F_1 \arrow[ur, two heads] \arrow[rr, dashed, "f_1"'] & & F_0 \arrow[r, two heads, "\epsilon"] & A \arrow[r] & 0 \\
\cdots \arrow[ur] & & & & \mathrm{Ker}\, f_1 \arrow[ur, hook] & & & & &
\end{tikzcd}$$

For a right exact functor $F$, we can define its **left derived functors** $L_iF(-)$. We can describe them on objects by the following: for a module $A$, we pick a projective resolution $P \longrightarrow A$ and put $L_iF(A) := H_i(F(P))$. We proceed in a similar way for the **right derived functors** $R_iF(-)$ of a left exact functor $F$: for a module $A$, we pick an injective resolution $A \longrightarrow I$ and put $R^iF(A) := H^i(F(I))$. We can show that the definition does not depend on the chosen resolution, or more precisely, if we fix a resolution for each module to have well defined derived functors, then the functors obtained by changing the resolutions are naturally isomorphic.

## 1.3 Group (Co)homology

For a fixed ring $R$ and a fixed $R$-module $M$, we denote by $\mathrm{Tor}_*^R(M,-)$ the left derived functors of $- \otimes_R M$ and by $\mathrm{Ext}_R^*(-,M)$ the right derived functors of $\mathrm{Hom}_R(M,-)$. For a group $G$, we define its $n$th (co)homology groups with coefficients in $M$ to be:

$$H^n(G,M) := \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},M)$$
$$H_n(G,M) := \mathrm{Tor}^{\mathbb{Z}G}_n(M,\mathbb{Z})$$

Functoriality of (co)homology plays a central role in everyday life. We first start with homology, the situation for cohomology being similar. Let $\mathcal{C}$ denote the category consisting of pairs $(G,M)$ where $G$ is a group and $M$ a $G$-module, with morphisms $(\phi,f):(G,M)\longrightarrow(G',M')$ where $\phi:G\longrightarrow G'$ is a group homomorphism and $f:M\longrightarrow M'$ a map of abelian groups such that $f(gm)=\phi(g)f(m)$ for $g\in G$ and $m\in M$. By taking projective resolutions $F$ and $F'$ of $\mathbb{Z}$ over $\mathbb{Z}G$ and $\mathbb{Z}G'$ respectively, a bit of homological algebra gives us an augmentation preserving $G$-chain map $\tau:F\longrightarrow F'$, well-defined up to homotopy such that $\tau(gx)=\phi(g)\tau(x)$ for $g\in G$ and $x\in F$. In the end $\tau\otimes f$ induces a well defined map on homology, denoted $(\phi,f)_*$. This makes $H_*$ a covariant functor on $\mathcal{C}$.

In the case of cohomology, we define a new category $\mathcal{D}$ with the same objects but where a morphism $(G,M)\longrightarrow(G',M')$ consists of a pair $(\phi:G\longrightarrow G',\ f:M'\longrightarrow M)$. For such a morphism, similar constructions on the chain level induce a morphism on cohomology $(\phi,f)^*:H^*(G',M')\longrightarrow H^*(G,M)$, thus turning $H^*$ into a contravariant functor on $\mathcal{D}$.

For the sake of concreteness, we here provide two popular examples of maps induced by functoriality. For a subgroup $H<G$, the inclusion $i$ induces a map

$$\mathrm{res}^G_H := H^*(i,\mathrm{id}):H^*(G,M)\longrightarrow H^*(H,M)$$

where we look at $M$ as a $H$-module by forgetting parts of the $G$-action.

For a fixed $g\in G$, the morphism $(H,M)\longrightarrow(gHg^{-1},M)$ in $\mathcal{D}$ defined by $(h\mapsto ghg^{-1},\ m\mapsto gm)$ gives an isomorphism

$$g^*:H^*(gHg^{-1},M)\longrightarrow H^*(H,M).$$

# 2 (Co)homology via Topology

When computing the (co)homology of a group $G$, difficulties arise from finding a nice projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Indeed, even though we can always choose the **standard resolution** $F_\bullet\longrightarrow\mathbb{Z}$ with $F_i=\mathbb{Z}[G^{i+1}]$ and differentials $(g_0,\dots g_n)\mapsto\sum_{i=0}^n(-1)^i(g_0,\dots,\widehat{g_i},\dots,g_n)$, the actual computations do not give any satisfying result in the majority of cases.

## 2.1 Eilenberg-MacLane Spaces

Let $G$ be a group. We can construct an Eilenberg-MacLane $BG$ (such that $\pi_1(BG)=G$ and $\pi_n(BG)=0$ if $n\neq 1$), called a **classifying space**, as the quotient of $\Delta$-complex, built from the simplices $[g_0,\dots,g_n]$, $g_i\in G$ (glued together in the natural way), by the $G$-action. Such a model $X$ of $K(G,1)$ is a CW-complex, so there exists a simply connected universal cover $\widetilde{X}$ on which $G$ acts freely. The **augmented chain complex** of $\widetilde{X}$:

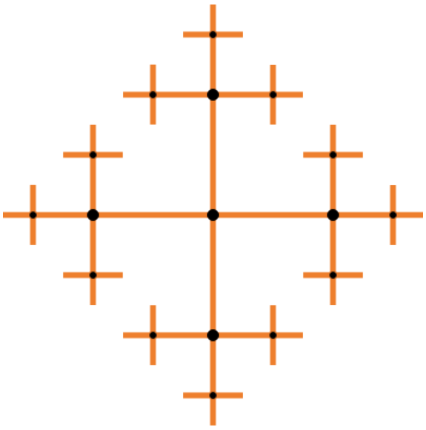
$$\cdots\longrightarrow C_2(\widetilde{X})\longrightarrow C_1(\widetilde{X})\longrightarrow C_0(\widetilde{X})\longrightarrow\mathbb{Z}\longrightarrow 0$$

is therefore a free (thus projective) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, where we denote by $C_i(\widetilde{X})$ the singular $i$-chains.

As an application, we can compute the (co)homology of the free group $F(S)$ on a set $S$ of generators. A little bit of algebraic topology shows that $\pi_1(\bigvee_{s\in S}S^1)\simeq F(S)$ and that the universal cover of $\bigvee_{s\in S}S^1$ may be constructed as the simplicial complex whose vertices are the elements of $F(S)$ and whose 1-simplices are the pairs $\{g,gs\}$ for $g\in F(S)$ and $s\in S$. The augmented complex is then:

$$0\longrightarrow\mathbb{Z}F(S)\xrightarrow{\partial}\mathbb{Z}F(S)\xrightarrow{\epsilon}\mathbb{Z}\longrightarrow 0$$

where $\partial(\{g,gs\})=s-1$ and $\epsilon(g)=1$ is the **augmentation map**. In the case of two generators, we have the following picture of the universal cover:



From the chain complex, we see that the (co)homology groups of degree $r > 1$ are zero. By a direct computation from the definitions we get that $H_0(F(S), \mathbb{Z}) = H^0(F(S), \mathbb{Z}) = \mathbb{Z}$ and $H_1(F(S), \mathbb{Z}) = H^1(F(S), \mathbb{Z}) = \mathbb{Z}^n$ if $|S| = n$.

## 2.2 Groups Acting on Spheres

Another way of connecting algebra and topology is given by groups acting on spaces. In particular, we can find a nice resolution to compute the (co)homology of a group $G$ acting on a $G$-complex $X$ (a CW-complex on which $G$ acts freely by permuting the cells) homeomorphic to an odd dimensional sphere $S^{2n-1}$. We denote by $C_i(X)$ the singular $i$-chains. Because of the free action, each $C_i(X)$ is a free $G$-module, and we can show that we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow C_{2n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

By splicing such sequences, we get a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of period $2n$:

$$\cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow C_{2n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For example, $G = \mathbb{Z}/n \simeq \langle t \rangle$ ($t^n = 1$) acts freely by rotations on a circle made of $n$ vertices and $n$ 1-cells. This gives a resolution:

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where $t-1$ is the multiplication by $t-1$ and $N$ is the multiplication by $N = 1 + t + t^2 + \ldots + t^{n-1}$. Direct computations then give:

$$H_i(G,\, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \text{ even} \\ \mathbb{Z}/n & \text{for } i \text{ odd} \end{cases} \quad \text{and} \quad H^i(G,\, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0 \\ \mathbb{Z}/n & \text{for } i \neq 0 \text{ even} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

For a finite group $G$ and a $G$-module $M$, we introduce the **Tate cohomology groups** $\widehat{H}^*$ defined by:

$$\widehat{H}^i(G, M) := \begin{cases} H^i(G, M) & \text{for } i \geq 1 \\ \text{Coker } N & \text{for } i = 0 \\ \text{Ker } N & \text{for } i = -1 \\ H_{-(i+1)}(G, M) & \text{for } n \leq -2 \end{cases}$$

where $N : M_G \longrightarrow M^G$ is induced by $M \longrightarrow M$, $m \mapsto \sum_{g \in G} gm$.

We will say that a group $G$ has **period cohomology** of period $d$ if we have isomorphisms $\widehat{H}^n(G, M) \simeq \widehat{H}^{n+d}(G, M)$. As we have seen above, if a group $G$ acts on a $G$-complex homeomorphic to an odd sphere, it has periodic cohomology. Conversely, Swan proved in [9] that a group $G$ with cohomology of period $q$ over $\mathbb{Z}$ can act freely and simplicially on a finite dimensional simplicial homotopy $(q-1)$-sphere of dimension $(q-1)$ (in fact, the result holds in a more general setting modulo Serre classes, but we refer to the original paper for the details).

# 3 The Cohomology Ring

Giving extra structure on algebraic invariants helps to refine them and often gives a better understanding of the original object. We present a brief overview of the ring structure on cohomology. More explicitly, for a group $G$ and a commutative ring $R$, we endow

$$H^*(G,R) := \bigoplus_{k\in\mathbb{N}} H^k(G,R)$$

with a multiplication, turning it into a graded ring.

Let $G$ be a group and $R$ a commutative ring that we turn into a $G$-module with trivial action for simplicity. For $P_*$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, we define the **cross-product** on the co-chain level by:

$$\begin{array}{rccc} \times: & \mathrm{Hom}_G(P_*,R) \otimes_{\mathbb{Z}} \mathrm{Hom}_G(P_*,R) & \longrightarrow & \mathrm{Hom}_{G\times G}(P_* \otimes_{\mathbb{Z}} P_*, R \otimes_{\mathbb{Z}} R) \\ & f \otimes f' & \longmapsto & \Big[(x\otimes x') \mapsto (-1)^{|f'||x|} f(x) \otimes f'(x')\Big] \end{array}$$

We can show that $P \otimes P$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G \times G]$ and that the product of two cocycles is a cocycle whose cohomology class depends only on the classes of the given cocycles. Skipping the details, there is an induced **cohomology cross-product**:

$$H^r(G,R) \otimes H^s(G,R) \longrightarrow H^{r+s}(G\times G, R\otimes R).$$

By functoriality, the maps $d : G \longrightarrow G \times G$, $g \mapsto (g,g)$ and $R \otimes R \longrightarrow R$ give rise to a morphism:

$$H^{r+s}(G\times G, R\otimes R) \longrightarrow H^{r+s}(G,R).$$

We finally define the **cup-product** by the composition:

$$- \smile - : H^r(G,R) \otimes H^s(G,R) \longrightarrow H^{r+s}(G\times G, R\otimes R) \longrightarrow H^{r+s}(G,R)$$

which makes $H^*(G,R)$ a graded $R$-algebra.

Pictorially, we read the following diagram to make explicit computations:

$$\begin{tikzcd}
H^r(G,R) \otimes H^s(G,R) \arrow[r] \arrow[d, equal] \arrow[rr, bend left, "- \smile -"] & H^{r+s}(G\times G, R\otimes R) \arrow[r] \arrow[d, equal] & H^{r+s}(G,R) \arrow[d, equal] \\
H^r(\mathrm{Hom}_G(P,R)) \otimes H^s(\mathrm{Hom}_G(P,R)) \arrow[r, "- \times -"'] & H^{r+s}(\mathrm{Hom}_G(P\otimes P, R\otimes R)) \arrow[r, "H^*(\mathrm{Hom}_G(\Delta_*, \cdot))"'] & H^{r+s}(\mathrm{Hom}_G(P,R))
\end{tikzcd}$$

This additional structure is of great help in a number of situations. Stealing from algebraic topology, a standard example would be to distinguish $S^2 \vee S^4$ and $\mathbb{CP}^2$. Furthermore, it is one of the reasons for the multiplicative structure on pages of some spectral sequences (see below) which turns out to be vital in many situations to actually compute the differentials.

The cohomology ring has be studied in various settings and more could be said about it. For brevity, we will just mention that it is Noetherian whenever the coefficient module is a commutative Noetherian ring.

# 4 Spectral Sequences

Computing the (co)homology of a group quickly gets out of hand. Indeed, even for reasonably nice groups such as $S_3$ or $D_8$ one cannot hope to compute their (co)homology from simple manipulations. A general idea in mathematics is to break down objects into simpler parts on which computations are more approachable. The spectral sequences we will consider in this section fit into this approach, informally we will start from know (co)homology groups and bootstrap up to our desired result. Spectral sequences were first introduced by Jean Leray as a way to compute sheaf cohomology but the subject has been developed and is now widely used in various computations. For an encyclopedic reference on a collection of spectral sequences, we advise to check McCleary's book [5].

Let $G$ be a group. As we want to work on something easier than $G$, a natural idea is to pick a normal subgroup $N$ in $G$. This allows us to write down a short exact sequence of groups:

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

Now, because $N$ and $G/N$ are "smaller" than $G$, we can perhaps assume that we know how to compute their (co)homology groups. We can ask if there is a way to compute the (co)homology of $G$ from there. It turns out that their is an algorithmic way that leads to the desired result up to extension problems: in our case, the computational tool is called the **Lyndon-Hochshild-Serre spectral sequence** associated to the short exact sequence.

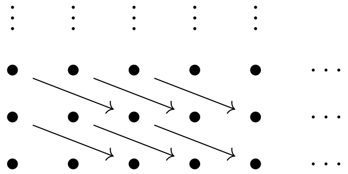
The spectral sequence is an infinite sequence of bigraded abelian groups

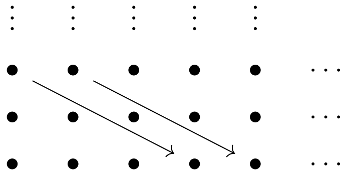
$$E_0, E_1, E_2, \ldots$$

called **pages**. Each page $\{E_r^{p,q}\}_{p,q}$ has morphisms (**differentials**) of bidegree $(r, 1-r)$

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1},$$

and is related to the adjacent ones by $E_{r+1} = H(E_r, d_r)$. We will only consider **first quadrant** spectral sequences, ie. such that $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$. Finally, in the case of homology, the differentials have bidegree $(-r, r-1)$ and we often swap the indexes (we write $E^r_{p,q}$). Pictorially we can represent the pages as follows (we have drawn parts of the $E_2$-page and $E_3$-page of a cohomological spectral sequence):





Let us fix $(p,q)$ for a moment. Because the groups oustide the first quadrant are 0, for some $r$ large enough the differentials at $E_r^{p,q}$ will be 0. Now, $E_{r+1}^{p,q}$ is defined to be the cohomology at $E_r^{p,q}$, ie. it is $E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1} = E_r^{p,q}/0 = E_r^{p,q}$. In the end, for every pair $(p,q)$, there is some $r$ large enough such that:

$$E_r^{p,q} = E_{r+1}^{p,q} = \ldots =: E_\infty^{p,q}.$$

We will say that the spectral sequence $\{E_r^{p,q}\}_{r,p,q}$ **converges** to a graded module $H^\bullet$ if there exists a **filtration**

$$0 = F^* H^* \subset \ldots \subset F^1 H^* \subset F^0 H^* = H^*$$

such that the **associated graded complex** $\{G^p H^{p+q}\}_{p,q} := \{F^p H^{p+q} / F^{p+1} H^{p+q}\}_{p,q}$ of $H$ satisfies

$$E_\infty^{p,q} \simeq G^p H^{p+q} \qquad \forall p, q.$$

We shall write $E_r^{p,q} \Rightarrow H_\bullet$.

There are several ways of constructing the Lyndon-Hochshild-Serre (LHS) spectral sequence. We refer to [10], [4], [1] or [8] for the technical details. Different points of view give different constructions. When looking at the cohomology of $G$ as the derived functors of the invariant functor $-^G$, the LHS spectral sequence is a special case of Grothendieck spectral sequence for the composition of functors (from the fact that $A^G = (A^N)^{G/N}$). When computing cohomology from projective resolutions, it arises from a double complex. We can even recover it from the Serre spectral sequence of a fibration. In the end, the $E_2$-page is the same and is often taken as the starting point. From the definitions above, we describe the LHS spectral sequence with:

$$E_2^{p,q} = H^p(G/N, H^q(N,M)) \Rightarrow H^{p+q}(G,M)$$

Having introduced a general computational tool, we take that opportunity to describe another common point of algebraic topology and group cohomology. We recall that a (Hurewicz) **fibration** is a continuous mapping $p : E \longrightarrow B$ satisfying the homotopy lifting property with respect to any space (or CW complexes for a Serre fibration). The **homotopy lifting poperty** with respect to a space $X$ is described by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\tilde{f}_0"] \arrow[d, "\mathrm{id}\times\{0\}"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B
\end{tikzcd}$$

We define a fiber $F$ by taking a point $b \in B$ and letting $F := p^{-1}(b)$. Another choice of point gives homotopy equivalent fibers so that we can speak of the fiber. We often write a fibration as $F \longrightarrow E \longrightarrow B$. The Serre spectral sequence associated to a Serre fibration is the following:

$$E_2^{p,q} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

As we briefly discussed in the introduction, the cohomology of a group $G$ might be computed as the ordinary cohomology of an Eilenberg-MacLane space $K(G,1)$. Starting from of short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

we would like to construct a fibration of the form

$$K(N,1) \longrightarrow K(G,1) \longrightarrow K(Q,1)$$

where we use $K(-,1)$ as both the general idea and a model of an Eilenberg-MacLane space (which are all homotopy equivalent anyway).

We see that $f : B \longrightarrow C$ induces a map $Bf : K(G,1) \longrightarrow K(Q,1)$. We define the **mapping path space** of $Bf$ to be $E_{Bf} := \{(x,p) \in K(G,1) \times K(Q,1)^I \mid Bf(x) = p(0)\}$. We can prove that $E_{Bf}$ is homotopy equivalent to $K(G,1)$ (by embedding $K(G,1)$ in $E_{Bf}$ with $x \mapsto (x, c_{Bf(x)})$ where $c_{Bf(x)}$ is the constant map at $Bf(x)$ and retracting $E_{Bf}$ by contracting the paths). On top of that, we get a fibration:

$$\begin{tikzcd}
E_{Bf} \arrow[d, "(x,p)\mapsto p(1)"] \\
K(Q,1)
\end{tikzcd}$$

with **homotopy fiber** $F_{Bf}$. That is to say, we have the fibration:

$$F_{Bf} \longrightarrow E_{Bf} \simeq K(G,1) \longrightarrow K(Q,1)$$

In fact, $F_{Bf}$ is a $K(N,1)$. Indeed, the long exact sequence of homotopy reduces to:

$$0 \longrightarrow \pi_1(F_{Bf}) \longrightarrow B \xrightarrow{f} C \longrightarrow \pi_0(F_{bf}) \longrightarrow 0$$

And we see that $\pi_1(F_{Bf}) \simeq N$ and $\pi_n(F_{Bf}) = 0 \; \forall n \neq 1$.

# A A Higher Point of View

Cohomology theories arise in various settings: group cohomology, singular cohomology of CW complexes, de Rham cohomology, etc. It turns out that theses theories can be unified under a general definition of "cohomology". In this section, we try to unwind the general constructions to arrive at a more familiar form.

This appendix is apart from the rest of this report and our approach will be somewhat terse and informal, mainly following the nLab article on cohomology [6].

Lets start with the general definition before applying it a more specific case of our interest. The cohomology theories emerge from $(\infty, 1)$-categories, that is to say higher categories where all $k$-morphisms are reversible for $k > 1$. Let $\mathbf{H}$ be such a category. For two objects $A$ and $X$ of $\mathbf{H}$, the $(\infty, 1)$-categorical hom-space $\mathbf{H}(X, A)$ is a $\infty$-groupoid (where every morphism is reversible). We will be interested in the set of connected components $\pi_0\mathbf{H}(X, A)$, which we can also see as the hom-set in the homotopy category $\mathrm{Ho}_{\mathbf{H}}$ of $\mathbf{H}$ (where we identify the 1-morphisms that are connected by a 2-morphism). The objects $(c : X \longrightarrow A) \in \mathbf{H}(X, A)$ are called cocycles on $X$ with coefficients in $A$, and the morphisms $(\lambda : c_1 \longrightarrow c_2)$ are the coboundaries. Two cocycles connected by a coboundary are said to be cohomologous, and the equivalence classes $[c] \in \pi_0\mathbf{H}(X, A)$ are the cohomology classes. In the end, we define the cohomology set to be:

$$H(X, A) := \pi_0\mathbf{H}(X, A) = \mathrm{Ho}_{\mathbf{H}}(X, A).$$

Before going astray, we stop developing more abstract nonsense here. In fact, we do not need anymore details to glimpse some properties.

For instance, for a sufficiently nice topological space $X$ (eg. a CW complex), we can use the fact that the $n$th cohomology is the set of homotopic functions from $X$ to an Eilenberg-MacLane space. Explicitly, we have:

$$H^n(X, \mathbb{Z}) = [X, K(\mathbb{Z}, n)]$$

which is exactly our definition where $\mathbf{H}$ is the $(\infty, 1)$-category of topological spaces, ie. **Top**. Because the $K(\mathbb{Z}, n)$ can be grouped together to form a ring spectrum, our definition gives a hint towards a ring structure on $H^*(X, \mathbb{Z})$, which we indeed find with the cup product.

The natural $(\infty, 1)$-category for group cohomology is denoted $\infty$-**Grpd**. It is the $(\infty, 1)$-category of $\infty$-groupoids, ie $(\infty, 0)$-categories. We will only study the case of a discrete group $G$ and an abelian discrete group $A$ with trivial $G$-action (to avoid unnecessary difficulties coming from local coefficients). The main bridge between our higher world and our previous constructions by chain complexes is the following isomorphism:

$$H^n_{\mathrm{Grp}}(G, A) \;\simeq\; \pi_0\, \mathbf{sSet}(\overline{W}G,\ \mathrm{DK}(A[n])).$$

We will not prove it here, but rather explain what the involved objects are and how they fit together.

Firstly, we denoted by **sSet** the category of simplicial sets. Simplicial complexes appear naturally in topology for their geometric realizations are simple examples of nice topological spaces: a point, a segment, a triangle, etc. It turns out that the underlying concept is fundamental in homotopy theory. We generalize our visual intuition by defining the simplex category $\Delta$ consisting of objects $[n] := \{0, 1, \ldots, n\}$ and morphisms $[n] \longrightarrow [m]$ that are order preserving functions. A simplicial set is then defined to be a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. In the end, we have the functor category $\mathbf{sSet} := [\Delta^{\mathrm{op}}, \mathbf{Set}]$.

Secondly, we can define (in fact, this is a shortcut which avoids technical details):

$$\overline{W}G = \left( \cdots \longrightarrow G \times G \times G \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; G \times G \rightrightarrows G \longrightarrow * \right)$$

as the standard bar construction.

Thirdly, DK denotes the Dold-Kan correspondence, which asserts that there is an equivalence of categories between abelian simplicial groups and connective (with zeros in negative degrees) chain complexes of abelian groups. We also see $A[n]$ as the chain complex with $A$ concentrated in degree $n$.

We can derive known properties of low-dimensional cohomology groups. For the sake of concreteness, we write down the example of $H^1$. For a morphism $c : \overline{W}G \longrightarrow \mathrm{DK}(A[1])$, we have in degree 1:

$$c_1 : \left( * \xrightarrow{g} * \right) \longmapsto \left( * \xrightarrow{c(g) \in A} * \right)$$

and in degree 2:

$$\begin{tikzcd}
 & & * \arrow[dr, "g_2"] & & & & * \arrow[dr, "c(g_2)"] & \\
c_2 : & * \arrow[ur, "g_1"] \arrow[rr, "g_1 g_2"'] & & * & \longmapsto & * \arrow[ur, "c(g_1)"] \arrow[rr, "c(g_1 g_2)"'] & & *
\end{tikzcd}$$

so that

$$\begin{aligned} H^1(G, A) = \pi_0\, \mathbf{sSet}(\overline{W}G,\ \mathrm{DK}(A[n])) &= \{c : G \longrightarrow A \text{ s.t. } c(g_1)c(g_2) = c(g_1 g_2)\} \\ &= \mathrm{Hom}_{\mathrm{Grp}}(G, A). \end{aligned}$$

# References

[1] Kenneth S. Brown, *Cohomology of groups*, Springer, 1982.

[2] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, 1956.

[3] W. Hurewicz, *Beiträge zur topologie der deformationen. iv. asphärische räume*, Nederl. Akad. Wetensch. Proc. **39** (1936), 215–224.

[4] J.P. May, *A primer on spectral sequences*.

[5] John McCleary, *A user's guide to spectral sequences*, Cambridge University Press, 2001.

[6] nLab, *Cohomology*, January 2016.

[7] ———, *Eilenberg-mac lane spectrum*, July 2016.

[8] Joseph J. Rotman, *An introduction to homological algebra*, Springer, 2009.

[9] Richard G. Swan, *Periodic resolution for finite groups*, Annals of Mathematics **62** (1960), 267–291.

[10] Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.