

Group cohomology and control of p -fusion

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Abstract We show that if an inclusion of finite groups $H \leq G$ of index prime to p induces a homeomorphism of mod p cohomology varieties, or equivalently an F -isomorphism in mod p cohomology, then H controls p -fusion in G , if p is odd. This generalizes classical results of Quillen who proved this when H is a Sylow p -subgroup, and furthermore implies a hitherto difficult result of Mislin about cohomology isomorphisms. For $p = 2$ we give analogous results, at the cost of replacing mod p cohomology with higher chromatic cohomology theories.

The results are consequences of a general algebraic theorem we prove, that says that isomorphisms between p -fusion systems over the same finite p -group are detected on elementary abelian p -groups if p odd and abelian 2-groups of exponent at most 4 if $p = 2$.

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1 Introduction

The variety of the mod p cohomology ring of a finite group was first studied by Quillen in his fundamental 1971 paper [36], and has been a central tool in

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group cohomology since then. The variety describes the mod p group cohomology ring up to F -isomorphism, i.e., a ring homomorphism with nilpotent kernel and where every element in the target raised to a p^k th power lies in the image; see [36, Proposition B.8-9] and Remark 4.1.

Quillen's first application of the theory was to show in [35] that if the Sylow p -subgroup inclusion $S \leq G$ induces an F -isomorphism on mod p cohomology then S controls p -fusion in G , if p is odd, which in this case means that G is p -nilpotent. Quillen's result has subsequently been revisited in a number of contexts [10, 11, 16, 20, 25], however all retaining the hypothesis that S is a Sylow p -subgroup in G .

The main goal of this paper is to considerably strengthen Quillen's result by replacing S by an arbitrary subgroup H of G containing S , thereby moving past p -nilpotent groups to all finite groups. We recall that for $S \leq H \leq G$, H is said to *control p -fusion* in G , if pairs of tuples of elements of S are conjugate in H if they are conjugate in G , or equivalently if for all p -subgroups $P, Q \leq S$, $N_H(P, Q)/C_H(P)$ equals $N_G(P, Q)/C_G(P)$ as homomorphisms from P to Q .

Theorem A (F -isomorphism implies control of p -fusion, p odd) *Let $\iota: H \leq G$ be an inclusion of finite groups of index prime to p , p an odd prime, and consider the induced map on mod p group cohomology $\iota^*: H^*(G; \mathbb{F}_p) \rightarrow H^*(H; \mathbb{F}_p)$. If for each $x \in H^*(H; \mathbb{F}_p)$, $x^{p^k} \in \text{im}(\iota^*)$ for some $k \geq 0$, then H controls p -fusion in G .*

Recall that ι^* is injective by an easy transfer argument [14, Proposition 4.2.5], since $p \nmid |G : H|$. Hence, the condition above, that for each $x \in H^*(H; \mathbb{F}_p)$ there exists $k \geq 0$ with $x^{p^k} \in \text{im}(\iota^*)$, is in fact equivalent to ι^* being an F -isomorphism. Note that by the classical 1956 Cartan–Eilenberg stable elements formula [12, XII.10.1], ι^* is an (actual) isomorphism if H controls p -fusion in G , so the converse also holds.

The assumption in Theorem A that H and G share a common Sylow p -subgroup is necessary, as the inclusion $C_p \rightarrow C_{p^2}$ shows. Likewise the assumption that p is odd is necessary, as Quillen's original example $Q_8 < 2A_4 = Q_8 \rtimes C_3$ shows. Stronger yet, we show in Example 4.2 that for any n there exists an inclusion $H \leq G$ of odd index with different 2-fusion but which induces a mod 2 cohomology isomorphism modulo the class of n -nilpotent unstable modules $\mathcal{N}il_n$ [40, Chap. 6]; F -isomorphism means isomorphism modulo the largest class $\mathcal{N}il_1$.

Our proof of Theorem A is purely algebraic: By [36, Proposition 10.9(ii) \Rightarrow (i)] (or the algebraic reference [1]) F -isomorphism in mod p group cohomology implies control of fusion on elementary abelian subgroups. Thus, Theorem A follows from the following group theoretic statement, which is of independent interest. For p odd it says that if H controls p -fusion in G on

elementary abelian p -subgroups then it in fact controls p -fusion. We formulate and prove the statement in terms of fusion systems, and refer the reader for example to [2] for definitions and information about these—we also recap the essential definitions in Sect. 2.

Theorem B (Small exponent abelian p -subgroups control p -fusion) *Let $\mathcal{G} \leq \mathcal{F}$ be two saturated fusion systems on the same finite p -group S . Suppose that $\text{Hom}_{\mathcal{G}}(A, B) = \text{Hom}_{\mathcal{F}}(A, B)$ for all $A, B \leq S$ with A, B elementary abelian if p is odd, and abelian of exponent at most 4 if $p = 2$. Then $\mathcal{G} = \mathcal{F}$.*

Our proof of this theorem is rather short. In outline we use Alperin's Fusion Theorem to reduce to a situation where we can apply results of J.G. Thompson on p' -automorphisms of p -groups [17, Chap. 5.3]. Consequently, our proof of Theorem A is also relatively elementary. In particular, at odd primes, we obtain a comparatively simple algebraic proof of Mislin's Theorem. This theorem states that, for a homomorphism $\varphi: H \rightarrow G$ of finite groups, which induces an isomorphism in mod p group cohomology, $|\ker(\varphi)|$ and $|G : \varphi(H)|$ are coprime to p and $\varphi(H)$ controls p -fusion in G . Here the first part is a 1978 theorem of Jackowski [26, Theorem 1.3]. (Jackowski gave a topological argument, but a short algebraic proof exists via Tate cohomology; see [4, Theorem 5.16.1] with \mathbb{Z} replaced by $\mathbb{Z}_{(p)}$.) So the proof of Mislin's Theorem reduces quickly to the situation that φ is an inclusion of finite groups of index prime to p , where the statement follows from Theorem A if p is odd. Mislin's original proof of his theorem uses the Dwyer–Zabrodsky theorem [13] in algebraic topology, whose proof again relies on Lannes' theory [27], extending Miller's proof of the Sullivan conjecture [30]. In the early 1990s, for example at the 1994 Banff conference on representation theory, Alperin made the highly publicized challenge to find a purely algebraic proof of Mislin's theorem, and this was pursued by many authors. Symonds [41], following an idea of Robinson [39, §7], provided an algebraic reduction of the problem to a statement about cohomology of trivial source modules, which he then proved topologically. Algebraic proofs were finally completed independently by Hida [21] and Okuyama [33], who gave algebraic proofs of Symonds' statement, through quite delicate arguments in modular representation theory. (See also e.g., [1] and [42].)

We now come to a further application of Theorem B. As remarked above, the assumptions in Theorem A that p is odd and $|G : H|$ is prime to p are both in fact necessary. Switching from mod p cohomology to generalized cohomology theories, we can however combine the methods of Theorem B with Hopkins–Kuhn–Ravenel (HKR) generalized character theory [22, 23] to obtain a statement that holds for all primes, and that also avoids the assumption that H and G share a common Sylow p -subgroup.

Theorem C (Chromatic group cohomology isomorphism implies control of p -fusion) *Let $\varphi: H \rightarrow G$ be a homomorphism of finite groups, and let $E(n)$ denote height n Morava E -theory at a fixed prime p . Suppose that φ induces an isomorphism*

$$\varphi^*: E(n)^*(BG) \left[\frac{1}{p} \right] \xrightarrow{\sim} E(n)^*(BH) \left[\frac{1}{p} \right]$$

for some $n \geq \mathrm{rk}_p(G)$. Then $|\ker(\varphi)|$ and $|G : \varphi(H)|$ are prime to p , and $\varphi(H)$ controls p -fusion in G .

In fact our proof works not just for $E(n)$ but for any height n cohomology theory satisfying the assumptions listed in [23, Theorem C]. We recall that for height n Morava E -theory, $E(n)^*(pt) = W(\mathbb{F}_{p^n})[[w_1, \dots, w_{n-1}]]\langle u, u^{-1} \rangle$, with $W(\mathbb{F}_{p^n})$ the Witt vectors over \mathbb{F}_{p^n} , i.e., the unramified extension of degree n of the p -adic integers, $|w_i| = 0$ and $|u| = -2$. As usual, the notation $[\frac{1}{p}]$ means that we invert p after taking cohomology, producing a \mathbb{Q}_p -algebra.

The converse to Theorem C is clear, e.g., by the standard Cartan–Eilenberg stable elements formula and the fact that a mod p cohomology isomorphism of spaces induces an $E(n)^*$ -isomorphism. Theorem C also provides a new proof of a strong form of Mislin’s theorem, assuming only isomorphism in large degrees. This proof is valid at all primes, but replaces the reliance on Quillen’s variety theory by the (currently) less algebraic HKR character theory—the proof mirrors that of Atiyah’s 1961 p -nilpotence criterion [3, 35], replacing K -theory by higher chromatic $E(n)$ -theories; see Remark 4.3.

A p -rank restriction in Theorem C is indeed necessary: $\mathbb{F}_{p^2} \rtimes \mathbb{F}_{p^2}^\times < (\mathbb{F}_{p^2} \rtimes \mathbb{F}_{p^2}^\times) \rtimes \mathrm{Aut}(\mathbb{F}_{p^2})$ is an example of an inclusion of groups of index prime to p , for p odd, which is an $E(1)^*[\frac{1}{p}]$ -equivalence, by HKR character theory (3.2), but with different p -fusion; the same example with \mathbb{F}_{p^2} replaced by \mathbb{F}_{2^3} works for $p = 2$. We speculate that the bound $n \geq \mathrm{rk}_p(G)$ we give may be close to optimal, but we do not know an example to this effect.

Finally, we remark that isomorphism on $E(n)^*$ is equivalent to isomorphism on n th Morava K -theory $K(n)^*$, whereas an $E(n)^*[\frac{1}{p}]$ -isomorphism is a priori significantly weaker—see Remark 4.4 for a variety interpretation of Theorem C and Remark 4.5 for the connection to other stable homotopy theory results.

To prove Theorem C we need the following variant of Theorem B, where we drop the assumption of a common Sylow p -subgroup, but on the other hand assume the same fusion on all abelian p -subgroups—it again appears to be new, even in special cases.

Theorem D (Abelian p -subgroups control fusion) *Assume that a finite group homomorphism $\varphi: H \rightarrow G$ induces a bijection*

$$\mathrm{Rep}(A, H) \xrightarrow{\sim} \mathrm{Rep}(A, G)$$

for all finite abelian p -groups A with $\mathrm{rk}_p(A) \leq \mathrm{rk}_p(G)$. Then $|\ker(\varphi)|$ and $|G : \varphi(H)|$ are prime to p , and $\varphi(H)$ controls p -fusion in G . More generally, suppose that \mathcal{F} and \mathcal{G} are saturated fusion systems on finite p -groups S and T respectively, and that $\varphi: T \rightarrow S$ is a fusion preserving homomorphism inducing a bijection $\mathrm{Rep}(A, \mathcal{G}) \xrightarrow{\sim} \mathrm{Rep}(A, \mathcal{F})$ for any finite abelian p -group A with $\mathrm{rk}_p(A) \leq \mathrm{rk}_p(S)$. Then φ induces an isomorphism from T to S and \mathcal{G} to \mathcal{F} .

Here $\mathrm{Rep}(A, G)$ denotes the quotient of $\mathrm{Hom}(A, G)$ where we identify φ with $c_g \circ \varphi$ for all $g \in G$, and likewise $\mathrm{Rep}(A, \mathcal{F})$ is the quotient of $\mathrm{Hom}(A, S)$, identifying two morphisms if they differ by a morphism in \mathcal{F} ; we spell out what the assumptions of the theorem mean in Lemma 2.6.

Finally, we remark that Theorems A and C can be formulated in terms of the fusions systems of the groups, and they should hold for abstract fusion systems as well. Indeed, as is clear from our proofs, the only missing piece is a reference for the Quillen stratification and the HKR character theorem in that context—we will however not pursue this here.

2 p' -automorphisms of p -groups and proofs of Theorems B and D

The goal of this section is to prove Theorems B and D by group theoretic methods, combining manipulations with fusion systems with results of J.G. Thompson on p' -automorphisms of p -groups, which can by now be found in textbooks.

Thompson's *critical subgroup theorem* [15, Lemma 2.8.2] (see also the textbook reference [17, Theorem 5.3.11]) says that for any finite p -group P there exists a characteristic subgroup C of P such that $C/Z(C)$ is elementary abelian, $[P, C] \leq Z(C)$, $C_P(C) = Z(C)$, and every nontrivial p' -automorphism of P restricts to a non-trivial p' -automorphism of C . Our main classical group theoretic tool in this paper is a variant of that theorem, where instead of a critical subgroup we use a certain characteristic subgroup of P of small exponent and consider its maximal abelian subgroups.

Theorem 2.1 (Small exponent abelian subgroups detect p' -automorphisms) *Let P be a finite p -group. There exists a characteristic subgroup D of P , of exponent p if p is odd and exponent 4 if $p = 2$, such that $[D, P] \leq Z(D)$, and such that every non-trivial p' -automorphism of P restricts to a non-trivial*

automorphism of D . Furthermore, for any such D and any maximal (with respect to inclusion) abelian subgroup A of D we have $A \trianglelefteq P$ and $C_{\text{Aut}(P)}(A)$ is a p -group.

Note that the example of the extra-special group p_+^{1+2} shows that an abelian characteristic subgroup that detects p' -automorphisms need not exist.

Proof of Theorem 2.1 Taking $D = \Omega_1(C)$, the subgroup generated by elements of order p of a critical subgroup C , produces such a subgroup D as in the theorem, for p odd, as proved in [17, Theorem 5.3.13]. For $p = 2$ the claim holds for $D = \Omega_2(C)$, the subgroup of C generated by elements of order at most 2^2 ; we establish this fact in Lemma 2.2 below.

For the last part, let A be a maximal abelian subgroup of D with respect to inclusion. Since $[A, P] \leq Z(D) \leq A$ it follows that $A \trianglelefteq P$. Furthermore, if $B \leq C_{\text{Aut}(P)}(A)$ is a p' -group, then $A \times B$ acts on P and thus on D . Since A is maximal abelian it follows that $C_D(A) = A$, and in particular B acts trivially on $C_D(A)$; Thompson's $A \times B$ -lemma [17, Theorem 5.3.4] now says that $[D, B] = 1$ and so $B = 1$, and we conclude that $C_{\text{Aut}(P)}(A)$ is a p -group as wanted. \square

We now provide a proof of the postponed lemma for $p = 2$.

Lemma 2.2 *Let P be a 2-group such that $P/Z(P)$ is elementary abelian. Then for all $x, y \in P$, $(xy)^4 = x^4y^4$ and in particular $\Omega_2(P)$ is of exponent at most 4. Furthermore if B is a p' -group of automorphisms of P with $[\Omega_2(P), B] = 1$, then $B = 1$.*

Proof Note that $(xy)^2 = x^2(x^{-1}yxy^{-1})y^2$ with all three factors in $Z(P)$, so

$$\begin{aligned} (xy)^4 &= x^4y^4(x^{-1}yxy^{-1})^2 = x^4y^4(x^{-1}(x^{-1}yxy^{-1})yxy^{-1}) \\ &= x^4y^4(x^{-2}yx^2y^{-1}) = x^4y^4. \end{aligned}$$

For the last statement about p' -automorphisms we follow [17, Theorem 5.3.10]. Let P be a minimal counterexample. If Q is a proper B -invariant subgroup of P then $Q/(Q \cap Z(P))$ is elementary abelian and $Q \cap Z(P) \leq Z(Q)$, so $Q/Z(Q)$ is elementary abelian. Moreover, $\Omega_2(Q) \leq \Omega_2(P)$ and thus $[\Omega_2(Q), B] = 1$. So, as P is a minimal counterexample, $[Q, B] = 1$. By [17, Theorem 5.2.4], P is non-abelian. So in particular, $Z(P)$ is a proper characteristic subgroup of P and thus $[Z(P), B] = 1$. We now show that $[P, B] \leq \Omega_2(P)$: Suppose $x \in P$ and $b \in B$, and note that $x^4 \in Z(P)$, as $P/Z(P)$ is elementary abelian, and thus $(x^4)^b = x^4$, as $[Z(P), B] = 1$. Hence $[x, b]^4 = (x^{-1}x^b)^4 = x^{-4}(x^4)^b = x^{-4}x^4 = 1$ as wanted, where we also used

the first part of the lemma. By assumption $[\Omega_2(P), B] = 1$, so in particular $[[P, B], B] = 1$ by the above, and we conclude that $[P, B] = 1$, by [17, Theorem 5.3.6]. \square

In the case where \mathcal{F} is the fusion system of $G = S \rtimes K$, with $p \nmid |K|$, Theorem B follows directly from Theorem 2.1, as the action of elements of K on S is detected by small abelian subgroups of S , but the proof of the general statement requires more work, and here fusion systems enter in a more prominent way. The arguments can be translated into the special case of ordinary finite groups, but doing so provides no essential simplifications, and indeed, from our perspective, the arguments are considerably shorter and more transparent in the setup of fusion systems.

Recall that a *saturated fusion system* \mathcal{F} on a finite p -group S [9, Definition 1.2], [2, Proposition I.2.5] is a category whose objects are the subgroups of S , and morphisms are group monomorphisms satisfying axioms which mimic those satisfied by morphisms induced by conjugation in some ambient group G . More precisely, conjugation by elements in S need to be in the category, every map needs to factor as an isomorphism followed by an inclusion, and furthermore two non-trivial conditions need to be satisfied, called the Sylow and extension axiom, which we recall below together with some terminology. We refer to [2] and [9] for detailed information, and also direct the reader to Puig's original work [34], where terminology however differs. A subgroup $Q \leq S$ is called *fully \mathcal{F} -normalized* if $|N_S(Q)|$ is maximal among \mathcal{F} -conjugates of Q , it is called *fully \mathcal{F} -centralized* if the corresponding property holds for the centralizer, and it is called *\mathcal{F} -centric* if $C_S(Q') = Z(Q')$ for all \mathcal{F} -conjugates Q' of Q . The *Sylow axiom* says that if Q is fully \mathcal{F} -normalized then it is fully \mathcal{F} -centralized and $\text{Aut}_S(Q)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(Q)$. (Here $\text{Aut}_S(Q)$ means the automorphisms of Q induced by elements in S .) The *extension axiom* says that any morphism $\varphi: Q \rightarrow S$ with $\varphi(Q)$ fully \mathcal{F} -centralized extends to

$$N_\varphi = \{g \in N_S(Q) \mid {}^\varphi(c_g|_Q) \in \text{Aut}_S(\varphi(Q))\}.$$

The first tool we need is the following variant of the extension axiom.

Lemma 2.3 *Fix a saturated fusion system \mathcal{F} on S and let $\varphi: P \rightarrow S$ be any monomorphism (not necessarily in \mathcal{F}). For $Q \trianglelefteq P$ and $\psi = \varphi|_Q$ the following hold.*

1. $N_\psi \geq P$ and ${}^\psi \text{Aut}_P(Q) = \text{Aut}_{\varphi(P)}(\varphi(Q))$.
2. If $\psi \in \mathcal{F}$ and $\varphi(Q)$ is fully \mathcal{F} -centralized then ψ extends to $\hat{\psi} \in \text{Hom}_{\mathcal{F}}(P, \varphi(P)C_S(\varphi(Q)))$.

Proof For (1) we calculate, for any $g \in P$ and $x \in \varphi(Q)$,

$$(\psi c_g)(x) = \psi \circ c_g \circ \psi^{-1}(x) = \psi(g\psi^{-1}(x)g^{-1}) = \varphi(g)x\varphi(g^{-1}) = c_{\varphi(g)}(x),$$

from which it is clear that $N_\psi \geq P$ and ${}^\psi \text{Aut}_P(Q) = \text{Aut}_{\varphi(P)}(\varphi(Q))$.

For (2) note that the extension axioms imply that ψ extends to $\hat{\psi} \in \text{Hom}_{\mathcal{F}}(P, S)$. And, since $\text{Aut}_{\varphi(P)}(\varphi(Q)) = {}^\psi \text{Aut}_P(Q) = \text{Aut}_{\hat{\psi}(P)}(\varphi(Q))$, where the last equality is by applying (1) with $\hat{\psi}$ in place of φ , we conclude that $\hat{\psi}(P) \leq \varphi(P)C_S(\varphi(Q))$ as wanted. \square

For the purpose of the next proof, recall that a proper subgroup H of a finite group G is called *strongly p -embedded* if p divides the order of H and, for all $g \in G \setminus H$, $H \cap {}^g H$ has order prime to p . Provided p divides $|G|$, one easily shows that H is strongly p -embedded in G if and only if H contains a Sylow p -subgroup S of G such that $N_G(R) \leq H$ for every $1 \neq R \leq S$ (see for example [18, Lemma 17.10] or [37, Proposition 5.2]); in particular an overgroup of a strongly p -embedded subgroup is again strongly p -embedded, if it is a proper subgroup. (Groups with strongly embedded subgroups play a central role in many aspects of local group theory, and in particular they show up in connection with Alperin's fusion theorem [2, Theorem I.3.6], though we shall only indirectly need them in that capacity here.)

We now give the key step in deducing Theorem B from Theorem 2.1, providing a way to show that the fusion in \mathcal{F} and \mathcal{G} agree on all subgroups P by downward induction starting with S .

Main Lemma 2.4 *Let $\mathcal{G} \leq \mathcal{F}$ be two saturated fusion systems on the same finite p -group S , and $P \leq S$ an \mathcal{F} -centric and fully \mathcal{F} -normalized subgroup, with $\text{Aut}_{\mathcal{F}}(R) = \text{Aut}_{\mathcal{G}}(R)$ for every $P < R \leq N_S(P)$. Suppose that there exists a subgroup $Q \trianglelefteq P$ with $\text{Hom}_{\mathcal{F}}(Q, S) = \text{Hom}_{\mathcal{G}}(Q, S)$. Then $\text{Aut}_{\mathcal{F}}(P) = \langle \text{Aut}_{\mathcal{G}}(P), C_{\text{Aut}_{\mathcal{F}}(P)}(Q) \rangle$.*

Proof To ease the notation set $G = \text{Aut}_{\mathcal{F}}(P)$, $H = \text{Aut}_{\mathcal{G}}(P)$, and $\overline{G} = G/\text{Inn}(P)$, and denote by \overline{U} the image in \overline{G} of any subgroup $U \leq G$. We want to show that $G = \langle H, C_G(Q) \rangle$.

Step 1: We first assume in addition that

$$C_S(\xi(Q)) \leq P \quad \text{for all } \xi \in H \quad (*)$$

and show that $G = HC_G(Q)$. Let $\gamma \in G$ be arbitrary; set $\psi = (\gamma^{-1})|_{\gamma(Q)} \in \text{Hom}_{\mathcal{F}}(\gamma(Q), Q)$. Then $\psi \in \text{Hom}_{\mathcal{G}}(\gamma(Q), Q)$ by assumption. We claim that Q is fully centralized in \mathcal{G} , and postpone the proof to Lemma 2.5 below, since it is a general statement. Granted this, Lemma 2.3(2), applied to γ^{-1} and \mathcal{G} in the roles of φ and \mathcal{F} , implies that we can extend $\psi: \gamma(Q) \rightarrow Q$ to

$\hat{\psi}: P \rightarrow PC_S(Q)$ in \mathcal{G} , and, as $C_S(Q) \leq P$ by assumption $(*)$, we conclude that $\hat{\psi} \in H$. Since $\gamma \circ \hat{\psi} \in C_G(Q)$, we have $\gamma \in C_G(Q)H$, and, as γ was arbitrary, this yields $G = HC_G(Q)$ as required.

Step 2: If $P = S$ assumption $(*)$ is automatically satisfied and the lemma follows from Step 1; likewise we are done if $H = G$. In this step we show that if $P < S$ and $H < G$ then \bar{H} is strongly p -embedded in \bar{G} . Consider $P < R \leq N_S(P)$. For $\varphi \in N_G(\text{Aut}_R(P))$ it follows from the extension axiom that φ extends to $\hat{\varphi} \in \text{Hom}_{\mathcal{F}}(R, S)$, since P is fully \mathcal{F} -normalized and $R \leq N_\varphi$. Furthermore, by Lemma 2.3(1), $\text{Aut}_R(P) = {}^\varphi \text{Aut}_R(P) = \text{Aut}_{\hat{\varphi}(R)}(P)$, so since $C_S(P) \leq P$ by \mathcal{F} -centricity of P , we have $\hat{\varphi}(R) = R$. It then follows from our hypothesis that $\hat{\varphi} \in \text{Aut}_{\mathcal{G}}(R)$ and thus $\varphi \in H$. We conclude that \bar{H} is strongly p -embedded in \bar{G} .

Step 3: Finally set $H_0 = \langle H, C_G(Q) \rangle$, and suppose for contradiction that there exists $\chi \in G \setminus H_0$. Then by Step 2, \bar{H}_0 is a strongly p -embedded subgroup of \bar{G} , so in particular $\text{Aut}_S(P) \cap {}^\chi H_0 = \text{Inn}(P)$ as $\text{Aut}_S(P) \leq H_0$. Note that $C_G(\chi(Q)) = {}^\chi C_G(Q) \leq {}^\chi H_0$, so $C_{\text{Aut}_S(P)}(\chi(Q)) \leq \text{Aut}_S(P) \cap {}^\chi H_0 = \text{Inn}(P)$. Hence $N_S(P) \cap C_S(\chi(Q)) \leq P$, using that P is centric. Now, as $C_S(\chi(Q))P$ is a p -group, $C_S(\chi(Q)) \leq P$ (see [17, Theorem 2.3.4] for this elementary property of finite p -groups). Note that $\xi \circ \chi \in G \setminus H_0$ for any $\xi \in H$; so as χ was arbitrary the argument actually shows that $C_S(\xi(\chi(Q))) \leq P$ for any $\xi \in H$. But now $(*)$ holds with $\chi(Q)$ in place of Q . Observe also that $\text{Hom}_{\mathcal{F}}(\chi(Q), S) = \text{Hom}_{\mathcal{G}}(\chi(Q), S)$. For if $\varphi \in \text{Hom}_{\mathcal{F}}(\chi(Q), S)$ then $\varphi \circ \chi$ and χ^{-1} are in $\text{Hom}_{\mathcal{F}}(Q, S) = \text{Hom}_{\mathcal{G}}(Q, S)$, so $\varphi = (\varphi \circ \chi) \circ \chi^{-1} \in \text{Hom}_{\mathcal{G}}(\chi(Q), S)$. Therefore Step 1 gives that $G = HC_G(\chi(Q))$. As $C_G(\chi(Q))$ is conjugate to $C_G(Q)$ in G , it follows that $C_G(\chi(Q))$ is conjugate to $C_G(Q)$ by an element of H and thus $G = HC_G(Q)$. This is a contradiction, and we conclude that $G = H_0$ as wanted. \square

We next prove the postponed lemma.

Lemma 2.5 *Let \mathcal{F} be a saturated fusion system on S and suppose that $Q \trianglelefteq P \leq S$, with P fully \mathcal{F} -normalized, and $C_S(\xi(Q)) \leq P$ for all $\xi \in \text{Aut}_{\mathcal{F}}(P)$. Then Q is fully \mathcal{F} -centralized.*

Proof By [28, Lemma 2.6] we may choose $\alpha: N_S(Q) \rightarrow S$ in \mathcal{F} such that $\alpha(Q)$ is fully normalized. Furthermore, as P is fully normalized, again by [28, Lemma 2.6], there is $\beta \in \text{Hom}_{\mathcal{F}}(N_S(\alpha(P)), N_S(P))$ such that $\beta(\alpha(P)) = P$. Then

$$\beta(C_S(\alpha(Q)) \cap N_S(\alpha(P))) \leq C_S(\beta(\alpha(Q))) \leq P = \beta(\alpha(P))$$

where the second inclusion follows by assumption as $\beta \circ \alpha$ restricts to an element of $\text{Aut}_{\mathcal{F}}(P)$. This yields $C_S(\alpha(Q)) \cap N_S(\alpha(P)) \leq \alpha(P)$, so

$$N_{C_S(\alpha(Q))\alpha(P)}(\alpha(P)) = (C_S(\alpha(Q)) \cap N_S(\alpha(P)))\alpha(P) = \alpha(P).$$

Thus, as $C_S(\alpha(Q))\alpha(P)$ is a p -group, it follows from [17, Theorem 2.3.3(iii) and Theorem 2.3.4] that $C_S(\alpha(Q)) \leq \alpha(P)$. Hence, $C_S(\alpha(Q)) = C_{\alpha(P)}(\alpha(Q)) = \alpha(C_P(Q)) = \alpha(C_S(Q))$ where the last equality holds since our assumption gives $C_S(Q) \leq P$. It follows $|C_S(\alpha(Q))| = |\alpha(C_S(Q))| = |C_S(Q)|$; so Q is fully \mathcal{F} -centralized as $\alpha(Q)$ is fully \mathcal{F} -centralized. \square

Proof of Theorem B By Alperin's fusion theorem, \mathcal{F} is generated by \mathcal{F} -automorphisms of fully \mathcal{F} -normalized and \mathcal{F} -centric subgroups; see [2, Theorem I.3.6] (in fact we only need " \mathcal{F} -essential" subgroups and S). We want to show that $\text{Aut}_{\mathcal{G}}(P) = \text{Aut}_{\mathcal{F}}(P)$ for all $P \leq S$; by downward induction on the order we can assume that $\text{Aut}_{\mathcal{G}}(R) = \text{Aut}_{\mathcal{F}}(R)$ for all subgroups $R \leq S$ with $|R| > |P|$, and by the fusion theorem we can furthermore assume that P is \mathcal{F} -centric and fully \mathcal{F} -normalized. Now choose a characteristic subgroup D of P as described in Theorem 2.1, and a maximal abelian subgroup A of D , and recall that the theorem tells us that $A \trianglelefteq P$ and that $C_{\text{Aut}_{\mathcal{F}}(P)}(A)$ is a p -group. As P is fully \mathcal{F} -normalized, $\text{Aut}_S(P)$ is a Sylow p -subgroup of $\text{Aut}_{\mathcal{F}}(P)$, so if we replace A by a conjugate of A under $\text{Aut}_{\mathcal{F}}(P)$, we can arrange that $C_{\text{Aut}_{\mathcal{F}}(P)}(A) \leq \text{Aut}_S(P) \leq \text{Aut}_{\mathcal{G}}(P)$. But A also satisfies the assumptions on Q in Lemma 2.4, so $\text{Aut}_{\mathcal{F}}(P) = \langle \text{Aut}_{\mathcal{G}}(P), C_{\text{Aut}_{\mathcal{F}}(P)}(A) \rangle$, and we conclude that $\text{Aut}_{\mathcal{G}}(P) = \text{Aut}_{\mathcal{F}}(P)$ as wanted. \square

We now head towards a proof of Theorem D. Recall that for Q a group and \mathcal{F} a fusion system on S we define $\text{Rep}(Q, \mathcal{F}) = \text{Hom}(Q, S)/\mathcal{F}$ as the quotient of $\text{Hom}(Q, S)$ under \mathcal{F} -conjugation, i.e., where we identify $\varphi \in \text{Hom}(Q, S)$ with $\alpha \circ \varphi$, for all $\alpha \in \text{Hom}_{\mathcal{F}}(\varphi(Q), S)$. The proof of Theorem D reduces quickly to the case that \mathcal{G} is a subsystem of \mathcal{F} . We first make explicit what the assumption in Theorem D then means, and state this as a lemma.

Lemma 2.6 *Let \mathcal{F} be a fusion system on a finite p -group S and let \mathcal{G} be a sub-fusion system of \mathcal{F} on $T \leq S$. Suppose Q is a finite p -group (not necessarily a subgroup of S). The induced map $\text{Rep}(Q, \mathcal{G}) \rightarrow \text{Rep}(Q, \mathcal{F})$ is surjective if and only if every epimorphic image of Q in S is \mathcal{F} -conjugate to a subgroup of T . It is injective, if and only if \mathcal{G} controls fusion on the epimorphic images of Q in T , i.e., for any epimorphic image $Q' \leq T$ of Q we have $\text{Hom}_{\mathcal{F}}(Q', T) = \text{Hom}_{\mathcal{G}}(Q', T)$.*

The next lemma, together with Theorem B, will easily imply Theorem D.

Lemma 2.7 *Let \mathcal{F} be a saturated fusion system on a finite p -group S and let \mathcal{G} be a saturated subsystem of \mathcal{F} on $T \leq S$. Suppose that there exists an \mathcal{F} -centric subgroup $Q \leq T$ with $\text{Hom}_{\mathcal{F}}(Q, T) = \text{Hom}_{\mathcal{G}}(Q, T)$. Then $\text{Aut}_{\mathcal{F}}(T) = \text{Aut}_{\mathcal{G}}(T)$ and $T = S$.*

Proof As T is a finite p -group, there is a finite chain

$$Q = T_0 \triangleleft T_1 \triangleleft \cdots \triangleleft T_n = T$$

with $T_{i+1} = N_T(T_i)$ for $0 \leq i < n$. Note that, as Q is \mathcal{F} -centric, every T_i is \mathcal{F} -centric and thus also \mathcal{G} -centric. We want to show that $\text{Aut}_{\mathcal{F}}(T) = \text{Aut}_{\mathcal{G}}(T)$ by proving that

$$\text{Hom}_{\mathcal{F}}(T_i, T) = \text{Hom}_{\mathcal{G}}(T_i, T), \quad \text{for all } 0 \leq i \leq n, \quad (**)$$

by induction on i . For $i = 0$ the claim is true by assumption. Let now $0 \leq i < n$ such that $\text{Hom}_{\mathcal{F}}(T_i, T) = \text{Hom}_{\mathcal{G}}(T_i, T)$. Let $\gamma \in \text{Hom}_{\mathcal{F}}(T_{i+1}, T)$. Then $\psi = \gamma|_{T_i} \in \text{Hom}_{\mathcal{F}}(T_i, T) = \text{Hom}_{\mathcal{G}}(T_i, T)$. As T_i is \mathcal{G} -centric, $\gamma(T_i)$ is fully \mathcal{G} -centralized and $C_T(\gamma(T_i)) \leq \gamma(T_i)$; so by Lemma 2.3(2), applied to γ and \mathcal{G} in the roles of φ and \mathcal{F} , ψ extends to $\hat{\psi} \in \text{Hom}_{\mathcal{G}}(T_{i+1}, \gamma(T_{i+1}))$. Then $\hat{\psi}^{-1} \circ \gamma \in C_{\text{Aut}_{\mathcal{F}}(T_{i+1})}(T_i)$ and, by [9, Proposition A.8] or [2, Lemma I.5.6], $C_{\text{Aut}_{\mathcal{F}}(T_{i+1})}(T_i) = \text{Aut}_{Z(T_i)}(T_{i+1})$. We conclude that $\hat{\psi}^{-1} \circ \gamma \in \text{Aut}_{Z(T_i)}(T_{i+1}) \leq \text{Aut}_{\mathcal{G}}(T_{i+1})$ and thus $\gamma \in \text{Hom}_{\mathcal{G}}(T_{i+1}, T)$, i.e., $(**)$ holds. So

$$\text{Aut}_{\mathcal{G}}(T) = \text{Hom}_{\mathcal{G}}(T, T) = \text{Hom}_{\mathcal{F}}(T, T) = \text{Aut}_{\mathcal{F}}(T).$$

If $\text{Aut}_{\mathcal{F}}(T) = \text{Aut}_{\mathcal{G}}(T)$ then, in particular, $\text{Aut}_{\mathcal{F}}(T)/\text{Inn}(T)$ has order prime to p , by the Sylow axiom for \mathcal{G} , and so $\text{Aut}_S(T) = \text{Inn}(T)$. Since $Q \leq T$ is \mathcal{F} -centric, this implies that $N_S(T) = T$, and thus $S = T$. \square

Proof of Theorem D We only prove the claim about fusion systems, as the claim about groups is a special case. First, it is obvious that $T \rightarrow S$ has to be a monomorphism, since if an element is conjugate to the trivial element, it is trivial. Hence, we may consider \mathcal{G} as a subsystem of \mathcal{F} . Choose a subgroup $A \leq T$ such that A is of maximal order among the abelian subgroups of T . The assumptions, together with Lemma 2.6, imply that every abelian subgroup of S is \mathcal{F} -conjugate to a subgroup of T , so A is of maximal order among the abelian subgroups of S , and hence \mathcal{F} -centric. Moreover, again by Lemma 2.6, $\text{Hom}_{\mathcal{G}}(A, T) = \text{Hom}_{\mathcal{F}}(A, T)$. Lemma 2.7 now shows that $T = S$. This reduces us to a special case of the setup of Theorem B, and the result follows. \square

3 Proofs of Theorems A and C

Proof of Theorem A By Theorem B we just need to verify that an F -isomorphism on cohomology rings implies that H controls fusion in G on elementary abelian p -groups. However this is the statement of [36, Proposition 10.9(ii) \Rightarrow (i)] (see also [1]). \square

Before proving Theorem C we state a lemma explaining the condition on n , whose proof is elementary and seems best left to the reader. Below \mathbb{Z}_p denotes the p -adic integers.

Lemma 3.1 *For a homomorphism $\varphi: H \rightarrow G$ of finite groups, and a fixed natural number n , $\text{Rep}(\mathbb{Z}_p^n, H) \xrightarrow{\sim} \text{Rep}(\mathbb{Z}_p^n, G)$ if and only if $\text{Rep}(A, H) \xrightarrow{\sim} \text{Rep}(A, G)$ for all finite abelian p -groups A with $\text{rk}_p(A) = n$. Furthermore, isomorphism for a fixed positive $n \geq \min\{\text{rk}_p(G), \text{rk}_p(H) + 1\}$ implies $\text{rk}_p(G) = \text{rk}_p(H)$ and isomorphism for all n .*

In further preparation for the proof of Theorem C, we briefly recall the HKR character theorem [23, Theorem C]: For any multiplicative cohomology theory E and finite group G , taking E^* -cohomology induces a map

$$\text{Rep}(\mathbb{Z}_p^n, G) \rightarrow \text{Hom}_{E^*\text{-alg}}(E^*(BG), E_{\text{cont}}^*(B\mathbb{Z}_p^n))$$

with $E_{\text{cont}}^*(B\mathbb{Z}_p^n) = \text{colim}_r E^*(B(\mathbb{Z}/p^r)^n)$, since any $\alpha: \mathbb{Z}_p^n \rightarrow G$ factors canonically through $(\mathbb{Z}/p^r)^n$ for r large. By adjunction we can view this as an E^* -algebra homomorphism

$$E^*(BG) \rightarrow \prod_{\text{Rep}(\mathbb{Z}_p^n, G)} E_{\text{cont}}^*(B\mathbb{Z}_p^n) \quad (3.1)$$

where the right-hand side is $E_{\text{cont}}^*(B\mathbb{Z}_p^n)$ -valued functions on the finite set $\text{Rep}(\mathbb{Z}_p^n, G)$, with point-wise multiplication.

The map (3.1) is the n -character map and the HKR character theorem [23, Theorem C] says that, for certain E , this becomes an isomorphism after suitable localization. More precisely, assume that E is Morava E -theory $E(n)$, so $E^*(BS^1) \cong E^*[[x]]$, $|x| = 2$, and define $L(E^*)$ to be the ring of fractions of $E_{\text{cont}}^*(B\mathbb{Z}_p^n)$ obtained by inverting $\alpha^*(x)$ for all non-zero $\alpha \in \text{Hom}_{\text{cont}}(\mathbb{Z}_p^n, S^1) \cong (\mathbb{Z}/p^\infty)^n$. Then, by [23, Theorem C], $L(E^*)$ is faithfully flat over $E(n)^*[\frac{1}{p}]$ (and in particular non-zero) and (3.1) induces an isomorphism

$$L(E^*) \otimes_{E(n)^*[\frac{1}{p}]} E^*(BG) \left[\frac{1}{p} \right] \xrightarrow{\sim} \prod_{\text{Rep}(\mathbb{Z}_p^n, G)} L(E^*) \quad (3.2)$$

Proof of Theorem C By the assumption of the theorem and the HKR character isomorphism (3.2) we have an isomorphism

$$\prod_{\text{Rep}(\mathbb{Z}_p^n, G)} L(E^*) \xrightarrow{\sim} \prod_{\text{Rep}(\mathbb{Z}_p^n, H)} L(E^*) \quad (3.3)$$

given by precomposing with the natural map $\text{Rep}(\mathbb{Z}_p^n, H) \rightarrow \text{Rep}(\mathbb{Z}_p^n, G)$. Since $L(E^*) \neq 0$ we conclude that $\text{Rep}(\mathbb{Z}_p^n, H) \rightarrow \text{Rep}(\mathbb{Z}_p^n, G)$ is an isomorphism. By the assumption on n and Lemma 3.1 this implies that $\text{Rep}(A, H) \rightarrow \text{Rep}(A, G)$ is an isomorphism for all finite abelian groups, and Theorem C now follows from Theorem D. \square

4 Variations on the results and further comments

In this final section we elaborate on some supplementary results alluded to in the introduction.

Remark 4.1 (A variety version of Theorem A) In Theorem A we can replace the assumption of F -isomorphism by the assumption that the map $\iota^*: H^*(G; \mathbb{F}_p) \rightarrow H^*(H; \mathbb{F}_p)$ induces a bijection of maximal ideals, by referencing [36, Proposition 10.9(iii) \Rightarrow (i)], and noting that the maximal ideal spectrum of $H^*(G; \mathbb{F}_p)$ identifies with $\text{Hom}_{\mathbb{F}_p\text{-alg}}(H^*(G; \mathbb{F}_p), \mathbb{F}_p) = \text{Hom}_{\text{rings}}(H^*(G; \mathbb{F}_p), \mathbb{F}_p)$.

In general a finite morphism $f: A \rightarrow B$ of finitely generated \mathbb{F}_p -algebras is an F -isomorphism if and only if it induces a *variety isomorphism*, i.e., a bijection $\text{Hom}_{\text{rings}}(B, \Omega) \xrightarrow{\sim} \text{Hom}_{\text{rings}}(A, \Omega)$ for all algebraically closed fields Ω [36, Proposition B.8-9]. But to get the same fusion on elementary abelian p -subgroups we in fact just need a bijection on $\text{Hom}_{\text{rings}}(-, \Omega)$, for some proper field extension Ω of \mathbb{F}_p , by properties of the Quillen stratification; see [36, §9-10] and also [14, §9.1].

Example 4.2 (An isomorphism modulo $\mathcal{N}il_n$ for $p = 2$ which does not control p -fusion) For any n , let $G_n = (2A_4)^n$, $P_n = (Q_8)^n$ and $H_n = \ker(\psi)$, where $\psi: G_n \rightarrow G_n/P_n \cong (C_3)^n \rightarrow C_3$ is given by $(g_1, \dots, g_n) \mapsto g_1 \dots g_n$. Note that H_n does not control p -fusion in G_n . We however claim that the restriction $H^*(G_n; \mathbb{F}_2) \rightarrow H^*(H_n; \mathbb{F}_2)$ is an isomorphism modulo $\mathcal{N}il_n$, as defined in [40, Chap. 6], hence showing that Theorem A fails severely for $p = 2$ (F -isomorphism is equivalent to isomorphism modulo $\mathcal{N}il_1$): Recall that $H^*(Q_8; \mathbb{F}_2) \cong H^{<4}(Q_8; \mathbb{F}_2) \otimes \mathbb{F}_2[z]$, with $|z| = 4$, where the action of $2A_4/Q_8 \cong C_3$ on $\mathbb{F}_2[z]$ is trivial, while on $H^{<4}(Q_8; \mathbb{F}_2)$ it is trivial in degrees 0 and 3, and degrees 1 and 2 consists of the two dimensional irreducible $\mathbb{F}_2 C_3$ -module V . Since G_n and H_n both have Sylow 2-subgroup P_n , the restriction map $H^*(G_n; \mathbb{F}_2) \rightarrow H^*(H_n; \mathbb{F}_2)$ is injective, and the cokernel is a tensor product of $\mathbb{F}_2[z_1, \dots, z_n]$ with a certain finite module M , given as the sum of the non-trivial irreducible G_n/P_n -representations on $H^{<4}(Q_8; \mathbb{F}_2)^{\otimes n}$ which restrict trivially to H_n/P_n . Using the definition of $\mathcal{N}il_m$ [40, Chap. 6], the largest m for which the restriction map is an isomorphism modulo $\mathcal{N}il_m$ therefore is the first degree where M is non-zero. To

determine this degree we extend coefficients to \mathbb{F}_4 and use Frobenius reciprocity $\mathrm{Hom}_{\mathbb{F}_4(H_n/P_n)}(\mathbb{F}_4, -) \cong \mathrm{Hom}_{\mathbb{F}_4(G_n/P_n)}((\mathbb{F}_4)^{\uparrow_{H_n}^{G_n}}, -)$, and note that $(\mathbb{F}_4)^{\uparrow_{H_n}^{G_n}} \cong (\mathbb{F}_4 \otimes \cdots \otimes \mathbb{F}_4) \oplus (\omega \otimes \cdots \otimes \omega) \oplus (\bar{\omega} \otimes \cdots \otimes \bar{\omega})$, for \mathbb{F}_4 , ω and $\bar{\omega}$ the three 1-dimensional $\mathbb{F}_4 C_3$ -modules. In this notation, we have to locate the first copy of $\omega \otimes \cdots \otimes \omega$ or $\bar{\omega} \otimes \cdots \otimes \bar{\omega}$ in $(H^{<4}(Q_8; \mathbb{F}_4))^{\otimes n}$. This occurs for the first time in degree n , where there is a summand $V \otimes \cdots \otimes V$, which over \mathbb{F}_4 is $(\omega \oplus \bar{\omega}) \otimes \cdots \otimes (\omega \oplus \bar{\omega})$, completing the proof of the claim.

Remark 4.3 (A generalization of Mislin's theorem via Theorem C) A notion of equivalence stronger than F -isomorphism, and in fact also than that of Example 4.2, is isomorphism in large degrees. If a homomorphism $\varphi: H \rightarrow G$ induces an isomorphism in mod p cohomology in large degrees, we can use Theorem C to see that $|\ker(\varphi)|$ and $|G : \varphi(H)|$ are coprime to p and that $\varphi(H)$ controls p -fusion in G , providing a new proof of a strengthening of Mislin's theorem first obtained in [32, Corollary 3.4] (cf. also [5, Theorem 1.1]): By finiteness of group cohomology the induced map between E^2 -terms of $E(n)^*$ -Atiyah–Hirzebruch spectral sequences [6, Theorem 12.2] has kernel and cokernel a finite p -group in each total degree. We therefore deduce an isomorphism $E(n)^*(BG)[\frac{1}{p}] \xrightarrow{\sim} E(n)^*(BH)[\frac{1}{p}]$ by spectral sequence comparison [6, Theorem 7.2], and the claim now follows from Theorem C.

It is perhaps interesting to note that this proof structurally mirrors Atiyah's 1961 proof of his p -nilpotency criterion [3], [35, Theorem 1.3], which says that a Sylow inclusion $S < G$ controls p -fusion if it induces an isomorphism in mod p cohomology in sufficiently high dimension: Reinterpreting [35, p. 362], Atiyah uses his version of the Atiyah–Hirzebruch spectral sequence [3, Theorem 5.1] to conclude that $K^*(BG; \mathbb{Z}_p)[\frac{1}{p}] \xrightarrow{\sim} K^*(BS; \mathbb{Z}_p)[\frac{1}{p}]$. It now follows from the Atiyah–Segal completion theorem [3, Theorem 7.2] that S and G have the same fusion on cyclic p -subgroups, and hence the same p -fusion by [24, Satz IV.4.9].

Remark 4.4 (A variety version of Theorem C and the role of inverting p) Also in Theorem C it is enough to assume a variety isomorphism: If $\varphi^*: E(n)^*(BG)[\frac{1}{p}] \rightarrow E(n)^*(BH)[\frac{1}{p}]$ induces a bijection on $\mathrm{Hom}_{\mathrm{rings}}(-, \Omega)$ for all algebraically closed fields Ω , then the same holds after extending scalars along $E(n)^*[\frac{1}{p}] \rightarrow L(E^*)$, by the universal property of the tensor product. Hence (3.2) shows that (3.3) induces a bijection $\coprod_{\mathrm{Rep}(\mathbb{Z}_p^n, H)} \mathrm{Hom}_{\mathrm{rings}}(L(E^*), \Omega) \xrightarrow{\sim} \coprod_{\mathrm{Rep}(\mathbb{Z}_p^n, G)} \mathrm{Hom}_{\mathrm{rings}}(L(E^*), \Omega)$ for any algebraically closed field Ω , so $\mathrm{Rep}(\mathbb{Z}_p^n, H) \xrightarrow{\sim} \mathrm{Rep}(\mathbb{Z}_p^n, G)$, and the conclusion of Theorem C follows from Theorem D as above.

In [19, §3] Greenlees–Strickland explain how the variety of $E(n)^*(BG)[\frac{1}{p}]$ constitutes a ‘zeroth pure stratum’ of a chromatic stratification of the formal spectrum of $E(n)^*(BG)$. Hence having an isomorphism on $E(n)^*(-)[\frac{1}{p}]$ is a priori significantly weaker than having isomorphism on $E(n)^*(-)$ or the formal spectrum $\mathrm{Spf}(E(n)^*(-))$.

Remark 4.5 (Theorem C in relationship to other results in stable homotopy theory) To illuminate the assumptions in Theorem C we note that a map induces an isomorphism on $E(n)$ (without inverting p) if and only if it induces an isomorphism on the corresponding uncompleted Johnson–Wilson theory, or isomorphism on $K(i)$ for all $i \leq n$ (see [38, Theorem 2.1] and [29, Lecture 23]). This in turn happens if and only if it induces isomorphism on just $K(n)$, by a result of Bousfield [8, Theorem 1.1].

Homotopy theorists may wonder if there exists a ‘purely homotopic’ proof of Theorem C. We do not know such a proof, but combining Mislin’s original theorem [31] with some deep results in homotopy theory, one can get a weaker statement that $E(n)^*$ -isomorphism for a quite large n (and without inverting p) implies that H controls p -fusion in G . We briefly explain this: Bousfield proved in 1982 a ‘ $K(n)$ -Whitehead theorem’ stating that a map between spaces which is an isomorphism on $K(n)^*$ also induces an isomorphism on $H^i(-; \mathbb{F}_p)$ for $i \leq n$ (see [7, Example 8.4] and [8, Theorem 1.4]). The claim now follows since it is possible to give a large constant n , depending on the Sylow subgroup, such that isomorphism on $H^i(-; \mathbb{F}_p)$ for $i \leq n$ implies isomorphism on $H^*(-; \mathbb{F}_p)$, e.g., using a result of Symonds [43, Proposition 10.2] that say that the generators and relations of group cohomology are at most in degree $2k^2$, where k is the minimal dimension of a faithful complex representation of G . Observe the bound needs to depend on more than the p -rank: For any n we can pick p such that $2n \mid p - 1$. In this case $\mathbb{F}_p \rtimes C_n < \mathbb{F}_p \rtimes C_{2n}$ induces an isomorphism on $H^i(-; \mathbb{F}_p)$ for $i < 2n - 1$ without controlling p -fusion. This is in contrast to the Huppert–Thompson–Tate p -nilpotency criterion [44], which states that an inclusion of a Sylow p -subgroup that induces isomorphism on $H^1(-; \mathbb{F}_p)$ controls p -fusion.

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