Coherent cohomology

Definition

Let $X$ be an algebraic stack and $\mathcal{F}$ a quasi-coherent sheaf on $X$. The coherent cohomology of $\mathcal{F}$ is the right derived functors of global sections:

$$H^s(X, \mathcal{F}) = H^s R\Gamma(\mathcal{F}).$$

**Warning:** I may need hypotheses on $X$, but I will be vague about this.

If $X$ is derived Deligne-Mumford stack, we have a descent spectral sequence

$$H^s(X, \pi_t \mathcal{O}_X) \Longrightarrow \pi_{t-s} R\Gamma(\mathcal{O}_X).$$
Čech complexes

Suppose $X \to Y$ is faithfully flat. We have the simplicial bar construction

$$\epsilon : X_\bullet \to Y.$$  

We get a spectral sequence

$$\pi^s H^t(X_\bullet, \epsilon^* \mathcal{F}) \Rightarrow H^{s+t}(Y, \mathcal{F}).$$

If $U$ is affine, $H^s(U, \mathcal{F}) = 0$ for $s > 0$.

If $X = \bigsqcup U_i$ is where $\mathcal{U} = \{U_i\}$ is a finite affine cover of $Y$ separated, we get an isomorphism with coherent cohomology and Čech cohomology

$$H^s(Y, \mathcal{F}) \cong \check{H}(\mathcal{U}, \mathcal{F}).$$

Comodules and comodule Ext

Let $\mathcal{M}$ be a stack and suppose $\text{Spec}(A) \to \mathcal{M}$ is a flat presentation with the property that

$$\text{Spec}(A) \times_\mathcal{M} \text{Spec}(A) \cong \text{Spec}(\Lambda).$$

Then $(A, \Gamma)$ is a Hopf algebroid and we have

$$H^s(X, \mathcal{F}) \cong \text{Ext}^s_A(\Lambda, M)$$

where $M = \epsilon^* \mathcal{F}$ is the comodule obtained from $\mathcal{F}$.

**Example:** $\mathcal{M} = X \times_G EG$ with $X = \text{Spec}(A)$ and $G = \text{Spec}(\Lambda)$. Here $\Gamma = A \otimes \Lambda$. 
Let $G : \text{Spec}(L) \to \mathcal{M}_{\text{fg}}$ classify the formal group of the universal formal group law. Then $E(L, G) = \text{MUP}$ is periodic complex cobordism. We have

$$\text{Spec}(L) \times_{\mathcal{M}_{\text{fg}}} \text{Spec}(L) = \text{Spec}(W) = \text{Spec}(\text{MUP}_0 \text{MUP})$$

where

$$W = L[\{a_0^\pm 1, a_1, a_2, \ldots\}] .$$

Then

$$H^s(\mathcal{M}_{\text{fg}}, \omega^\otimes t) \cong \text{Ext}^s_W(L, \text{MUP}_{2t}) \cong \text{Ext}^s_{\text{MUP}_* \text{MUP}}(\Sigma^{2t} \text{MUP}_*, \text{MUP}_*) .$$

This is not really the $E_2$-term of the ANSS so we must talk about:

**Example**

1. The Lazard ring $L$ is graded:
   $$x +_{F_\mu} y = \mu^{-1}(\mu x) +_{F} (\mu y) .$$

2. $W = L[\{a_0^\pm 1, a_1, \ldots\}]$ is graded:
   $$(\phi_\mu)(x) = \mu^{-1}\phi(\mu x) .$$

3. Weierstrass curves: $x \mapsto \mu^{-2} x$, $y \mapsto \mu^{-3} y .
Let $H = \text{Spec}(\Lambda)$ be an affine group scheme with an action of $\mathbb{G}_m$ and let

$$G = H \rtimes \mathbb{G}_m = \text{Spec}(\Lambda[\mu^{\pm 1}])$$

be the semi-direct product. Let $X = \text{Spec}(A)$ be an affine right $G$-scheme. Then $(A_*, \Gamma_* = A_* \otimes \Lambda_*)$ is a graded Hopf algebroid.

$$H^s(X \times_G E\mathbb{G}_m, F) = \text{Ext}^s_{\Gamma_*}(A_*, M_*).$$

where $M_* = F(\text{Spec}(A) \to X \times_G E\mathbb{G}_m)$.

1. $H^s(M_{fg}, \omega \otimes t) \cong \text{Ext}^s_{MU, MU}(\Sigma^{2t}MU_*, MU_*);$  
2. $H^s(M_{\text{Weier}}, \omega \otimes t) \cong \text{Ext}^s_\Lambda(\Sigma^{2t}A_*, A_*).$  
3. $H^*\left(\mathbb{A}^{n+1} \times \mathbb{G}_m, E\mathbb{G}_m, F \right) = M_0.$

**Derived push-forward**

Given $f : X \to Y$ and a sheaf $F$ on $X$, then $f_*F$ is the sheaf on $Y$ associated to

$$U \mapsto H^0(U \times_Y X, F).$$

If $F$ is quasi-coherent and $f$ is quasi-compact, $f_*F$ is quasi-coherent.

There is a composite functor spectral sequence

$$H^s(Y, R^tf_*F) \Longrightarrow H^{s+t}(X, F).$$

If higher cohomology on $Y$ is zero:

$$H^0(Y, R^tf_*F) \cong H^t(X, F).$$
Example: projective space

Let $S_\ast = \mathbb{Z}[x_0, \ldots, x_n]$ with $|x_i| = 1$.

**Theorem**

$$H^t(\mathbb{P}^n, \mathcal{O}(\ast)) \cong \begin{cases} S_\ast & t = 0; \\ S_\ast/(x_0^\infty, \ldots, x_n^\infty) & t = n. \end{cases}$$

We examine the diagram

$$
\begin{array}{ccc}
\mathbb{A}^{n+1} & \xrightarrow{j} & \mathbb{A}^{n+1} \\
\downarrow & & \downarrow \\
\mathbb{P}^n & \xrightarrow{i} & \mathbb{A}^{n+1} \times_{\mathbb{G}_m} \mathbb{E}_{\mathbb{G}_m}.
\end{array}
$$

We must calculate (the global sections) of

$$Rj_*j^*\mathcal{O}_{\mathbb{A}^{n+1}} \sim Rj_*j^*S_\ast.$$ 

Example: the affine case

Let $X = \text{Spec}(R)$ and $j : U \to X$ the open defined by an ideal $I = (a_1, \ldots, a_n)$. If $\mathcal{F}$ is defined by the module $M$, then $j_*j^*\mathcal{F}$ is defined by $K$ where there is an exact sequence

$$0 \to K \to \prod_s M[\frac{1}{a_s}] \to \prod_{s < t} M[\frac{1}{a_s a_t}].$$

There is also an exact sequence

$$0 \to \Gamma_I M \to M \to K \to 0$$

where

$$\Gamma_I M = \{ x \in M \mid I^n x = 0 \text{ for some } n \}.$$
Local cohomology

If \( i : U \to Y \) is the open complement of a closed sub-stack \( Z \subseteq Y \) define local cohomology by the fiber sequence

\[
R \Gamma_Z F \to F \to R i_* i^* F.
\]

If \( X = \text{Spec}(A) \) and \( Z \) is define by \( I = (a_1, \ldots, a_k) \) local cohomology can be computed by the Koszul complex

\[
M \to \prod_s M[\frac{1}{a_s}] \to \prod_{s < t} M[\frac{1}{a_s a_t}] \to \cdots \to M[\frac{1}{a_1 \cdots a_k}] \to 0.
\]

Since \( x_0, \ldots, x_n \in S_* \) is a regular sequence:

\[
R^{n+1} \Gamma_{\{0\}} S_* = S_*/(x_0^\infty, \ldots, x_n^\infty)
\]

and \( R^t \Gamma_{\{0\}} S_* = 0, \ t \neq n + 1. \)