Explicit solutions to some optimal variance stopping problems

Jesper Lund Pedersen*1

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen, Denmark

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The stopping problem with variance as the optimality criterion is introduced. Due to the variance criterion, smooth fit cannot be applied directly. The problem is solved by embedding it into tractable auxiliary optimal stopping problems, where smooth fit is used to obtain explicit, optimal solutions. Optimal strategies are presented in closed form for several examples. A characteristic feature is that the optimal stopping boundaries depend on the initial value of the gain process, i.e. the state space of the gain process does not split into one continuation set and one stopping set.

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1. Introduction

This paper introduces an optimal variance stopping problem. The objective is to maximize – over a family of stopping times – the variance of the gain process. Closed-form solutions of optimal stopping times are presented under various diffusions for the gain process. The diffusions include geometric Brownian motion (Section 3), square-root process (Section 4) and Jacobi diffusion (Section 5).

This paper differs from traditional optimal stopping results in that the objective function is involving a nonlinear term coming from the variance criterion, meaning that the standard optimal stopping theory (see Peskir and Shiryaev [11], Chapter 1) is not directly applicable to the variance stopping problem. The main obstacle is the inability to apply smooth fit due to this nonlinear term. The basic proof of optimality of the variance stopping problem rests on embedding the problem into a tractable auxiliary optimal stopping problem, where smooth fit is employed to obtain explicit solutions. The relationship between the solutions of the two problems is then investigated. In the examples of the variance stopping problem, the optimal stopping boundaries (see Theorems 3.2, 4.1 and 5.1) depend on the initial value of the gain process, i.e. the state space of the gain process does not split into one continuation set and one stopping set.

An optimal stopping problem can be represented by a linear function of the expected value of the gain processes. An example is the optimal stopping problem (2.2) in the next section, which is a linear function of the first and second moment of the stopped gain process. The variance stopping problem (see problem (2.1) in the next section) is a quadratic

*Email: jesper@math.ku.dk
function of the first and second moment of the stopped gain process. Thus, the variance
criterion can be viewed as an example of a quadratic framework in contrast to a linear
framework.

In the literature, variance-optimizing problems have been formulated as stochastic
control problems. The traditional dynamic programming approach cannot be used
directly in these problems, again due to the nonlinear term in the criterion function.
Richardson [12] and Duffie and Richardson [4] derived optimal solutions for variance
problems based primarily on a projection theorem. Li and Ng [9] (discrete time) and
Zhou and Li [15] (continuous time) solved a mean-variance problem using an embedding
technique.

2. Problem formulation

The formulation of the variance stopping problem is presented in this section, as well as
some preliminary results.

Throughout this paper, \((B_t)\) is a standard Brownian motion. The process \((X_t)\) is a linear
diffusion (see Borodin and Salminen [1]) determined by the stochastic differential
equation

\[
dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dB_t
\]

with \(X_0 = x\) under the measure \(\mathbb{P}_x\). Under \(\mathbb{P}_x\), the variance of a random variable \(Y\) is
denoted by \(\text{Var}_x(Y) = \mathbb{E}_x[(Y - \mathbb{E}_x[Y])^2]\).

The mathematical formulation of the variance stopping problem is then

\[
\mathbb{V}_*(x) = \sup_{\tau} \text{Var}_x(X_\tau), \tag{2.1}
\]

where the supremum is taken over all stopping times \(\tau\) for the process \((X_t)\) satisfying that
\(\mathbb{E}_x(X_{\tau}^2) < \infty\). The problem is to determine an optimal stopping time \(\tau(x)\) for which the
supremum is attained and moreover to compute the value function \(\mathbb{V}_*(x)\). The standard
optimal stopping theory cannot be applied directly to deal with the latter’s nonlinear term
of the criterion function \(\text{Var}_x(X_\tau) = \mathbb{E}_x[X_{\tau}^2] - \mathbb{E}_x[X_\tau]^2\). The idea to solving problem (2.1)
is to embed the problem into an auxiliary optimal stopping problem. For similar
considerations, see the proof of Theorems 8 and 10 in Shiryaev [14] (Chapter 4).

To be specific, the auxiliary optimal stopping problem for a constant \(c\) is

\[
V^{(c)}(x) = \sup_{\tau} \mathbb{E}_x[(X_{\tau} - c)^2], \tag{2.2}
\]

where the supremum is taken over all stopping times \(\tau\) for the process \((X_t)\). The optimal
stopping time that solves this problem is denoted as \(\tau^{(c)}\). Before proceeding, some
additional notation is required. In general,

\[
\tau_b = \inf\{t > 0 : X_t \geq b\} \tag{2.3}
\]
denotes the first hitting time to a level \(b\). In the examples below (see Propositions 3.1, 4.2
and 5.3), the optimal stopping time \(\tau^{(c)}\) is equal to \(\tau^{(c)}_{b^{(c)}}\), where \(b^{(c)}\) is the optimal stopping
boundary.

The following result verifies that the solution of optimal stopping problem (2.2) leads
to the solution of variance stopping problem (2.1).
Theorem 2.1. Let $X_0 = x$ be given and fixed. If $c^*(x)$ is a constant such that for stopping problem (2.2), the value function $V(c^*(x))(x)$ is finite and the optimal stopping time $\tau_{c^*(x)}$ satisfies

$$c^*(x) = \mathbb{E}_x[X_{\tau_{c^*(x)}}],$$

then $\tau_{c^*(x)}$ is also an optimal stopping time for the variance stopping problem (2.1).

Proof. The assumptions ensure that $\mathbb{E}_x[X_{\tau_{c^*(x)}}] < \infty$ and equation (2.4) is well defined. Let $\tau$ be any stopping for $X_t$ satisfying $\mathbb{E}_x[X_{\tau}^2] < \infty$, it then follows that

$$\text{Var}_x(X_{\tau}) = \mathbb{E}_x[(X_{\tau} - \mathbb{E}_x[X_{\tau}])^2] = \mathbb{E}_x[(X_{\tau} - c^*(x))^2] - (\mathbb{E}_x[X_{\tau}] - c^*(x))^2 \leq \mathbb{E}_x[(X_{\tau_{c^*(x)}} - c^*(x))^2] = \text{Var}_x(X_{\tau_{c^*(x)}}).$$

Finally, taking supremum over the given family of stopping times yields

$$\nabla^*_x(x) \leq \text{Var}_x(X_{\tau_{c^*(x)}}) \leq \nabla^*_x(x)$$

and $\tau^*_x(x) = \tau_{c^*(x)}$ is an optimal stopping time for the variance stopping problem (2.1). □

Remark 2.2.

1. By virtue of the above theorem, solving the variance stopping problem (2.1) is to first solve the stopping problem (2.2). Once, the stopping problem is solved, the solution to the variance stopping problem might be obtained by finding a solution to equation (2.4).
2. A basic approach to solving the stopping problem is to use free-boundary problem and smooth fit (see e.g. Pedersen [10] or Peskir and Shiryaev [11]). Since the gain function is quadratic, the gain process has to be at least non-recurrent for the problem to be non-trivial. Indeed, if $X_t$ is recurrent, any first hitting time to a level $b$ in the state space is always finite and the value function must be equal to the supremum of $(b - c)^2$ for $b$ in the state space.
3. A heuristic argument on homogeneity of the problem shows that optimal stopping boundaries are straightlines. This observation together with equation (2.4) indicate that an optimal stopping time for variance stopping problem (2.1) is given in terms of straightline stopping boundaries that depend on the initial value of the process.

3. Geometric Brownian motion

In this section, the gain process is a geometric Brownian motion. Samuelson [13] introduced the geometric Brownian motion into financial modelling and it is used as a basis for the Black–Scholes analysis.

Let $(X_t)$ denote a geometric Brownian motion satisfying the stochastic differential equation

$$dX_t = \mu X_t \, dt + \sigma X_t \, dB_t, \quad X_0 = x > 0$$

(3.1)
with \( \mu \in \mathbb{R} \) and \( \sigma > 0 \). The distribution of \( X_t \) is a log-normal where the variance is given by

\[
\text{Var}_x(X_t) = \mathbb{E}_x[(X_t - \mathbb{E}_x[X_t])^2] = (e^{(\sigma^2 + 2\mu)t} - e^{2\mu t}) \cdot x^2.
\]

If \( \sigma^2 + 2\mu < 0 \), then \( \text{Var}_x(X_t) \to 0 \) for \( t \to \infty \) and as a function of \( t \) the variance has a maximum. The maximum is given by

\[
\sup_{t \geq 0} \text{Var}_x(X_t) = \text{Var}_x(X_{t_*}) = -\frac{\sigma^2}{\sigma^2 + 2\mu} \left( \frac{2\mu}{\sigma^2 + 2\mu} \right)^{2\mu/\sigma^2} x^2,
\]

where the maximum is attained at \( t_* = \log(2\mu/(\sigma^2 + 2\mu))/\sigma^2 \). Moreover, for \( \sigma^2 + 2\mu \geq 0 \), the value function of the variance stopping problem is infinite.

As it is remarked in Remark 2.2, the derivation of the solution to the variance stopping problem (2.1) relies on the solution of the auxiliary optimal stopping problem (2.2). The optimal stopping time of the stopping problem (2.2) is given in the next proposition.

**Proposition 3.1.** Consider the optimal stopping problem (2.2) for \( c > 0 \), where the process \( X_t \) is the geometric Brownian motion given in (3.1). Assume that \( \sigma^2 + 2\mu \geq 0 \). Then, an optimal stopping time is given by

\[
\tau^{(c)} = \inf\{t > 0 : X_t \geq 4\mu c/(\sigma^2 + 2\mu)\}.
\]

**Proof.** Let \( c > 0 \) be given and fixed. The hitting probability is (see Borodin and Salminen [1])

\[
P_x(\tau_b < \infty) = \left(\frac{x}{b}\right)^{1-2\mu/\sigma^2} \quad \text{for} \quad x < b,
\]

where the first hitting time \( \tau_b \) is defined in (2.3). Because \( X_\infty = 0 \), \( X_{\tau_b} \) takes the two values 0 or \( b \) and hence, for \( 0 < x \leq 2c < b \), using the hitting probability gives

\[
V(x) \geq \mathbb{E}_x[(X_{\tau_b} - c)^2]
= c^2 P_x(\tau_b = \infty) + (b - c)^2 P_x(\tau_b < \infty)
= c^2 + (b^2 - 2bc) P_x(\tau_b < \infty)
> c^2
\geq (c - x)^2.
\]

Therefore, the interval \((0, 2c]\) is a subset of the continuation set.

This observation together with Remark 2.2 shows that the continuation set might be of the form \((0, b^{(c)}]\), where the optimal stopping boundary \( b^{(c)} > 2c \) is to be found. Standard arguments based on the strong Markov property is used to formulate the following free-boundary problem for the value function and optimal boundary

\[
\frac{1}{2} \sigma^2 x^2 V''(x) + \mu x V'(x) = 0 \quad \text{for} \quad 0 < x < b
\]

\[
V(x) = (x - c)^2 \quad \text{for} \quad x \geq b \quad \text{(instantaneous stopping)}
\]

\[
V(0) = c^2
\]

\[
V'(b) = 2(b - c) \quad \text{(smooth fit)}.
\]
The general solution of the differential equation is given by $V(x) = A + Bx^{1-2\mu/\sigma^2}$. The three conditions determine the constants $A$, $B$ and $b$ uniquely and it is straightforward to verify that the solution to the free-boundary problem is given by

$$V(x) = \begin{cases} 
    c^2 + (b^{(c)})^2 - 2b^{(c)}c \left( \frac{x}{b^{(c)}} \right)^{1-2\mu/\sigma^2} & \text{for } 0 \leq x < b^{(c)} \\
    (x - c)^2 & \text{for } x \geq b^{(c)}
\end{cases},$$

where the boundary is given by $b^{(c)} = 4\mu c/(\sigma^2 + 2\mu)$. Note that $b^{(c)} > 2c$ and $V(\cdot)$ is $C^2$ on $(0, b^{(c)}) \cup (b^{(c)}, \infty)$ but only $C^1$ at $b^{(c)}$.

It remains to show that

$$\tau^{(c)} = \tau_{b^{(c)}} = \inf\{t > 0 : X_t \geq b^{(c)}\}$$

is an optimal stopping. The problem is to verify that

$$V^{(c)}(x) \leq V(x) = E_x[(X_{\tau^{(c)}} - c)^2].$$

By the definition of the optimal stopping problem, it then follows that $\tau^{(c)}$ will be the optimal stopping time. Using the hitting probability in (3.2) gives

$$E_x[(X_{\tau^{(c)}} - c)^2] = c^2 + (b^{(c)})^2 - 2b^{(c)}c P_x(\tau^{(c)} < \infty) = V(x).$$

This is the equality in (3.3). The properties of $V(\cdot)$ ensure that the Itô formula can be applied in its standard form and together with $(1/2)\sigma^2 x^2 V''(x) + \mu x V'(x) \leq 0$ yield

$$0 \leq V(X_t) = V(x) + \int_0^t \sigma X_s V'(X_s) dB_s + \int_0^t \left( \frac{1}{2} \sigma^2 X_s^2 V''(X_s) + \mu X_s V'(X_s) \right) 1_{\{X_s \neq b^{(c)}\}} ds$$

$$\leq V(x) + \int_0^t \sigma X_s V'(X_s) dB_s,$$

where the latter stochastic integral is a local martingale. The computations show that the stochastic integral is bounded from below by $-V(x)$ and therefore is a supermartingale. Using further that $V(x) \geq (x - c)^2$, an application of optional sampling theorem shows that $V^{(c)}(x) \leq V(x)$. This is the inequality in (3.3) and the proof is complete.

The solution to the variance stopping problem for geometric Brownian motion is stated in Theorem 3.2. The theorem shows that the optimal stopping boundary depends linearly on the initial value of the process.

THEOREM 3.2. Consider the variance stopping problem (2.1), where the process $X_t$ is the geometric Brownian motion given in (3.1). Assume that $\sigma^2 + 2\mu < 0$. For $X_0 = x > 0$, an optimal stopping time is given by (see Figure 1)

$$\tau_*(x) = \inf\{t > 0 : X_t = (4\mu/\sigma^2 + 2\mu)^{\sigma^2/(\sigma^2 - 2\mu)} \cdot x\}.$$  (3.4)

The associated value function is given by

$$\mathbb{V}_*(x) = \left( \frac{4\mu}{\sigma^2 + 2\mu} \right)^{(\sigma^2 + 2\mu)/(\sigma^2 - 2\mu)} - \left( \frac{4\mu}{\sigma^2 + 2\mu} \right)^{4\mu/(\sigma^2 - 2\mu)} \cdot x^2.$$
The proof is based on the approach described in Remark 2.2. Recall that $X_t$ takes the two values 0 or $b$, and using the hitting probability in equation (3.2), the mean value is given by

$$E_x[X_{t^*}] = bP_x(\tau_b < \infty) = b\left(\frac{x}{b}\right)^{1-2\mu/\sigma^2}$$

for $x < b$.

From Proposition 3.1, $b^{(c)} = 4\mu c/(\sigma^2 + 2\mu)$ is the optimal stopping boundary of stopping problem (2.2) and the associated optimal stopping time is $\tau^{(c)} = \tau_{b^{(c)}}$. If $c^*(x)$ solves the equation (see Figure 2)

$$c = E_x[X_{\tau^{(c)}}] = b^{(c)}\left(\frac{x}{b^{(c)}}\right)^{1-2\mu/\sigma^2} \wedge x,$$

then by Theorem 2.1, $\tau^{(c^*(x))}$ is an optimal solution to the variance stopping problem. Consequently, the optimal stopping boundary $b^*(x) = b^{(c^*(x))}$ for the variance stopping problem solves the equation

$$b = \frac{4\mu c}{\sigma^2 + 2\mu} = \frac{4\mu b}{\sigma^2 + 2\mu} \left(\frac{x}{b}\right)^{1-2\mu/\sigma^2}.$$
and the solution is given by

\[ b_*(x) = \left( \frac{4\mu}{\sigma^2 + 2\mu} \right)^{\sigma^2/(\sigma^2 - 2\mu)} x > x. \]

This implies that \( \tau_*(x) = \tau^{(c_m(x))} = \tau_{b_*(x)} \) is optimal. The value function can be computed by the following formula:

\[ \forall x = \text{Var}_{x}(X_{\tau_*(x)}) = b_*(x)^2 \mathbb{P}_x(\tau_*(x) < \infty) \mathbb{P}_x(\tau_*(x) = \infty). \]  

(3.5)

This completes the proof. \( \square \)

**Remark 3.3.** A minor change of equation (2.4) in Theorem 2.1 gives the solution to a variance stopping problem with a given mean. Let \( m \) be given and fixed constant, then the optimal stopping variance problem with given mean \( m \) is

\[ \sup \limits_{\tau} \text{Var}_{x}(X_{\tau}) \quad \text{subject to} \quad \mathbb{E}_x[\tau] = m. \]  

(3.6)

Consider stopping problem (2.2), if \( c_m(x) \) is a constant such that \( V^{(c_m(x))}(x) < \infty \) and

\[ m = \mathbb{E}_x[X_{\tau^{(c_m(x))}}], \]  

(3.7)

then \( \tau^{(c_m(x))} \) is an optimal solution to problem (3.6). Indeed, if \( \tau \) is any stopping time satisfying \( \mathbb{E}_x[X_{\tau}] = m \), then as in the proof of Theorem 2.1

\[ \text{Var}_{x}(X_{\tau}) = \mathbb{E}_x[(X_{\tau} - \mathbb{E}_x[X_{\tau}])^2] \]

\[ \leq \mathbb{E}_x[(X_{\tau^{(c_m(x))}} - c_m(x))^2] - (m - c_m(x))^2 \]

\[ = \text{Var}_{x}(X_{\tau^{(c_m(x))}}). \]

Let \( X_t \) be the geometric Brownian motion in Theorem 3.2 and note that \( X_t \) is a supermartingale which implies that \( x \geq \mathbb{E}_x[X_{\tau_0}] \). Let \( 0 < m \leq x \) be given and fixed. Then, \( c_m(x) \) solves equation (3.7) given by (see Figure 2)

\[ m = b^{(c)}(x) \left( \frac{x}{b^{(c)}} \right)^{1 - 2\mu / \sigma^2}, \]

where \( b^{(c)} = 4\mu c/(\sigma^2 + 2\mu) \) is the stopping boundary given in Proposition 3.1. The solution is

\[ c_m(x) = m \frac{\sigma^2 + 2\mu}{4\mu} \left( \frac{x}{m} \right)^{1 - \sigma^2/(2\mu)} \]

and the optimal stopping boundary is given by \( b^{(c_m(x))} = m(x/m)^{1 - \sigma^2/(2\mu)} \). Thus,

\[ \tau^{(c_m(x))} = \inf\{ t > 0 : X_t = m(x/m)^{1 - \sigma^2/(2\mu)} \} \]

is the optimal stopping time for the variance stopping problem with given mean \( m \).

### 4. Square-root process

In this section, the gain process is a square-root process. Feller [6] introduced the process as a model of population growth. In the finance literature, it is referred to as the CIR process (Cox et al. [2]).
Let \((X_t)\) denote a square-root process satisfying the stochastic differential equation
\[
dX_t = -\mu X_t \, dt + \sigma \sqrt{X_t} \, dB_t, \quad X_0 = x > 0,
\]
where \(\mu \in \mathbb{R}\) and \(\sigma > 0\) are constants. The state space is \([0, \infty)\) and if \(\mu > 0\), the boundary point 0 is absorbing and is hit in finite time. The variance of \(X_t\) is given by
\[
\text{Var}_x(X_t) = \frac{\sigma^2}{\mu} \left( e^{-\mu t} - e^{-2\mu t} \right) x.
\]
If \(\mu > 0\), then \(\text{Var}_x(X_t) \to 0\) for \(t \to \infty\) and hence the variance stopping problem might be finite.

The solutions to the variance stopping problem and the stopping problem are given below together with main ideas of the proofs.

**Theorem 4.1.** Consider the variance stopping problem (2.1), where the process \(X_t\) is the square-root process given in (4.1). Assume that \(\mu, \sigma > 0\). For \(X_0 = x > 0\), an optimal stopping time is given by
\[
\tau^*_x = \inf \{ t > 0 : X_t = b^*_x \},
\]
where \(b^*_x\) is the unique solution to the equation for \(b > x\) (see Figure 3)
\[
b^*_x = \frac{e^{2\mu b/\sigma^2} - 2e^{2\mu x/\sigma^2} + 1}{e^{2\mu b/\sigma^2} - e^{2\mu x/\sigma^2}} = \frac{\sigma^2}{\mu} \left( 1 - e^{-2\mu b/\sigma^2} \right).
\]

The associated value function is given by
\[
\mathbb{V}^*_x(x) = b^*_x(x) \left( \frac{e^{2\mu x/\sigma^2} - 1}{e^{2\mu b/\sigma^2} - 1} \right)^2.
\]

![Figure 3. A drawing of the optimal stopping boundary \(x \mapsto b^*_x(x)\) that is the solution to equation (4.2).](image-url)
Proof. The hitting probability to a level $b$ and the mean value of $X_{\tau_b}$ are

$$P(\tau_b < \infty) = \frac{e^{2\mu x/\sigma^2} - 1}{e^{2\mu b/\sigma^2} - 1} \quad \text{and} \quad E[X_{\tau_b}] = b \frac{e^{2\mu x/\sigma^2} - 1}{e^{2\mu b/\sigma^2} - 1}$$

for $x < b$. The process $X_t$ is a supermartingale and hence $x \geq E[X_{\tau_b}]$.

For stopping problem (2.2), the stopping boundary $b^{(c)}$ is the solution to equation (4.4), see Proposition 4.2. The boundary satisfies that $b^{(c)} \geq 2c$ and $c \mapsto b^{(c)}$ is increasing. Note that $c \mapsto E[X_{\tau^{(c)}}]$ is decreasing since $X_t$ is a supermartingale. Then, there is $c_\gamma \leq x/2$ such that $b^{(c_\gamma)} = x$ and $b^{(c)} > x$ for all $c > c_\gamma$. Therefore, there is a unique solution $c_\gamma(x)$ to the equation

$$c = E[X_{\tau^{(c)}}] = b^{(c)} \frac{(1 - x)^{1-2\mu/\sigma^2} - 1}{(1 - b^{(c)})^{1-2\mu/\sigma^2} - 1} \quad \text{for} \quad c > c_\gamma. \quad (4.3)$$

By Theorem 2.1, $\tau(x) = \tau^{(c_\gamma(x))}$ is optimal for the variance stopping problem. The associated stopping boundary is $b^e(x) = b^{(c_\gamma(x))} > x$ and combining equations (4.3) and (4.4) gives that $b^e(x)$ is the unique solution to equation (4.2). The value function can be computed by (3.5) and this completes the proof. \hfill \Box

Proposition 4.2. Consider the optimal stopping problem (2.2) for $c > 0$, where the process $X_t$ is the square-root process given in (4.1). Assume that $\mu, \sigma > 0$. Then, an optimal stopping time is given by

$$\tau^{(c)} = \inf\{ t > 0 : X_t \equiv b^{(c)} \},$$

where $b^{(c)}$ is the unique solution to the equation for $b > 2c$

$$(b - c) \left(1 - e^{-2\mu b/\sigma^2}\right) = \frac{\mu}{\sigma^2}\left(b^2 - 2bc\right). \quad (4.4)$$

Proof. Let $c > 0$ be given and fixed. By the same arguments as in the geometric Brownian motion case (see the proof of Proposition 3.1), the interval $(0, 2c]$ is a subset of continuation region. Therefore, the optimal stopping boundary $b^{(c)}$ is strictly greater than $2c$ and hence equation (4.4) only has to be considered for $b \geq 2c$. Define the function

$$f(b) = \log(b - c) + \log\left(1 - e^{2\mu b/\sigma^2}\right) - \log(b^2 - 2bc) - \log(\mu/\sigma^2)$$

for $b > 2c$. Taking log on both sides of equation (4.4) and then subtracting the left-hand side, one has that equation (4.4) is equivalent to the equation $f(b) = 0$ for $b > 2c$. Note that

$$f'(b) = \left[\frac{1}{b - c} - \frac{1}{b - 2c}\right] + \frac{2\mu}{\sigma^2}\left[\frac{1}{e^{2\mu b/\sigma^2} - 1} - \frac{1}{2\mu b/\sigma^2}\right] < 0$$

for $b > 2c$ (note that both brackets $[]$ are strictly negative). Hence, $f(b)$ is strictly decreasing, it increases to $\infty$ as $b$ goes down to $2c$ and it decreases to $-\infty$ as $b$ goes up to $\infty$. Therefore, equation (4.4) has a unique solution. In fact, the solution $b^{(c)}$ is increasing in $c$. 
The value function \( V(c) \) and the optimal boundary \( b(c) \) might solve the free-boundary problem

\[
\frac{1}{2} \sigma^2 xV''(x) - \mu xV'(x) = 0 \quad \text{for} \quad 0 < x < b
\]

\[
V(x) = (x - c)^2 \quad \text{for} \quad x \geq b \quad \text{(instantaneous stopping)}
\]

\[
V(0) = c^2
\]

\[
V'(b) = 2(b - c) \quad \text{(smooth fit)}.
\]

The general solution of the differential equation is given by

\[
V(x) = A + B(e^{2\mu x/\sigma^2 - 1}).
\]

The three conditions determine the constants \( A, B \) and \( b \) uniquely and it is straightforward to verify that the solution to the free-boundary problem is given by

\[
V(x) = \begin{cases} 
  c^2 + (b^2 - 2bc) \frac{e^{2\mu x/\sigma^2 - 1}}{e^{2\mu x/\sigma^2 - 1}} & \text{for} \quad 0 < x < b \\
  (x - c)^2 & \text{for} \quad x \geq b
\end{cases}
\]

where \( b \) is the solution to (4.4) for \( b > 2c \).

It remains to verify that the candidate indeed is the solution to the stopping problem. A verification argument employing the Itô formula as in the proof of Proposition 3.1 will provide a formal proof. The details are omitted.

5. Jacobi diffusion

The gain process in this section is a Jacobi process. The Jacobi process has been studied particularly in relations to models in genetics, see e.g. Karlin and Taylor [8] and Ethier and Kurtz [5]. In some cases, the Jacobi diffusion process and the square-root belong to the statistically applicable class of Pearson diffusion (see Forman and Sørensen [7]).

Let \( X_t \) denote a Jacobi process satisfying the stochastic differential equation

\[
dX_t = -\mu X_t dt + \sigma \sqrt{X_t(1 - X_t)} dB_t, \quad X_0 = x \in (0, 1),
\]

with \( \mu \in \mathbb{R} \) and \( \sigma > 0 \) being constants. The state space of the process is \([0,1]\). If \( \mu > 0 \), the boundary point 0 is absorbing and is hit in finite time. Moreover, the variance of \( X_t \) is

\[
\text{Var}_x(X_t) = e^{-2\mu t}(e^{-\sigma^2 t} - 1) \cdot x^2 + \frac{\sigma^2}{\sigma^2 + 2\mu} e^{-\mu t}(1 - e^{-(\sigma^2 + 2\mu)t} \cdot x).
\]

If \( \mu > 0 \), then \( \text{Var}_x(X_t) \to 0 \) for \( t \to \infty \), which indicates that the variance stopping problem might be finite in this case.

Theorem 5.1 presents the optimal solution to the variance stopping problem in terms of the optimal stopping boundary, which solves an equation that depends on the initial value of the process.

**Theorem 5.1.** Consider the variance stopping problem (2.1), where the process \( X_t \) is the Jacobi diffusion given in (5.1). Assume that \( \mu, \sigma > 0 \). For \( X_0 = x \in (0, 1) \), an optimal stopping time is given by

\[
\tau_*(x) = \inf\{t > 0 : X_t = b_*(x)\},
\]
where \( b_*(x) \) is the unique solution to the equation for \( x < b < 1 \)

\[
\begin{cases}
   b_*(1-b)^{1-2\mu/\sigma^2} - 2(1-x)^{1-2\mu/\sigma^2} + 1 = \frac{2\sigma^2}{\sigma^2 - 2\mu}((1-b)^{2\mu/\sigma^2} - (1-b)) & \text{if } 2\mu \neq \sigma^2 \\
   b_*(\log(1-b) - 2\log(1-x)) - (1-x)^{1-2\mu/\sigma^2} = -2(1-b)\log(1-b) & \text{if } 2\mu = \sigma^2.
\end{cases}
\]

The associated value function is given by

\[
\mathbb{V}_*(x) = \begin{cases}
   b_*(x)^2 \left( (1-x)^{1-2\mu/\sigma^2} - 1 \right) \left( (1-b_*(x))^{1-2\mu/\sigma^2} - (1-x)^{1-2\mu/\sigma^2} \right) & \text{if } 2\mu \neq \sigma^2 \\
   b_*(x) \frac{\log(1-x)(\log(1-b_*(x)) - \log(1-x))}{\log^2(1-b_*(x))} & \text{if } 2\mu = \sigma^2.
\end{cases}
\]

**Proof.** Let \( x \in (0, 1) \) be given and fixed and consider the case \( 2\mu \neq \sigma^2 \). The other case \( 2\mu = \sigma^2 \) can be dealt with analogously and details in this direction will be omitted.

The hitting probability to a level \( b \) is (see Delbaen and Shirakawa [3])

\[
P_x(\tau_b < \infty) = \frac{(1-x)^{1-2\mu/\sigma^2} - 1}{(1-b)^{1-2\mu/\sigma^2} - 1} \quad \text{for } x < b
\]

and the mean value of \( X_{\tau_b} \) is

\[
E_x[X_{\tau_b}] = \frac{b_*(1-x)^{1-2\mu/\sigma^2} - 1}{(1-b)^{1-2\mu/\sigma^2} - 1} \quad \text{for } x < b.
\]

Moreover, \( x \geq E_x[X_{\tau_b}] \), since \( X_t \) is a supermartingale.

Proposition 5.3 provides that the optimal stopping boundary \( b^{(c)} \) of stopping problem (2.2) is the solution to equation (5.5). Then, \( c \mapsto b^{(c)} \) is increasing, and since \( b^{(c)} \geq 2c \), there is a constant \( c_x \) such that \( b^{(c_x)} = x \) and \( b^{(c)} > x \) for all \( c > c_x \). Furthermore, \( c \mapsto E_x[X_{\tau^{(c)}}] \) is decreasing, due to that \( X_t \) is a supermartingale. Let \( c_*(x) \) be the solution to the equation

\[
c = E_x[X_{\tau^{(c)}}] = \frac{(1-x)^{1-2\mu/\sigma^2} - 1}{(1-b^{(c)})^{1-2\mu/\sigma^2} - 1} \quad \text{for } c > c_x.
\]

Note that there exists a unique solution to the equation, since the left-hand side is increasing and the right-hand side is decreasing. By Theorem 2.1, \( \tau_*(x) = \tau^{(c_x(x))} \) is optimal for the variance stopping problem and \( b_*(x) = b^{(c_x(x))} > x \) is the associated stopping boundary which is the unique solution to equation (5.2) by combining equations (5.3) and (5.5). The calculation of the optimal variance is straightforward using (3.5) and this completes the proof.

**Remark 5.2.**

1. If \( \mu/\sigma^2 = 1 \), then equation (5.2) has the explicit solution \( b_*(x) = (1/2)(1 + x) \) and \( \tau_*(x) = \inf\{t > 0 : X_t = (1/2)(1 + x)\} \) is an optimal stopping time of the variance stopping problem.
2. Let $Y_t = 1 - X_t$, then $Y_t$ is the solution to the stochastic differential equation

$$dY_t = \mu(Y_t)\,dt + \sigma\sqrt{Y_t(Y_t)}\,dB_t, \quad Y_0 = y \in (0, 1).$$

If $\mu > 0$, the boundary point 1 is absorbing and is hit in finite time. Then, the problem

$$\mathbb{V}_*(y) = \sup_{\tau} \mathbb{V}_\tau(y) = \sup_{\tau} \mathbb{V}_{1-y}(X_\tau)$$

has the solution (see Figure 4)

$$\tau_*(y) = \inf\{t > 0 : X_t = b_*(1 - y)\} = \inf\{t > 0 : Y_t = 1 - b_*(1 - y)\}, \quad (5.4)$$

where $b_*(\cdot)$ is the solution to equation (5.2). In this case, the optimal stopping boundary satisfies $1 - b_*(1 - y) < y$.

The solution to the optimal-stopping problem is given in Proposition 5.3 and then the remainder of this section is devoted to the main ideas of its proof.

**PROPOSITION 5.3.** Consider the optimal stopping problem (2.2) for $0 < c < 1/2$, where the process $X_t$ is the Jacobi diffusion given in (5.1). Assume that $\mu, \sigma > 0$. Then, an optimal stopping time is given by

$$\tau^*(c) = \inf\{t > 0 : X_t \equiv b^*(c)\},$$

where $b^*(c)$ is the unique solution to the equation for $2c < b < 1$

$$2(b - c) = \begin{cases} \frac{b^2 - 2bc}{1 - b} \cdot \frac{2\mu/\sigma^2 - 1}{1 - (1 - b)^{2\mu/\sigma^2 - 1}} & \text{if } 2\mu \neq \sigma^2, \\ \frac{b^2 - 2bc}{1 - b} \cdot \frac{1}{-\log(1 - b)} & \text{if } 2\mu = \sigma^2. \end{cases} \quad (5.5)$$

![Figure 4](image_url)  

*Figure 4.* A drawing of the optimal stopping time (5.4) for two paths of the Jacobi diffusion, where the boundary point 1 is absorbing.
Proof. Let $0 < c < 1/2$ be given and fixed and assume that $2\mu \neq \sigma^2$. The case $2\mu = \sigma^2$ can be dealt with by the same method. As in the two previous examples, the interval $(0, 2c]$ is a subset of the continuation region. Therefore, the optimal stopping boundary $b^{(c)}$ is strictly greater than $2c$ and hence equation (5.5) only has to be considered for $2c < b < 1$. Define the function

$$f(b) = \log(2(b - c)) - \log(b^2 - 2bc) + \log(1 - b) + \log\left(\frac{1 - (1 - b)^{2\mu/\sigma^2 - 1}}{2\mu/\sigma^2 - 1}\right)$$

for $2c < b < 1$. Taking log on both sides of equation (5.5), and then subtracting the left-hand side, one has equation (5.5) to be equivalent to the equation $f(b) = 0$ for $b > 2c$. Note that

$$f'(b) = \left[\frac{1}{b - c} - \frac{1}{b - 2c}\right] + \frac{1}{1 - b} \left[\frac{2\mu/\sigma^2 - 1}{(1 - b)^{1 - 2\mu/\sigma^2} - 1} - \frac{1}{b}\right] < 0$$

for $2c < b < 1$ (note that both brackets $[\ ]$ are strictly negative). Hence, $f(b)$ is strictly decreasing, it increases to $\infty$ as $b$ goes down to $2c$ and it decreases to $-\infty$ as $b$ goes up to $1$. Therefore, equation (5.5) has a unique solution. Note that the solution $b^{(c)}$ is increasing as a function of $c$.

The proof will only be sketched, since the arguments are the same as in the proof of Proposition 3.1. The value function $V^{(c)}(\cdot)$ and the optimal boundary $b^{(c)}$ might solve the free-boundary problem

$$\frac{1}{2}\sigma^2 x(1 - x)V''(x) - \mu x V'(x) = 0 \quad \text{for } 0 < x < b < 1$$

$$V(x) = (x - c)^2 \quad \text{for } x \geq b \quad \text{(instantaneous stopping)}$$

$$V(0) = c^2$$

$$V'(b) = 2(b - c) \quad \text{(smooth fit)}.$$

The general solution of the differential equation is given by

$$V(x) = A + B((1 - x)^{1 - 2\mu/\sigma^2} - 1).$$

The three conditions determine the constants $A$, $B$ and $b$ uniquely, and it is straightforward to verify that the solution of the free-boundary problem is given by

$$V(x) = \begin{cases} 
  c^2 + (b^2 - 2bc) \frac{(1 - x)^{1 - 2\mu/\sigma^2} - 1}{(1 - b)^{1 - 2\mu/\sigma^2} - 1} & \text{if } 0 \leq x < b \\
  (x - c)^2 & \text{if } x \geq b
\end{cases},$$

where $b$ is the solution to equation (5.5) for $2c < b < 1$.

It remains to verify that the candidate indeed is the solution to the stopping problem. A verification argument employing the Itô formula as in the proof of Proposition 3.1 will provide a formal proof, but this is omitted.

Note
1. http://www.math.ku.dk/~jesper
References