Discounted Optimal Stopping Problems
for the Maximum Process

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The maximality principle [6] is shown to be valid in some examples of discounted optimal stopping problems for the maximum process. In each of these examples explicit formulas for the value functions are derived and the optimal stopping times are displayed. In particular, in the framework of the Black-Scholes model, the fair prices of two lookback options with infinite horizon are calculated. The main aim of the paper is to show that in each considered example the optimal stopping boundary satisfies the maximality principle and that the value function can be determined explicitly.

1. Introduction

The main purpose of the paper is to illustrate by examples that the maximality principle [6] remains valid for discounted optimal stopping problems involving the maximum process associated with a one-dimensional time-homogeneous diffusion. This is done by solving the problems explicitly (see Remark 2.2 and 2.6 below). The main interest for such a class of optimal stopping problems comes from option pricing theory in Mathematical Finance, and an example below is related to that. The motivation of this problem was the conjecture in the paper of Peskir [6] that the maximality principle holds in discounted optimal stopping problems for the maximum process (see also the paper of Graversen & Peskir [4]). For completeness, the method and the ideas are recalled here.

Let \((X_t, \mathbb{P}_x)\) denote a non-negative diffusion associated with the infinitesimal operator on \((0, \infty)\) given by

\[ L_X = \mu(x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} \]

where \(x \mapsto \mu(x)\) and \(x \mapsto \sigma(x) > 0\) are assumed to be continuous for \(x > 0\). Denote by \((S_t)\) the maximum process associated with \((X_t)\) given by

\[ S_t = \left( \max_{0 \leq r \leq t} X_r \right) \vee s \]

started at \(s \geq x\) under \(\mathbb{P}_{x,s} := \mathbb{P}_x\). Let the discounting rate \(x \mapsto \lambda(x) \geq 0\) be a continuous function, and let us define the functional

\[ \Lambda_t = \int_0^t \lambda(X_u) \, du. \]

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The optimal stopping problem, which is considered below, has the value function given by

\[ V_s(x, s) = \sup_{\tau} E_{x,s} \left( e^{-\Lambda \tau} \left( S_\tau - D(X_\tau) \right)^+ \right) \]

where the supremum is taken over all finite stopping times \( \tau \) for \( (X_t) \), and the cost function \( x \mapsto D(x) \) is a non-negative \( C^1 \)-function. An example (the Russian option) of problem (1.2) was solved in the framework of option pricing theory by Shepp & Shiryaev [9], [10], when the diffusion \( (X_t) \) is a geometric Brownian motion, \( \lambda(x) \) is a positive constant and \( D(x) \equiv 0 \) (see also [1] and [3]).

Let \((\hat{X}_t)\) be the killed diffusion at rate \( \lambda(\cdot) \) of \((X_t)\) [8]. If a new point \( \Delta \) is adjoined to the state space \( I = [0, \infty) \), and we set \( I_\Delta = [0, \infty) \cup \{\Delta\} \), the (homogeneous) transition function of the process \((\hat{X}_t)\) is given by

\[ \hat{P}_t(x, A) = E_x \left( e^{-\Lambda t} 1_A(\hat{X}_t) \right) \]

and the probability that \((\hat{X}_t)\) started at \( x \) gets killed at time \( t \) is

\[ \hat{P}_t(x, \{\Delta\}) = 1 - E_x \left( e^{-\Lambda t} \right) \]

The killed process \((\hat{X}_t)\) corresponds to the killing of the paths of \((X_t)\) at the rate \( \lambda \), and at the time of killing the process \((\hat{X}_t)\) takes the value \( \Delta \) and stays in \( \Delta \). The infinitesimal operator of \((\hat{X}_t)\) is given by

\[ L_{\hat{X}} = \mu(x) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2}{\partial x^2} - \lambda(x) = L_X - \lambda(x). \]

Note that in the special case of constant killing rate \( \lambda(x) \equiv \lambda > 0 \), the killing time is a random variable \( T \) independent of \((X_t)\) and has the exponential distribution of parameter \( \lambda \).

By the foregoing, it follows that the problem (1.2) reduces to the problem

\[ V_s(x, s) = \sup_{\tau} E_{x,s} \left( (\hat{S}_\tau - D(\hat{X}_\tau))^+ \right) \]

and, since the point \( \Delta \) cannot affect the value of \( X_t \) and due to the specific form of \( S_t \), we may take \( \hat{S}_t = S_t \). From the reduced problem (1.3) we are led to think that the value function \( V_s \) solves the equation (see [6])

\[ L_{\hat{X}} V(x, s) = 0 \quad \text{for} \quad g_s(s) < x < s \]

where \( s \mapsto g_s(s) < s \) is the optimal stopping boundary, and that the stopping time

\[ \tau_{g_s} = \inf \{ t > 0 : X_t \leq g_s(S_t) \} \]

is optimal, i.e. \( V_s(x, s) = E_{x,s} \left( \exp \left( -\Lambda \tau_{g_s} \right) \left( S_{\tau_{g_s}} - D(X_{\tau_{g_s}}) \right)^+ \right) \). For this reason, it is natural to think that the value function \( V_s \) and the optimal stopping boundary \( g_s \) solve the system

\[ \begin{align*}
L_{\hat{X}} V(x, s) &= \lambda(x) V(x, s) \quad \text{for} \quad g(s) < x < s \\
V(x, s) \bigg|_{x=g(s)+} &= s - D(g(s)) \quad \text{(instantaneous stopping)} \\
\frac{\partial V}{\partial x}(x, s) \bigg|_{x=g(s)+} &= -D'(g(s)) \quad \text{(smooth fit)} \\
\frac{\partial V}{\partial s}(x, s) \bigg|_{x=s-} &= 0 \quad \text{(normal reflection)}
\end{align*} \]

with \( L_{\hat{X}} \) as in (1.1) and where (1.5) and (1.6) are only apply when \( g(s) > 0 \).
The system (1.4)-(1.7) is a free boundary problem, and has no unique solution. But the maximality principle enables us to pick up the optimal stopping boundary \( g \) among all possible ones in a unique way, that is, the optimal stopping boundary \( s \mapsto g(s) \) is the maximal solution which stays below and never hits the diagonal in \( \mathbb{R}^2 \) (see Figure 1 and 2). This solution will be called the maximal solution in the sequel. Note that in general this system may have no simple solution. Thus, the system defines the boundary function \( s \mapsto g(s) \) implicitly and this is the main technical difficulty to verify the maximality principle for the problem (1.2) in full generality.

Example 1 is a simple example with past-depending discounting where the diffusion is a reflected Brownian motion. In Example 2 the diffusion is the square of the Bessel process and the optimal stopping problem is of the same type as Example 1. The main emphasis in these examples is on the explicit expressions obtained. The fair price of the perpetual lookback option with fixed/floating strike is calculated in Example 3 in the framework of Black-Scholes model. The optimal stopping boundary for the perpetual lookback option with fixed strike is rather nontrivial, thus showing the full power of the maximality principle.

2. Examples

In this section we explicitly solve some discounted optimal stopping problems for the maximum process by applying the technique described in the first section.

Throughout \((B_t)\) denotes a standard Brownian motion started at zero under \(P\).

**Example 1. Reflected Brownian motion.**

The main emphasis of this example (and the next) is on the closed formulas obtained for the value function and the optimal stopping time.

Let the diffusion \((X_t)\) be a reflected Brownian motion, i.e. \((X_t) = (|B_t + x|)\) started at \(x \geq 0\) under \(P_x\). The infinitesimal operator of \((X_t)\) on \((0, \infty)\) is given by

\[
L_x = \frac{1}{2} \frac{\partial^2}{\partial x^2}.
\]

In the setting of Section 1 with discounting rate \(\lambda(x) = x^{-2}\) and cost function \(D(x) \equiv 0\) the optimal stopping problem (1.2) is given by

\[
V^*(x, s) = \sup_{\tau} E_{x, s}(e^{-\Lambda_\tau} S_\tau)
\]

for \(0 < x \leq s\) where the functional \(\Lambda_t\) is given by

\[
\Lambda_t = \int_0^t |X_u|^{-2} du.
\]

We shall now solve the problem (2.2).

The first step is to solve the system (1.4)-(1.7) with \(L_x\) as in (2.1). The particular choice of the discounting rate \(\lambda(\cdot)\) makes (1.4) of Cauchy-type, and the general solution is

\[
V(x, s) = A(s) x^2 + B(s) x^{-1} \quad \text{for } g(s) < x < s
\]

where \(s \mapsto A(s)\) and \(s \mapsto B(s)\) are unknown functions. The instantaneous stopping condition (1.5) and the smooth fit condition (1.6) imply that

\[
A(s) = \frac{1}{3} g(s)^{-2} s \quad \text{and} \quad B(s) = \frac{2}{3} g(s) s
\]
so that \( V(x, s) = \frac{1}{3} g(s)^{-2} s x^2 + \frac{2}{3} g(s) s x^{-1} \) for \( g(s) < x < s \). Finally, the normal reflection condition (1.7) implies that \( s \mapsto g(s) \) satisfies the differential equation

\[
(2.3) \quad g'(s) = \left[ \frac{1}{2} \left( \frac{s}{g(s)} \right)^2 + \left( \frac{s}{g(s)} \right)^{-1} \right] / \left[ \left( \frac{s}{g(s)} \right)^3 - 1 \right].
\]

Instead of a long analysis of the first-order nonlinear differential equation (2.3), observe that \( g(s) = \beta s \) with \( \beta = (1/4)^{1/3} \) is a solution and \( g \) will be our candidate for the optimal stopping boundary, i.e. \( g \) should be the maximal solution (see Figure 1 below). Thus, the estimated candidate for the value function \( V_* \) in (2.2) is

\[
V(x, s) = \frac{1}{3} \beta^{-2} s^{-1} x^2 + \frac{2}{3} \beta s^2 x^{-1}
\]

for \( \beta s < x \leq s \) and the candidate for the optimal stopping time \( \tau_* \) is

\[
\tau = \inf \{ t > 0 : X_t \leq \beta S_t \}.
\]

Formulating the estimated formulas in the following proposition, the last step is to apply the Itô formula to prove the correctness of the proposition.

\[
\begin{align*}
V_* & (x, s) = \begin{cases} 
\frac{1}{3} \beta^{-2} s^{-1} x^2 + \frac{2}{3} \beta s^2 x^{-1} & \text{if } \beta s < x \leq s \\
 s & \text{if } 0 < x \leq \beta s 
\end{cases} \\
\tau_* & = \inf \{ t > 0 : X_t \leq \beta S_t \}
\end{align*}
\]

where \( \beta = (1/4)^{1/3} \).

**Figure 1.** A computer drawing of solutions of the differential equation (2.3). The bold line is the maximal solution which stays below and never hits the diagonal in \( \mathbb{R}^2 \). By the maximality principle, this solution equals \( g_* \).
Lebesgue measure zero, we have the equality
\[ V_{\tau}(t) = V(x, s) \]
\[ \int_0^t \lambda(X_u)e^{-\Lambda u} V(X_u, S_u) \, du + \int_0^t \frac{\partial V}{\partial x}(X_u, S_u) \, dB_u + \int_0^t e^{-\Lambda u} \frac{\partial V}{\partial s}(X_u, S_u) \, dS_u + \frac{1}{2} \int_0^t e^{-\Lambda u} \frac{\partial^2 V}{\partial x^2}(X_u, S_u) \, du. \]
The integral with respect to \( dS_u \) is identically zero, since the increment \( \Delta S_u \) is zero outside the diagonal in \( \mathbb{R}^2 \), while at the diagonal, \( V \) satisfies the normal reflection condition (1.7). Thus, we have
\[ e^{-\Lambda t} V(X_t, S_t) = V(x, s) + M_t + \int_0^t e^{-\Lambda u} \left( \mathbf{L}_x V(X_u, S_u) - \lambda(X_u)V(X_u, S_u) \right) \, du \]
where the process \( (M_t) \) is the continuous local martingale given by
\[ M_t = \int_0^t e^{-\Lambda u} \frac{\partial V}{\partial x}(X_u, S_u) \, dB_u. \]
Using that \( \mathbf{L}_x V(x, s) - \lambda(x)V(x, s) \leq 0 \) for \( 0 < x < g(s) \) and \( \mathbf{L}_x V(x, s) - \lambda(x)V(x, s) = 0 \) for \( g(s) < x < s \), and the fact that the set of all \( u > 0 \) for which \( X_u \) is either \( g(S_u) \) or \( S_u \) is of Lebesgue measure zero, we have the equality
\[ e^{-\Lambda \tau \wedge t} V(X_{\tau \wedge t}, S_{\tau \wedge t}) = V(x, s) + M_{\tau \wedge t} \]
and the inequality
\[ e^{-\Lambda t} V(X_t, S_t) \leq V(x, s) + M_t. \]
Let \( \tau \) be any stopping time for \( (X_t) \). Choose a localization \( \{\sigma_k\}_{k \geq 1} \) for \( (M_t) \) of bounded stopping times. Clearly, \( V(x, s) \geq s \) for all \( 0 < x \leq s \) and from (2.7) we get
\[ E_{x,s}(e^{-\Lambda \tau \wedge \sigma_k} S_{\tau \wedge \sigma_k}) \leq E_{x,s}(e^{-\Lambda \tau \wedge \sigma_k} V(X_{\tau \wedge \sigma_k}, S_{\tau \wedge \sigma_k})) \]
\[ \leq V(x, s) + E_{x,s}(M_{\tau \wedge \sigma_k}) = V(x, s) \]
for all \( k \geq 1 \). Letting \( k \to \infty \), it is immediately seen by Fatou’s lemma that
\[ E_{x,s}(e^{-\Lambda \tau} S_{\tau}) \leq V(x, s). \]
Taking supremum over all stopping times \( \tau \) for \( (X_t) \) we obtain
\[ V_s(x, s) \leq V(x, s). \]

Finally, to prove equality in (2.8) and that the value function \( V \) and the optimal stopping time \( \tau_* \) are given by (2.4) and (2.5) respectively, it is enough to prove that
\[ V(x, s) = E_{x,s}(e^{-\Lambda \tau_*} S_{\tau_*}). \]
By (2.6) and the definition of the stopping time \( \tau_* \), we have
\[ e^{-\Lambda \tau_*} S_{\tau_*} = e^{-\Lambda \tau} V(X_{\tau_*}, S_{\tau_*}) = V(x, s) + M_{\tau_*} \]
so the proof will be completed if we show that
\[ E_{x,s}(M_{\tau_*}) = 0. \]
By Doob’s optional sampling theorem and Burkholder-Davis-Gundy’s inequality for continuous local martingales, in order to prove (2.10) it is enough to show that

\begin{equation}
E_{x,s}\left(\sqrt{\int_{0}^{\tau^{*}}\left(e^{-\Lambda u}\frac{\partial V}{\partial x}(X_u, S_u)\right)^2 du}\right) := J < \infty.
\end{equation}

For this we compute

\[
\frac{\partial V}{\partial x}(x, s) = \frac{2}{3}\beta^{-1}\left(x - \frac{\beta s}{x}\right)^2 \leq \frac{2}{3}(\beta^2 - \beta)
\]

for \(\beta s < x < s\). Inserting this into (2.11) we get

\[
J = E_{x,s}\left(\sqrt{\int_{0}^{\tau^{*}}\left(e^{-\Lambda u}\frac{\partial V}{\partial x}(X_u, S_u)\right)^2 du}\right)
\leq \frac{2}{3}(\beta^2 - \beta) E_{x,s}(\sqrt{\tau^{*}}) < \infty
\]

provided that \(E_{x,s}(\sqrt{\tau^{*}}) < \infty\), which is known to be true (see [5]). The proof is completed. □

Remark 2.2. Up to (2.8) in the proof we did not make any use of the specific form of the optimal stopping boundary \(s \mapsto g^{*}(s)\) and the corresponding value function \(V^{*}\). Let \(g\) solve the equation (2.3) and stay below the diagonal in \(\mathbb{R}^2\). Let \(V_{g}\) be the corresponding function which solve the system (1.4)-(1.7). With exactly the same arguments as in the proof (2.8) it follows that \(V_{s} \leq V_{g}\). We also have that \(g \mapsto V_{g}\) is (strictly) decreasing. Therefore \(s \mapsto g^{*}(s)\) is the maximal solution, and thus this example illustrate the validity of the maximality principle.

Example 2. Bessel process.
This example is of the same type as Example 1. Thus, the results in this example will only be postulated, since the computations and proofs are almost the same as in Example 1.

Let \((X_t)\) be the square of a Bessel process of dimension \(\alpha > 0\) (see [7]) satisfying the stochastic differential equation

\[
dX_t = \alpha \, dt + 2\sqrt{X_t} \, dB_t.
\]

The infinitesimal operator of \((X_t)\) on \((0, \infty)\) is given by

\begin{equation}
L_X = \alpha \frac{\partial}{\partial x} + 2x \frac{\partial^2}{\partial x^2}.
\end{equation}

With discounting rate \(\lambda(x) = rx^{-1}\), where \(r > 0\) is a constant and cost function \(D(x) \equiv 0\), the optimal stopping problem (1.2) is given by

\begin{equation}
V^{*}(x, s) = \sup_{\tau} E_{x,s}(e^{-\Lambda_{\tau}} S_{\tau})
\end{equation}

for \(0 < x \leq s\), where the functional \(\Lambda_{\tau}\) is given by

\[
\Lambda_{\tau} = r \int_{0}^{\tau} X_u^{-1} du.
\]

Again, by the choice of the discounting rate \(\lambda(\cdot)\), (1.4) with \(L_X\) as in (2.12) is of Cauchy-type, and it is possible to find the solution to the system (1.4)-(1.7) which satisfies the maximality principle. It turns out that the optimal stopping boundary \(s \mapsto g^{*}(s)\) is a linear function (see Remark 2.4 below). The result is stated in the following theorem.
**Theorem 2.3.** Consider the optimal stopping problem (2.13). Let $\gamma_1 < \gamma_2$ be the two roots of the quadratic equation

$$2 \gamma^2 - (2 - \alpha) \gamma - r = 0$$

that is

$$\gamma_{1,2} = \frac{(2 - \alpha) \pm \sqrt{(2 - \alpha)^2 + 8r}}{4}.$$

If $r > \alpha$, then $\gamma_1 < 0$ and $\gamma_2 > 1$ and the value function $V_\ast$ is given by

$$V_\ast(x, s) = \begin{cases} 
\gamma_2 - \gamma_1 (\frac{x}{\beta s})^\gamma & \text{if } \beta s < x \leq s \\
\gamma_2 - \gamma_1 & \text{if } 0 < x \leq \beta s
\end{cases}$$

and the optimal stopping time $\tau_\ast$ is given by

$$\tau_\ast = \inf \{ t > 0 : X_t \leq \beta S_t \}$$

where $\beta$ is a constant given by

$$\beta = \left( \frac{1 - 1/\gamma_2}{1 - 1/\gamma_1} \right)^{1/(\gamma_2 - \gamma_1)}.$$

For $r \leq \alpha$ we have $V_\ast(x, s) = \infty$ and it is never optimal to stop.

**Remark 2.4.** The boundary function $s \mapsto g(s)$ in the system (1.4)-(1.7) satisfies the differential equation

$$g'(s) = \left[ \frac{1}{\gamma_2 \left( \frac{s}{g(s)} \right)^\gamma} - \frac{1}{\gamma_1 \left( \frac{s}{g(s)} \right)^\gamma} \right] \left[ \left( \frac{s}{g(s)} \right)^{\gamma + 1} - \left( \frac{s}{g(s)} \right)^{\gamma - 1} \right]$$

and the optimal stopping boundary $g_\ast(s) = \beta s$ is the maximal solution. This shows the validity of the maximality principle in this example.

**Example 3. Perpetual lookback options.**

In the framework of the standard Black-Scholes model under the equivalent martingale measure we shall consider two examples of pricing an American option with infinite horizon. Thus, the diffusion $(X_t)$ is a geometric Brownian motion satisfying the stochastic differential equation

$$dX_t = rX_t \, dt + \sigma X_t \, dB_t$$

where $r > 0$ and $\sigma > 0$ are two given constants. The infinitesimal operator of $(X_t)$ on $(0, \infty)$ is given by

$$L_X = r x \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2}.$$

Let us consider two lookback options.

**Option 1.** The payment function of the lookback option with fixed strike (called ‘option on extrema’ in [2]) is given by

$$f_t = e^{-\lambda t} (S_t - K)^+$$

where $\lambda > 0$ and $K \geq 0$ are two constants. For $K = 0$ it is the Russian option [9]. Under these assumptions, the fair price of the perpetual lookback option with fixed strike is, according to the general option pricing theory, the value of the optimal stopping problem

$$V_\ast(x, s) = \sup_{\tau} E_{x,s}(e^{-r\tau} f_\tau)$$
for $0 < x \leq s$. With the notation of Section 1, with discounting rate $\lambda(x) \equiv r + \lambda$ and cost function $D(x) \equiv K$, we can rewrite (2.16) as follows

$$(2.17) \quad V_s(x, s) = \sup_{\tau} \mathbb{E}_{x,s} \left( e^{-\lambda \tau} \left( S_{\tau} - K \right)^+ \right)$$

for $0 < x \leq s$, where the functional $\Lambda_t$ is given by

$$\Lambda_t = (r + \lambda) t.$$

The first step in solving problem (2.17) is to find all solutions to the system (1.4)-(1.7) with $L_X$ as in (2.15), and straightforward computations give that the solutions are

$$(2.18) \quad V(x, s) = \frac{s - K}{\gamma_2 - \gamma_1} \left[ \gamma_2 \left( \frac{x}{g(s)} \right)^{\gamma_1} - \gamma_1 \left( \frac{x}{g(s)} \right)^{\gamma_2} \right]$$

for $g(s) < x < s$, where $s \mapsto g(s)$ is to satisfy the differential equation

$$(2.19) \quad g'(s) = \left[ \frac{1}{\gamma_2} \left( \frac{s}{g(s)} \right)^{\gamma_2} - \frac{1}{\gamma_1} \left( \frac{s}{g(s)} \right)^{\gamma_1} \right] \gamma_1 \left( \frac{s}{g(s)} \right)^{\gamma_2} - \gamma_2 \left( \frac{s}{g(s)} \right)^{\gamma_1} \right]$$

and $\gamma_1 < 0$ and $\gamma_2 > 1$ are the two roots of the quadratic equation

$$\frac{1}{2} \sigma^2 \gamma^2 + (r - \frac{1}{2} \sigma^2) \gamma - (r + \lambda) = 0$$

that is,

$$(2.20) \quad \gamma_{1,2} = \frac{1}{2} - \frac{r}{\sigma^2} \pm \sqrt{\left( \frac{1}{2} + \frac{r}{\sigma^2} \right)^2 + \frac{2\lambda}{\sigma^2}}.$$

If $K = 0$, we see from Proposition 2.7 below (with $\kappa = 0$) that $g_*(s) = \beta s$ is the maximal solution to (2.19) which stays below and never hits the diagonal in $\mathbb{R}^2$, where $\beta$ is given by

$$\beta = \left( \frac{1 - \frac{1}{\gamma_2}}{1 - \frac{1}{\gamma_1}} \right)^{1/(\gamma_2 - \gamma_1)}.$$

If $K > 0$, by Picard’s method of successive approximations, we can establish the existence of the solution $g_*$ to (2.19) such that $g_*(s) < s$ and $|g_*(s) - \beta s| \to 0$ for $s \to \infty$. The proof of this statement is very technical and will be omitted (see [4] for a similar proof). The maximal solution should be $g_*$ (see Figure 2 below). Thus, we guess that the candidate for the optimal stopping time $\tau_*$ is

$$\tau_* = \inf \{ t > 0 : X_t \leq g_*(S_t) \}$$

and the candidate for the value function $V_*$ is given in (2.18) for $K < x \leq s$. If the process starts at $0 < x \leq s < K$, and $\tau_* = \tau_K + \tau_* \circ \theta_{\tau_K}$ is optimal, we have, by strong Markov property, that

$$V_*(x, s) = \mathbb{E}_x \left( \exp(-\Lambda_{\tau_K}) \right) V_*(K, K)$$

where

$$\tau_K = \inf \{ t > 0 : X_t = K \}.$$

It is well-known (see [7]) that

$$\mathbb{E}_x \left( \exp(-\Lambda_{\tau_K}) \right) = (x/K)^{\gamma_2}.$$

Let us formulate the estimated formulas in the following theorem.
Figure 2. A computer drawing of solutions of the differential equation (2.19). The bold line is the maximal solution which stays below and never hits the diagonal in $\mathbb{R}^2$. By the maximality principle, this solution equals $g_*$. The stipple line is the function $s \mapsto \beta s$.

**Theorem 2.5.** The fair price of the perpetual lookback option with fixed strike defined by (2.17) is given by

$$V_*(x,s) = \begin{cases} 
  s - K & \text{if } s > K \text{ and } g_*(s) < x \leq s \\
  (x/K)^{\gamma_2} V_*(K,K) & \text{if } 0 < x \leq s \leq K \\
  s - K & \text{if } s > K \text{ and } 0 < x \leq g_*(s)
\end{cases}$$

where

$$V_*(K,K) = \lim_{s \downarrow K} V_*(K,s).$$

The optimal stopping time is given by

$$\tau_* = \inf \{ t > 0 : X_t \leq g_*(S_t) \}$$

where $s \mapsto g_*(s)$ is the solution of the differential equation

$$g'(s) = \left[ \frac{1}{\gamma_2} \left( \frac{s}{g(s)} \right)^{\gamma_2} - \frac{1}{\gamma_1} \left( \frac{s}{g(s)} \right)^{\gamma_1} \right] / \left[ s - K \left( \frac{s}{g(s)} \right)^{\gamma_2} - \left( \frac{s}{g(s)} \right)^{\gamma_1} \right]$$

such that $g_*(s) < s$ for all $s > 0$ and

$$|g_*(s) - \beta s| \to 0 \text{ for } s \to \infty$$

where $\beta$ is a constant given by

$$\beta = \left( \frac{1 - 1/\gamma_2}{1 - 1/\gamma_1} \right)^{1/(\gamma_2 - \gamma_1)}$$

and $\gamma_1$ and $\gamma_2$ are defined in (2.20).
**Proof.** For $K = 0$ this follows directly from Proposition 2.7 below (with $\kappa = 0$). Assume that $K > 0$ and that the following computations are under $\mathbf{P}_{x,s}$. Denote the function on the right-hand side in the theorem by $V(x, s)$. Applying the Itô formula to the process $e^{-\lambda t}V(X_t, S_t)$, by the same arguments as in the proof of Theorem 2.1, we obtain

$$e^{-\lambda t}V(X_t, S_t) = V(x, s) + M_t + \int_0^t e^{-\lambda u} \left( \mathbf{L}_x V(X_u, S_u) - \lambda(X_u)V(X_u, S_u) \right) du$$

by means of (2.14), where the process $(M_t)$ is the continuous local martingale given by

$$M_t = \sigma \int_0^t e^{-\lambda u} \frac{\partial V}{\partial x}(X_u, S_u) X_u dB_u.$$

Similarly to the proof of Theorem 2.1 we have the equality

$$(2.21) \quad e^{-\Lambda_{\tau,\lambda} t} V(X_{\tau,\lambda}, S_{\tau,\lambda}) = V(x, s) + M_{\tau,\lambda}$$

and the inequality

$$(2.22) \quad V_{*}(x, s) \leq V(x, s).$$

It is enough to show by (2.21) that

$$\mathbf{E}_{x,s}(M_{\tau}) = 0$$

to prove equality in (2.22). It is easily seen by (2.21) that $(M_{\tau,\lambda})$ is bounded from below by $-V(x, s)$. Let $\varepsilon > 0$ be given and hence there exists $s'$ such that

$$\tau_* \leq \tau_\varepsilon' + \tau' \circ \theta_{\tau_*},$$

where $\tau'$ is a stopping time given by

$$\tau' = \inf \{ t > 0 : X_t \leq (\beta S_t - \varepsilon) \}.$$  

The upper bound for $(M_{\tau,\lambda})$ is then

$$M_{\tau,\lambda} \leq e^{-\Lambda_{\tau,\lambda} t} V(X_{\tau,\lambda}, S_{\tau,\lambda}) \leq e^{-\Lambda_{\tau,\lambda} t} V(S_{\tau,\lambda}, S_{\tau,\lambda}) \leq \sup_{t \leq \tau_*} e^{-\lambda t} V(S_t, S_t)$$

$$\leq \sup_{t \leq \tau_* + \tau' \circ \theta_{\tau_*}} e^{-\lambda t} V(S_t, S_t) \leq \left\{ \max_{t \leq \tau_*} e^{-\lambda t} V(S_t, S_t) \right\} \vee \left\{ \sup_{\tau_* \leq t \leq \tau' \circ \theta_{\tau_*}} e^{-\lambda t} V(S_t, S_t) \right\}$$

$$\leq k_1 \vee \left\{ \sup_{\tau_* \leq t \leq \tau' \circ \theta_{\tau_*}} e^{-\lambda t} \frac{S_t - K}{\gamma_2 - \gamma_1} \left( \gamma_2 \left( \frac{S_t}{g_* (S_t)} \right)^{\gamma_1} - \gamma_1 \left( \frac{S_t}{g_* (S_t)} \right)^{\gamma_2} \right) \right\}$$

$$\leq k_1 \vee \left\{ \sup_{\tau_* \leq t \leq \tau' \circ \theta_{\tau_*}} e^{-\lambda t} \frac{S_t}{\gamma_2 - \gamma_1} \left( \gamma - \gamma_1 \left( \frac{s'}{\beta s' - \varepsilon} \right)^{\gamma_2} \right) \right\} \leq k_1 + k_2 \sup_{t > 0} e^{-\lambda t} S_t$$

where $k_1$ and $k_2$ are two constants. The variable

$$\sup_{t > 0} e^{-\lambda t} S_t$$

is integrable (see [9]) and therefore $(M_{\tau,\lambda})$ is uniformly integrable and hence we can conclude that

$$\mathbf{E}_{x,s}(M_{\tau}) = 0.$$  

The proof is complete.
Remark 2.6. By the same arguments as in Remark 2.2 we see that \( s \mapsto g_\ast(s) \) is the maximal solution. Moreover, we do not see how the problem could be solved without the maximality principle.

Option 2. The payment function of the lookback option with floating strike (called ‘partial lookback’ in [2]) is given by

\[
h_t = e^{-\lambda t} (S_t - \kappa X_t)^+
\]

where \( \lambda > 0 \) and \( \kappa \geq 0 \) are constants. When \( \kappa = 0 \) it is the Russian option [9]. The fair price of the perpetual lookback option with floating strike is the value of the optimal stopping problem

\[
V_s(x, s) = \sup_{\tau} \mathbb{E}_{x,s} \left( e^{-r\tau} h_\tau \right)
\]

for \( 0 < x \leq s \). The only change from the lookback option with fixed strike is the cost function \( D(x) = \kappa x \) and we have the optimal stopping problem

\[
(2.23) \quad V_s(x, s) = \sup_{\tau} \mathbb{E}_{x,s} \left( e^{-\Lambda_\tau} (S_\tau - \kappa X_\tau)^+ \right).
\]

The problem (2.23) was solved in [10] in the special case \( \kappa = 0 \) (see also [9] and [3]) and in the general case the following proposition was proved in [1].

Proposition 2.7. The fair price of the perpetual lookback option with floating strike defined by (2.23) is given by

\[
V_s(x, s) = \begin{cases} 
x/\beta \\ 
\gamma_2 - \gamma_1 \\
\gamma_2 - \gamma_1 \\
\gamma_1 - \kappa \beta \left( 1 - 1/\gamma_1 \right) \\
\beta S_t - \kappa X_t \\
\gamma_1 - \kappa \beta \left( 1 - 1/\gamma_2 \right)
\end{cases} \left[ \left( \gamma_2 - \kappa \beta \left( \gamma_2 - 1 \right) \right) \left( \frac{x}{\beta s} \right)^{\gamma_1 - 1} - \left( \gamma_1 - \kappa \beta \left( \gamma_1 - 1 \right) \right) \left( \frac{x}{\beta s} \right)^{\gamma_2 - 1} \right] \quad \text{if } \beta s < x \leq s \\
0 < x \leq \beta s
\]

and the optimal stopping time \( \tau_\ast \) is given by

\[
\tau_\ast = \inf \{ t > 0 : X_t \leq \beta S_t \}
\]

where \( \beta \) is the unique solution to the equation

\[
\beta^{\gamma_2 - \gamma_1} = \frac{1 - 1/\gamma_2}{1 - 1/\gamma_1} \cdot \frac{1 - \kappa \beta (1 - 1/\gamma_1)}{1 - \kappa \beta (1 - 1/\gamma_2)}
\]

and \( \gamma_1 \) and \( \gamma_2 \) are defined in (2.20).

Remark 2.8. It is easily checked that the value function \( V_s \) and the optimal stopping boundary \( s \mapsto g_\ast(s) \) solve the system (1.4)-(1.7) with \( L_X \) in (2.15) which satisfies the maximality principle. However, in this case it is not the most natural method to solve the problem (2.23). Instead, by Girsanov’s change of measure, the two-dimensional problem can be reduced to a one-dimensional problem. The one-dimensional problem can be solved by different methods, but in any of the methods it is crucial that the process \( (S_t/X_t) \) is a new diffusion which is a special property of the geometric Brownian motion (see [10], [3] and [1]).

References


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