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First passage times of general sequences of random vectors: A large deviations approach [1]

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Abstract

Suppose $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ is a sequence of random variables such that the probability law of Y_n/n satisfies the large deviation principle and suppose $A \subset \mathbb{R}^d$. Let $T(A) = \inf\{n\colon Y_n \in A\}$ be the first passage time and, to obtain a suitable scaling, let $T^\varepsilon(A) = \varepsilon \inf\{n\colon Y_n \in A/\varepsilon\}$. We consider the asymptotic behavior of $T^\varepsilon(A)$ as $\varepsilon \to 0$. We show that the the probability law of $T^\varepsilon(A)$ satisfies the large deviation principle; in particular, $\boldsymbol{P}\{T^\varepsilon(A) \in C\} \approx \exp\{-\inf_{\tau \in C} I_A(\tau)/\varepsilon\}$ as $\varepsilon \to 0$, where $I_A(\cdot)$ is a large deviation rate function and C is any open or closed subset of $[0, \infty)$. We then establish conditional laws of large numbers for the normalized first passage time $T^\varepsilon(A)$ and normalized first passage place $Y^\varepsilon_{T^\varepsilon(A)}$. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $Y_1, Y_2, \ldots$ be a sequence of random variables taking values in $\mathbb{R}^d$. For any subset A of $\mathbb{R}^d$, let $T(A) = \inf\{n\colon Y_n \in A\}$ be the *first passage time*, i.e., the first time that the sequence $Y_1, Y_2, \ldots$ hits the set A. The purpose of this article is to study the distributional properties of $T(A)$ and, in effect, to determine the limiting behavior of $T(A)$ as the set A drifts to infinity, or, more precisely, the limiting behavior of

$$T^\varepsilon(A) = \varepsilon \inf\{n\colon Y^\varepsilon_n \in A\} \text{ as } \varepsilon \to 0 \qquad \text{where } Y^\varepsilon_n = \varepsilon Y_n.$$

Problems of this general type were first studied in the context of collective risk theory by Lundberg (1909). Letting $Y_t = ct - X_t$, where $\{X_t\}_{t \geqslant 0}$ is a compound Poisson process and c is a positive constant, he considered $\boldsymbol{P}\{Y_t < -1/\varepsilon, \text{ some } t\}$, namely the probability that the process $\{Y_t\}_{t \geqslant 0}$ *ever* hits the negative halfline $(-\infty, -1/\varepsilon)$. This is

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equivalent to $\boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}$, where $A=(-\infty,-1)$. A well-known result due to Cramér states that if $\{Y_t\}_{t\geqslant 0}$ has positive drift, then for certain constants C and R,

$$\boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}\sim C\mathrm{e}^{-R/\varepsilon} \quad \text{as } \varepsilon\to 0, \tag{1.1}$$

where R is identified as the nonzero element of the two-point set $\{\alpha\colon \Lambda(\alpha)=0\}$ and Λ is the logarithmic moment generating function; see Cramér (1954).

Extensions of Cramér's estimate have been widely studied, particularly in the setting of random variables taking values in $\mathbb{R}^1$. An extension to the d-dimensional setting has been given in Collamore (1996a), where it is shown under certain regularity conditions that if A is *any* open subset of $\mathbb{R}^d$, then

$$\lim_{\varepsilon\to 0}\varepsilon\log \boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}=-\inf_{x\in A}\tilde{I}(x), \tag{1.2}$$

where $\tilde{I}$ is the support function of the d-dimensional surface $\{\alpha\colon \Lambda(\alpha)\leqslant 0\}$. This limiting result is shown to hold, moreover, for general sequences $\{Y_n\}_{n\in\mathbb{Z}_+}$, provided that the probability law of Y_n/n satisfies the large deviation principle. [Various one-dimensional results for general sequences have been established by other authors; see Grandell (1991), Nyrhinen (1994) and references therein.]

While the above results describe $\boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}$ as $\varepsilon\to 0$, they give little insight into the actual distribution of $T^{\varepsilon}(A)$. In fact, it is quite easy to construct examples of sequences having the same exponential decay rates in Eq. (1.1), but for which the actual distributions of $T^{\varepsilon}(A)$ are very different. It is of interest to develop refinements of Eqs. (1.1) and (1.2) which yield an improved characterization of $T^{\varepsilon}(A)$.

In the setting of Eq. (1.1), such refinements have been given by von Bahr (1974) and Siegmund (1975). They have shown that if $Y_t=ct-X_t$, where $\{X_t\}_{t\geqslant 0}$ is a compound Poisson process, or if $\{Y_t\}_{t\geqslant 0}$ is a more general process, and if A is the half-line $(-\infty,-1)$, then

$$\boldsymbol{P}\{T^{\varepsilon}(A)\leqslant\tau(\varepsilon)\}\sim C\mathrm{e}^{-R/\varepsilon}\Phi(y) \quad \text{as } \varepsilon\to 0, \tag{1.3}$$

where $\Phi(\cdot)$ denotes the standard Normal distribution function, $\tau(\varepsilon)=\beta_1+\beta_2 y\sqrt{\varepsilon}$, and C,R,β_1 and β_2 are constants. Eq. (1.3) gives the same asymptotic decay for $\boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}$ as was given in Eq. (1.1), but it also shows that, conditioned on $\{T^{\varepsilon}(A)<\infty\}$, a proper rescaling of $T^{\varepsilon}(A)$ converges to a Normal distribution. We note that other relevant one-dimensional theorems have been developed by Segerdahl (1955); Martin-Löf (1986), who has established large deviation results, e.g. for $\boldsymbol{P}\{T^{\varepsilon}(A)\leqslant\tau_0\}$ as $\varepsilon\to 0$; and very recently by Nyrhinen (1998), who, under a technical condition on the lower bound, has established more complete large deviation results for general sequences $Y_1,Y_2,\ldots\subset\mathbb{R}^1$.

Our interest is in developing related limit theorems, but from a viewpoint more general than has been considered in the works of von Bahr, Siegmund, Martin-Löf and Nyrhinen. We are particularly interested in developing such theorems in the setting of the basic large deviations results given, for example, in Varadhan (1984), Ney and Nummelin (1987a, b) and Ellis (1984). Specifically, our objective is to study the case where A is a *general* subset of $\mathbb{R}^d$ and $Y_1,Y_2,\ldots$ a general sequence of random variables for which the probability law of Y_n/n satisfies the large deviation principle.

Under certain regularity conditions on $\{Y_n\}_{n\in\mathbb{Z}_+}\subset\mathbb{R}^d$ and $A\subset\mathbb{R}^d$, we show

$$\limsup_{\varepsilon\to 0}\varepsilon\log \boldsymbol{P}\{T^{\varepsilon}(A)\in F\}\leqslant -\inf_{\tau\in F} I_A(\tau)$$

$$\text{for all sets } F \text{ which are closed in } [0,\infty) \tag{1.4}$$

and

$$\liminf_{\varepsilon\to 0}\varepsilon\log \boldsymbol{P}\{T^{\varepsilon}(A)\in G\}\geqslant -\inf_{\tau\in G} I_A(\tau)$$

$$\text{for all sets } G \text{ which are open in } [0,\infty). \tag{1.5}$$

Thus, the probability law of $T^{\varepsilon}(A)$ satisfies the large deviation principle with rate function $I_A(\cdot)$. We show that Eqs. (1.4) and (1.5) hold quite generally, namely, when A is any subset of $\mathbb{R}^d$ and when $Y_1,Y_2,\ldots$ are the sums of an i.i.d. sequence of random variables, or the additive functions of a Markov chain, or a sequence satisfying the conditions of the Gärtner–Ellis theorem. The proofs of Eqs. (1.4) and (1.5) will rely on large deviations estimates, as $\varepsilon\to 0$, for *joint* probabilities of the form

$$\boldsymbol{P}\{(Y_m^{\varepsilon},Y_n^{\varepsilon})\in\mathfrak{A}\quad\text{some } (m,n)\in\mathfrak{C}/\varepsilon\},$$

where $\mathfrak{A}\subset\mathbb{R}^{2d}$, $\mathfrak{C}\subset\{(\tau_u,\tau_v)\colon\ \tau_v\geqslant\tau_u\geqslant 0\}$, and $\{Y_n\}_{n\in\mathbb{Z}_+}$ is a general sequence for which the probability law of Y_n/n satisfies the large deviation principle. See Theorem 4.2 below.

If $A\subset\mathbb{R}^d$ is convex, then the form of the function $I_A(\cdot)$ in Eqs. (1.4) and (1.5) suggests that there should be a most likely normalized first passage time, in the sense that we should have $T^{\varepsilon}(A)\approx\rho$ for some positive constant ρ. To this end, we show

$$\lim_{\varepsilon\to 0}\boldsymbol{P}\{|T^{\varepsilon}(A)-\rho|>\gamma|T^{\varepsilon}(A)<\infty\}=0\quad\text{for all } \gamma>0, \tag{1.6}$$

for a certain constant $\rho>0$. We also establish an analogous result for the normalized first passage place, $Y^{\varepsilon}_{T^{\varepsilon}(A)}$, namely,

$$\lim_{\varepsilon\to 0}\boldsymbol{P}\{||Y^{\varepsilon}_{T^{\varepsilon}(A)}-x_0||>\gamma|T^{\varepsilon}(A)<\infty\}=0\quad\text{for all } \gamma>0, \tag{1.7}$$

for a certain point x_0 which lies on the boundary of A. Hence, conditioned on the event $\{T^{\varepsilon}(A)<\infty\}$, $T^{\varepsilon}(A)$ converges in probability to ρ and $Y^{\varepsilon}_{T^{\varepsilon}(A)}$ converges in probability to x_0.

We note that large deviations theorems having a similar form to Eqs. (1.6) and (1.7) have been developed in various other settings. For example, the exit from a domain of a perturbed dynamical system near a point of stable equilibrium has been studied by Freidlin and Wentzell (1984), who have shown under certain circumstances that there is a most likely exit point. Also, certain large exceedance results have been established for Lévy processes $\subset\mathbb{R}^d$ by Dembo et al. (1994). These last results have recently been extended beyond the i.i.d. or Lévy setting by Zajic (1995).

2. Statement of results

Let $Y_1, Y_2, \dots$ be a sequence of random variables taking values in $\mathbb{R}^d$. Let $Y_n^\varepsilon = \varepsilon Y_n$ for all $\varepsilon > 0$ and all $n \in \mathbb{Z}_+$.

Our objective is to study

$$T^\varepsilon(A) = \varepsilon \inf\{n\colon\ Y_n^\varepsilon \in A\}$$

for general sets $A \subset \mathbb{R}^d$ and, particularly, to determine the limiting behavior of $T^\varepsilon(A)$ as $\varepsilon \to 0$.

First we introduce some further notation. Let

$$\Lambda(\alpha) = \lim_{n\to\infty} n^{-1} \log \boldsymbol{E} \exp\{\langle \alpha, Y_n\rangle\} \quad \text{for all } \alpha \in \mathbb{R}^d;$$

$$Z_{m,n} = (Y_m, Y_n - Y_m), \quad Z_{m,n}^\varepsilon = \varepsilon Z_{m,n} \quad \text{for all } n \geqslant m;$$

$$\Lambda_{m,n}(\alpha) = \log \boldsymbol{E} \exp\{\langle \alpha, Z_{m,n}\rangle\} \quad \text{for all } \alpha \in \mathbb{R}^{2d} \text{ and } n \geqslant m;$$

$$\Lambda_r(\alpha) = \lim_{\substack{n\to\infty \\ m/n \to r}} n^{-1} \log \boldsymbol{E} \exp\{\langle \alpha, Z_{m,n}\rangle\} \quad \text{for all } \alpha \in \mathbb{R}^{2d};$$

$$\mathscr{L}_a f = \{x\colon\ f(x) \leqslant a\} \quad \text{for any } f\colon \mathbb{R}^d \to \mathbb{R};$$

$$\text{cone } S = \{\lambda x\colon\ \lambda \geqslant 0, x \in S\};$$

and

$$\mathscr{B}_\delta = \left\{ x\colon\ \inf_{y \in \mathscr{L}_0 \Lambda^*} ||x - y|| < \delta \right\}.$$

(It is assumed that the limits in the definitions of Λ and Λ_r exist.) For any set S, let $\operatorname{ri} S, \partial S$ denote the relative interior, relative boundary of S, respectively; and for any function f, let f^*, $\operatorname{dom} f$, $\operatorname{cl} f$, $0^+ f$ denote the convex conjugate of f, the domain of f, the closure of f, and the recession function of f, respectively. [For definitions, see Rockafellar (1970).]

The following regularity conditions will be imposed on the sequence $\{Y_n\}_{n \in \mathbb{Z}_+}$ and the set A.

Hypotheses: (H0) The probability law of Y_n/n satisfies the large deviation principle with a rate function $I = \Lambda^*$, and Λ is differentiable at every point in its domain.

(H1) For each $r \in [0, 1]$ and $\alpha_u, \alpha_v \in \mathbb{R}^d$, $\Lambda_r(\alpha_u, \alpha_v) = r\Lambda(\alpha_u) + (1 - r)\Lambda(\alpha_v)$.

(H2) For some $\delta > 0$, $\operatorname{cl} A \cap \operatorname{cone} \mathscr{B}_\delta = \emptyset$.

To consider the nature of these hypotheses in the context of some standard examples of sequences $\{Y_n\}_{n \in \mathbb{Z}_+}$ satisfying the large deviation principle, suppose for example that $Y_n = X_1 + \cdots + X_n$, where $\{X_i\}_{i \in \mathbb{Z}_+}$ is an i.i.d. sequence of random variables. Then, by Cramér's theorem, the probability law of Y_n/n satisfies the large deviation principle as long as

$$\Lambda(\alpha) \equiv \lim_{n\to\infty} n^{-1} \log \boldsymbol{E} \exp\{\langle \alpha, Y_n\rangle\} = \log \boldsymbol{E} \exp\{\langle \alpha, X_1\rangle\} \tag{2.1}$$

is finite in a neighborhood of the origin. Since $0 \in \operatorname{dom} \Lambda$, a slightly stronger condition would be to assume that $\operatorname{dom} \Lambda$ is open. As the right-hand side of Eq. (2.1) is differentiable on the interior of its domain, "$\operatorname{dom} \Lambda$ open" would also imply that Λ is differentiable on its full domain. Hence, "$\operatorname{dom} \Lambda$ open" is sufficient to imply (H0). Next, observe by independence

$$\begin{aligned} \Lambda_r(\alpha_u, \alpha_v) &\equiv \lim_{\substack{n\to\infty,\\ m/n\to r}} n^{-1} \log \boldsymbol{E} \exp\{\langle \alpha_u, Y_m \rangle + \langle \alpha_v, Y_n - Y_m \rangle\} \\ &= \lim_{\substack{n\to\infty,\\ m/n\to r}} n^{-1} \{\log(\boldsymbol{E} \exp \langle \alpha_u, X_1 \rangle)^m + \log(\boldsymbol{E} \exp \langle \alpha_v, X_1 \rangle)^{n-m}\} \\ &= r\Lambda(\alpha_u) + (1-r)\Lambda(\alpha_v). \end{aligned} \tag{2.2}$$

Therefore, (H1) *always* holds. Finally, note that for i.i.d. sums $\mathscr{L}_0 \Lambda^* = \{\boldsymbol{E}X_1\}$. Hence, (H2) holds as long as the set A avoids an arbitrarily thin δ-cone about the mean ray $\{\lambda \boldsymbol{E}X_1 \colon \lambda \geqslant 0\}$, that is, as long as the mean of the process is directed *away* from the set A.

If $\{Y_n\}_{n\in\mathbb{Z}_+}$ is a Markov-additive process as defined in Ney and Nummelin (1987a, b), then the situation is analogous, that is, (H0), (H1) and (H2) hold as long as the domain of Λ is open and the set A avoids a thin δ-cone about the relevant mean vector. The situation is also analogous for general sequences satisfying the conditions of the Gärtner–Ellis theorem, except that in this case we do not automatically have (H1).

In our first result, we consider the decay of $\boldsymbol{P}\{T^\varepsilon(A) \in C\}$ as $\varepsilon \to 0$, where $C \subset [0,\infty)$. We show that this probability decays exponentially in ε, i.e.,

$$\boldsymbol{P}\{T^\varepsilon(A) \in C\} \approx \exp\left\{-\inf_{\tau\in C} I_A(\tau)/\varepsilon\right\},$$

with rate of decay described by a function $I_A(\cdot)$ defined as follows.

Definitions. (i) For any set $A \subset \mathbb{R}^d$, define $I_A : [0,\infty) \to [0,\infty)$ by

$$I_A(\tau) = \inf\left\{\tau\Lambda^*\left(\frac{x}{\tau}\right) : x \in A\right\} \quad \text{for all } \tau > 0,$$

and $I_A(0) = \inf\{(0^+\Lambda^*)(x) \colon x \in A\}$, where $0^+\Lambda^*$ is the recession function of Λ^*.

With slight abuse of notation, we will also write $\bar{I}_A(\cdot)$ for $I_{\operatorname{cl} A}(\cdot)$ and $\underline{I}_A(\cdot)$ for $I_{\operatorname{int} A}(\cdot)$.

(ii) For any set $C \in [0,\infty)$, define $J_C \colon \mathbb{R}^d \to [0,\infty)$ by

$$J_C(x) = \inf\left\{\tau\Lambda^*\left(\frac{x}{\tau}\right) : \tau \in C\right\} \quad \text{if 0 is not a limit point of } C,$$

and $J_C(x) = \min\{\inf_{\tau\in C-\{0\}} \tau\Lambda^*\left(\frac{x}{\tau}\right), (0^+\Lambda^*)(x)\}$ if 0 is a limit point of C.

Definition. A set A will be called a *semi-cone* if $x \in \partial A \Rightarrow \{\lambda x \colon \lambda > 1\} \subset \operatorname{int} A$, that is, the ray generated by any point on the relative boundary of A is an *interior* ray of A.

Theorem 1. *Let* $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ *be a sequence of random variables satisfying* (H0) *and* (H1), *and let* $A \subset \mathbb{R}^d$ *be a set satisfying* (H2).

(i) *Upper bound. For any set F which is closed in* $[0,\infty)$,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \boldsymbol{P}\{T^{\varepsilon}(A)\in F\} \leqslant -\inf_{\tau\in F} \bar{I}_A(\tau). \tag{2.3}$$

(ii) *Lower bound. If A is a semi-cone, then for any set G which is open in* $[0,\infty)$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \boldsymbol{P}\{T^{\varepsilon}(A)\in G\} \geqslant -\inf_{\tau\in G} \underline{I}_A(\tau). \tag{2.4}$$

Remark 2.1. (i) If $0\in \operatorname{dom}\Lambda^*$, then it is not necessary to take the infimum at the point $\tau=0$ (Rockafellar, 1970, Theorem 8.5). Likewise, if A and C are convex and A is open and intersects $\operatorname{ri}(\bigcup_{\tau\in C}\tau \operatorname{dom}\Lambda^*)$, then we do not need to take the infimum at $\tau=0$ (see Remark 3.3).

(ii) If $\operatorname{cone} A\subset \operatorname{dom}\Lambda^*$, then $\bar{I}_A(\tau)=\underline{I}_A(\tau)$ for all $\tau>0$. Therefore, under this assumption, the probability law of $T^{\varepsilon}(A)$ satisfies the large deviation principle with rate function I_A.

(iii) If $C\subset[0,\infty)$ is an interval and A is a convex open semi-cone intersecting $\operatorname{ri}(\bigcup_{\tau\in C}\tau \operatorname{dom}\Lambda^*)$, then Theorem 1 reduces to

$$\lim_{\varepsilon\to 0} \varepsilon \log \boldsymbol{P}\{T^{\varepsilon}(A)\in C\} = -\inf_{\tau\in C} I_A(\tau). \tag{2.5}$$

In the special case $C=[0,\infty)$, Eq. (2.5) describes the probability that the sequence $\{Y_n^{\varepsilon}\}$ *ever* hits the set A, which is Theorem 2.1 of Collamore (1996a). We note that Theorem 2.1 of Collamore (1996a) is proved under slighly weaker conditions; in particular, if $C=[0,\tau_0)$ for some $0<\tau_0\leqslant\infty$, then (H1) and the assumption that A is a semi-cone can be dropped.

(iv) In general, the condition that A is a semi-cone *cannot* be dropped to obtain the stated lower bound. However, if $A\subset\mathbb{R}^d$ is convex and the point x_0 given below in Lemma 2.2(i) is an *exposed point*, in the sense that the ray joining 0 to x_0 does not intersect $\operatorname{cl} A$ except at x_0, then this condition can be dropped.

Theorem 1 suggests that if $I_A(\tau)$ is minimized for a *unique* $\tau=\rho$, then the most likely normalized first passage time should be $T^{\varepsilon}(A)\approx\rho$.

Lemma 2.2. $Y_1,Y_2,\ldots\subset\mathbb{R}^d$ *be a sequence of random variables having a differentiable logarithmic moment generating function*, Λ, *and let* $A\subset\mathbb{R}^d$ *be a convex set satisfying* (H2), $A\cap \operatorname{ri}\operatorname{cone}(\operatorname{dom}\Lambda^*)\neq\emptyset$. *Then*:

(i) $\inf_{x\in \operatorname{cl}A} J_{[0,\infty)}(x)$ *is achieved over* $\operatorname{cl}A$ *at a unique point* $x_0\in\partial A$.

(ii) *At some point* α_0 *on the zero-set* $\{\alpha\colon \Lambda(\alpha)=0\}$, *the gradient vector of* Λ *points in the direction of* x_0, *that is,*

$$x_0=\rho\nabla\Lambda(\alpha_0) \quad \textit{for some constant } \rho>0.$$

(iii) $\inf_{\tau\in[0,\infty)} I_A(\tau)$ *is achieved over* $[0,\infty)$ *at the unique point* ρ *given in* (ii).

A stronger version of this lemma will be proved below in Theorems 3.4, 3.7 and 6.1. Also see Remarks 3.3 and 3.5 and the discussion just prior to Theorem 6.1.

Theorem 2. *Let* $Y_1,Y_2,\ldots\subset\mathbb{R}^d$ *be a sequence of random variables satisfying* (H0) *and* (H1). *Let A be a convex open set satisfying* (H2), $A\cap \operatorname{ri}\operatorname{cone}(\operatorname{dom}\Lambda^*)\neq\emptyset$. *Then* *for any* $\gamma > 0$,

$$\lim_{\varepsilon\to 0} \boldsymbol{P}\{|T^\varepsilon(A) - \rho| > \gamma \,|T^\varepsilon(A) < \infty\} = 0, \tag{2.6}$$

where ρ *is the positive constant appearing in Lemma* 2.2(iii).

Since the rate function in Eq. (2.5) for the interval $C = [0, \infty)$ is

$$\inf_{\tau\in[0,\infty)} I_A(\tau) \equiv \inf_{x\in A} \min\left\{ \inf_{\tau>0} \tau\Lambda^*\left(\frac{x}{\tau}\right), (0^+\Lambda^*)(x) \right\} \equiv \inf_{x\in A} J_{[0,\infty)}(x),$$

another natural consequence of Lemma 2.2 is the following.

Theorem 3. *Let* $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ *be a sequence of random variables satisfying* (H0) *and* (H1). *Let* A *be a convex open set satisfying* (H2), $A \cap \mathrm{ri\ cone}\,(\mathrm{dom}\,\Lambda^*) \neq \emptyset$. *Then for any* $\gamma > 0$,

$$\lim_{\varepsilon\to 0} \boldsymbol{P}\left\{ ||Y^\varepsilon_{T^\varepsilon(A)} - x_0|| > \gamma \,|T^\varepsilon(A) < \infty \right\} = 0, \tag{2.7}$$

where x_0 *is the element of* ∂A *appearing in Lemma* 2.2(i).

Remark 2.3. Some of the conditions in Theorems 1–3 can be slightly weakened, as follows.

(i) Let

$$T^\varepsilon_N(A) = \varepsilon \inf\{n \geqslant N:\ Y^\varepsilon_n \in A\},$$

that is, the first time after an initial time N that $\{Y^\varepsilon_n\}_{n\in\mathbb{Z}_+}$ hits A. If A is a convex open set and N is suitably large, then Theorem 1(i), 2 and 3 hold for $T^\varepsilon_N(A)$ without assuming (H1). If A is a general set, then these theorems hold for $T^\varepsilon_{N_0}(A)$, some $N_0 \geqslant 1$, with the weaker condition (H1′) of Collamore (1996a,b) in place of (H1). For details, see Collamore (1996b).

(ii) If A is a general set, then Theorems 2 and 3 hold provided that: (a) $\inf_{x\in A} J_{[0,\infty)}(x)$ is achieved over $\mathrm{cl}\,A$ at a *unique* point x_0, and (b) the infimum in the definition of $J_{[0,\infty)}$ is the same over $\mathrm{int}\,A$ as it is over $\mathrm{cl}\,A$ [as is the case, e.g., when A is open and contained in $\mathrm{int\ cone}\ (\mathrm{dom}\,\Lambda^*)$].

Example 2.4. First we consider the classical ruin model studied, e.g. in Cramér (1954), namely assume

$$Y_t = ct - \sum_{i=1}^{N(t)} X_i, \tag{2.8}$$

where $N(t)$ is a Poisson(λ) process, $\{X_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{R}^1$ is an independent sequence of random variables, and $c - \lambda \boldsymbol{E}X_1 = \boldsymbol{E}Y_1 > 0$. For simplicity, assume that the distribution of X_i is exponential (θ). Let A denote the interval $(-\infty, -1)$ and consider

$$T^\varepsilon(A) = \varepsilon \inf\left\{ n \in \mathbb{Z}_+:\ Y_n < -\frac{1}{\varepsilon} \right\}. \tag{2.9}$$

The logarithmic moment generating function for the discrete sequence $\{Y_n\}_{n\in\mathbb{Z}_+}$ is

$$\Lambda(\alpha)=\begin{cases} -\dfrac{\lambda\theta\alpha}{1+\theta\alpha}+c\alpha & \text{for all } \alpha>-\dfrac{1}{\theta}, \\ \infty & \text{otherwise.}\end{cases} \tag{2.10}$$

It follows that

$$I_A(\tau)=\tau\Lambda^*\left(\frac{x}{\tau}\right)\Big]_{x=-1}=\left(\sqrt{\frac{1+c\tau}{\theta}}-\sqrt{\lambda\tau}\right)^2 \quad \text{for all } \tau>0, \tag{2.11}$$

and $\Lambda^*(0)<\infty$. By Theorem 1 and Remark 2.1, we may then evaluate $\lim_{\varepsilon\to0}\varepsilon\log \boldsymbol{P}\{T^\varepsilon(A)\in C\}$ by minimizing the function $I_A(\tau)$ over $\tau\in C-\{0\}$ [more precisely, over $(\operatorname{cl} C-\{0\})$ to obtain an upper bound and over $(\operatorname{int} C-\{0\})$ to obtain a lower bound, where C is regarded as a subset of $[0,\infty)$]. Since $\{\alpha\colon \Lambda(\alpha)=0\}=\{0,(\lambda\theta-c)/c\theta\}$, we obtain by Theorem 2 that the *most likely* normalized first passage time is

$$\rho=-\frac{1}{\nabla\Lambda(\alpha_0)}\Big]_{\alpha_0=(\lambda\theta-c)/c\theta}=\frac{\lambda\theta}{c^2-c\lambda\theta}, \tag{2.12}$$

and, as expected, this is the minimum of $I_A(\cdot)$ over $\tau\in(0,\infty)$.

To contrast these results with those of von Bahr (1974) and Siegmund (1975), now let $T^\varepsilon(A)$ be defined as in Eq. (2.9) but with $\{Y_t\}_{t\geqslant0}$ in place of $\{Y_n\}_{n\in\mathbb{Z}_+}$. Then by Theorem 2 of Siegmund (1975),

$$\boldsymbol{P}\{T^\varepsilon(A)\leqslant\tau(\varepsilon)\}\sim C\mathrm{e}^{-R/\varepsilon}\Phi(y) \quad \text{as } \varepsilon\to0, \tag{2.13}$$

where $\Phi(\cdot)$ is the standard Normal distribution function, C and $R=(c-\lambda\theta)/c\theta$ are constants, and

$$\tau(\varepsilon)=\frac{\lambda\theta}{c^2-c\lambda\theta}+\sqrt{\frac{2\lambda\theta^2}{(c-\lambda\theta)^3}}\,y\sqrt{\varepsilon}. \tag{2.14}$$

[The values in Eq. (2.14) are obtained by computing moments associated with the sequences $\{X_i\}$ and $\{T_i\}$, where T_i is the ith interarrival time of the Poisson process $N(t)$.]

By setting $y=0$, it is evident from Eqs. (2.13) and (2.14) that the most likely normalized first passage time as $\varepsilon\to0$ is $\rho=\lambda\theta/(c^2-c\lambda\theta)$, which is in agreement with Eq. (2.12). But the difference between Eq. (2.13) and our results is that Eq. (2.13) studies the variation of $T^\varepsilon(A)$ about ρ for intervals which *decrease* after normalization by $C\sqrt{\varepsilon}$ as $\varepsilon\to0$; this results in a limiting Normal distribution. Our results study the variation of $T^\varepsilon(A)$, e.g. over intervals whose distance away from ρ after normalization is *fixed*; this leads to exponentially small probabilities with decay characterized by a certain large deviation rate function.

A primary advantage of our approach lies in its generality and, especially, its ability to handle general sets $A\subset\mathbb{R}^d$ where $d>1$. A very simple multidimensional example is the following.

Example 2.5. Let $Y_n=X_1+\cdots+X_n$, where $\{X_i\}_{i\in\mathbb{Z}_+}\subset\mathbb{R}^d$ in an i.i.d. Normal sequence with mean vector μ and positive definite covariance S, and let $A\subset\mathbb{R}^d$ be an open semi-cone which is disjoint from a thin δ-cone about μ.

The logarithmic moment generating function of $\{Y_n\}_{n\in\mathbb{Z}_+}$ is

$$\Lambda(\alpha)=\langle\alpha,\mu\rangle+\tfrac{1}{2}\langle\alpha,S\alpha\rangle. \tag{2.15}$$

It follows that

$$I_A(\tau)=\inf_{x\in A}\tau\Lambda^*\left(\frac{x}{\tau}\right)=\inf_{x\in A}\left[\frac{\tau}{2}\left\langle\left(\frac{x}{\tau}-\mu\right),\ S^{-1}\left(\frac{x}{\tau}-\mu\right)\right\rangle\right] \quad \text{for all} \quad \tau>0. \tag{2.16}$$

By Theorem 1 and Remark 2.1, we may then evaluate $\lim_{\varepsilon\to 0}\varepsilon\log \boldsymbol{P}\{T^{\varepsilon}(A)\in C\}$ by minimizing the function $I_A(\tau)$ over $\tau\in C-\{0\}$.

For example, if $\mu=-(1/\sqrt{d})(1,\dots,1)$, $S=I$ [the identity matrix], and $A=\{(x_1,\dots,x_d)\colon x_i>1\}$, then $I_A(\tau)=(\tau/2)\cdot d(1/\tau+(1/\sqrt{d}))^2$, which, among other things, has a minimum value at $\rho=\sqrt{d}$. The existence and computation of this minimum value (corresponding to the most likely normalized first passage time) can also be obtained from Theorem 2. By symmetry, the unique element $x_0\in\partial A$ of Lemma 2.2(i) is $(1,\dots,1)$. Then the element $\alpha_0\in\partial(\mathscr{L}_0\Lambda)$ of Lemma 2.2(ii) is specified by the condition that $\nabla\Lambda(\alpha)$ is parallel to x_0. By Eq. (2.15) it follows that $\alpha_0=(2/\sqrt{d})(1,\dots,1)$ and then $\nabla\Lambda(\alpha_0)=(1/\sqrt{d})(1,\dots,1)$. Therefore, by Theorem 2, the most likely normalized first passage time is $\rho=\sqrt{d}$.

3. Preliminary results from convex analysis

Notation

$$\mathscr{H}^{+}(\alpha,t)=\{x\in\mathbb{R}^d\colon \langle\alpha,x\rangle\geqslant t\} \quad \text{for all } \alpha\in\mathbb{R}^d \text{ and } t\in\mathbb{R};$$

$$\mathscr{H}^{-}(\alpha,t)=\{x\in\mathbb{R}^d\colon \langle\alpha,x\rangle\leqslant t\} \quad \text{for all } \alpha\in\mathbb{R}^d \text{ and } t\in\mathbb{R};$$

$$S+T=\{s+t\colon s\in S,\ t\in T\} \quad \text{for all sets } S \text{ and } T.$$

For any set S, let $\operatorname{ri} S$, ∂S denote the relative interior of S, relative boundary of S, respectively.

For any function f, let $f^*(\cdot)$, $\operatorname{dom} f$, $\operatorname{cl} f$, $\operatorname{epi} f$, and $\partial f(\cdot)$ denote the convex conjugate of f, the domain of f, the closure of f, the epigraph of f, and the subgradient set of f, respectively.

For any set S, let 0^+S denote the recession cone of S; and for any function f, let $0^+f(\cdot)$ denote the recession function of f. (For definitions, see Rockafellar, 1970.)

Our main objective in this section is to develop the convexity properties of the following two functions.

Definitions. Let Λ denote the logarithmic moment generating function, as introduced in Section 2.

(i) For any convex set $\mathrm{C}\subset[0,\infty)$, let

$$\Gamma_{\mathrm{C}}(\alpha)=\sup_{\tau\in\mathrm{C}}\tau\Lambda(\alpha).$$

(ii) For any convex set $\mathfrak{C} \subset \{(\tau_u, \tau_v)\colon \tau_v \geqslant \tau_u \geqslant 0\}$, let

$$\Gamma_{\mathfrak{C}}(\alpha_u, \alpha_v) = \sup_{(\tau_u,\tau_v)\in\mathfrak{C}} \{\tau_u \Lambda(\alpha_u) + (\tau_v - \tau_u)\Lambda(\alpha_v)\}.$$

In the next two theorems, we establish the relevance of the functions $\Gamma_{\mathrm{C}}(\cdot)$ and $\Gamma_{\mathfrak{C}}(\cdot)$ by relating them to the rate functions $I_A(\cdot)$ and $J_\tau(\cdot)$ introduced just prior to Theorem 1.

Theorem 3.1. *Let Ξ be a convex set contained in the positive orthant $\{(\xi_1, \ldots, \xi_k)\colon \xi_1 > 0, \ldots, \xi_k > 0\}$.*
(i) *If $f\colon \mathbb{R}^d \to \mathbb{R}$ is a convex function, then the function*

$$F(x_1, \ldots, x_k) = \inf_{(\xi_1,\ldots,\xi_k)\in\Xi} \left\{ \xi_1 f\left(\frac{x_1}{\xi_1}\right) + \cdots + \xi_k f\left(\frac{x_k}{\xi_k}\right) \right\}$$

is also convex.
(ii) *If $f\colon \mathbb{R}^d \to \mathbb{R}$ is a closed convex function, then the convex conjugate of*

$$F(\alpha_1, \ldots, \alpha_k) = \sup_{(\xi_1,\ldots,\xi_k)\in\Xi} \{\xi_1 f(\alpha_1) + \cdots + \xi_k f(\alpha_k)\}$$

is $\mathrm{cl}\, G$, *where*

$$G(x_1, \ldots, x_k) = \inf_{(\xi_1,\ldots,\xi_k)\in\Xi} \left\{ \xi_1 f^*\left(\frac{x_1}{\xi_1}\right) + \cdots + \xi_k f^*\left(\frac{x_k}{\xi_k}\right) \right\}.$$

Proof. (i) Define

$$F_\xi(x_1, \ldots, x_k) = \xi_1 f\left(\frac{x_1}{\xi_1}\right) + \cdots + \xi_k f\left(\frac{x_k}{\xi_k}\right) \quad \text{and} \quad \mathfrak{F} = \bigcup_{(\xi_1,\ldots,\xi_k)\in\Xi} \mathrm{epi}\, F_\xi.$$

Then evidently

$$F(x) = \inf\{\mu\colon (x, \mu) \in \mathfrak{F}\}. \tag{3.1}$$

To show that F is convex, note that the epigraph of $x \to \lambda f(x/\lambda)$ is $\lambda(\mathrm{epi}\, f)$, for all $\lambda > 0$. Letting

$$\mathfrak{F}_i = \{(x_1, \ldots, x_k, \mu)\colon (x_i, \mu) \in \mathrm{epi}\, f \text{ and } x_j = 0 \text{ for } j \neq i\} \subset \mathbb{R}^{kd+1}, \tag{3.2}$$

it follows that

$$\mathrm{epi}\, F_\xi = \xi_1 \mathfrak{F}_1 + \cdots + \xi_k \mathfrak{F}_k. \tag{3.3}$$

Now let $\mathfrak{f}_u, \mathfrak{f}_v \in \mathfrak{F}$ and $0 < \lambda < 1$. Then by the definition of $\mathfrak{F}$ and Eq. (3.3): $\mathfrak{f}_u = \xi_1^{(u)} \mathfrak{f}_1^{(u)} + \cdots + \xi_k^{(u)} \mathfrak{f}_k^{(u)}$ for some $\xi^{(u)} \in \Xi$ and $\mathfrak{f}_i^{(u)} \in \mathfrak{F}_i$, $i = 1, \ldots, k$; and similarly $\mathfrak{f}_v = \xi_1^{(v)} \mathfrak{f}_1^{(v)} + \cdots + \xi_k^{(v)} \mathfrak{f}_k^{(v)}$ for some $\xi^{(v)} \in \Xi$ and $\mathfrak{f}_i^{(v)} \in \mathfrak{F}_i$, $i = 1, \ldots, k$. Then

$$\begin{aligned} \lambda \mathfrak{f}_u + (1-\lambda)\mathfrak{f}_v &= (\lambda \xi_1^{(u)} \mathfrak{f}_1^{(u)} + (1-\lambda)\xi_1^{(v)} \mathfrak{f}_1^{(v)}) + \cdots \\ &= \xi_1 \left(\frac{\lambda \xi_1^{(u)}}{\xi_1} \mathfrak{f}_1^{(u)} + \frac{(1-\lambda)\xi_1^{(v)}}{\xi_1} \mathfrak{f}_1^{(v)} \right) + \cdots, \end{aligned} \tag{3.4}$$

where $\xi_1 = \lambda\xi_1^{(u)} + (1-\lambda)\xi_1^{(v)}$ and so on for $\xi_2,\ldots,\xi_k$. On the last line of Eq. (3.4), the two scalars inside the brackets sum to one; hence the convexity of $\mathfrak{F}_1$ implies that this quantity in brackets is an element of $\mathfrak{F}_1$, and so on for the indices $2,\ldots,k$. Also, the convexity of Ξ implies that $(\xi_1,\ldots,\xi_k)\in\Xi$. Therefore,

$$\lambda\mathfrak{f}_u + (1-\lambda)\mathfrak{f}_v \in \xi_1\mathfrak{F}_1 + \cdots + \xi_k\mathfrak{F}_k = \operatorname{epi} F_\xi \in \mathfrak{F}. \tag{3.5}$$

We conclude that $\mathfrak{F}$ is convex. The convexity of F then follows from Eq. (3.1) and Theorem 5.3 of Rockafellar (1970).

(ii) Define

$$F_\xi(\alpha_1,\ldots,\alpha_k) = \xi_1 f(\alpha_1) + \cdots + \xi_k f(\alpha_k). \tag{3.6}$$

Then the convex conjugate of F_ξ is

$$F_\xi^*(x_1,\ldots,x_k) = \xi_1 f^*\left(\frac{x_1}{\xi_1}\right) + \cdots + \xi_k f^*\left(\frac{x_k}{\xi_k}\right), \tag{3.7}$$

and an affine function $h: x \to \langle\alpha, x\rangle - \mu$ minorizes $G \Leftrightarrow h$ minorizes F_ξ^* for all $\xi\in\Xi$. By definition of the convex conjugate and Theorem 12.2 of Rockafellar (1970), this occurs $\Leftrightarrow (\alpha,\mu)\in\operatorname{epi} F_\xi$ for all $\xi\in\Xi$; in other words, $\Leftrightarrow (\alpha,\mu)\in\operatorname{epi} F$. Since G is convex, by (i), we conclude $F = G^*$. Hence $F^* = \operatorname{cl} G$ (Rockafellar, 1970, Theorem 12.2). □

Next we identify the function $\operatorname{cl} G$ of the previous theorem.

Theorem 3.2. *Let $f: \mathbb{R}^d \to \mathbb{R}$ be a closed proper convex function with $f(0)>0$, and let Ξ be a convex set contained in the positive orthant $\{(\xi_1,\ldots,\xi_k): \xi_1>0,\ldots,\xi_k>0\}$. Let $(f\lambda)(x) = \lambda f(x/\lambda)$, for all $\lambda>0$ and $x\in\mathbb{R}^d$, and let*

$$F(x_1,\ldots,x_k) = \inf_{(\xi_1,\ldots,\xi_k)\in\Xi} \{(\xi_1 f)(x_1) + \cdots + (\xi_k f)(x_k)\}.$$

Then

$$\operatorname{cl} F(x_1,\ldots,x_k) = \inf_{(\xi_1,\ldots,\xi_k)\in\tilde{\Xi}} \{(\xi_1 f)(x_1) + \cdots + (\xi_k f)(x_k)\},$$

where $\tilde{\Xi} = \operatorname{cl}\Xi$ but with each $\xi_i = 0$ replaced with $\xi_i = 0^+$ [so that the infimum is taken in this case over $0^+ f$, the recession function of f].

Proof. Let $K\subset\mathbb{R}^{d+2}$ be the convex cone generated by $\{(1,y): y\in\operatorname{epi} f\}$. Since f is a closed proper convex function, it follows that

$$\operatorname{cl} K = \{(\lambda,y): \lambda>0,\ y\in\lambda(\operatorname{epi} f)\}\cup\{(0,y): y\in 0^+(\operatorname{epi} f)\} \tag{3.8}$$

(Rockafellar, 1970, Theorem 8.2). Define

$$H = \{(\xi_1,y_1,\ldots,\xi_k,y_k): (\xi_1,\ldots,\xi_k)\in\Xi \text{ and } y_i\in\mathbb{R}^{d+1},\ i=1,\ldots,k\}$$

and

$$L = (K\times\cdots\times K)\cap H \subset \mathbb{R}^{k(d+2)}.$$

We study the image of the convex set L under the transformation

$$\mathscr{A}\colon (\xi_1, y_1, \ldots, \xi_k, y_k) \to (x_1, \ldots, x_k, \mu_1 + \cdots + \mu_k),\ \xi_i \in \mathbb{R} \quad \text{and}$$
$$y_i = (x_i, \mu_i) \in \mathbb{R}^d \times \mathbb{R}.$$

It follows directly from the definitions that

$$\mathscr{A}(L) = \{(x_1, \ldots, x_k, \mu_1 + \cdots + \mu_k)\colon (x_i, \mu_i) \in \xi_i(\operatorname{epi} f) \text{ and } (\xi_1, \ldots, \xi_k) \in \Xi\},$$
$$\operatorname{cl}(\mathscr{A}(L)) = \operatorname{cl}(\operatorname{epi} F). \tag{3.9}$$

Since $\operatorname{cl} L = (\operatorname{cl} K \times \cdots \times \operatorname{cl} K) \cap \operatorname{cl} H$, these definitions and Eq. (3.8) also imply

$$\mathscr{A}(\operatorname{cl} L) = \{(x_1, \ldots, x_k, \mu_1 + \cdots + \mu_k)\colon (x_i, \mu_i) \in \xi_i(\operatorname{epi} f) \text{ and } (\xi_1, \ldots, \xi_k) \in \tilde{\Xi}\}, \tag{3.10}$$

where $\tilde{\Xi} = \operatorname{cl} \Xi$ but with $\xi_i = 0$ replaced with $\xi_i = 0^+$ [so that for such ξ_i we take $(x_i, \mu_i) \in 0^+(\operatorname{epi} f)$, the recession cone of $\operatorname{epi} f$]. Finally, note $\operatorname{cl} \mathscr{A}(L) = \mathscr{A}(\operatorname{cl} L)$ (Rockafellar, 1970, Theorem 9.1, since $f(0) \neq 0$ implies that the only point of $0^+(\operatorname{cl} L)$ which is mapped by $\mathscr{A}$ to zero is zero itself). Thus we conclude

$$\operatorname{cl}(\operatorname{epi} F) = \{(x_1, \ldots, x_k, \mu_1 + \cdots + \mu_k)\colon (x_i, \mu_i) \in \xi_i(\operatorname{epi} f) \text{ and } (\xi_1, \ldots, \xi_k) \in \tilde{\Xi}\}. \tag{3.11}$$

Since $\operatorname{epi}(\xi_i f)$ is $\xi_i(\operatorname{epi} f), \xi_i \neq 0$, and $0^+(\operatorname{epi} f)$ is $\operatorname{epi}(0^+ f)$, the theorem follows from (3.11). □

Remark 3.3. We now apply Theorems 3.1 and 3.2 to relate Γ_C^* and $\Gamma_{\mathfrak{C}}^*$ to the rate functions I_A and J_C.

(i) Suppose $C \subset [0, \infty)$ is convex. If 0 is a limit point of C, then it follows from Theorems 3.1 and 3.2 that

$$\Gamma_C^*(x) = \operatorname{cl}\left\{\inf_{\tau \in C} \tau \Lambda^*\left(\frac{x}{\tau}\right)\right\} = \min\left\{\inf_{\tau \in C - \{0\}} \tau \Lambda^*\left(\frac{x}{\tau}\right),\ (0^+ \Lambda^*)(x)\right\}. \tag{3.12}$$

If 0 is not a limit point of C, then $(0^+ \Lambda^*)(x)$ may be dropped from the infimum on the right of Eq. (3.12). Thus we obtain

$$\Gamma_C^*(x) = J_C(x) \quad \text{and} \quad \inf_{x \in A} \Gamma_C^*(x) = \inf_{\tau \in \operatorname{cl} C} I_A(\tau), \tag{3.13}$$

for any $A \subset \mathbb{R}^d$.

Under certain circumstances, it is not necessary to include the recession function when taking the infimum on the right of the second equation of Eq. (3.13). For example, if A is a convex open set intersecting $\operatorname{ri} \mathfrak{D}_C$, where

$$\mathfrak{D}_C \equiv \bigcup_{\tau \in C} \tau \operatorname{dom} \Lambda^*,$$

then $\inf_{x \in A} \Gamma_C^*(x) = \inf_{x \in A \cap \operatorname{ri} \mathfrak{D}_C} \Gamma_C^*(x)$. Since $\Gamma_C^*(x) = \inf_{\tau \in C} \tau \Lambda^*(x/\tau)$ on $\operatorname{ri} \mathfrak{D}_C$, by Eq. (3.12), we see that we do not need to include the recession function when computing $\inf_{\tau \in C} I_A(\tau)$ in this case.

(ii) Let $\mathfrak{C} \subset \{(\tau_u, \tau_v): \tau_v \geqslant \tau_u \geqslant 0\}$ be convex, and let

$$\mathscr{J}(x_u, x_v) = \inf_{(\tau_u, \tau_v) \in \mathfrak{C}} \left\{ \tau_u \Lambda^* \left(\frac{x_u}{\tau_u} \right) + (\tau_v - \tau_u) \Lambda^* \left(\frac{x_v}{\tau_v - \tau_u} \right) \right\}.$$

Then $\Gamma^*_{\mathfrak{C}}(x) = \mathrm{cl}\, \mathscr{J}(x)$. The closure can be removed e.g. if $\mathrm{cl}\, \mathfrak{C}$ does not intersect the x_u-axis or the x_v-axis; otherwise, the infimum must be taken in a slightly broader sense, as described in Theorem 3.2.

Theorem 3.4. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a closed proper convex function, and let $\mathscr{E}$ be a subset of $\mathbb{R}^d$. Assume $\mathscr{D} \cap \mathscr{L}_\alpha f$ is nonempty and bounded for some α and some $\mathscr{D} \supset \mathscr{E}$. Then*:

(i) *There exists a point $x_0 \in \mathrm{cl}\, \mathscr{E}$ such that* $\inf_{x \in \mathscr{E}} f(x) = f(x_0)$.

(ii) *If $\mathscr{E}$ intersects* $\mathrm{ri}(\mathrm{dom}\, f)$ *and either* (a) *$\mathscr{E}$ is convex or* (b) $\mathrm{cl}\, \mathscr{E} \cap \partial(\mathrm{dom}\, f) = \emptyset$, *then there exists a point $\alpha_0 \in \partial f(x_0)$.*

(iii) *If $\mathscr{E}$ is a convex set intersecting* $\mathrm{ri}(\mathrm{dom}\, f)$, *then the point α_0 in* (ii) *actually determines a separating hyperplane. That is, if $a = \inf_{x \in \mathscr{E}} f(x)$, then for some $t \in \mathbb{R}$ we have $\mathscr{E} \subset \mathscr{H}^+(\alpha_0, t)$ and $\mathscr{L}_a f \subset \mathscr{H}^-(\alpha_0, t)$.*

Proof. (i) Let $\tilde{f} = f$ on $\mathrm{cl}\, \mathscr{D}$ and $\tilde{f} = \infty$ on $(\mathrm{cl}\, \mathscr{D})^{\mathrm{c}}$. Then $\mathscr{L}_\alpha \tilde{f}$ is compact for all α (Rockafellar, 1970, Corollary 8.7.1). Hence (i) follows from the lower semicontinuity of $\tilde{f}$.

(ii) and (iii) For the convex case, see Lemma 3.7 of Collamore (1996b) or Lemma 3.2 of Collamore (1996a). (These carry over with minor modifications to the slightly more general problem stated here.) For the nonconvex case [where $\mathrm{cl}\, \mathscr{E} \cap \partial(\mathrm{dom}\, f) = \emptyset$], see Theorem 23.4 of Rockafellar (1970). □

Remark 3.5. (i) In Theorem 3.4 it is assumed that $\mathscr{D} \cap \mathscr{L}_\alpha f$ is bounded for some α and some $\mathscr{D}$. We now discuss the nature of this hypothesis in the context of the functions Γ^*_C and $\Gamma^*_{\mathfrak{C}}$ and the hypotheses (H0)–(H2).

Under hypothesis (H0), the logarithmic moment generating function, Λ, is assumed to be differentiable. Hence Λ^* is essentially strictly convex (Rockafellar, 1970, Theorem 26.3), which implies that $\mathscr{L}_0 \Lambda^*$ is compact. If C and $\mathfrak{C}$ are convex, it follows by Theorem 3.1 that $\mathscr{L}_0 \Gamma^*_C = \{\tau x: \tau \in \mathrm{cl}\, C,\ x \in \mathscr{L}_0 \Lambda^*\}$ and $\mathscr{L}_0 \Gamma^*_{\mathfrak{C}} = \{(\xi_u x_u, (\xi_v - \xi_u) x_v): (\xi_u, \xi_v) \in \mathrm{cl}\, \mathfrak{C},\ (x_u, x_v) \in \mathscr{L}_0 \Lambda^*\}$. Hence the zero level sets of Γ^*_C and $\Gamma^*_{\mathfrak{C}}$ are bounded for bounded convex intervals C and $\mathfrak{C}$. Thus, for such intervals, Theorem 3.4 holds with no restriction on $\mathscr{E}$.

If the interval $C \subset [0, \infty)$ is unbounded, then $\Gamma^*_C \geqslant \Gamma^*_{[0,\infty)}$ has compact level sets on $(\mathrm{cone}\, \mathscr{B}_\delta)^{\mathrm{c}}$, for any $\delta > 0$. To demonstrate this fact, we note by Lemma 3.1 of Collamore (1996a) that

$$\inf\{\Gamma^*_{[0,\infty)}(x): x \in (\mathrm{cone}\, \mathscr{B}_\delta)^{\mathrm{c}} \text{ and } \|x\| = 1\} = t \quad \text{for some } t > 0. \tag{3.14}$$

Also, by definition,

$$\Gamma^*_{[0,\infty)}(x) = \sup_{\alpha \in \mathbb{R}^d} \{\langle \alpha, x \rangle - \mathbf{1}_{\mathscr{L}_0 \Lambda}(\alpha)\} = \sup_{\alpha \in \mathscr{L}_0 \Lambda} \langle \alpha, x \rangle, \tag{3.15}$$

where $\mathbf{1}_{\mathscr{L}_0 \Lambda}(\cdot)$ is the indicator function on $\mathscr{L}_0 \Lambda$. Hence $\Gamma^*_{[0,\infty)}(\lambda x) = \lambda \Gamma^*_{[0,\infty)}(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^d$, i.e., $\Gamma^*_{[0,\infty)}$ is a positively homogeneous function. Using the positive homogeneity of $\Gamma^*_{[0,\infty)}$ in conjunction with Eq. (3.14), we obtain that for any given $a < \infty$,

$$\inf\{\Gamma^*_{[0,\infty)}(x)\colon x \in (\operatorname{cone} \mathscr{B}_\delta)^{\mathrm{c}} \text{ and } \|x\| \geqslant K\} \geqslant a \quad \text{for a sufficiently large constant } K. \tag{3.16}$$

We conclude that Lemma 3.4 applies for any set $\mathscr{E} = A$, where A satisfies hypothesis (H2).

(ii) If $\mathscr{E} = A$, where A satisfies hypothesis (H2), then $A \subset (\operatorname{cone} \mathscr{B}_\delta)^{\mathrm{c}}$. Also, if $C \subset [0,\infty)$ is convex, then by Theorem 3.1 we have $\mathscr{L}_0 \Gamma^*_C = \{\tau x\colon \tau \in \operatorname{cl} C,\ x \in \mathscr{L}_0 \Lambda^*\} \subset \operatorname{cone} \mathscr{B}_\delta$. Therefore, it follows by the convexity of Γ^*_C that x_0 is a *boundary* point of A.

To motivate our next result, note by Theorem 23.5 of Rockafellar (1970) that $\alpha_0 \in \partial f(x_0) \Leftrightarrow x_0 \in \partial f^*(\alpha_0)$. It is therefore of interest to characterize the set $\partial f^*(\alpha_0)$. Next we do this when f^* is the function $F(\alpha) = \sup_{\xi \in \Xi}\{\xi_1 f(\alpha_1) + \cdots + \xi_k f(\alpha_k)\}$ given earlier in Theorem 3.1.

Theorem 3.6. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a convex function which is differentiable on its domain; let Ξ be a convex set contained in the positive orthant $\{(\xi_1, \ldots, \xi_k)\colon \xi_1 > 0, \ldots, \xi_k > 0\}$; and let $F : \mathbb{R}^{kd} \to \mathbb{R}$ be defined by*

$$F(\alpha) = \sup_{\xi \in \Xi} f_\xi(\alpha),$$

where

$$f_\xi(\alpha_1, \ldots, \alpha_k) = \xi_1 f(\alpha_1) + \cdots + \xi_k f(\alpha_k) \quad \text{for } \alpha_1, \ldots, \alpha_k \in \mathbb{R}^d.$$

Assume $\nabla f(\alpha_i)$ exists and is nonzero for each i, and assume F is finite and lower semicontinuous at α. Then

$$\partial F(\alpha) = \bigcup_{\xi \in \Xi_\alpha} \nabla f_\xi(\alpha), \tag{3.17}$$

where $\Xi_\alpha = \{\xi \in \operatorname{cl} \Xi\colon f_\xi(\alpha) = F(\alpha)\}$.

Proof. Let

$$\mathfrak{F}_\alpha = \bigcup_{\xi \in \Xi_\alpha} \nabla f_\xi(\alpha)$$

[the set given on the right of Eq. (3.17)], and define neighborhoods of the index set Ξ_α and of $\mathfrak{F}_\alpha$ as follows. For any $\delta \geqslant 0$, let

$$\Xi_\alpha^{(\delta)} = \{\xi \in \operatorname{cl} \Xi\colon f_\xi(\alpha) \geqslant F(\alpha) - \delta\} \quad \text{and} \quad \mathfrak{F}_\alpha^{(\delta)} = \bigcup_{\xi \in \Xi_\alpha^{(\delta)}} \{\nabla f_\xi(\tilde{\alpha})\colon \|\tilde{\alpha} - \alpha\| \leqslant \delta\}.$$

($\supset$) Assume $x\in\mathfrak{F}_\alpha$ and show $x\in\partial F(\alpha)$. If $x\in\mathfrak{F}_\alpha$, then $x=\nabla f_\xi(\alpha)$ for some $\xi\in\Xi_\alpha$. Hence

$$\sup_{\tilde\alpha\in\mathbb{R}^{kd}}\{\langle\tilde\alpha,x\rangle-f_\xi(\tilde\alpha)\}=\{\langle\alpha,x\rangle-f_\xi(\alpha)\} \tag{3.18}$$

(Rockafellar, 1970, Theorem 23.5). Since the definition of F implies $F(\tilde\alpha)\geqslant f_\xi(\tilde\alpha)$ for all $\tilde\alpha$; and the definition of Ξ_α implies $F(\alpha)=f_\xi(\alpha)$ for $\xi\in\Xi_\alpha$; it follows that

$$\sup_{\tilde\alpha\in\mathbb{R}^{kd}}\{\langle\tilde\alpha,x\rangle-F(\tilde\alpha)\}=\{\langle\alpha,x\rangle-F(\alpha)\}. \tag{3.19}$$

Therefore, $x\in\partial F(\alpha)$ (Rockafellar, 1970, Theorem 23.5).

($\subset$) Assume $x\notin\mathfrak{F}_\alpha$ and show $x\notin\partial F(\alpha)$. Consider the set $\mathfrak{F}_\alpha^{(\delta)}$ as $\delta\downarrow 0$. Note first that $\{\nabla f(\tilde\alpha)\colon \|\tilde\alpha-\alpha\|\leqslant\delta\}$ decreases to $\{(\nabla f(\alpha_1),\ldots,\nabla f(\alpha_k))\}$ as $\delta\downarrow 0$ (Rockafellar, 1970, Corollary 25.5.1); and by assumption the elements $\nabla f(\alpha_i)$ are nonzero for all i. It follows that

$$\mathfrak{F}_\alpha^{(\delta)}=\{(\xi_1\nabla f(\tilde\alpha_1),\ldots,\xi_k\nabla f(\tilde\alpha_k))\colon \xi\in\Xi_\alpha^{(\delta)} \text{ and } \|\tilde\alpha-\alpha\|\leqslant\delta\}$$

decreases to

$$\mathfrak{F}_\alpha=\{(\xi_1\nabla f(\alpha_1),\ldots,\xi_k\nabla f(\alpha_k))\colon \xi\in\Xi_\alpha\}.$$

It is easily verified that Ξ_α is convex, hence so is $\mathfrak{F}_\alpha$. Thus, we conclude

$$\operatorname{conv}\mathfrak{F}_\alpha^{(\delta)}\downarrow\operatorname{conv}\mathfrak{F}_\alpha=\mathfrak{F}_\alpha \quad \text{as } \delta\downarrow 0. \tag{3.20}$$

Therefore, $x\notin\mathfrak{F}_\alpha\Rightarrow x\notin\operatorname{conv}\mathfrak{F}_\alpha^{(\delta)}$ for $\delta\leqslant$ some δ_0.

Fix $\delta\leqslant\delta_0$. Then $\{x\}$ and $\operatorname{conv}\mathfrak{F}_\delta^{(\alpha)}$ are disjoint convex sets; consequently, there exists a strongly separating hyperplane; i.e.

$$\operatorname{conv}\mathfrak{F}_\alpha^{(\delta)}\subset\mathscr{H}^-(z,t-\varepsilon) \quad\text{and}\quad \{x\}\subset\mathscr{H}^+(z,t) \tag{3.21}$$

for some $z\in\mathbb{R}^{kd}$, $t\in\mathbb{R}$, and $\varepsilon>0$. Consider the derivative of F in the direction of z. By definition this is

$$F'(\alpha;z)\equiv\lim_{\lambda\downarrow 0}\frac{F(\alpha+\lambda z)-F(\alpha)}{\lambda}. \tag{3.22}$$

Next observe that for $\lambda\geqslant 0$ sufficiently small:

$$F(\alpha+\lambda z)\equiv\sup_{\xi\in\Xi}f_\xi(\alpha+\lambda z)=\sup_{\xi\in\Xi_\alpha^{(\delta)}}f_\xi(\alpha+\lambda z). \tag{3.23}$$

[Otherwise $G(\tilde\alpha)\equiv\sup\{f_\xi(\tilde\alpha)\colon \xi\in\Xi-\Xi_\alpha^{(\delta)}\}$ would satisfy

$$G(\alpha+\lambda_i z)=F(\alpha+\lambda_i z) \text{ along a sequence } \lambda_i\downarrow 0.$$

Also, by definition of $\Xi_\alpha^{(\delta)}$: $G(\alpha)\leqslant F(\alpha)-\delta$. Since F is lower semicontinuous at α, it would follow that G is not convex. But G is a supremum of convex functions and hence G is convex. Contradiction.] It follows by Eqs. (3.22) and (3.23) that

$$F'(\alpha;z)\leqslant\lim_{\lambda\downarrow 0}\sup_{\alpha\in\Xi_\alpha^{(\delta)}}\left[\frac{f_\xi(\alpha+\lambda z)-f_\xi(\alpha)}{\lambda}\right]. \tag{3.24}$$

By the mean value theorem, the quantity in brackets in Eq. (3.24) is $\langle \nabla f_\xi(\tilde{\alpha}), z\rangle$ for some $\tilde{\alpha} \in [\alpha, \alpha+\lambda z]$; and if λ is sufficiently small, then it follows by the definition of $\mathfrak{F}_\alpha^{(\delta)}$ that $\nabla f_\xi(\tilde{\alpha}) \in \mathfrak{F}_\alpha^{(\delta)}$. Therefore, by Eqs. (3.21) and (3.24) we obtain

$$F'(\alpha; z) \leqslant t - \varepsilon. \tag{3.25}$$

Hence by Eqs. (3.21) and (3.25): $F'(\alpha;z) < \langle x, z\rangle$. This implies $x \notin \partial F(\alpha)$ (Rockafellar, 1970, Theorem 23.2). $\square$

Of particular interest are the properties of

$$\inf_{x\in A} \Gamma^*_{[0,\infty)}(x) = \inf_{x\in A} \min\left\{\inf_{\tau>0} \Lambda^*\left(\frac{x}{\tau}\right),\ (0^+\Lambda^*)(x)\right\} = \inf_{\tau\in[0,\infty)} I_A(\tau),$$

namely, the rate function in Eq. (2.5) corresponding to the probability that the sequence $Y_1, Y_2, \ldots$ *ever* hits the set $A \subset \mathbb{R}^d$.

Theorem 3.7. *Let $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ be a sequence of random variables having a differentiable logarithmic moment generating function, Λ, and let $A \subset \mathbb{R}^d$ be a convex set satisfying* (H2), $A \cap \operatorname{ri}\operatorname{cone}(\operatorname{dom} \Lambda^*) \neq \emptyset$. *Let x_0 and α_0 be given as in Theorem* 3.4 *when $f = \Gamma^*_{[0,\infty)}$ and $\mathscr{E} = A$. Then*:
(i) $\alpha_0 \in \partial(\mathscr{L}_0\Lambda)$ *and* $\Lambda(\alpha_0) = 0$.
(ii) *There exists a constant $\rho > 0$ such that $x_0 = \rho\nabla\Lambda(\alpha_0)$.*
(iii) *The element x_0 is unique.*

Proof. (i) Note $\Gamma_{[0,\infty)}(\alpha) \equiv \sup_{\tau\in[0,\infty)} \tau\Lambda(\alpha) = \mathbf{1}_{\mathscr{L}_0\Lambda}(\alpha)$, where $\mathbf{1}_{\mathscr{L}_0\Lambda}(\cdot)$ is the indicator function on the set $\mathscr{L}_0\Lambda$. Hence

$$\Gamma^*_{[0,\infty)}(x_0) \equiv \sup_{\alpha\in\mathbb{R}^d} \{\langle \alpha, x_0\rangle - \mathbf{1}_{\mathscr{L}_0\Lambda}(\alpha)\} = \sup_{\alpha\in\mathscr{L}_0\Lambda} \langle \alpha, x_0\rangle. \tag{3.26}$$

Since (H2) $\Rightarrow x_0 \neq 0$, the supremum on the right can only be achieved on the boundary of $\mathscr{L}_0\Lambda$. Hence $\alpha_0 \in \partial(\mathscr{L}_0\Lambda)$. Since Λ is differentiable at α_0, it follows that $\Lambda(\alpha_0) = 0$.

(ii) This follows from Theorem 3.6. The constant ρ is positive since (H2) $\Rightarrow x_0 \neq 0$.

(iii) Let $x_0^{(1)}, x_0^{(2)}$ be two such elements, and let $\alpha_0 \in \partial\Gamma^*_{[0,\infty)}(x_0^{(2)})$ denote the element obtained in Lemma 3.4(ii) which corresponds to $x_0^{(2)}$. Let $a = \inf_{x\in \operatorname{cl} A} \Gamma^*_{[0,\infty)}(x)$.

Since $\{x_0^{(1)}, x_0^{(2)}\} \subset \mathscr{L}_a\Gamma^*_{[0,\infty)} \cap \operatorname{cl} A$, it follows that *both* $x_0^{(1)}$ and $x_0^{(2)}$ lie on the hyperplane given in Theorem 3.4(iii) which separates $\mathscr{L}_a\Gamma^*_{[0,\infty)}$ and $\operatorname{cl} A$. From this fact, together with the fact that α_0 *achieves* the supremum on the right of Eq. (3.26), we obtain

$$\langle \alpha_0, x_0^{(1)}\rangle = \langle \alpha_0, x_0^{(2)}\rangle = \sup_{\alpha\in\mathscr{L}_0\Lambda} \langle \alpha, x_0\rangle. \tag{3.27}$$

Thus, both $x_0^{(1)}$ and $x_0^{(2)}$ belong to the normal cone to $\mathscr{L}_0\Lambda$ at α_0. This implies

$$x_0^{(i)} = \rho_i \nabla\Lambda(\alpha_0),\ i = 1,2 \quad \text{for certain positive constants } \rho_1, \rho_2 \tag{3.28}$$

[Rockafellar (1970), Corollary 23.7.1). This corollary is applicable since (H2) $\Rightarrow \Lambda^*(0) > 0$, hence $\inf_\alpha \Lambda(\alpha) < 0$.] Therefore

$$x_0^{(1)} = \rho_1(\nabla\Lambda(\alpha_0)) = \rho_1(\rho_2^{-1}x_0^{(2)}). \tag{3.29}$$

Next observe by Eq. (3.26) that $\Gamma^*_{[0,\infty)}(x) = \sup_{\alpha \in \mathscr{L}_0\Lambda} \langle \alpha, x\rangle$, which shows that the function $\Gamma^*_{[0,\infty)}$ is positively homogeneous, i.e., $\Gamma^*_{[0,\infty)}(\lambda x) = \lambda\Gamma^*_{[0,\infty)}(x)$ for all λ, x. Hence by Eq. (3.29)

$$\Gamma^*_{[0,\infty)}(x_0^{(1)}) = \frac{\rho_1}{\rho_2}\Gamma^*_{[0,\infty)}(x_0^{(2)}). \tag{3.30}$$

Since $x_0^{(1)}$ and $x_0^{(2)}$ both minimize $\Gamma^*_{[0,\infty)}$ over $\mathrm{cl}\, A$, it follows from Eq. (3.30) that $(\rho_1/\rho_2) = 1$, and by Eq. (3.29) this implies $x_0^{(1)} = x_0^{(2)}$. □

4. Estimates for occupation probabilities

For any set $A \subset \mathbb{R}^d$, let

$$\mathbb{P}^\varepsilon_{\mathrm{C}}(A) = \boldsymbol{P}\{Y_n^\varepsilon \in A,\ n \in C/\varepsilon\} \quad \text{for all convex } C \subset [0,\infty).$$

Thus, e.g. if $C = (\tau_1, \tau_2)$, then $\mathbb{P}^\varepsilon_{\mathrm{C}}(\mathrm{A})$ is the probability that the normalized sequence $\{Y_n^\varepsilon\}$ hits $A \subset \mathbb{R}^d$ at some time during the interval $\varepsilon^{-1}(\tau_1, \tau_2)$. For any set $\mathfrak{A} \subset \mathbb{R}^{2d}$, let

$$\mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A}) = \boldsymbol{P}\{Z^\varepsilon_{m,n} \in \mathfrak{A},\ (m,n) \in \mathfrak{C}/\varepsilon\} \quad \text{for all convex } \mathfrak{C} \subset \{(\tau_u, \tau_v)\colon\ \tau_v \geqslant \tau_u \geqslant 0\}.$$

Thus, e.g. if $\mathfrak{C} = (\tau_1, \tau_2) \times (\zeta_1, \zeta_2)$, then $\mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A})$ is the probability that the normalized sequence $Z^\varepsilon_{m,n} \equiv (Y^\varepsilon_m, Y^\varepsilon_n - Y^\varepsilon_m)$ hits $\mathfrak{A} \subset \mathbb{R}^{2d}$ at some time during the interval $\mathfrak{C}/\varepsilon$, i.e. for some $m \in \varepsilon^{-1}(\tau_1, \tau_2)$ and some $n \in \varepsilon^{-1}(\zeta_1, \zeta_2)$.

In this section we derive estimates for the "occupation probabilities" $\mathbb{P}^\varepsilon_{\mathrm{C}}(A)$ and $\mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A})$. Asymptotics for the hitting time $T^\varepsilon(A)$, i.e. the *first* time $\{Y_n^\varepsilon\}$ hits A, will follow directly from these estimates.

Notation. First we recall the definitions of Γ_{C} and $\Gamma_{\mathfrak{C}}$ from the previous section. For any convex set $C \in [0,\infty)$, let

$$\Gamma_{\mathrm{C}}(\alpha) = \sup_{\tau \in C} \tau\Lambda(\alpha) \quad \text{for all } \alpha \in \mathbb{R}^d;$$

and for any convex set $\mathfrak{C} \subset \{(\tau_u, \tau_v)\colon\ \tau_v \geqslant \tau_u \geqslant 0\}$, let

$$\Gamma_{\mathfrak{C}}(\alpha_u, \alpha_v) = \sup_{(\tau_u, \tau_v) \in \mathfrak{C}} \{\tau_u\Lambda(\alpha_u) + (\tau_v - \tau_u)\Lambda(\alpha_v)\} \quad \text{for all } \alpha_u, \alpha_v \in \mathbb{R}^d.$$

Also let

$$\mathscr{H}_{\mathfrak{C}}(\alpha, a) = \text{the open half-space } \{x \in \mathbb{R}^{2d}\colon\ \langle \alpha, x\rangle > (a + \Gamma_{\mathfrak{C}}(\alpha))\}$$
$$\text{for all } \alpha \in \mathbb{R}^{2d},\ a \in \mathbb{R};$$

$$\text{proj}(\mathfrak{A}) = \{x_u \in \mathbb{R}^d\colon\ (x_u, x_v) \in \mathfrak{A}\} \cup \{x_u + x_v \in \mathbb{R}^d\colon\ (x_u, x_v) \in \mathfrak{A}\}$$

for any set $\mathfrak{A} \subset \mathbb{R}^{2d}$.

Theorem 4.1. *Let* $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ *be a sequence of random variables satisfying* (H0) *and* (H1), *and let* $A \subset \mathbb{R}^d$ *be a set satisfying* (H2). *Let* C *be a convex subset of* $[0, \infty)$. *Then*

(i) *Upper bound*:

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\mathrm{C}}^{\varepsilon}(A) \leqslant - \inf_{x \in \mathrm{cl}\, A} \Gamma_{\mathrm{C}}^{*}(x). \tag{4.1}$$

(ii) *Lower bound*:

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\mathrm{C}}^{\varepsilon}(A) \geqslant - \inf_{x \in \mathrm{int}\, A} \Gamma_{\mathrm{C}}^{*}(x). \tag{4.2}$$

Theorem 4.2. *Let* $Y_1, Y_2, \ldots \subset \mathbb{R}^d$ *be a sequence of random variables satisfying* (H0) *and* (H1), *and let* $\mathfrak{A} \subset \mathbb{R}^{2d}$ *be a set such that* proj($\mathfrak{A}$) *satisfies* (H2). *Let* $\mathfrak{C}$ *be a convex subset of* $\{(\tau_u, \tau_v)\colon\ \tau_v \geqslant \tau_u \geqslant 0\}$, *and assume* $(0,0) \notin \mathrm{cl}\, \mathfrak{C}$. *Then*

(i) *Upper bound*:

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\mathfrak{C}}^{\varepsilon}(\mathfrak{A}) \leqslant - \inf_{x \in \mathrm{cl}\, \mathfrak{A}} \Gamma_{\mathfrak{C}}^{*}(x). \tag{4.3}$$

(ii) *Lower bound*:

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\mathfrak{C}}^{\varepsilon}(\mathfrak{A}) \geqslant - \inf_{x \in \mathrm{int}\, \mathfrak{A}} \Gamma_{\mathfrak{C}}^{*}(x). \tag{4.4}$$

First we turn to the proof of Theorem 4.2 and then indicate how this proof can be modified to establish Theorem 4.1. The proof of the upper bound of Theorem 4.2 is dependent upon the following.

Lemma 4.3. *Let* $Y_1, Y_2, \ldots$ *be a sequence of random variables satisfying* (H0) *and* (H1). *Let* $\mathfrak{C}$ *be a bounded convex subset of* $\{(\tau_u, \tau_v)\colon\ \tau_v \geqslant \tau_u \geqslant 0\}$, *and assume* $(0,0) \notin \mathrm{cl}\, \mathfrak{C}$. *Then*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\mathfrak{C}}^{\varepsilon}\{\mathscr{H}_{\mathfrak{C}}(\alpha, a)\} \leqslant -a. \tag{4.5}$$

Proof of Lemma 4.3. Let $\mu_{m,n}$ denote the probability law of $Z_{m,n}$, and define a transformed measure $\tilde{\mu}_{m,n}$ as follows:

$$\tilde{\mu}_{m,n}(F) = \int_F \exp\{\langle \alpha, z\rangle - \Lambda_{m,n}(\alpha)\}\, \mathrm{d}\mu_{m,n}(z) \tag{4.6}$$

for all measurable sets $F \subset \mathbb{R}^{2d}$. Then by definition,

$$\begin{aligned} \boldsymbol{P}\{Z_{m,n}^{\varepsilon} \in \mathscr{H}_{\mathfrak{C}}(\alpha, a)\} &= \int_{\varepsilon^{-1}\mathscr{H}_{\mathfrak{C}}(\alpha,a)} \exp\{-(\langle \alpha, z\rangle - \Lambda_{m,n}(\alpha))\}\, \mathrm{d}\tilde{\mu}_{m,n}(z) \\ &= \boldsymbol{E}[\exp\{-(\langle \alpha, \tilde{Z}_{m,n}\rangle - \Lambda_{m,n}(\alpha))\}; \tilde{Z}_{m,n}^{\varepsilon} \in \mathscr{H}_{\mathfrak{C}}(\alpha, a)], \end{aligned} \tag{4.7}$$

where $\tilde{Z}_{m,n}$ is a random variable having distribution $\tilde{\mu}_{m,n}$ and $\tilde{Z}^{\varepsilon}_{m,n} = \varepsilon \tilde{Z}_{m,n}$. We replace $\Lambda_{m,n}$ with a limiting generating function, $\Lambda_{m/n}$, by introducing the ratio

$$\mathfrak{R}_{m,n} = \exp\{\Lambda_{m,n}(\alpha) - n\Lambda_{m/n}(\alpha)\}; \tag{4.8}$$

then Eq. (4.7) becomes

$$\boldsymbol{P}\{Z^{\varepsilon}_{m,n} \in \mathscr{H}_{\mathfrak{C}}(\alpha,a)\} = \mathfrak{R}_{m,n}\,\boldsymbol{E}[\exp\{-(\langle \alpha, \tilde{Z}_{m,n}\rangle - n\Lambda_{m/n}(\alpha))\};\ \tilde{Z}^{\varepsilon}_{m,n} \in \mathscr{H}_{\mathfrak{C}}(\alpha,a)]. \tag{4.9}$$

The utility of this last representation is then evident from the following result, where it is shown that the random variable in this last expectation is *deterministically bounded* over $\{\tilde{Z}^{\varepsilon}_{m,n} \in \mathscr{H}_{\mathfrak{C}}(\alpha,a)\}$ for $(m,n) \in \mathfrak{C}/\varepsilon$.

Sublemma 1: *If* $(m,n) \in \mathfrak{C}/\varepsilon$ *and* $\tilde{Z}^{\varepsilon}_{m,n} \in \mathscr{H}_{\mathfrak{C}}(\alpha,a)$, *then*

$$\{\langle \alpha, \tilde{Z}_{m,n}\rangle - n\Lambda_{m/n}(\alpha)\} > \frac{a}{\varepsilon}. \tag{4.10}$$

Proof. By definition,

$$\begin{aligned}\tilde{Z}^{\varepsilon}_{m,n} \in \mathscr{H}_{\mathfrak{C}}(\alpha,a) &\Leftrightarrow \{\langle \alpha, \tilde{Z}^{\varepsilon}_{m,n}\rangle - \Gamma_{\mathfrak{C}}(\alpha)\} > a \\ &\Leftrightarrow \left\{\langle \alpha, \tilde{Z}_{m,n}\rangle - \frac{1}{\varepsilon}\Gamma_{\mathfrak{C}}(\alpha)\right\} > \frac{a}{\varepsilon}.\end{aligned} \tag{4.11}$$

Thus, the proof will be complete as soon as we show that, on the right-hand side of Eq. (4.11), $\varepsilon^{-1}\Gamma_{\mathfrak{C}}(\alpha)$ can be replaced with $n\Lambda_{m/n}(\alpha)$. To this end, observe that by (H1),

$$n\Lambda_{m/n}(\alpha) = m\Lambda(\alpha_u) + (n-m)\Lambda(\alpha_v) \quad \text{where } \alpha = (\alpha_u, \alpha_v). \tag{4.12}$$

Thus $(m,n) \in \mathfrak{C}/\varepsilon$ implies

$$n\Lambda_{m/n}(\alpha) \leqslant \varepsilon^{-1} \sup_{(\tau_u,\tau_v)\in\mathfrak{C}} \{\tau_u \Lambda(\alpha_u) + (\tau_v - \tau_u)\Lambda(\alpha_v)\} \equiv \varepsilon^{-1}\Gamma_{\mathfrak{C}}(\alpha). \tag{4.13}$$

Substituting this inequality on the right-hand side of Eq. (4.11) gives

$$\{\langle \alpha, \tilde{Z}_{m,n}\rangle - n\Lambda_{m/n}(\alpha)\} > \frac{a}{\varepsilon}. \qquad \square$$

By Sublemma 1 and Eq. (4.9),

$$\boldsymbol{P}\{Z^{\varepsilon}_{m,n} \in \mathscr{H}_{\mathfrak{C}}(\alpha,a)\} \leqslant \mathfrak{R}_{m,n}\,\mathrm{e}^{-a/\varepsilon} \quad \text{for } (m,n) \in \mathfrak{C}/\varepsilon. \tag{4.14}$$

Consequently, the probability that $Z^{\varepsilon}_{m,n}$ enters $\mathscr{H}_{\mathfrak{C}}(\alpha,a)$ at some time $(m,n) \in \mathfrak{C}/\varepsilon$ is

$$\mathbb{P}^{\varepsilon}_{\mathfrak{C}}\{\mathscr{H}_{\mathfrak{C}}(\alpha,a)\} \leqslant \mathrm{e}^{-a/\varepsilon} \sum_{(m,n)\in\mathfrak{C}/\varepsilon} \mathfrak{R}_{m,n}. \tag{4.15}$$

It follows that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}}\{\mathscr{H}_{\mathfrak{C}}(\alpha,a)\} \leqslant -a + \limsup_{\varepsilon\to 0} \max_{(m,n)\in\mathfrak{C}/\varepsilon} \{\varepsilon \log \mathfrak{R}_{m,n}\}. \tag{4.16}$$

Finally, the lemma is obtained by showing that the ratio $\mathfrak{R}_{m,n}$ can, in a suitable sense, be neglected.

Sublemma 2: $\limsup_{\varepsilon\to 0} \max_{(m,n)\in\mathfrak{C}/\varepsilon}\{\varepsilon\log\mathfrak{R}_{m,n}\}=0$.

Proof. Suppose false. Then there exists a sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}_+}$ with $\varepsilon_i\to 0$ as $i\to\infty$ and

$$\varepsilon_i\log\mathfrak{R}_{m_i,n_i}\geqslant t>0 \quad \text{some } (m_i,n_i)\in\mathfrak{C}/\varepsilon_i. \tag{4.17}$$

Note: $(m_i,n_i)\in\mathfrak{C}/\varepsilon_i$, where $\mathfrak{C}\subset\{(\tau_u,\tau_v)\colon\ \tau_v\geqslant\tau_u\geqslant 0\}$ is bounded and does not have $(0,0)$ as a limit point. It follows that along a subsequence

$$n_i\to\infty \text{ and } \frac{m_i}{n_i}\to r \text{ as } i\to\infty \quad \text{for some constant } r\in[0,1]. \tag{4.18}$$

Then, along this subsequence,

$$\lim_{i\to\infty}\frac{\Lambda_{m_i,n_i}(\alpha)}{n_i}=\lim_{\substack{n_i\to\infty,\\ m_i/n_i\to r}}\frac{1}{n_i}\log\boldsymbol{E}\exp\{\langle\alpha,Z_{m_i,n_i}\rangle\}\equiv\Lambda_r(\alpha). \tag{4.19}$$

Also, by (H1) and Eq. (4.18),

$$\lim_{i\to\infty}\Lambda_{m_i/n_i}(\alpha)=\lim_{i\to\infty}\left\{\frac{m_i}{n_i}\Lambda(\alpha_u)+\left(1-\frac{m_i}{n_i}\right)\Lambda(\alpha_v)\right\}=\Lambda_r(\alpha). \tag{4.20}$$

By Eqs. (4.19) and (4.20) it follows that

$$\limsup_{i\to\infty}\varepsilon_i\log\mathfrak{R}_{m_i,n_i}\equiv\limsup_{i\to\infty}\varepsilon_i n_i\left\{\frac{\Lambda_{m_i,n_i}(\alpha)}{n_i}-\Lambda_{m_i/n_i}(\alpha)\right\}=0 \tag{4.21}$$

[since $\varepsilon_i(m_i,n_i)\in\varepsilon_i(\varepsilon_i^{-1}\mathfrak{C})=\mathfrak{C}$ implies $\{\varepsilon_i n_i\}_{i\in\mathbb{Z}_+}$ is bounded]. But Eq. (4.21) contradicts Eq. (4.17). □

Proof of Theorem 4.2 Upper bound:
Step 1. *The upper bound holds under the assumption that* $\mathfrak{A}$ *and* $\mathfrak{C}$ *are bounded.*

Proof. Let $a<\inf_{x\in\operatorname{cl}\mathfrak{A}}\Gamma^*_{\mathfrak{C}}(x)$. Then for any $x\in\operatorname{cl}\mathfrak{A}$,

$$\sup_{\alpha\in\mathbb{R}^{2d}}\{\langle\alpha,x\rangle-\Gamma_{\mathfrak{C}}(\alpha)\}\equiv\Gamma^*_{\mathfrak{C}}(x)>a; \tag{4.22}$$

hence for some $\alpha_x\in\mathbb{R}^{2d}$,

$$x\in\mathscr{H}_{\mathfrak{C}}(\alpha_x,a)\equiv\{z:\ \langle\alpha_x,z\rangle-\Gamma_{\mathfrak{C}}(\alpha)>a\}. \tag{4.23}$$

By Eq. (4.23), $\{\mathscr{H}_{\mathfrak{C}}(\alpha_x,x)\}_{x\in\operatorname{cl}\mathfrak{A}}$ is an open cover for the compact set $\operatorname{cl}\mathfrak{A}$; hence there exists a finite subcover: $\mathscr{H}_{\mathfrak{C}}(\alpha_{x_1},a),\ldots,\mathscr{H}_{\mathfrak{C}}(\alpha_{x_l},a)$; and

$$\mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A})\leqslant\sum_{i=1}^{l}\mathbb{P}^{\varepsilon}_{\mathfrak{C}}\{\mathscr{H}_{\mathfrak{C}}(\alpha_{x_i},a)\}. \tag{4.24}$$

By Lemma 4.3,

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}^{\varepsilon}_{\mathfrak{C}}\{\mathscr{H}_{\mathfrak{C}}(\alpha_{x_i},a)\}\leqslant -a \quad \text{for each } i. \tag{4.25}$$

Consequently, by Eq. (4.24),

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A}) \leqslant -a. \tag{4.26}$$

The desired upper bound is then obtained by letting $a \uparrow \inf_{x\in \mathrm{cl}\,\mathfrak{A}} \Gamma^*_{\mathfrak{C}}(x)$.

Step 2. *The upper bound can be extended to the case where $\mathfrak{A}$ and $\mathfrak{C}$ are possibly unbounded.*

Proof. Let a be a finite constant such that $a \leqslant \inf_{x\in \mathrm{cl}\,\mathfrak{A}} \Gamma^*_{\mathfrak{C}}(x)$. For $R, K < \infty$, define

$$\mathfrak{A}_R = \mathfrak{A} \cap \{(x_u, x_v)\colon\ \|x_u\| \leqslant R,\ \|x_u + x_v\| \leqslant R\}$$

and

$$\mathfrak{C}_K = \mathfrak{C} \cap ([0,K] \times [0,K]).$$

Since $\mathfrak{A}_R$ and $\mathfrak{C}_K$ are bounded, it follows by Step 1 that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}_K}(\mathfrak{A}_R) \leqslant -\inf_{x\in \mathrm{cl}\,\mathfrak{A}_R} \Gamma^*_{\mathfrak{C}_K}(x) \leqslant -\inf_{x\in \mathrm{cl}\,\mathfrak{A}} \Gamma^*_{\mathfrak{C}}(x) \leqslant -a \tag{4.27}$$

for any $R, K < \infty$. (The second inequality holds because $\mathfrak{C}_K \subset \mathfrak{C} \Rightarrow \Gamma_{\mathfrak{C}_K} \leqslant \Gamma_{\mathfrak{C}}$, hence $\Gamma^*_{\mathfrak{C}_K} \geqslant \Gamma^*_{\mathfrak{C}}$.) We need to show that the bounded sets $\mathfrak{A}_R$ and $\mathfrak{C}_K$ on the left of Eq. (4.27) may be replaced with the possibly unbounded sets $\mathfrak{A}$ and $\mathfrak{C}$.

For this purpose, observe

$$\begin{aligned} Z^{\varepsilon}_{m,n} \in \mathfrak{A} \cap \mathfrak{A}^{\mathrm{c}}_R &\Leftrightarrow (Y^{\varepsilon}_m, Y^{\varepsilon}_n - Y^{\varepsilon}_m) \in \{(x_u, x_v)\colon\ (x_u, x_v) \in \mathfrak{A},\ \|x_u\| > R \text{ or } \|x_u + x_v\| > R\} \\ &\Leftrightarrow (Y^{\varepsilon}_m, Y^{\varepsilon}_n) \in \{(x_u, x_u + x_v)\colon\ (x_u, x_v) \in \mathfrak{A},\ \|x_u\| > R \text{ or } \|x_u + x_v\| > R\}. \end{aligned}$$

By the definition of proj($\mathfrak{A}$) it follows that

$$Z^{\varepsilon}_{m,n} \in \mathfrak{A} \cap \mathfrak{A}^{\mathrm{c}}_R \Rightarrow Y^{\varepsilon}_i \in \mathrm{proj}(\mathfrak{A}) \cap B^{\mathrm{c}}_{0,R}, \quad \text{either } i = m \text{ or } i = n \tag{4.28}$$

(where $B_{0,R}$ is a ball of radius R about the origin). Hence the event $\{Z^{\varepsilon}_{m,n} \in (\mathfrak{A} \cap \mathfrak{A}^{\mathrm{c}}_R),\ (m,n) \in \mathfrak{C}_K/\varepsilon\}$ is contained in the event $\{Y^{\varepsilon}_i \in \mathrm{proj}(\mathfrak{A}) \cap B^{\mathrm{c}}_{0,R},\ i \in [0, K/\varepsilon]\}$. The evaluation of the probability of this last event may then be handled by applying Eq. (4.14) of Collamore (1996a). Namely, since $a < \infty$ and proj($\mathfrak{A}$) satisfies (H2):

$$\limsup_{\varepsilon\to 0} \varepsilon \log \boldsymbol{P}\{Y^{\varepsilon}_i \in \mathrm{proj}(\mathfrak{A}) \cap B^{\mathrm{c}}_{0,R},\quad i \in [0, K/\varepsilon]\} \leqslant -a, \tag{4.29}$$

sufficiently large R. Consequently,

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}_K}(\mathfrak{A} \cap \mathfrak{A}^{\mathrm{c}}_R) \leqslant -a, \tag{4.30}$$

sufficiently large R. Finally, observe that the event $\{Z^{\varepsilon}_{m,n} \in \mathfrak{A},\ (m,n) \in \mathfrak{C}_K/\varepsilon\}$ is the union of the events $\{Z^{\varepsilon}_{m,n} \in \mathfrak{A}_R,\ (m,n) \in \mathfrak{C}_K/\varepsilon\}$ and $\{Z^{\varepsilon}_{m,n} \in \mathfrak{A} \cap \mathfrak{A}^{\mathrm{c}}_R,\ (m,n) \in \mathfrak{C}_K/\varepsilon\}$. Therefore $\mathbb{P}^{\varepsilon}_{\mathfrak{C}_K}(\mathfrak{A}) \leqslant \mathbb{P}^{\varepsilon}_{\mathfrak{C}_K}(\mathfrak{A}_R) + \mathbb{P}^{\varepsilon}_{\mathfrak{C}_K}(\mathfrak{A} \cap \mathfrak{A}^{\mathrm{c}}_R)$. It follows by Eqs. (4.27) and (4.30) that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}_K}(\mathfrak{A}) \leqslant -a. \tag{4.31}$$

It remains to show that $\mathfrak{C}_K$ may likewise be extended to $\mathfrak{C}$. By an argument similar to the one given in Eq. (4.28), $Z^\varepsilon_{m,n} \in \mathfrak{A} \Rightarrow \{Y^\varepsilon_m, Y^\varepsilon_n\} \in \text{proj}(\mathfrak{A})$. Hence $\mathbb{P}^\varepsilon_{\mathfrak{C}\cap\mathfrak{C}^c_K}(\mathfrak{A}) \equiv \boldsymbol{P}\{Z^\varepsilon_{m,n} \in \mathfrak{A},\ (m,n) \in \varepsilon^{-1}(\mathfrak{C} \cap \mathfrak{C}^c_K)\}$ is bounded above by $\boldsymbol{P}\{Y^\varepsilon_i \in \text{proj}(\mathfrak{A}),\ i \in [K/\varepsilon, \infty)\}$. The evaluation of this last probability may be handled by applying Eq. (4.7) of Collamore (1996a). Namely, since $a < \infty$ and proj($\mathfrak{A}$) satisfies (H2):

$$\limsup_{\varepsilon\to 0} \varepsilon \log \boldsymbol{P}\{Y^\varepsilon_i \in \text{proj}(\mathfrak{A}), \quad i \in [K/\varepsilon, \infty)\} \leqslant -a, \tag{4.32}$$

sufficiently large K. Hence

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^\varepsilon_{\mathfrak{C}\cap\mathfrak{C}^c_K}(\mathfrak{A}) \leqslant -a, \tag{4.33}$$

sufficiently large K. Since $\mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A}) \leqslant \mathbb{P}^\varepsilon_{\mathfrak{C}_K}(\mathfrak{A}) + \mathbb{P}^\varepsilon_{\mathfrak{C}\cap\mathfrak{C}^c_K}(\mathfrak{A})$, it follows by Eqs. (4.31) and (4.33) that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A}) \leqslant -a. \tag{4.34}$$

Finally, the desired upper bound is obtained by letting $a \uparrow \inf_{x\in \text{cl}\,\mathfrak{A}} \Gamma^*_{\mathfrak{C}}(x)$. □

Lower bound: Fix $(\tau_u, \tau_v) \in \text{int}\,\mathfrak{C}$, and construct a sequence $\{\tilde{Z}_\varepsilon\}_{\varepsilon>0} \subset \mathbb{R}^{2d}$ as follows: for each $\varepsilon > 0$ let

$$\tilde{Z}_\varepsilon = Z_{m_\varepsilon, n_\varepsilon} \quad \text{where } m_\varepsilon = \lfloor \tau_u/\varepsilon \rfloor \text{ and } n_\varepsilon = \lfloor \tau_v/\varepsilon \rfloor,$$

and where $\lfloor x \rfloor$ denotes the greatest integer $\leqslant x$. The sequence $\{\tilde{Z}_\varepsilon\}_{\varepsilon>0}$ has been constructed from elements of the original sequence, $\{Z_{m,n}\}_{m,n\in\mathbb{Z}_+}$. Its generating function is

$$\tilde{\Lambda}(\alpha) = \lim_{\varepsilon\to 0} \varepsilon \log \boldsymbol{E}[\exp\{\langle \alpha, \tilde{Z}_\varepsilon \rangle\}] = \tau_v \lim_{\varepsilon\to 0} \frac{1}{\lfloor \tau_v/\varepsilon \rfloor} \log \boldsymbol{E}[\exp\{\langle \alpha, Z_{\lfloor \tau_u/\varepsilon \rfloor, \lfloor \tau_v/\varepsilon \rfloor} \rangle\}]. \tag{4.35}$$

The limit on the right can be simplified by applying (H1). Since $\lfloor \tau_v/\varepsilon \rfloor \to \infty$ and $\lfloor \tau_u/\varepsilon \rfloor / \lfloor \tau_v/\varepsilon \rfloor \to \tau_u/\tau_v$ as $\varepsilon \to 0$, the right-hand side of Eq. (4.35) can be identified as $\tau_v \Lambda_{\tau_u/\tau_v}(\alpha)$. Hence, by (H1) and Eq. (4.35),

$$\tilde{\Lambda}(\alpha) = \tau_v \left[\frac{\tau_u}{\tau_v} \Lambda(\alpha_u) + \left(1 - \frac{\tau_u}{\tau_v}\right) \Lambda(\alpha_v) \right] \quad \text{where } \alpha = (\alpha_u, \alpha_v). \tag{4.36}$$

By (H0) and the Gärtner–Ellis theorem (Dembo and Zeitouni, 1993, Theorem 2.3.6(c)), it follows that the probability law of $\varepsilon\tilde{Z}_\varepsilon$ satisfies the large deviation principle with rate function

$$\begin{aligned}
\tilde{\Lambda}^*(x_u, x_v) &= \sup_{\alpha_u, \alpha_v \in \mathbb{R}^d} [\langle \alpha_u, x_u \rangle + \langle \alpha_v, x_v \rangle - \tau_u \Lambda(\alpha_u) - (\tau_v - \tau_u)\Lambda(\alpha_v)] \\
&= \tau_u \sup_{\alpha\in\mathbb{R}^d} \left[\left\langle \alpha, \frac{x_u}{\tau_u} \right\rangle - \Lambda(\alpha) \right] + (\tau_v - \tau_u) \sup_{\alpha\in\mathbb{R}^d} \left[\left\langle \alpha, \frac{x_v}{\tau_v - \tau_u} \right\rangle - \Lambda(\alpha) \right] \\
&= \tau_u \Lambda^* \left(\frac{x_u}{\tau_u} \right) + (\tau_v - \tau_u) \Lambda^* \left(\frac{x_v}{\tau_v - \tau_u} \right).
\end{aligned} \tag{4.37}$$

Next observe

$$\mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A}) \equiv \boldsymbol{P}\{Z^{\varepsilon}_{m,n} \in \mathfrak{A},\ (m,n) \in \mathfrak{C}/\varepsilon\} \geqslant \boldsymbol{P}\{\tilde{Z}_{\varepsilon} \in \mathfrak{A}\}, \tag{4.38}$$

where by definition $\tilde{Z}_{\varepsilon} = Z_{m_{\varepsilon},n_{\varepsilon}}$, $m_{\varepsilon} = \lfloor \tau_u/\varepsilon \rfloor$, $n_{\varepsilon} = \lfloor \tau_v/\varepsilon \rfloor$, and where ε is sufficiently small so that the operation $\lfloor \cdot \rfloor$ does not cause $(m_{\varepsilon}, n_{\varepsilon})$ to jump outside of the interval $\mathfrak{C}/\varepsilon \supset \{(\tau_u/\varepsilon, \tau_v/\varepsilon)\}$. Applying the large deviation lower bound to the right-hand side of Eq. (4.38) yields

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A}) \geqslant -\inf_{z \in \operatorname{int} \mathfrak{A}} \tilde{\Lambda}^*(z) \geqslant -\tilde{\Lambda}^*(x) \quad \text{for any } x \in \operatorname{int} \mathfrak{A}. \tag{4.39}$$

Hence by Eq. (4.37),

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A}) \geqslant -\left[\tau_u \Lambda^*\left(\frac{x_u}{\tau_u}\right) + (\tau_v - \tau_u)\Lambda^*\left(\frac{x_v}{\tau_v - \tau_u}\right)\right], \tag{4.40}$$

for any $x = (x_u, x_v) \in \operatorname{int} \mathfrak{A}$. Taking the supremum in Eq. (4.40) over all $(\tau_u, \tau_v) \in \operatorname{int} \mathfrak{C}$, then applying Theorem 3.1, and finally taking the supremum over all $x \in \operatorname{int} \mathfrak{A} - \partial(\operatorname{dom} \Gamma^*_{\mathfrak{C}})$, we obtain

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A}) \geqslant -\inf_{x \in \operatorname{int} \mathfrak{A} - \partial(\operatorname{dom} \Gamma^*_{\mathfrak{C}})} \Gamma^*_{\mathfrak{C}}(x). \tag{4.41}$$

As $\operatorname{int} \mathfrak{A}$ is open and $\Gamma^*_{\mathfrak{C}}$ convex, the extension of the infimum in Eq. (4.41) to all elements of $\operatorname{int} \mathfrak{A}$ can then be handled as in the discussion following Eq. (4.9) of Collamore (1996a). Thus the required lower bound follows from Eq. (4.41). □

Remark 4.4. In Theorem 4.2 it is assumed that $\mathfrak{C} \subset \{(\tau_u, \tau_v)\colon \tau_v \geqslant \tau_u \geqslant 0\}$. Now suppose $\mathfrak{C} \subset \{(\tau_u, \tau_v)\colon \tau_u \geqslant 0,\ \tau_v \geqslant 0\}$, and assume $(0,0) \notin \operatorname{cl} \mathfrak{C}$. Put

$$\mathfrak{C}_+ = \mathfrak{C} \cap \{(\tau_u, \tau_v)\colon \tau_v \geqslant \tau_u\} \quad \text{and} \quad \mathfrak{C}_- = \mathfrak{C} \cap \{(\tau_u, \tau_v)\colon \tau_v < \tau_u\}.$$

Then by Theorem 4.2 we have

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\mathfrak{C}_+}(\mathfrak{A}) \approx -\inf_{x \in \mathfrak{A}} \Gamma^*_{\mathfrak{C}_+}(x) \tag{4.42}$$

and similarly

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_{\tilde{\mathfrak{C}}_-}(\tilde{\mathfrak{A}}) \approx -\inf_{x \in \tilde{\mathfrak{A}}} \Gamma^*_{\tilde{\mathfrak{C}}_-}(x), \tag{4.43}$$

where $\tilde{\mathfrak{C}}_- = \{(\tau_v, \tau_u)\colon (\tau_u, \tau_v) \in \mathfrak{C}_-\}$ and $\tilde{\mathfrak{A}} = \{(x_u + x_v, -x_v)\colon (x_u, x_v) \in \mathfrak{A}\}$. If we extend the definition of $Z_{m,n}$ in the natural way to $\{(m,n)\colon n < m\}$, then

$$Z_{m,n} \equiv (Y_m, Y_n - Y_m) \in \mathfrak{A} \;\Leftrightarrow\; (Y_n, Y_m - Y_n) \in \tilde{\mathfrak{A}} \;\Leftrightarrow\; Z_{n,m} \in \tilde{\mathfrak{A}},$$

implying $\mathbb{P}^{\varepsilon}_{\mathfrak{C}_-}(\mathfrak{A}) = \mathbb{P}^{\varepsilon}_{\tilde{\mathfrak{C}}_-}(\tilde{\mathfrak{A}})$. Thus, it follows by Eqs. (4.42) and (4.43) that

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}^{\varepsilon}_{\mathfrak{C}}(\mathfrak{A}) \approx -\min\left\{\inf_{x \in \mathfrak{A}} \Gamma^*_{\mathfrak{C}_+}(x),\ \inf_{x \in \tilde{\mathfrak{A}}} \Gamma^*_{\tilde{\mathfrak{C}}_-}(x)\right\} \tag{4.44}$$

[where "≈" may be replaced by the usual upper and lower bounds].

Proof of Theorem 4.1. Suppose C is bounded, and let $\mathfrak{C}=C\times D$, where D is chosen such that $C\times D\subset\{(\tau_u,\tau_v)\colon\ \tau_v\geqslant\tau_u\geqslant 0\}$ and $(0,0)\notin \mathrm{cl}\,(C\times D)$. Let $\alpha=(\tilde{\alpha},0)$. Then an application of Lemma 4.3 yields

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}^{\varepsilon}_{\mathrm{C}}\{\mathscr{H}_C(\tilde{\alpha},a)\}\leqslant -a, \tag{4.45}$$

where $\mathscr{H}_C(\tilde{\alpha},a)=\{x\colon\ \langle\tilde{\alpha},x\rangle>(a+\Gamma_{\mathrm{C}}(\tilde{\alpha}))\}$.

Approximate $\mathbb{P}^{\varepsilon}_{\mathrm{C}}(A)$ with $\sum_{i=1}^{k}\mathbb{P}^{\varepsilon}_{\mathrm{C}}\{\mathscr{H}_C(\alpha_{x_i},a)\}$, in the sense of (4.24), and use (4.45) to determine an upper bound for $\mathbb{P}^{\varepsilon}_{\mathrm{C}}(A)$.

The proof of Theorem 4.1 then follows Theorem 4.2, so we omit the details. □

5. Proof of Theorem 1

Upper bound: First assume that $F\subset[0,\infty)$ is compact. For all $\delta>0$ and all $\tau\in[0,\infty)$, let

$$B_\delta(\tau)=\{\zeta\in[0,\infty)\colon\ |\zeta-\tau|<\delta\}\quad\text{and}\quad B_\delta(F)=\bigcup_{\tau\in F}B_\delta(\tau).$$

To apply Theorem 4.1, note

$$\boldsymbol{P}\{T^{\varepsilon}(A)\in B_\delta(\tau)\}\leqslant\mathbb{P}^{\varepsilon}_{B_\delta(\tau)}(A); \tag{5.1}$$

on the left we have the probability that $\{Y^{\varepsilon}_n\}$ *first* hits A during the time interval $B_\delta(\tau)/\varepsilon$ and on the right the probability that $\{Y^{\varepsilon}_n\}$ *ever* hits A during that interval. Hence

$$\limsup_{\varepsilon\to 0}\varepsilon\log\boldsymbol{P}\{T^{\varepsilon}(A)\in B_\delta(\tau)\}\leqslant -\inf_{x\in\mathrm{cl}\,A}\Gamma^*_{B_\delta(\tau)}(x)=-\inf_{\tilde{\tau}\in\mathrm{cl}\,B_\delta(\tau)}\bar{I}_A(\tilde{\tau}) \tag{5.2}$$

by Theorem 4.1 and then Theorems 3.1 and 3.2. Next observe that $\{B_\delta(\tau)\}_{\tau\in F}$ is an open cover for F; hence there exists a finite subcover; and by applying Eq. (5.2) to the elements of this subcover we obtain

$$\limsup_{\varepsilon\to 0}\varepsilon\log\boldsymbol{P}\{T^{\varepsilon}(A)\in F\}\leqslant -\inf_{\tau\in\mathrm{cl}\,B_\delta(F)}\bar{I}_A(\tau). \tag{5.3}$$

It remains to show

$$\inf_{\tau\in\mathrm{cl}\,B_\delta(F)}\bar{I}_A(\tau)\uparrow\inf_{\tau\in F}\bar{I}_A(\tau)\quad\text{as }\delta\downarrow 0. \tag{5.4}$$

Assume false. Then for each $i\in\mathbb{Z}_+$ there exists $x_i\in\mathrm{cl}\,A$ and $\tau_i\in B_{1/i}(F)$ such that

$$\lim_{i\to\infty}\tau_i\Lambda^*\left(\frac{x_i}{\tau_i}\right)<\inf_{\tau\in F}\bar{I}_A(\tau). \tag{5.5}$$

Then F is compact $\Rightarrow$ along a subsequence $\tau_i\to\tau_0\in F$. Next we observe that similarly $x_i\to x_0\in\mathrm{cl}\,A$. For this purpose, note: $\tau_i\Lambda^*(x_i/\tau_i)\geqslant\Gamma^*_{[0,\infty)}(x_i)$ (Theorem 3.1). Since the restriction of $\Gamma^*_{[0,\infty)}$ to $\mathrm{cl}\,A$ has compact level sets (by hypothesis (H2) and Remark 3.5(i)), it follows that $\{x_i\}$ is bounded. Hence along a subsequence $x_i\to x_0\in\mathrm{cl}\,A$. If $\tau_0\neq 0$, then by the lower semicontinuity of Λ^*,

$$\tau_0\Lambda^*\left(\frac{x_0}{\tau_0}\right)\leqslant\lim_{i\to\infty}\tau_i\Lambda^*\left(\frac{x_i}{\tau_i}\right)\quad\text{as }i\to\infty. \tag{5.6}$$

This shows that Eq. (5.5) is impossible in this case. On the other hand, if $\tau_0 = 0$, then observe

$$\left(\frac{x_i}{\tau_i}, \Lambda^*\left(\frac{x_i}{\tau_i}\right)\right) \in \operatorname{epi} \Lambda^* \quad \text{for all } i, \tag{5.7}$$

where $(\operatorname{epi} \Lambda^*)$ is the epigraph of Λ^*. By Theorem 8.2 of Rockafellar (1970), it follows that

$$\left(x_0, \lim_{i\to\infty} \tau_i \Lambda^*\left(\frac{x_i}{\tau_i}\right)\right) = \lim_{i\to\infty} \tau_i \left(\frac{x_i}{\tau_i}, \Lambda^*\left(\frac{x_i}{\tau_i}\right)\right) \in 0^+(\operatorname{epi} \Lambda^*). \tag{5.8}$$

Hence, by definition of the recession function, the limit on the left of Eq. (5.5) is $\geqslant (0^+\Lambda^*)(x_0) \geqslant \bar{I}_A(0)$, and so Eq. (5.5) is once again impossible.

By Eqs. (5.3) and (5.4) we conclude that the upper bound holds for all compact sets $F \subset [0,\infty)$. Finally, the extension to closed but unbounded sets may be handled by applying Eq. (4.7) of Collamore (1996a). □

Lower bound: First assume that G is an interval which is open in $[0,\infty)$. Thus $G = (\tau_1, \tau_2)$, where $0 \leqslant \tau_1 < \tau_2 \leqslant \infty$, or $G = [\tau_1, \tau_2)$, where $\tau_1 = 0$ and $0 < \tau_2 \leqslant \infty$.

Let $[\zeta_1, \zeta_2] \subset (\tau_1, \tau_2)$, and let $\mathfrak{C} = [0, \tau_1] \times (\zeta_1, \zeta_2)$.

Let

$$\mathfrak{D}_C = \operatorname{dom} \Gamma_C^* \quad \text{for all intervals } C \subset [0,\infty);$$

$$\mathfrak{S}_\delta = \{y\colon \|y - x\| < \delta \text{ for some } x \in \partial\mathfrak{D}_{(\zeta_1,\zeta_2)}\} \quad \text{for all } \delta > 0;$$

$$A_\delta = \operatorname{int}(A - \mathfrak{S}_\delta) \quad \text{for all } \delta > 0;$$

$$\mathfrak{A}_\delta = \{(x_u, x_v)\colon x_u \in A,\ x_u + x_v \in A_\delta\} \quad \text{for all } \delta > 0;$$

$$\mathfrak{M}_{\mathscr{E}} = \left\{x_0 \in \operatorname{cl}\mathscr{E}\colon \Gamma^*_{(\zeta_1,\zeta_2)}(x_0) = \inf_{x\in\mathscr{E}} \Gamma^*_{(\zeta_1,\zeta_2)}(x)\right\} \quad \text{for all sets } \mathscr{E} \subset \mathbb{R}^d.$$

Note that the open set $A_\delta \uparrow [\operatorname{int} A - \partial\mathfrak{D}_{(\zeta_1,\zeta_2)}]$ as $\delta \downarrow 0$.

Consider

(i) $\mathbb{P}^\varepsilon_{(\zeta_1,\zeta_2)}(A_\delta) = \boldsymbol{P}\{Y_n^\varepsilon \in A_\delta,\ n \in \varepsilon^{-1}(\zeta_1,\zeta_2)\}$,

(ii) $\mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A}_\delta) = \boldsymbol{P}\{(Y_m^\varepsilon, Y_n^\varepsilon) \in A \times A_\delta,\ m \in \varepsilon^{-1}[0,\tau_1] \text{ and } n \in \varepsilon^{-1}(\zeta_1,\zeta_2)\}$.

The quantity given in (i) is the probability that $\{Y_n^\varepsilon\}_{n\in\mathbb{Z}_+}$ hits A_δ during the interval $\varepsilon^{-1}(\zeta_1,\zeta_2)$. The quantity given in (ii) is the probability that $\{Y_n^\varepsilon\}_{n\in\mathbb{Z}_+}$ hits A during the interval $\varepsilon^{-1}[0,\tau_1]$ and then A_δ during the interval $\varepsilon^{-1}(\zeta_1,\zeta_2)$. If we subtract (ii) from (i), we obtain the probability that $\{Y_n^\varepsilon\}_{n\in\mathbb{Z}_+}$ hits A_δ during the interval $\varepsilon^{-1}(\zeta_1,\zeta_2)$ but *does not* hit A during the prior interval $\varepsilon^{-1}[0,\tau_1]$. Since $A_\delta \subset A$, this is a lower bound for the probability that $\{Y_n^\varepsilon\}$ *first* hits A during the interval $\varepsilon^{-1}G \supset \varepsilon^{-1}(\zeta_1,\zeta_2)$. In other words,

$$\boldsymbol{P}\{T^\varepsilon(A) \in G\} \geqslant \mathbb{P}^\varepsilon_{(\zeta_1,\zeta_2)}(A_\delta) - \mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A}_\delta). \tag{5.9}$$

As $\varepsilon \to 0$, the exponential rate of decay of $\mathbb{P}^\varepsilon_{(\zeta_1,\zeta_2)}(A_\delta)$ is $\leqslant \{\varepsilon^{-1} \inf_{x\in A_\delta} \Gamma^*_{(\zeta_1,\zeta_2)}(x)\}$, by Theorem 4.1(ii), while the exponential rate of decay of $\mathbb{P}^\varepsilon_{\mathfrak{C}}(\mathfrak{A}_\delta)$ is $\geqslant \{\varepsilon^{-1} \inf_{x\in \operatorname{cl}\mathfrak{A}_\delta}$

$\Gamma^*_{\mathfrak{C}}(x)\}$, by Theorem 4.2(i). The next lemma shows that this decay is actually dominated by the first term on the right of Eq. (5.9).

Lemma 5.1. *Assume* $\inf_{x\in A_{\tilde\delta}} \Gamma^*_{(\zeta_1,\zeta_2)}(x)$ *is finite for some* $\tilde\delta > 0$. *Then there exists a positive real number* δ_0 *such that*

$$\inf_{x\in \mathrm{cl}\,\mathfrak{A}_\delta} \Gamma^*_{\mathfrak{C}}(x) > \inf_{x\in A_\delta} \Gamma^*_{(\zeta_1,\zeta_2)}(x) \quad \textit{for all } 0<\delta\leqslant\delta_0. \tag{5.10}$$

Proof of Lemma 5.1. Since A_δ increases in size as $\delta\to 0$, the assumption of the lemma implies

$$A_\delta \cap \mathrm{dom}\,\Gamma^*_{(\zeta_1,\zeta_2)} \neq \emptyset \quad \text{for all } 0<\delta\leqslant\tilde\delta. \tag{5.11}$$

Since the elements of the collection $\{A_\delta\}_{\delta>0}$ have been constructed to be disjoint from $\partial\mathfrak{D}_{(\zeta_1,\zeta_2)}$ (the relative boundary of the domain of $\Gamma^*_{(\zeta_1,\zeta_2)}$), it follows by Eq. (5.11) that

$$A_\delta \cap \mathrm{ri}\,\mathfrak{D}_{(\zeta_1,\zeta_2)} \neq \emptyset \quad \text{for all } 0<\delta\leqslant\tilde\delta. \tag{5.12}$$

Hence the conditions of Theorem 3.4(i) and (ii) are satisfied with $f=\Gamma^*_{(\zeta_1,\zeta_2)}$, $\mathscr{E}=A_\delta$, and $0<\delta\leqslant\tilde\delta$. From now on, we will assume that δ has been chosen in the interval $(0,\tilde\delta]$, so that this is true.

Also, let $(x_{0,u},x_{0,v})\in\mathbb{R}^d\times\mathbb{R}^d$ be an element obtained by Theorem 3.4(i) with $f=\Gamma^*_{\mathfrak{C}}$ and $\mathscr{E}=\mathfrak{A}_\delta$.

We begin by relating $\Gamma^*_{\mathfrak{C}}$ to $\Gamma^*_{(\zeta_1,\zeta_2)}$.

Step 1. (i) *For any* $\alpha\in\mathbb{R}^d$, $\Gamma_{\mathfrak{C}}(\alpha,\alpha)=\Gamma_{(\zeta_1,\zeta_2)}(\alpha)$.

(ii) *For any* $x_u,x_v\in\mathbb{R}^d$, $\Gamma^*_{\mathfrak{C}}(x_u,x_v)\geqslant\Gamma^*_{(\zeta_1,\zeta_2)}(x_u+x_v)$.

Proof. By definition

$$\Gamma_{\mathfrak{C}}(\alpha,\alpha)=\sup_{(\tau_u,\tau_v)\in\mathfrak{C}}\{\tau_u\Lambda(\alpha)+(\tau_v-\tau_u)\Lambda(\alpha)\}=\sup_{\tau_v\in(\zeta_1,\zeta_2)}\tau_v\Lambda(\alpha)=\Gamma_{(\zeta_1,\zeta_2)}(\alpha),$$

hence

$$\Gamma^*_{\mathfrak{C}}(x_u,x_v)\geqslant\sup_{(\alpha,\alpha)\in\mathbb{R}^{2d}}\{\langle\alpha,x_u\rangle+\langle\alpha,x_v\rangle-\Gamma_{\mathfrak{C}}(\alpha,\alpha)\}=\Gamma^*_{(\zeta_1,\zeta_2)}(x_u+x_v). \qquad \square$$

Step 2. $x_{0,u}+x_{0,v}\notin\mathfrak{M}_{A_\delta} \Rightarrow \inf_{x\in\mathrm{cl}\,\mathfrak{A}_\delta}\Gamma^*_{\mathfrak{C}}(x)>\inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x)$.

Proof. Note $(x_{0,u},x_{0,v})\in\mathrm{cl}\,\mathfrak{A}_\delta=\mathrm{cl}\{(x_u,x_v):\ (x_u,x_u+x_v)\in A\times A_\delta\}\Rightarrow x_{0,u}+x_{0,v}\in\mathrm{cl}\,A_\delta$. Hence, if $x_{0,u}+x_{0,v}\notin\mathfrak{M}_{A_\delta}$, then

$$\Gamma^*_{(\zeta_1,\zeta_2)}(x_{0,u}+x_{0,v})>\inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x). \tag{5.13}$$

Consequently $\Gamma^*_{\mathfrak{C}}(x_{0,u},x_{0,v})>\inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x)$ (Step 1(ii)). By the choice of $(x_{0,u},x_{0,v})$ it follows that

$$\inf_{x\in\mathrm{cl}\,\mathfrak{A}_\delta}\Gamma^*_{\mathfrak{C}}(x)>\inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x). \qquad \square$$

This establishes the lemma for the case $x_{0,u}+x_{0,v}\notin\mathfrak{M}_{A_\delta}$ and we turn next to the general case. The proof of the lemma for the general case is reliant upon the following.

Step 3. *Suppose* $x_{0,u}+x_{0,v}\in\mathfrak{M}_{A_\delta}$. *Then*

$$\inf_{x\in \operatorname{cl}\mathfrak{A}_\delta}\Gamma^*_{\mathfrak{C}}(x)\leqslant \inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x)\Rightarrow x_{0,u}=cx_{0,v}\quad \textit{for some constant } c\in\left(0,\frac{\tau_1}{\zeta_1-\tau_1}\right].$$

Proof. Let $x_0=x_{0,u}+x_{0,v}$. Then $x_0\in\mathfrak{M}_{A_\delta}$, i.e. x_0 satisfies Theorem 3.4(i) with $f=\Gamma^*_{(\zeta_1,\zeta_2)}$ and $\mathscr{E}=A_\delta$. Let α_0 be an element which satisfies Theorem 3.4(ii) with $f=\Gamma^*_{(\zeta_1,\zeta_2)}$ and $\mathscr{E}=A_\delta$. Since x_0,α_0 satisfy Theorem 3.4(i) and (ii), it follows by Theorem 23.5 of Rockafellar (1970) that

$$\inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x)=\Gamma^*_{(\zeta_1,\zeta_2)}(x_0)=\{\langle\alpha_0,x_0\rangle-\Gamma_{(\zeta_1,\zeta_2)}(\alpha_0)\}. \tag{5.14}$$

Therefore, if we assume

$$\inf_{x\in \operatorname{cl}\mathfrak{A}_\delta}\Gamma^*_{\mathfrak{C}}(x)\leqslant \inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x),$$

then it follows that

$$\inf_{x\in \operatorname{cl}\mathfrak{A}_\delta}\Gamma^*_{\mathfrak{C}}(x)\leqslant\{\langle\alpha_0,x_0\rangle-\Gamma_{(\zeta_1,\zeta_2)}(\alpha_0)\}. \tag{5.15}$$

The left-hand side of Eq. (5.15) can be identified as $\Gamma^*_{\mathfrak{C}}(x_{0,u},x_{0,v})$, since $(x_{0,u},x_{0,v})$ was chosen as an element at which $\Gamma^*_{\mathfrak{C}}$ attains its infimum over $\operatorname{cl}\mathfrak{A}_\delta$. The right-hand side of Eq. (5.15) can be identified as $\{\langle\alpha_0,x_{0,u}\rangle+\langle\alpha_0,x_{0,v}\rangle-\Gamma_{\mathfrak{C}}(\alpha_0,\alpha_0)\}$, since by definition $x_0=x_{0,u}+x_{0,v}$, and by Step 1(i) we have $\Gamma_{\mathfrak{C}}(\alpha_0,\alpha_0)=\Gamma_{(\zeta_1,\zeta_2)}(\alpha_0)$. Hence, Eq. (5.15) gives

$$\Gamma^*_{\mathfrak{C}}(x_{0,u},x_{0,v})\leqslant\{\langle\alpha_0,x_{0,u}\rangle+\langle\alpha_0,x_{0,v}\rangle-\Gamma_{\mathfrak{C}}(\alpha_0,\alpha_0)\}, \tag{5.16}$$

implying $(x_{0,u},x_{0,v})\in\partial\Gamma_{\mathfrak{C}}(\alpha_0,\alpha_0)$ (Rockafellar, 1970, Theorem 23.5). By Theorem 3.6 we then obtain

$$(x_{0,u},x_{0,v})=(\tau_u\nabla\Lambda(\alpha_0),(\tau_v-\tau_u)\nabla\Lambda(\alpha_0)) \tag{5.17}$$

for some $(\tau_u,\tau_v)\in\operatorname{cl}\mathfrak{C}=[0,\tau_1]\times[\zeta_1,\zeta_2]$. Finally note $(x_{0,u},x_{0,v})\in\operatorname{cl}\mathfrak{A}_\delta\Rightarrow x_{0,u}\in\operatorname{cl}A$; then (H2) $\Rightarrow 0\notin\operatorname{cl}A\Rightarrow x_{0,u}\neq 0$. As a result, by Eq. (5.17) we obtain $x_{0,u}=cx_{0,v}$, where $c=\tau_u/(\tau_v-\tau_u)\in(0,\tau_1/(\zeta_1-\tau_1)]$. □

We are now prepared to establish the lemma.

Step 4. *If* $\delta\leqslant$ *some* δ_0, *then* $\inf_{x\in\operatorname{cl}\mathfrak{A}_\delta}\Gamma^*_{\mathfrak{C}}(x)>\inf_{x\in A_\delta}\Gamma^*_{(\zeta_1,\zeta_2)}(x)$.

Proof. Assume false. Then

$$\inf_{x\in \operatorname{cl}\mathfrak{A}_{\delta_i}}\Gamma^*_{\mathfrak{C}}(x)\leqslant \inf\nolimits_{x\in A_{\delta_i}}\Gamma^*_{(\zeta_1,\zeta_2)}(x) \tag{5.18}$$

for a sequence $\{\delta_i\}_{i\in\mathbb{Z}_+}$ where $\delta_i\to 0$ as $i\to\infty$. Along this sequence, it follows by Step 2 that

$$x_0^{(i)}\equiv x_{0,u}^{(i)}+x_{0,v}^{(i)}\in\mathfrak{M}_{A_{\delta_i}}. \tag{5.19}$$

Hence it follows by Step 3 that

$$x_{0,u}^{(i)} = c^{(i)} x_{0,v}^{(i)}, \quad \text{some constant } c^{(i)} \in \left(0, \frac{\tau_1}{\zeta_1 - \tau_1}\right]. \tag{5.20}$$

By combining Eqs. (5.19) and (5.20) we obtain

$$x_0^{(i)} = K^{(i)} x_{0,u}^{(i)}, \tag{5.21}$$

where $K^{(i)} \equiv (1 + 1/c^{(i)}) \in [\zeta_1/\tau_1, \infty)$. We study the limiting behavior of Eq. (5.21) as $i \to \infty$.

First consider $x_0^{(i)}$ as $i \to \infty$. Since $x_0^{(i)} \in \mathfrak{M}_{A_{\delta_i}}$,

$$\Gamma^*_{(\zeta_1,\zeta_2)}(x_0^{(i)}) = \inf_{x \in A_{\delta_i}} \Gamma^*_{(\zeta_1,\zeta_2)}(x) \downarrow \inf_{x \in \operatorname{int} A - \partial \mathfrak{D}_{(\zeta_1,\zeta_2)}} \Gamma^*_{(\zeta_1,\zeta_2)}(x) \quad \text{as } i \uparrow \infty. \tag{5.22}$$

Since $\Gamma^*_{(\zeta_1,\zeta_2)}$ has compact level sets on $\operatorname{cl} A$ (as in Remark 3.5(i)), it follows that the sequence $\{x_0^{(i)}\}$ is bounded. Hence $x_0^{(i)} \in \operatorname{cl} A_{\delta_i}$ converges (possibly after passing to a subsequence) to some point $x_0 \in \operatorname{cl} A$. Furthermore, by Eq. (5.22) and the lower semicontinuity of $\Gamma^*_{(\zeta_1,\zeta_2)}$,

$$\Gamma^*_{(\zeta_1,\zeta_2)}(x_0) = \inf_{x \in \operatorname{int} A - \partial \mathfrak{D}_{(\zeta_1,\zeta_2)}} \Gamma^*_{(\zeta_1,\zeta_2)}(x). \tag{5.23}$$

The infimum on the right of Eq. (5.23) can be extended to all elements of $(\operatorname{int} A)$ as in the discussion following Eq. (4.9) of Collamore (1996a). Hence $x_0 \in \mathfrak{M}_{\operatorname{int} A}$. We conclude that x_0 is actually a *boundary* point of A (Remark 3.5(ii)).

Next consider $x_{0,u}^{(i)}$ as $i \to \infty$. Since $\{x_0^{(i)}\}$ is bounded and $K^{(i)} \geqslant \zeta_1/\tau_1 > 1$, it follows by Eq. (5.21) that $\{x_{0,u}^{(i)}\}$ is likewise bounded. Hence $x_{0,u}^{(i)} \in \operatorname{cl} A$ converges (possibly after passing to a subsequence) to some point $x_{0,u} \in \operatorname{cl} A$.

Going back to Eq. (5.21) and letting $i \to \infty$, we now obtain

$$x_0 = K x_{0,u} \quad \text{where } x_0 \in \partial A,\ x_{0,u} \in \operatorname{cl} A \quad \text{and} \quad K \geqslant \frac{\zeta_1}{\tau_1} > 1. \tag{5.24}$$

Then $x_{0,u} \in \operatorname{cl} A \Rightarrow \lambda x_{0,u} \in \partial A$ for some $0 < \lambda \leqslant 1$, and if A is a semi-cone, this implies $x_0 = (K/\lambda)\lambda x_{0,u}$ is an *interior* point of A. We have reached a contradiction. □

By Lemma 5.1 and the discussion following Eq. (5.9),

$$\liminf_{\varepsilon \to 0} \varepsilon \log \boldsymbol{P}\{T^{\varepsilon}(A) \in G\} \geqslant - \inf_{x \in A_\delta} \Gamma^*_{(\zeta_1,\zeta_2)}(x) \quad \text{for all } \delta \leqslant \text{some } \delta_0. \tag{5.25}$$

To obtain the required lower bound, let $\delta \downarrow 0$ and then let $(\zeta_1, \zeta_2) \uparrow G$. As $\delta \downarrow 0$, we have by definition that $A_\delta \uparrow \left[\operatorname{int} A - \partial \mathfrak{D}_{(\zeta_1,\zeta_2)}\right]$. As $(\zeta_1, \zeta_2) \uparrow G$, we have by Theorem 3.1 that $\operatorname{ri} \mathfrak{D}_{(\zeta_1,\zeta_2)} \uparrow \operatorname{ri} \mathfrak{D}_G$. Hence by Eq. (5.25) we obtain

$$\liminf_{\varepsilon \to 0} \varepsilon \log \boldsymbol{P}\{T^{\varepsilon}(A) \in G\} \geqslant - \inf_{x \in \operatorname{int} A - \partial \mathfrak{D}_G} \Gamma^*_G(x). \tag{5.26}$$

The infimum on the right of Eq. (5.26) can be extended to all elements of $(\operatorname{int} A)$ as in the discussion following Eq. (4.9) of Collamore (1996a). Thus, Eq. (5.26) implies

$$\liminf_{\varepsilon \to 0} \varepsilon \log \boldsymbol{P}\{T^{\varepsilon}(A) \in G\} \geqslant - \inf_{x \in \operatorname{int} A} \Gamma^*_G(x) = - \inf_{\tau \in G} \underline{I}_A(\tau), \tag{5.27}$$

the last step having been obtained by Theorems 3.1 and 3.2. This establishes the lower bound for open intervals $G \subset [0,\infty)$. Since any open subset of $[0,\infty)$ can be written as a countable union of such open intervals, the extension to general open sets follows immediately from Eq. (5.27). □

6. Proofs of Theorems 2 and 3

First we turn to the proof of Theorem 2, namely, to the identification of the most likely normalized first passage time.

To distinguish the most likely first passage time, we need to determine where $I_A(\tau)$ is minimized as a function of τ for convex sets $A \subset \mathbb{R}^d$. Since

$$\inf_{\tau\in C} I_A(\tau) = \inf_{x\in A} \Gamma_C^*(x) \quad \text{for all closed convex } C \subset [0,\infty) \tag{6.1}$$

(Remark 3.3), we may determine this by finding which *intervals* minimize the quantity on the right of Eq. (6.1), that is, which $C \subset [0,\infty)$ satisfy

$$\inf_{x\in A} \Gamma_C^*(x) = \min_{\tilde{C}\subset[0,\infty)} \left\{ \inf_{x\in A} \Gamma_{\tilde{C}}^*(x) \right\}. \tag{6.2}$$

The minimum on the right of Eq. (6.2) is actually $\inf_{x\in A} \Gamma_{[0,\infty)}^*(x) = \Gamma_{[0,\infty)}^*(x_0)$, for a unique point $x_0 \in \operatorname{cl} A$ (Theorems 3.4 and 3.7), and the infimum on the left can only achieve this value at x_0 [since at another $x \in \operatorname{cl} A$ we have $\Gamma_C^*(x) \geqslant \Gamma_{[0,\infty)}^*(x) > \Gamma_{[0,\infty)}^*(x_0)$). Thus, it is enough to show Eq. (6.2) locally at x_0, and this is the subject of the next theorem.

Theorem 6.1. *Suppose A is a convex set satisfying* (H2), $A \cap \operatorname{ri} \operatorname{cone}(\operatorname{dom} \Lambda^*) \neq \emptyset$, *and Λ is differentiable on its domain. Let x_0 and α_0 be given as in Theorem* 3.4 *when $f = \Gamma_{[0,\infty)}^*$ and $\mathscr{E} = A$, and let ρ be the constant given in Theorem* 3.7(ii). *Then for any convex $C \subset [0,\infty)$,*

$$\Gamma_C^*(x_0) = \min_{\tilde{C}\subset[0,\infty)} \Gamma_{\tilde{C}}^*(x_0) \;\Leftrightarrow\; \rho \in \operatorname{cl} C. \tag{6.3}$$

We remark that the minimum in Eq. (6.3) and in Step 1 below is over *all* convex $\tilde{C}$ such that $\operatorname{int} \tilde{C} = (\tau_1, \tau_2)$, where $0 \leqslant \tau_1 < \tau_2 \leqslant \infty$.

Proof of Theorem 6.1. We first identify the minimum value of $\Gamma_C^*(x_0)$ over $C \subset [0,\infty)$. Then we show that this minimum value is attained $\Leftrightarrow \rho \in \operatorname{cl} C$.

Step 1. $\min_{C\subset[0,\infty)} \Gamma_C^*(x_0) = \langle \alpha_0, x_0 \rangle$.

Proof. Note $\Gamma_C \leqslant \Gamma_{[0,\infty)}$ for $C \subset [0,\infty)$, hence $\Gamma_C^* \geqslant \Gamma_{[0,\infty)}^*$. Thus

$$\min_{C\subset[0,\infty)} \Gamma_C^*(x_0) = \Gamma_{[0,\infty)}^*(x_0). \tag{6.4}$$

Next observe that by definition $\Gamma^*_{[0,\infty)}(x_0) = \sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha,x_0\rangle - \mathbf{1}_{\mathscr{L}_0\Lambda}(\alpha)\}$, where $\mathbf{1}_{\mathscr{L}_0\Lambda}(\cdot)$ is the indicator function on $\mathscr{L}_0\Lambda$. Hence

$$\Gamma^*_{[0,\infty)}(x_0) = \{\langle\alpha,x_0\rangle - \mathbf{1}_{\mathscr{L}_0\Lambda}(\alpha)\}_{\alpha=\alpha_0} = \langle\alpha_0,x_0\rangle \tag{6.5}$$

(Rockafellar, 1970, Theorem 23.5, and Theorem 3.7(i) above).

In the remaining steps, we show that the minimum value obtained in Step 1 is *achieved* $\Leftrightarrow \rho\in \operatorname{cl} C$.

Step 2. *If* $\rho\in\operatorname{cl} C$, *then* $\Gamma^*_C(x_0) = \langle\alpha_0,x_0\rangle$.

Proof. Note $\rho\in\operatorname{cl} C \Rightarrow \sup_{\tau\in C}\tau\Lambda(\cdot)\geqslant\rho\Lambda(\cdot)$. Hence

$$\Gamma^*_C(x_0) \equiv \sup_{\alpha\in\mathbb{R}^d}\left\{\langle\alpha,x_0\rangle - \sup_{\tau\in C}\tau\Lambda(\alpha)\right\} \leqslant \sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha,x_0\rangle - \rho\Lambda(\alpha)\}. \tag{6.6}$$

Since $\nabla(\rho\Lambda)(\alpha_0) = \rho\nabla\Lambda(\alpha_0) = x_0$, it follows that

$$\Gamma^*_C(x_0) \leqslant \{\langle\alpha,x_0\rangle - \rho\Lambda(\alpha)\}_{\alpha=\alpha_0} = \langle\alpha_0,x_0\rangle \tag{6.7}$$

(Rockafellar, 1970, Theorem 23.5, and Theorem 3.7(i) above).

Step 3. *If* $\rho\notin\operatorname{cl} C$, *then* $\Gamma^*_C(x_0) > \langle\alpha_0,x_0\rangle$.

Proof. Since $C\subset\mathbb{R}$ is convex, we have $\operatorname{int} C = (\tau_1,\tau_2)$, where $0\leqslant\tau_1<\tau_2\leqslant\infty$. First consider the case $\tau_1,\tau_2>\rho$. Assume to the contrary that

$$\Gamma^*_C(x_0) \equiv \sup_{\alpha\in\mathbb{R}^d}\left\{\langle\alpha,x_0\rangle - \sup_{\tau\in C}\tau\Lambda(\alpha)\right\} = \langle\alpha_0,x_0\rangle \tag{6.8}$$

and derive a contradiction.
Note: $\nabla(\rho\Lambda)(\alpha_0) = \rho\nabla\Lambda(\alpha_0) = x_0$. Hence

$$\sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha,x_0\rangle - \rho\Lambda(\alpha)\} = \{\langle\alpha,x_0\rangle - \rho\Lambda(\alpha)\}_{\alpha=\alpha_0} = \langle\alpha_0,x_0\rangle \tag{6.9}$$

(Rockafellar, 1970, Theorem 23.5, and Theorem 3.7(i) above). Then, by Eqs. (6.8) and (6.9),

$$\max\left\{\sup_{\{\alpha:\Lambda(\alpha)\leqslant 0\}}\left\{\langle\alpha,x_0\rangle - \sup_{\tau\in C}\tau\Lambda(\alpha)\right\},\ \sup_{\{\alpha:\Lambda(\alpha)>0\}}\{\langle\alpha,x_0\rangle - \rho\Lambda(\alpha)\}\right\} \leqslant \langle\alpha_0,x_0\rangle. \tag{6.10}$$

Next observe

$$\tau_1\Lambda(\alpha) = \max_{i=1,2}\tau_i\Lambda(\alpha) = \sup_{\tau\in C}\tau\Lambda(\alpha) \quad \text{on } \{\alpha\colon \Lambda(\alpha)\leqslant 0\}, \tag{6.11}$$

and since $\tau_1\geqslant\rho$,

$$\tau_1\Lambda(\alpha)\geqslant\rho\Lambda(\alpha) \quad \text{on } \{\alpha\colon \Lambda(\alpha)>0\}. \tag{6.12}$$

By Eqs. (6.10)–(6.12), it follows that

$$\sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha,x_0\rangle-\tau_1\Lambda(\alpha)\}\leqslant\langle\alpha_0,x_0\rangle\,. \tag{6.13}$$

Since $\{\langle\alpha,x_0\rangle-\tau_1\Lambda(\alpha)\}_{\alpha=\alpha_0}=\langle\alpha_0,x_0\rangle$ (Theorem 3.7(i)), it then follows by Eq. (6.13) that

$$\sup_{\alpha\in\mathbb{R}^d}\{\langle\alpha,x_0\rangle-\tau_1\Lambda(\alpha)\}=\{\langle\alpha,x_0\rangle-\tau_1\Lambda(\alpha)\}_{\alpha=\alpha_0}. \tag{6.14}$$

Hence $x_0=\nabla(\tau_1\Lambda)(\alpha_0)$ (Rockafellar, 1970, Theorem 23.5), or $\nabla\Lambda(\alpha_0)=x_0/\tau_1$. But $\nabla\Lambda(\alpha_0)=x_0/\rho$ and $\tau_1,\tau_2>\rho$. We have reached a contradiction.

If $\tau_1,\tau_2<\rho$, then it can be shown under Eq. (6.8) that

$$\tau_2\Lambda(\alpha)=\max_{i=1,2}\tau_i\Lambda(\alpha)\quad\text{on }\{\alpha\colon\Lambda(\alpha)\geqslant 0\},\quad \tau_2\Lambda(\alpha)\geqslant\rho\Lambda(\alpha)\quad\text{on }\{\alpha\colon\Lambda(\alpha)<0\}, \tag{6.15}$$

and a repetition of the above argument then gives $\nabla\Lambda(\alpha_0)=x_0/\tau_2$, a contradiction.

This completes the proof of Step 3 and hence the theorem. □

Next we apply Theorem 6.1 to show that the most likely normalized first passage time is $T^\varepsilon(A)\approx\rho$.

Proof of Theorem 2. If $\{Y_n^\varepsilon\}_{n\in\mathbb{Z}_+}$ first hits A at a time *outside* of the interval $\varepsilon^{-1}[\rho-\gamma,\rho+\gamma]$, then either $\{Y_n^\varepsilon\}_{n\in\mathbb{Z}_+}$ first hits A during the interval $\varepsilon^{-1}[0,\rho-\gamma)$ or during the interval $\varepsilon^{-1}(\rho+\gamma,\infty)$. Thus

$$\boldsymbol{P}\{|T^\varepsilon(A)-\rho|>\gamma\text{ and }T^\varepsilon(A)<\infty\}=\boldsymbol{P}\{T^\varepsilon(A)\in[0,\rho-\gamma)\}+\boldsymbol{P}\{T^\varepsilon(A)\in(\rho+\gamma,\infty)\}. \tag{6.16}$$

Then $\boldsymbol{P}\{|T^\varepsilon(A)-\rho|>\gamma|T^\varepsilon(A)<\infty\}$ is obtained by dividing left- and right-hand sides by $\boldsymbol{P}\{T^\varepsilon(A)<\infty\}$. On the right side we have, for example,

$$\boldsymbol{P}\{T^\varepsilon(A)\in[0,\rho-\gamma)\}/\boldsymbol{P}\{T^\varepsilon(A)<\infty\}$$

and, by Theorem 1 and Remark 2.2,

$$\begin{aligned}&\limsup_{\varepsilon\to0}\varepsilon\log(\boldsymbol{P}\{T^\varepsilon(A)\in[0,\rho-\gamma)\}/\boldsymbol{P}\{T^\varepsilon(A)<\infty\})\\&\quad\leqslant-\inf_{\tau\in[0,\rho-\gamma)}\bar{I}_A(\tau)+\inf_{\tau\in[0,\infty)}\underline{I}_A(\tau)\\&\quad=-\inf_{x\in\operatorname{cl}A}\Gamma^*_{[0,\rho-\gamma)}(x)+\inf_{x\in\operatorname{cl}A}\Gamma^*_{[0,\infty)}(x).\end{aligned} \tag{6.17}$$

[The last step follows by Theorems 3.1 and 3.2. The last infimum has been extended from $\operatorname{int}A$ to $\operatorname{cl}A$ because A is assumed to be a convex open set intersecting $\operatorname{ri}(\operatorname{dom}\Gamma^*_{[0,\infty)})$.] By an analogous application of Theorem 1,

$$\begin{aligned}&\limsup_{\varepsilon\to0}\varepsilon\log(\boldsymbol{P}\{T^\varepsilon(A)\in(\rho+\gamma,\infty)\}/\boldsymbol{P}\{T^\varepsilon(A)<\infty\})\\&\quad\leqslant-\inf_{x\in\operatorname{cl}A}\Gamma^*_{(\rho+\gamma,\infty)}(x)+\inf_{x\in\operatorname{cl}A}\Gamma^*_{[0,\infty)}(x).\end{aligned} \tag{6.18}$$

Thus, dividing left- and right-hand sides of Eq. (6.16) by $\boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}$ and taking the limit as $\varepsilon\to 0$, we obtain by Eqs. (6.17) and (6.18),

$$\limsup_{\varepsilon\to 0} \varepsilon\log \boldsymbol{P}\{|T^{\varepsilon}(A)-\rho|>\gamma | T^{\varepsilon}(A)<\infty\}$$
$$\leqslant -\min\left\{\inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\rho-\gamma)}(x), \inf_{x\in \mathrm{cl}\,A} \Gamma^*_{(\rho+\gamma,\infty)}(x)\right\} + \inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\infty)}(x). \tag{6.19}$$

Assertion. $\min\{\inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\rho-\gamma)}(x),\ \inf_{x\in \mathrm{cl}\,A} \Gamma^*_{(\rho+\gamma,\infty)}(x)\} > \inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\infty)}(x)$.

Proof. First we show

$$\inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\rho-\gamma)}(x) > \inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\infty)}(x). \tag{6.20}$$

Let $\tilde{x}_0, x_0$ be given as in Theorem 3.4(i) when $\mathscr{E}=\mathrm{cl}\,A$ and $f=\Gamma^*_{[0,\rho-\gamma)}, \Gamma^*_{[0,\infty)}$, respectively.

If $\tilde{x}_0 \neq x_0$, then $\Gamma^*_{[0,\infty)}(\tilde{x}_0) > \Gamma^*_{[0,\infty)}(x_0)$, since x_0 is the *unique* element which minimizes $\Gamma^*_{[0,\infty)}$ over $\mathrm{cl}\,A$, by Theorem 3.7. Since $\Gamma_{[0,\rho-\gamma)} \leqslant \Gamma_{[0,\infty)} \Rightarrow \Gamma^*_{[0,\rho-\gamma)} \geqslant \Gamma^*_{[0,\infty)}$, it follows that $\Gamma^*_{[0,\rho-\gamma)}(\tilde{x}_0) > \Gamma^*_{[0,\infty)}(x_0)$.

If $\tilde{x}_0 = x_0$, then $\Gamma^*_{[0,\rho-\gamma)}(\tilde{x}_0) > \Gamma^*_{[0,\infty)}(x_0)$ by Theorem 6.1.

Thus, in either case, $\Gamma^*_{[0,\rho-\gamma)}(\tilde{x}_0) > \Gamma^*_{[0,\infty)}(x_0)$, and this implies

$$\inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\rho-\gamma)}(x) = \Gamma^*_{[0,\rho-\gamma)}(\tilde{x}_0) > \Gamma^*_{[0,\infty)}(x_0) = \inf_{x\in \mathrm{cl}\,A} \Gamma^*_{[0,\infty)}(x). \tag{6.21}$$

The proof of Eq. (6.20) with $\Gamma^*_{(\rho+\gamma,\infty)}$ in place of $\Gamma^*_{[0,\rho-\gamma)}$ is identical. □

By the Assertion and Eq. (6.19) we obtain

$$\limsup_{\varepsilon\to 0} \varepsilon\log \boldsymbol{P}\{|T^{\varepsilon}(A)-\rho|>\gamma | T^{\varepsilon}(A)<\infty\} \leqslant -t \quad \text{some } t>0, \tag{6.22}$$

which establishes the theorem. □

The technique used to prove Theorem 2 can be adapted to establish a law of large numbers for $Y_{T^{\varepsilon}(A)} =$ the place of first passage, as follows.

Proof of Theorem 3. Let x_0 be the element given in Theorem 3.4(i) when $f=\Gamma^*_{[0,\infty)}$ and $\mathscr{E}=A$, and let

$$A_{\gamma} = A \cap \{x\in\mathbb{R}^d\colon\ ||x-x_0||>\gamma\}$$

(a subset of A which omits a small γ-ball about x_0). Then, by definition of conditional expectation,

$$\begin{aligned}
&\boldsymbol{P}\{||Y^{\varepsilon}_{T^{\varepsilon}(A)}-x_0||>\gamma | T^{\varepsilon}(A)<\infty\}\\
&\quad = \boldsymbol{P}\{\text{first hitting } A \text{ at a point of } A_{\gamma}\}/\boldsymbol{P}\{\text{ever hitting } A\}\\
&\quad \leqslant \boldsymbol{P}\{\text{ever hitting } A_{\gamma}\}/\boldsymbol{P}\{\text{ever hitting } A\}\\
&\quad = \boldsymbol{P}\{T^{\varepsilon}(A_{\gamma})<\infty\}/\boldsymbol{P}\{T^{\varepsilon}(A)<\infty\}.
\end{aligned} \tag{6.23}$$

Hence

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \log \boldsymbol{P}\{||Y^{\varepsilon}_{T^{\varepsilon}(A)} - x_0|| > \gamma \mid T^{\varepsilon}(A) < \infty\} \\ \leqslant - \inf_{\tau\in[0,\infty)} \bar{I}_{A_\gamma}(\tau) + \inf_{\tau\in[0,\infty)} \underline{I}_A(\tau) \\ = - \inf_{x\in \mathrm{cl}\, A_\gamma} \Gamma^*_{[0,\infty)}(x) + \inf_{x\in\mathrm{cl}\, A} \Gamma^*_{[0,\infty)}(x) \end{aligned} \tag{6.24}$$

by Theorem 1 and Remark 2.2, and Theorems 3.1, 3.2, and the assumptions on A. The proof will be complete once we establish:

Assertion. $\inf_{x\in\mathrm{cl}\, A_\gamma} \Gamma^*_{[0,\infty)}(x) > \inf_{x\in\mathrm{cl}\, A} \Gamma^*_{[0,\infty)}(x)$.

Proof. If $\inf_{x\in\mathrm{cl}\, A_\gamma} \Gamma^*_{[0,\infty)}(x) = \infty$ the result is obvious, so from now on we will assume $\inf_{x\,\in\,\mathrm{cl} A_\gamma} \Gamma^*_{[0,\infty)}(x) < \infty$.

Form a sequence $\{x_i\}_{i\in\mathbb{Z}_+} \subset A_\gamma$ such that

$$\Gamma^*_{[0,\infty)}(x_i) \downarrow \inf_{x\in\mathrm{cl}\, A_\gamma} \Gamma^*_{[0,\infty)}(x) \quad \text{as } i \uparrow \infty. \tag{6.25}$$

Note that $\Gamma^*_{[0,\infty)}$ has compact level sets on $(\mathrm{cone}\, \mathscr{B}_\delta)^{\mathrm{c}}$ (as in Remark 3.5(i)) and $A \subset (\mathrm{cone}\, \mathscr{B}_\delta)^{\mathrm{c}}$ (hypothesis (H2)). Hence, the sequence $\{x_i\}_{i\in\mathbb{Z}_+}$ is bounded and, consequently, a subsequence of $\{x_i\}_{i\,\in\,\mathbb{Z}_+}$ converges to some $z \in \mathrm{cl}\, A_\gamma$. Since $\Gamma^*_{[0,\infty)}$ is lower semicontinuous,

$$\Gamma^*_{[0,\infty)}(z) \leqslant \liminf_{i\to\infty} \Gamma^*_{[0,\infty)}(x_i) = \inf_{x\in\mathrm{cl}\, A_\gamma} \Gamma^*_{[0,\infty)}(x). \tag{6.26}$$

Next observe $z \in \mathrm{cl}\, A_\gamma \Rightarrow z \neq x_0$. Since x_0 is the *unique* element which minimizes $\Gamma^*_{[0,\infty)}$ over $\mathrm{cl}\, A$, by Theorem 3.7, it follows that

$$\inf_{x\in\mathrm{cl}\, A} \Gamma^*_{[0,\infty)}(x) < \Gamma^*_{[0,\infty)}(z) \leqslant \inf_{x\in\mathrm{cl}\, A_\gamma} \Gamma^*_{[0,\infty)}(x). \qquad \square$$

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