Weekly notice #9

The lectures in week 13: We went through all the details of the rather technical proof of Hovedsætning (Main Theorem) 5.4 and after that we proved the uniqueness of the Lebesgue measure in $\mathbb{R}^k$. As mentioned, the existence of the measure will not be proved. After this, a solution to Exercise W7.4 was presented. Moreover, it was proved that for any Riemann integrable function $f :]a, b] \to \mathbb{R}$ there exist two Borel functions s, t such that $s \leq f \leq t$ and such that

$$\int_{]a,b]} s\, dm = \int_a^b f(x)\, dx = \int_{]a,b]} t\, dm.$$

In particular, $s = t$ m-a.e.

The lectures in week 14: Portions of §5.2 and §5.3 will be covered. After this, the Cantor set Z (Example 5.22) will be (briefly) discussed. We will then discuss various measure theoretic paradoxes, and we will finish (hopefully) with Vitali's Theorem (5.30).

Homework - to be handed in to the TA in week 15: Exercise 5.14 and 5.15. (See Exercise 5.8 for definitions.)

The problem sessions in week 15: W9.1, W9.2, W9.3, 5.7*, 5.8, 5.10, 5.11*, 5.32 (only 1°), 5.33*.

Problem W9.1. This is an exercise in σ-classes (Dynkin systems)!.

(i) Let $\mathbb{D}$ be a σ-class on a set X. Suppose that $A; B \in \mathbb{D}$ and that $A \subseteq B$. Show that $B \setminus A \in \mathbb{D}$.

(ii) Let $n \in \mathbb{N}$, set $X = \{1, 2, \ldots, n\}$, and consider the system of subsets

$$\mathbb{K} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots, \{1, 2, \ldots, n-1\}, X, \}.$$

Show that $\mathbb{K}(\mathbb{D}) = \mathcal{P}(X)$.

(iii) Consider here the system $\mathbb{K} = \{\{1, 2\}, \{1, 3\}\}$ on $X = \{1, 2, 3, 4\}$. Show that $\sigma(\mathbb{K}) = \mathcal{P}(X)$, and that $\mathbb{D}(\mathbb{K}) \neq \mathcal{P}(X)$, and conclude that $\mathbb{D}(\mathbb{K})$ is not a σ-algebra.

(iv) Let X and $\mathbb{K}$ be as in (iii). Construct two different measures, μ and ν, on $\mathcal{P}(X)$ such that $\mu(X) = \nu(X) < \infty$ and such that $\mu(K) = \nu(K)$ for all $K \in \mathbb{K}$.

Problem W9.2. Describe the Radon-measure on $\mathbb{R}$ corresponding to each of the following positive linear forms:

$$I_1(f) = f(4),\ I_2(f) = f(1) + 2f(2),\ = I_3(f) = \frac{1}{2}\int_0^1 f(x)\,dx + f(0) + f(1),$$

where $f \in C_c(\mathbb{R})$.

Problem W9.3. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be given by

$$f = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} 1_{[n,n+1[}.$$

Show that $f \in \mathcal{L}_{loc}(\mathbb{R}_{\geq 0})$, that the limit

$$\lim_{x\to\infty} \int_0^x f(t)\,dt$$

exists, but that f does not belong to $\mathcal{L}(\mathbb{R}_{\geq 0})$.