Mathematics 3MI | **Hans Plesner Jakobsen**
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Weekly notice #8

The lectures in week 12: This was more or less a guided tour through sections §4.3–4.6. Only a cursory knowledge of Sections §4.5–4.6 will be expected at the exam (you should be familiar with the material but not necessarily with the proofs).

The lectures in week 13: First it will be shown that the Riemann and Lebesgue integrals coincide when they are both defined (Problem W7.6). Then we will prove the uniqueness of the Lebesgue measure on $\mathbb{R}^k$ (§5.1) and continue with §5.2 and portions of §5.3.

Homework - to be handed in to the TA in week 14: 3MI-exam January 2001: Problem 3.

The problem sessions in week 14: A) First do exercise 4.38 from the book. Then prove "Sætning 4.24" in the case $J = \mathbb{N}$. State "Sætning 4.26" and "Sætning 4.28" for this special case. Discuss when and how the content of Exercise 4.38 may be generalized to doubly indexed sequences $\{a_{j,k}\}$ of complex numbers. [Hint: "Bemærkning 4.27".]

B) Do the following 3MI-exam problems: January '01: Problem 2; June '01: Problem 3; January '02: Problems 3, 7. Finally, do W8.1.

[An English version of the above exams can be found below. For a Danish version, use e.g. the link "Old 3MI-exams (in Danish)" from the course home page.]

Problem W8.1. Consider the function $f : \mathbb{R} \to \mathbb{R}_+$ given by

$$f(x) = \frac{1}{1+x^4}\ .$$

Let μ be the measure on $\mathbb{B}(\mathbb{R}$ given by

$$\mu(E) = \int_E f\ dm,$$

where m is the Lebesgue measure. Compute

$$\int_0^\infty x \, d\mu(x).$$

3MI-exam, January '01 - Problem 2 (10 points): Determine the limit

$$\lim_{n\to\infty} \int_{-\infty}^\infty \cos(\frac{x^2+7n}{n}) e^{-|x|} \, dx.$$

You must argue carefully for each step in the computation.

3MI-exam, January '01 - Problem 3 (20 points): Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1, & x \in [-1,0[, \\ -2 & x \in [0,4], \\ 0, & x \in]-\infty,-1[\cup]4,\infty[. \end{cases}$$

Let m be the usual Lebesgue measure on $\mathbb{R}$, let ε_0 be the Dirac measure in 0 on $\mathbb{R}$, and let μ denote the counting measure on $\mathbb{R}$.

(i) Argue that f is a Borel function.

(ii) Determine whether $f \in \mathcal{L}(m)$, determine whether $f \in \mathcal{L}(\varepsilon_0)$, and determine whether $f \in \mathcal{L}(\mu)$.

(iii) Determine those of the integrals below, that are defined (see question (ii))

$$\int_{\mathbb{R}} f \, dm, \quad \int_{\mathbb{R}} f \, d\varepsilon_0, \quad \int_{\mathbb{R}} f \, d\mu.$$

3MI-exam, June '01 - Problem 3 (20 points): For every Borel set $E \subseteq \mathbb{R}$ we define

$$\mu(E) = \int_E \frac{dm(x)}{1+x^2},$$

where m as usual denotes the Lebesgue measure on $\mathbb{R}$.

(i) Justify (e.g. by reference to a relevant place in the notes) that μ is a measure on $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$ and that

$$\int |x|^a \mu(x) = \int \frac{|x|^a dm(x)}{1+x^2} \qquad (\star)$$

for every $a \in \mathbb{R}$. (As usual, $0^a = \infty$ for $a < 0$, and $|x|^0 = 1$ for all x in $\mathbb{R}$.)

(ii) Show that the value of $(\star)$ is finite when $-1 < a < 1$.

3MI-exam, January '02 - Problem 3 (15 points): Set
$$F(t)=\int_0^\pi \sin(x+t\cos(x))dx, \quad t\in\mathbb{R}.$$
Justify that F is continuous and differentiable, and that
$$F'(t)=\int_0^\pi \cos(x+t\cos(x))\cos(x)dx, \quad t\in\mathbb{R}.$$

3MI-exam, January '02 - Problem 7 (25 points): Let $(X,\mathbb{E},\mu)$ be a measure space, let $f: X\to[0,\infty[$ be an $\mathbb{E}$-measurable function, and set
$$E_t=\{x\in X \mid f(x)\geq t\}, \quad t\in[0,\infty[.$$

a) Justify that $E_t\in\mathbb{E}$ for $t\in[0,\infty[$.

Consider the function
$$\varphi(t)=\mu(E_t), t\in[0,\infty[.$$

b) Show that φ is decreasing, that is,
$$\forall s,t\in[0,\infty[: s<t \Rightarrow \varphi(s)\geq\varphi(t).$$

c) Show that
$$\lim_{n\to\infty}\varphi(1/n)=\mu(X_0),$$
where $X_0=\{x\in X \mid f(x)>0\}$.