Mathematics 3MI **Hans Plesner Jakobsen**
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Weekly notice #5

The lectures in week 9: In the first hour, §2 was finished. §2.4 is left for yourselves to study. "Sætning 2.14" is good to know. It is always possible to avoid it by giving direct proofs (I did this in the second hour), but it is sometimes convenient to be able to refer to this proposition.

After §2 we embarked on §4 which will take us some weeks to get through. We reached the bottom of page 4.4 but the proof of Lemma 1 was rather short (and not directly equal to the one in the book.) Since it is important that you master *simple functions*, I have posed the proof of Lemma 1 as an exercise for the problem sessions. Observe that one point of Lemma 1 is that after we have *defined* the integral of simple functions by the $(*)$ at the bottom of page 4.2, we need to *prove*, among other things, (ii) for simple functions.

The lectures in week 10: We will continue with §4.

The problem sessions in week 11: Prove Lemma 1 on page 4.3. After this: W5.1*, W5.2, W5.3, and exercises <u>4.7</u>, 4.8, 4.11, and 4.13 from the book.

Problem W5.1. Let X be a metric space and let A be a Borel subset of X. Prove that $\mathbb{B}(X)_A = \mathbb{B}(A)$. Here, $\mathbb{B}(X)_A$ is the induced σ-algebra (c.f. §2.4). On the other hand, A is a metric space via the induced metric and hence $\mathbb{B}(A)$, the σ-algebra of Borel sets in the space A, is defined. [Hint: To prove that $\mathbb{B}(X)_A \subseteq \mathbb{B}(A)$, set

$$\tilde{\mathbb{E}} = \{E \in \mathbb{B}(X) \mid E \cap A \in \mathbb{B}(A)\}.$$

Show that the family of open sets in X, $\mathcal{G}_X$, is contained in $\tilde{\mathbb{E}}$ and after this, that $\mathbb{B}(X) = \tilde{\mathbb{E}}$. Then $\subseteq$ follows. For the inclusion $\mathbb{B}(X)_A \supseteq \mathbb{B}(A)$, first prove that $\mathcal{G}_A \subseteq \mathbb{B}(X)_A$.]

Problem W5.2. (i) Does there exist a sequence (f_n) of functions in $\mathcal{L}(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$ such that f_n converges pointwise to a function $f \in \mathcal{L}(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$, and such that

$$\int_{\mathbb{R}} f_n dm = 1/n, \qquad \int_{\mathbb{R}} f dm = 1?$$

(ii) Same question, but with $\mathcal{L}(\mathbb{R}, \mathbb{B}(\mathbb{R}), m)$ replaced by $\mathcal{M}^+(\mathbb{R}, \mathbb{B}(\mathbb{R}))$.

Problem W5.3. Let $(X, \mathbb{E}, \mu)$ be a measure space and let $s : X \mapsto \mathbb{R}^+$ be a simple measurable function. Prove that

$$\int_X s d\mu = \int_0^\infty \mu(s^{-1}(]t, \infty[))dt.$$

(The integral on the right hand side is either the (extended?) Riemann integral, or the Lebesgue integral - you decide!)