Mathematics 3MI — **Hans Plesner Jakobsen**
Department of Mathematics, I.M.F. — *February* 13 2003

Weekly notice #3

The lectures in week 7: §1 was finished and after that we started on §3 by introducing the concept of a **measure** μ on a σ-algebra $\mathbb{E}$. We finished the lectures by introducing the following slightly non-standard terminology: A set $N \in \mathbb{E}$ is a **null-set** if it has measure zero, i.e., if $\mu(N) = 0$. The reason for this is that the Danish name, "nulmængde", really is quite good. More generally, any subset (whether in $\mathbb{E}$ or not) of a set N of measure zero, is called a null-set.

The problem sessions in week 9: 2.1, 2.3, 2.4, <u>2.6</u>, <u>W3.1</u>, W3.2, W3.3*.

(The underlined problems are the most important ones whereas those with a * are more exceptional and/or challenging and should be done only if time permits.)

Problem W3.1.

(i) What does the statement "$f_n \to f$ ε_a-a.e. (almost everywhere)" mean, when ε_a denotes the Dirac measure on $\mathcal{P}(X)$ located at $a \in X$, and f_n and f denote functions on X?

(ii) Same question as in (i) but with ε_a replaced by the counting measure on X.

(iii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be *continous* functions and suppose that $f = g$ m-a.e. (m is the Lebesgue measure). Prove that $f = g$ everywhere.

(iv) Let the function $g : \mathbb{R} \to \mathbb{R}$ be given by: $g(x) = 0$ if $x \leq 0$ and $g(x) = 1$ if $x > 1$. Is there a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f = g$ m-a.e.?

Problem W3.2. Let (X, d) be a metric space and let $\mathbb{B}$ denote the Borel σ-algebra on X. Let $\mathcal{B}(X, \mathbb{C})$ and $\mathcal{C}(X, \mathbb{C})$ denote the sets of complex Borel functions and complex continuous functions on X, respectively. Explain why the following three facts hold true:

(i) If $f, g \in \mathcal{B}(X, \mathbb{C})$ and $\lambda \in \mathbb{C}$ then $\lambda \cdot f, f + g,$ and $fg \in \mathcal{B}(X, \mathbb{C})$.

(ii) If $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{B}(X, \mathbb{C})$ and if the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, then $f \in \mathcal{B}(X, \mathbb{C})$.

(iii) $\mathcal{C}(X,\mathbb{C}) \subseteq \mathcal{B}(X,\mathbb{C})$.

Actually, $\mathcal{B}(X,\mathbb{C})$ is the *smallest* family for which (i), (ii), and (iii) are true. Indeed,

Proposition W3. Let (X,d) be a metric space and let $\mathcal{A}$ be a family of complex functions on X such that:

(i) If $f,g \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ then $\lambda \cdot f, f+g, \text{ and } fg \in \mathcal{A}$.

(ii) If $(f_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{A}$ and if the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, then $f \in \mathcal{A}$.

(iii) $\mathcal{C}(X,\mathbb{C}) \subseteq \mathcal{A}$.

Then $\mathcal{B}(X,\mathbb{C}) \subseteq \mathcal{A}$.

You may begin to prove this proposition in Problem W3 below. The proof will be be completed in a problem on a later weekly notice.

Problem W3.3. Suppose $\mathcal{A}$ satisfies (i), (i), and (iii) in Proposition W3. Set $\mathbb{E} = \{A \subseteq X \mid \chi_A = 1_A \in \mathcal{A}\}$. Show that:

(i) $\mathbb{E}$ is a σ-algebra.

(ii) $\forall V^{\text{open}} \subseteq X : V^{\text{open}} \in \mathbb{E}$.

(Hint: Look at the functions $g_n(x) = \min(n \cdot dist(x, X \setminus V), 1)$)

(iii) $\mathbb{B} \subseteq \mathbb{E}$.