Weekly notice #13

Notice: The so-called Short List for the Essay Problem has now been posted. See the course home page or http://www.math.ku.dk/ ∼ jakobsen/essay-short.html directly

The lectures in week 18: First the Fubini Theorem was discussed. Then some more general observations about the transformation of integrals and, in particular, polar coordinates, were made and the volume of the n-sphere was computed. In the secont hour, the $\mathcal{L}_p$ and L_p spaces were introduced and the Hölder Inequality was proved (just barely).

The lectures in week 19: We will continue with Hölder's and Minkowski's Inequalities. Hopefully we will also cover the Fisher Completeness Theorem.

Homework - to be handed in to the TA in week 20: 3MI-Exam: January '02 Exercises 4 and 5. (see below.)

The problem sessions in week 20: W13.1, W13.2, W13.3, 7.10*, 7.12, 7.13, 7.15[†], 7.16, 7.17*, and 7.18*

(†) Hint for 7.15: Use Hölder's Inequality to show that if $r = tq + (1-t)s$, where $0 < t < 1$, then

$$\int_X |f|^r d\mu \leq \left(\int_X |f|^q d\mu\right)^t \left(\int_X |f|^s d\mu\right)^{1-t}.$$

Problem W13.1. Let $p \geq 1$, $n \in \mathbb{N}$, and $x_1, x_2, \ldots, x_n \in \mathbb{C}$. Show that

$$\left|\frac{x_1 + x_2 + \cdots + x_n}{n}\right|^p \leq \frac{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}{n}.$$

[Hint: Let $X\{1, 2, \ldots, n\}$ and let μ be the measure on X which is the counting measure divided by n. Use Hölder's Inequality on some relevant functions on this space.]

Problem W13.2. Let $(X, \mathbb{E}, \mu)$ be a measure space and let $1 \leq p < \infty$. Let $\mathcal{M}$ be a subset of $\mathcal{L}_p$. Show that f belongs to the closure of $\mathcal{M}$ (w.r.t. $\|\cdot\|_p$) if and only if there is a sequence $\{f_n\}_{n1}^{\infty}$ in $\mathcal{M}$ such that $f_n \to f$ pointwise μ-a.e. *and* in p-mean.

Problem W13.3. We consider here $\mathcal{L}_p([0,1])\mathcal{L}_p([0,1],\mathbb{B},m)$ spaces on the interval $[0,1]$ with the usual Lebesgue measure m.

(i) Show that $\|f\|_1 \leq \|f\|_2$ for all $f \in \mathcal{L}_2([0,1])$.
(ii) Explain why $\mathcal{L}_2([0,1]) \subseteq \mathcal{L}_1([0,1])$.
(iii) Let K be a positive real number. Show that $\{f \in \mathcal{L}_2([0,1]) \mid \|f\|_2 \leq K\}$ is a closed subset of $\mathcal{L}_1([0,1])$ w.r.t. $\|\cdot\|_1$. [Hint: Use W13.2 and Fatou's Lemma.]

3MI-Exam January 2002; Exercise 4: Show that

$$\left|\int_1^\infty \frac{f(x)}{x}\,dx\right| \leq \sqrt[4]{27}\left(\int_1^\infty |f(x)|^4\,dx\right)^{\frac{1}{4}}$$

for all $f \in \mathcal{L}_4([1,\infty[,m)$.

3MI-Exam January 2002; Exercise 5: Consider the function

$$f\sum_{n1}^{\infty}(-1)^{n-1}\sqrt{n}\cdot 1_{[n,n+\frac{1}{n^3}[}\ .$$

Show that $f \in \mathcal{L}_2(\mathbb{R},m)$. What is its norm?