Mathematics 3MI **Hans Plesner Jakobsen**
Department of Mathematics, I.M.F. *April* 5 2003

Weekly notice #10

The lectures in week 14: We covered §5.2 and some parts of §5.3 including the statement of the Riesz Representation Theorem. After that the translation invariance of the Lebesgue measure was discussed and we finished with Vitali's Theorem (5.30). A very brief discussion of the Banach-Tarski paradox in $\mathbb{R}^3$ was given. The Paradox says that it is possible, by the use of the Axiom of Choice, to cut a ball into 5 disjoint pieces and then to reassemble it by using only translations, rotations and reflexions, into 2 balls of the same size as the original. (Clearly, these pieces cannot be measurable.)

The lectures in week 15: Product measures will be defined and analyzed leading up to the theorems of Tonelli and Fubini. This corresponds to about the first half of §6. Sometimes during the lectures the by now twice postponed discussion of the Cantor set will take place.

Homework - to be handed in to the TA in week 16[1] or week 17: 3MI-exam January 2000; Exercise 7 (see below).

The problem sessions in week 16 (April 15) and week 17 (April 23): W10.1, 3MI-exam June 1999; Exercise 9 (see below), 6.2, 6.4*, 6.5, 6.6, 6.7*,and 6.9.

Problem W10.1. Let π be the measure on the set $X = \{0,1\} \times \{0,1\}$ given by

$$\pi(\{0,0\}) = 1,\ \pi(\{0,1\}) = 2,\ \pi(\{1,0\}) = 3,\ \pi(\{1,1\}) = 4.$$

Does there exist measures μ, ν on $\{0,1\}$ such that $\pi = \mu \otimes \nu$?

January 2000; Exercise 7. Let μ be the counting measure on $\mathbb{Z}$, let, as usual, m be the Lebesgue measure on $\mathbb{R}$, and set

$$E = \{(t,n) \in \mathbb{R} \times \mathbb{Z} \mid |t| + |n| \leq 2\}.$$

(i) Explain how it follows that E belongs to the product σ-algebra $\mathbb{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{Z})$.
(ii) Determine the section E^n for all $n \in \mathbb{Z}$.
(iii) Determine $(m \otimes \mu)(E)$, where $m \otimes \mu$ denotes the product measure on $\mathbb{R} \times \mathbb{Z}$.

[1]The TA Morten has promised that the class that meets in week 16 on Tuesday April 15 can hand in their homework until Wednesday April 23 at 12:00 noon.

June 1999; Exercise 9: Consider the subset E of $\mathbb{R}^2$ given by

$$E = \{(t, n) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}.$$

(i) Explain why E is a Borel set.
(ii) Determine the section E_x for all $x \in \mathbb{R}$.
(iii) With m as usual denoting the Lebesgue measure on $\mathbb{R}$ and with $\varepsilon_{1/2}$ being the Dirac measure in $1/2$, you are asked to determine $(\varepsilon_{1/2} \otimes m)(E)$.