Hermitian Symmetric Spaces and Their Unitary Highest Weight Modules

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The purpose of this article is to determine the set of unitarizable highest weight modules corresponding to Hermitian symmetric spaces of the noncompact type. The major step is that of proving unitarity at the “last possible place.” With this established the description of the full set of unitarizable highest weight modules follows by a straightforward tensor product argument combined with the main ingredients of the proof of the key theorem: Bernstein–Gelfand–Gelfand, and a diagrammatic representation of the set of positive noncompact roots.

INTRODUCTION

The purpose of this article is to determine the set of unitarizable highest weight modules corresponding to Hermitian symmetric spaces of the noncompact type. Specifically let \( g \) be a simple Lie algebra over \( \mathbb{R} \) and let \( g = \mathfrak{k} + \mathfrak{p} \) be a Cartan decomposition. By assumption \( \mathfrak{k} \) has a nontrivial center \( \eta = \mathfrak{r} \cdot \mathfrak{h}_0 \) and \( \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{r} \cdot \mathfrak{h}_0 \), where \( \mathfrak{t}_1 = [\mathfrak{t}, \mathfrak{t}] \). The modules \( W_\lambda \) considered are determined by a pair \((\mathfrak{a}_0, \lambda)\), where \( \mathfrak{a}_0 \) is \( \mathfrak{t}_1 \)-dominant and integral and \( \lambda \in \mathfrak{r}^* \). That is, \( \mathfrak{a} = (\mathfrak{a}_0, \lambda) \) determines a finite-dimensional \( \mathcal{H}(\mathfrak{t}_1) \)-module \( V_\lambda \) and \( W_\lambda \) is the irreducible quotient of \( \mathcal{H}(\mathfrak{g}^0) \otimes_{\mathcal{H}(\mathfrak{k}^0)} V_\lambda \), where \( \mathfrak{p}^+ = \{ z \in \mathfrak{p}^c | [h_0, z] = i z \} \).

\( W_\lambda \) may be represented as a space of \( V_\lambda \)-valued polynomials on \( \mathfrak{p}^+ \) and the \( \mathfrak{g} \)-invariant Hermitian form on \( W_\lambda \), restricted to a \( \mathfrak{t}_1 \)-invariant subspace of \( \mathfrak{d} \)-th order polynomials is a \( \mathfrak{d} \)-th order polynomial in \( \lambda \). By considering the set of first order polynomials on \( \mathfrak{p}^+ \) this leads to the idea of “the last possible place of unitariness” explicitly defined and determined in [6]. The main theorem we prove here is that the module at this last possible place indeed is unitarizable. From this the picture is completed by forming tensor products, along the lines of [4], of the unitary modules with the most singular, nontrivial, unitarizable module \( W_\lambda \) corresponding to \( \mathfrak{a}_0 = 0 \).

The main ingredients in the proof are Bernstein–Gelfand–Gelfand and a
diagrammatic representation of \( \Delta^*_+ \); the set of positive noncompact roots, which we develop here. The final steps in the proof then consists of describing certain subsets of \( \Delta^*_+ \) in terms of its diagram. For some of the classical groups the combinatorics connected with this has so far been too involved, and for those groups we have to rely on the proof of the Kashiwara–Vergne conjecture [3, 5–7, 13]. After this work was completed we have learned that Enright, Howe, and Wallach [10] have obtained the same result, see also Garland and Zuckerman [11], and [12]. Finally, the significant contribution by Parthasarathy [14] should be mentioned.

1. Notation

Let \( \mathfrak{g} \) be a simple Lie algebra over \( \mathbb{R} \) and \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) a Cartan decomposition of \( \mathfrak{g} \). We assume that \( \mathfrak{t} \) has a nonempty center \( \mathfrak{h} \); in this case \( \mathfrak{h} = \mathfrak{R} \cdot \mathfrak{h}_0 \) for an \( \mathfrak{h}_0 \in \mathfrak{h} \) whose eigenvalues under the adjoint action on \( \mathfrak{p}^C \) are \( \pm i \). Let

\[
\mathfrak{p}^+ = \{ z \in \mathfrak{p}^C | [\mathfrak{h}_0, z] = iz \}
\]

and

\[
\mathfrak{p}^- = \{ z \in \mathfrak{p}^C | [\mathfrak{h}_0, z] = -iz \}.
\]

Let \( \mathfrak{t}_1 = [\mathfrak{t}, \mathfrak{t}] \) and let \( \mathfrak{h} \) be a maximal Abelian subalgebra of \( \mathfrak{t} \). Then \( \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{R} \cdot \mathfrak{h}_0 \), \( \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{t}_1) \oplus \mathfrak{R} \cdot \mathfrak{h}_0 \), \( \mathfrak{h} \cap \mathfrak{t}_1 \) \( \mathfrak{c} \) is a Cartan subalgebra of \( \mathfrak{t}_1 \), and \( \mathfrak{h}^C \) is a Cartan subalgebra of \( \mathfrak{g}^C \). We let \( \sigma \) denote the conjugation in \( \mathfrak{g}^C \) relative to the real form \( \mathfrak{g} \) of \( \mathfrak{g}^C \). The sets of compact and noncompact roots of \( \mathfrak{g}^C \) relative to \( \mathfrak{h}^C \) are denoted \( \Delta^*_c \) and \( \Delta^*_n \), respectively. \( \Delta = \Delta^*_c \cup \Delta^*_n \). We choose an ordering of \( \Delta \) such that

\[
\mathfrak{p}^+ = \sum_{\alpha \in \Delta^*_+} \mathfrak{g}^\alpha,
\]

and set

\[
\mathfrak{g}^+ = \sum_{\alpha \in \Delta^*_+} \mathfrak{g}^\alpha,
\]

\[
\mathfrak{g}^- = \sum_{\alpha \in \Delta^*_+} \mathfrak{g}^\alpha,
\]

and

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Delta^*_+} \alpha.
\]

Throughout \( \beta \) denotes the unique simple noncompact root. For \( \gamma \in \Delta \) let \( H_\gamma \) be the unique element of \( \mathfrak{h} \cap (\mathfrak{g}^C, (\mathfrak{g}^C)^{-}\gamma) \) for which \( \gamma(H_\gamma) = 2 \). Then for all \( \gamma, \eta \) in \( \Delta \)

\[
\langle \gamma, \eta \rangle = \frac{2(\gamma, \eta)}{(\eta, \eta)} = \gamma_1(H_\eta), \quad \quad (1.1)
\]

where \( (\cdot, \cdot) \) is the bilinear form on \( \mathfrak{h}^C \) obtained from the Killing form of \( \mathfrak{g}^C \). The reflexion corresponding to \( \gamma \in \Delta \) is denoted by \( \sigma_\gamma \).

\[
\sigma_\gamma(\gamma_1) = \gamma_1 - \langle \gamma_1, \gamma \rangle \gamma.
\]

For \( \alpha \in \Delta^*_+ \) choose \( z_\alpha \in (\mathfrak{g}_\alpha)^* \) such that

\[
z_{\alpha} \cdot z_{\alpha}^* = H_{\alpha}.
\]

and let \( z_{-\alpha} = z_{\alpha}^* \). Following the notation of [8] we let \( \gamma_\nu \) denote the highest root. Then \( \gamma_\nu \in \Delta^*_+ \), and \( H_{\gamma_\nu} \in [\mathfrak{h} \cap \mathfrak{t}_1]^C \). Finally we let \( u - u^* \) be the antilinear antiautomorphism of \( \mathcal{H}(\mathfrak{g}^C) \) that extends the map \( x \to -x^* \) of \( \mathfrak{g}^C \).

2. Modules

Corresponding to the decomposition \( \mathcal{H}(\mathfrak{g}^C) = (\mathcal{H}(\mathfrak{t}_1^C) \mathfrak{g}^+ \ominus \mathcal{H}(\mathfrak{t}_1^C) \ominus \mathcal{H}(\mathfrak{h}^C)) \) we let, for \( u \in \mathcal{H}(\mathfrak{g}^C) \), \( \gamma(u) \) denote the unique element of \( \mathcal{H}(\mathfrak{h}^C) \) for which \( u - \gamma(u) \) is in \( \mathcal{H}(\mathfrak{g}^C) \mathfrak{g}^+ \ominus \mathcal{H}(\mathfrak{h}^C) \).

Let \( \chi \in (\mathfrak{h}^C)^* \). The Verma module \( M_\chi \) of highest weight \( \chi - \rho \) is defined to be \( M_\chi = \mathcal{H}(\mathfrak{g}^C)/L_{\chi - \rho} \), where \( L_{\chi - \rho} \) is the left ideal generated by the elements \( (H - \chi(H)) + \rho(H) \), \( H \in \mathfrak{h}^C \), and \( \mathfrak{g}^+ \). We denote the image of \( 1 \) in \( M_\chi \) by \( 1_{\chi - \rho} \), and the unique irreducible quotient is denoted by \( L_{\chi - \rho} \).

If \( \lambda \) is a dominant integral weight of \( \mathfrak{t}_1 \), and if \( \lambda \in \mathfrak{h}^C \) we denote by \( A = (A_\rho, \lambda) \) the linear functional on \( \mathfrak{h}^C \) given by

\[
A(\gamma) = \lambda.
\]

Further we let \( V_\lambda \) denote the irreducible finite-dimensional \( \mathcal{H}(\mathfrak{t}_1^C) \)-module of highest weight \( \lambda \). As \( \mathcal{H}(\mathfrak{t}_1^C) \)-modules, clearly \( V_\lambda = V_{\lambda - \rho} \).

The sesquilinear form \( B_\lambda \) on \( \mathcal{H}(\mathfrak{t}_1^C) \),

\[
B_\lambda(u, v) = A(\gamma(u^* v))
\]

is \( \mathfrak{g} \)-invariant. We let \( N_\lambda \) denote the kernel of \( B_\lambda \).

\[
N_\lambda = \{ u \in \mathcal{H}(\mathfrak{g}^C) \mid \forall v \in \mathcal{H}(\mathfrak{g}^C); A(\gamma(u^* v)) = 0 \},
\]

and set

\[
N_\lambda(t) = N_\lambda \cap \mathcal{H}(\mathfrak{t}_1^C).
\]

Clearly,

\[
I_\lambda \subseteq N_\lambda \quad \text{and} \quad I_\lambda(t) \subseteq N_\lambda(t).
\]
Let $J_A = I_A + \mathcal{H}(g^I) N_A(a)$ Since $\mathcal{H}(g^I) = \mathcal{H}(p^-) \mathcal{H}(t^I) \mathcal{H}(p^+)$ and $V_A = \mathcal{H}(t^I)/N_A(a)$ we have

**Lemma 2.1.** $\mathcal{H}(g^I) \otimes \mathcal{H}(\kappa \circ \rho)$, $V_A = \mathcal{H}(g^I)/J_A$.

Since $J_A \subseteq N_A$, $B_A$ gives rise to a $g$-invariant Hermitian form, also denoted by $B_A$, on $\mathcal{H}(g^I) \otimes \mathcal{H}(\kappa \circ \rho)$, $V_A$.

The unique irreducible quotient $W_A$ of $\mathcal{H}(g^I) \otimes \mathcal{H}(\kappa \circ \rho)$, $V_A$ is given as

$$W_A = \mathcal{H}(g^I)/N_A.$$  \hfill (2.5)

Any $g$-invariant Hermitian form on $W_A$ is proportional to $B_A$.

For further background information we refer to [8, Sect. 1; 4, Sect. 2].

### 3. Bernstein, Gelfand, and Gelfand

The only major theorem we shall be using describes the exact circumstances under which the irreducible quotient $L_n$ of one Verma module can occur in the Jordan-Hölder series $JH(M_\mu)$ of another. First some terminology.

**Definition 3.1.** Let $\chi, \psi \in (h^I)^*$. A sequence of roots $\gamma_1, \ldots, \gamma_k \in \Delta^+$ is said to satisfy condition (A) for the pair $(\chi, \psi)$ if

(i) $\chi = \sigma_n \cdots \sigma_1 \psi$.

(ii) Put $\chi_0 = \psi, \chi_i = \sigma_n \cdots \sigma_i \psi$. Then $\chi_{i-1} = \chi_i = n_i \gamma_i$, where $n_i \in \mathbb{N}$.

Observe that $n_i = \langle \chi_{i-1}, \gamma_i \rangle$.

**Theorem 3.2** [1, p. 42]. Let $\chi, \psi \in (h^I)^*$. Then $L_{\mu} \in JH(M_{\psi})$ if and only if there exists a sequence $\gamma_1, \ldots, \gamma_k \in \Delta^+$ satisfying condition (A) for the pair $(\chi, \psi)$.

In the present situation, $\Delta^+ = \Delta_{\mu}^+ \cup \Delta_{\mu}^-$, Through a series of elementary lemmas it will now be proved that if $\chi, \psi \in (h^I)^*$, if a sequence of roots from $\Delta^+$ satisfies condition (A) for the pair $(\chi, \psi)$, and if moreover $\chi$ is $\Gamma$-dominant, then there also exists a sequence of roots from $\Delta_{\mu}^+$ satisfying the condition for this pair.

The starting point is

**Lemma 3.3.** If $\gamma \in \Delta_{\mu}^+$, $\langle \chi, \gamma \rangle > 0$, and if $\gamma_1, \ldots, \gamma_k \in \Delta^+$ satisfies condition (A) for the pair $(\chi, \psi)$, then $\gamma_k \in \Delta_{\mu}^+$.  

**Proof.** Let $\omega = \gamma_1 \gamma_1 + \cdots + n_k \gamma_k$, and $\omega' = \omega - n_k \gamma_k$. By assumption, $n_k = \langle \chi - \omega', \gamma_k \rangle = \langle \chi, \gamma_k \rangle + 2n_k$.

Thus, $\langle \chi, \gamma_k \rangle = -n_k < 0$. \hfill \square

We now invoke the basic structure of the given situation through the following well-known fact:

**Lemma 3.4.** Let $\gamma \in \Delta$. The coefficient of $\beta$ in $\gamma$ is $-1$, $0$, or $1$.

In particular, if $\gamma \in \Delta_{\mu}^+$, the coefficient is 1.

**Corollary 3.5.** Let $\alpha \in \Delta_{\mu}$ and $\mu \in \Delta_{\mu}$. Then

$$\langle \mu, \alpha \rangle \in \{-1, 0, 1\} \quad \text{and} \quad \langle \alpha, \mu \rangle \in \{-2, -1, 0, 1, 2\}.$$  

**Proof.** $\sigma_n(\mu) = \mu - \langle \mu, \alpha \rangle \alpha$ and thus the first assertion is clear. Let $\langle \alpha, \mu \rangle = n$ and observe that $\alpha - n\mu \in \Delta$. Since $\langle \alpha - n\mu, \alpha \rangle = 2 - |n|$ it follows that $\sigma_n(\alpha - n\mu) = (|n| - 1)\alpha - n\mu$. \hfill \square

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be a sequence of roots in $\Delta^+$ that satisfies condition (A) for a pair $(\chi, \psi)$ of elements of $(h^I)^*$, let $i \leq k$, $k \geq 2$, and assume that $\gamma_{i-1} \in \Delta_{\mu}^+$ and $\gamma_i \in \Delta_{\mu}^+$. Thus

$$\chi_i = \sigma_n \gamma_i \chi_{i-1},$$

$$n_i = \langle \chi_{i-1}, \gamma_i \rangle = \langle \sigma_n \gamma_i, \chi_{i-1} \rangle.$$

We wish to replace the pair $(\gamma_{i-1}, \gamma_i)$ in the sequence by either a pair $(\gamma_{a}, \gamma_b)$ of positive roots such that $\gamma_{a} \in \Delta_{\mu}^+$ or by a single noncompact positive root $\gamma_c$ in such a way that the new sequence also satisfies condition (A) for $(\chi, \psi)$. This is in fact possible even in a more general context. In the present situation, however, it follows from Corollary 3.5 that only a few cases need to be described. We do this, and omit the simple verifications:

(i) $n_i = \langle \gamma_{a}, \gamma_{i-1} \rangle n_i > 0$:

$$\gamma_{a} = \sigma_n \gamma_{i-1} \gamma_{a} \gamma_{i-1}.$$

(ii) $n_i = \langle \gamma_{i-1}, \gamma_i \rangle n_i = 0$:

$$\gamma_i = \sigma_n \gamma_{i-1} \gamma_i.$$

(iii) $n_i = \langle \gamma_{a}, \gamma_{i-1} \rangle n_i > 0$:

$$\gamma_{a} = \sigma_n \gamma_{i-1} \gamma_{a} \gamma_{i-1}.$$
(iv) $n_i + \langle \gamma_{i-1}, \gamma_i \rangle n_{i-1} = 0$:
\[ \gamma_e = \sigma_n \gamma_{i-1}. \]
(v) $n_{i-1} + \langle \gamma_i, \gamma_{i-1} \rangle n_i < 0$ and $n_i + \langle \gamma_{i-1}, \gamma_i \rangle n_{i-1} < 0$:
\[ \langle \gamma_e, \gamma_i \rangle = \langle \sigma_n \gamma_{i-1}, \sigma_n \gamma_i \rangle. \]

Observe that in the last case, $\langle \gamma_i, \gamma_{i-1} \rangle = -2$ and $\langle \gamma_{i-1}, \gamma_i \rangle = -1$.

The net effect of this is that we may rearrange, perhaps even shorten, our sequence $\gamma_1, \ldots, \gamma_k$ in such a way that compact roots participating in it move towards $\gamma_e$ or disappear. If we are in the situation of Lemma 3.3 they can thus be made to disappear completely. Hence we may state

**Proposition 3.6.** Let $\chi, \psi \in \mathfrak{h}^{C*}$ and assume that the sequence $\gamma_1, \ldots, \gamma_k$ satisfies condition (A) for the pair $(\chi, \psi)$. If $\chi$ is $\mathfrak{k}$-dominant we may assume that $\gamma_i \in A^+_k, i = 1, \ldots, k$.

4. Concerning $A^+_k$

In the following we shall consider sets built up of elements from $A^+_k$. There is a pictorial way of representing these subsets which stems from a 2-dimensional diagram of $A^+_k$. This construction, which we present here, is quite analogous to, and easily derived from, the Dynkin diagram of $\mathfrak{d}$.

We stress that besides elementary facts about root systems, everything follows from Lemma 3.4.

Let $\Sigma$ denote the set of simple compact roots.

**Lemma 4.1.** Let $\alpha \in A^+_k$, let $\mu_i, \ldots, \mu_i$ be distinct elements of $\Sigma$, and assume that $\alpha + \mu_i \in A^+_k$ for all $j = 1, \ldots, i$. Then $i \leq 2$. If $i = 2$, $\alpha + \mu_i + \mu_j \in A^+_k$.

**Proof.** (i) Assume $i \geq 2$, and $\forall j: \langle \alpha, \mu_j \rangle = 0$. Since in this case $(\alpha + \mu_i, \alpha + \mu_j) = \langle \alpha, \alpha \rangle + \langle \mu_i, \mu_j \rangle$, $(\alpha + \mu_i, \alpha + \mu_j) = 2(\alpha, \alpha) = 2(\mu_i, \mu_j)$. Consider two distinct elements, $\mu_i$ and $\mu_j$, from the set $\{\mu_1, \ldots, \mu_i\}$, and let $\langle \alpha + \mu_i, \alpha + \mu_j \rangle = n$. By Lemma 3.4, $n = 0$, 1, or 2, but $n = 1$ is excluded since $\mu_i - \mu_j$ is not a root. It follows that $\langle \mu_i, \mu_j \rangle = \pm 2$ and thus, since the roots are simple, $\langle \mu_i, \mu_j \rangle = -2$. This, however, is not possible since by symmetry, $\langle \mu_i, \mu_i \rangle = -2$ and $\mu_i$ is not proportional to $\mu_j$.

Thus there can be at most one $\mu_i$ such that $\langle \alpha, \mu_i \rangle = 0$. In this case $\alpha$ is short, and $\alpha + \mu_j$ is long.

(ii) Assume $i \geq 3$, $\langle \alpha, \mu_j \rangle = 0$, $\langle \alpha, \mu_i \rangle \neq 0$, and $\langle \alpha, \mu_j \rangle \neq 0$. Let $n_2 = \langle \alpha, \mu_j \rangle$ and $n_3 = \langle \alpha, \mu_i \rangle$. By Corollary 3.5 it follows easily that root strings are of length at most 3 (cf. Proposition 6.2 in [6]). By (i), $\alpha$ is short and hence $n_2 = n_3 = -1$. Thus, by Corollary 3.5, $\langle \mu_i, \alpha \rangle = \langle \mu_j, \alpha \rangle = -1$, and, in particular, $\mu_i$ and $\mu_j$ are short. $\alpha + \mu_i$ and $\alpha + \mu_j$ are clearly long. It follows that $(\alpha + \mu_i, \alpha + \mu_j) = 1 + \langle \mu_i, \mu_j \rangle$, and as in (i) this implies that $\langle \mu_i, \mu_j \rangle = -1$.

Consider $\langle \alpha + \mu_i, \alpha + \mu_j \rangle$. By the previous observations this equals $\frac{1}{2}(\mu_i, \mu_j)$, and since it can only take the values 0 or 2, $\langle \mu_i, \mu_j \rangle \neq 0$. However, this leads to a contradiction of Lemma 3.4 since $\alpha + \mu_i + \mu_j \in A^+_k$ and $\langle \alpha + \mu_i, \alpha + \mu_j \rangle = -1$. In other words: if $\mu_i$ is present either $\mu_j$ or $\mu_j$ (or both) is not. If, $\mu_i$ is present the results concerning it and $\mu_j$ remain valid. In particular, $\langle \mu_i, \mu_j \rangle = -1$ and $\alpha + \mu_i + \mu_j$ as well as $\alpha + \mu_i + 2\mu_j$ are elements of $A^+_k$.

(iii) Assume $i \geq 2$ and $\langle \alpha, \mu_i \rangle = -2$. It follows from (ii) that $\langle \alpha, \mu_i \rangle = -n < 0$, and Corollary 3.5 implies that $\alpha$ is long and $\mu_i$ is short. An easy computation now gives that $\langle \alpha + \mu_i, \alpha + 2\mu_i \rangle > 0$ which contradicts Lemma 3.4. Thus, if $\langle \alpha, \mu_i \rangle = -2$, $i = 1$.

(iv) Assume $\langle \alpha, \mu_i \rangle = \langle \alpha, \mu_j \rangle = \langle \alpha, \mu_i \rangle = -1$. Obviously, $\alpha, \mu_i, \mu_j$ and $\mu_j$ are of equal length. By considering $\langle \alpha + \mu_i, \alpha + \mu_j \rangle$ it is seen that the compact roots are pairwise orthogonal. Since this implies that, e.g., $\alpha + \mu_i + \mu_j \in A^+_k$ and since $\langle \alpha + \mu_i + \mu_j, \alpha + \mu_j \rangle = -1$, we conclude that there can be at most two roots in $\Sigma$, $\mu_i$ and $\mu_j$, say, such that $\langle \alpha, \mu_i \rangle = \langle \alpha, \mu_j \rangle = -2$. In this case $\langle \mu_i, \mu_j \rangle = 0$ and $\alpha + \mu_i + \mu_j \in A^+_k$.

By considering the basis of $\mathfrak{d}$ consisting of the negatives of the elements in $\Sigma$, together with the previously highest root $\gamma_1$, we naturally obtain an analogous result concerning the possibilities of subtracting simple compact roots from a given $\alpha \in A^+_k$.

Now, $p^+$ is a highest weight module for $t^+_1$ and each root space is one-dimensional, hence any $\alpha \in A^+_k$ can be written as $\alpha = \alpha_1 + \mu_{\mu}$, where $\alpha_1 \in A^+_k$ and $\mu \in \Sigma$. This observation together with Lemma 4.1 leads directly to the construction of the diagram of $A^+_k$:

One begins with (say) $\beta$ and draws an arrow originating at $\beta$ for each simple root $\mu_i$ such that $\beta + \mu_i \in A^+_k$. Suppose for simplicity that $i = 2$ and consider $\beta + \mu_i$. By Lemma 4.1 $(\beta + \mu_i) + \mu_j \in A^+_k$. So there can be at most one more $\mu_i \in \Sigma$ such that $(\beta + \mu_i) + \mu_j \in A^+_k$.

It follows again from Lemma 4.1 that $\mu_i$, if it exists, is different from $\mu_j$, and $\beta + \mu_i + \mu_j \in A^+_k$. In this case one draws two arrows originating at $\beta + \mu_i$, one parallel to $\mu_j$ and with the same label, and another parallel to $\mu_i$ and labelled $\mu_j$. Similarly $\mu_j$ is drawn from $\beta + \mu_i$ in such a way that its endpoint coincides with the endpoint of the $\mu_j$, drawn from $\beta + \mu_j$. Continuing along these lines the diagram may easily be completed. In fact, the only situation which a priori might ruin this simple picture, that in which
we reach an \( a \in A_n^+ \) in the diagram and for reasons of the structure of the previously constructed part of the diagram are forced to make the same simple compact root \( \mu_0 \) point out from \( a \) in two different directions, is excluded by assumption. This observation is rather useful in the limitations it puts on the diagrams. We formulate it as a lemma which eliminates a situation that obviously would have to occur if the described phenomenon could take place.

**Lemma 4.2.** For no \( a \in A_n^+ \) does there exist three distinct roots, \( \mu_0, \mu_1, \) and \( \mu_2 \in \Sigma_c \) such that \( a - \mu_1, a - \mu_2, a - \mu_1 + \mu_0, \) and \( a - \mu_2 + \mu_0 \) all are elements of \( A_n^+ \).

**Proof.** We may assume that for no \( a' \) smaller than \( a \) does such a phenomenon occur. (\( a = \beta \) is excluded by assumption.) By excluding all other possibilities it follows from Lemma 4.1 that \( \langle a, \mu_1 \rangle = \langle a, \mu_1 \rangle = 1 \). (See Fig. 1.) Suppose \( \langle \mu_0, \mu_1 \rangle < 0 \), and consider \( a - \mu_1 \). It follows that we must be in case (ii) of the proof of Lemma 4.1. Thus \( a - \mu_1 - \mu_0 \) is a root and the same case then gives that \( \langle \mu_0, \mu_0 \rangle < 0 \). However, this implies that \( a - \mu_2 + \mu_0 \) is a root we quickly reach an \( a' \) smaller than \( a \) at which a phenomenon similar to the one in Fig. 1 takes place. This is contradictory to the original assumption and hence \( \langle \mu_0, \mu_1 \rangle = \langle \mu_0, \mu_1 \rangle = 0 \). But clearly \( \langle a, \mu_0 \rangle < 0 \) and thus \( \langle a - \mu_1 - \mu_0, \mu_0 \rangle < 0 \), which implies that three different roots originate at \( a - \mu_1 - \mu_2 \), and this is impossible.

We remark that the assumption that leads to Lemma 4.2; essentially that each element of \( A_n^+ \) should occur exactly once in the diagram, has been made in order to avoid having to deal with degenerate situations in the subsequent proofs. In other situations (e.g., for \( sp(n, \mathbb{R}) \)) it may well be natural to allow roots to occur more than once.

To further illustrate the simplicity of the construction and for future reference we present the resulting diagrams in the Appendix. That there are no more Hermitian symmetric spaces of the noncompact type than those listed is of course a classical result due to Cartan [2]. However, the criterion that one should be able to pick a root \( \beta \) in the Dynkin diagram such that the set of roots bigger than, or equal to, \( \beta \), can be represented by a 2-dimensional diagram as above also excludes all other Dynkin diagrams and all other choices of \( \beta \).

We now begin to collect some of the technical lemmas that will be needed in the following.

First we consider the \( \mathcal{H}(t_1^1) \) module \( p^r \otimes V_{\lambda_n} \), where \( V_{\lambda_n} \) is a finite-dimensional irreducible module of highest weight \( \lambda_n \). The highest weights of \( p^r \otimes V_{\lambda_n} \) are of the form \( \lambda_n - \alpha \) for certain \( \alpha \in A_n^+ \). We wish to describe these in terms of our diagrams. It follows from Proposition 7.3 in [6] and its proof (notably the last part) that

**Lemma 4.3.** Let \( a \in A_n^+ \), \( a_0 - a \) is a highest weight for the \( \mathcal{H}(t_1^1) \)-module \( p^r \otimes V_{\lambda_n} \) if and only if

(i) \( a_0 - a \) is \( t_i \)-dominant, and

(ii) if \( a = a_i + \mu \) with \( \mu \in \Sigma_c \) and \( a, \mu \in A_n^+ \), then \( \lambda_n(H_\mu) > 0 \).

**Lemma 4.4.** Let \( a \in A_n^+ \) and assume \( a - \mu_1 \in A_n^+ \) for \( \mu_1 \in \Sigma_c \); \( j = 1, \ldots, l \), and \( i \leq 2 \).

Then \( a_0 - a \) is a highest weight for the \( \mathcal{H}(t_1^1) \)-module \( p^r \otimes V_{\lambda_n} \) if and only if for all \( j = 1, \ldots, l \),

\[
\langle a_0, \mu_j \rangle \geq \max \{1, \langle a, \mu_j \rangle\}.
\]

**Proof.** \( a_0 - a \) is a highest weight if and only if \( \forall \mu \in A_n^+ : \langle a_0 - a, \mu \rangle \geq 0 \). In view of Lemma 4.3 the necessity is thus clear. As for the sufficiency, observe that since \( A_0 \) is dominant we need only consider those \( \mu \in A_n \) for which \( \langle a, \mu \rangle > 0 \). Let \( \mu \) be as such: \( \mu = \sum_{j=1}^l \gamma_j \mu_j \). It follows that there must be at least one \( \mu_j \) such that \( \gamma_j > 0 \) and \( a, \mu_j \) > 0. Hence for this simple root, \( a - \mu_j \in A_n^+ \). Let \( n_0 = \langle a, \mu_j \rangle \) and \( n = \langle a, \mu \rangle \). Then, by assumption,

\[
\langle a_0, \mu_j \rangle \geq \frac{2\langle a_0, \mu \rangle}{\langle a, \mu \rangle} \geq \frac{2\langle a_0, \mu \rangle}{\langle a, \mu \rangle} = n_0 \left( \frac{\langle \mu_0, \mu \rangle}{\langle \mu_0, \mu \rangle} \right).
\]

Now observe that by Corollary 3.5, \( \langle \gamma, a \rangle = 1 \) for any \( \gamma \in A_n \) for which \( \langle a, \mu \rangle > 0 \). Thus, \( \langle \mu_0, \mu_j \rangle / \langle \mu, \mu \rangle = n / n_0 \).

**Definition 4.4.** For \( a_0 \in A_n^+ \) we let

\[
C_{a_0}^+ = \{ a \in A_n^+ | a \geq a_0 \} \quad \text{and} \quad C_{a_0}^- = \{ a \in A_n^+ | a \leq a_0 \}.
\]

As is suggested by the way they appear in the diagram of \( A_n^+ \), we think of \( C_{a_0}^+ \) and \( C_{a_0}^- \) as the forward and backward cone, respectively, at \( a_0 \).
Let $\mathcal{A} = (A_\alpha, \lambda)$ and let $\alpha \in \mathcal{A}_\alpha^+$. Then, by definition,

$$\langle A, \alpha \rangle = \langle A_\alpha, \alpha \rangle + \lambda \left( \frac{\gamma_{i_1} \cdots \gamma_{i_r}}{\langle \alpha, \alpha \rangle} \right),$$

(4.1)

where $(\gamma_1, \gamma_r) = (\beta, \beta)$. Recall that $A_\alpha$ is always assumed to be $t_1$-dominant and integral.

**Lemma 4.6.** Let $\alpha_0, \alpha \in \mathcal{A}_\alpha^+$ and suppose that

$$\langle A_\alpha + \rho, a_\alpha \rangle + \lambda_0 \left( \frac{\gamma_{i_1} \cdots \gamma_{i_r}}{\langle \alpha_0, \alpha_0 \rangle} \right) = 1,$$

and

$$\langle A_\alpha + \rho, \alpha \rangle + \lambda \left( \frac{\gamma_{i_1} \cdots \gamma_{i_r}}{\langle \alpha, \alpha \rangle} \right) = n > 0.$$

If $\lambda < \lambda_0$, then $\alpha \notin \mathcal{C}_{\alpha_0}$.

**Proof.** By solving for $\lambda$ and $\lambda_0$, the inequality $\lambda < \lambda_0$ is seen to be equivalent to

$$2(A_\alpha + \rho, \alpha) > n(\alpha, \alpha) - \langle a_\alpha, a_\alpha \rangle + 2(A_\alpha + \rho, a_\alpha).$$

Suppose $\alpha = a_\alpha - \mu$ with $\mu = \sum r_i \rho_i$ and all $r_i \geq 0$. Unless $n = 1$ and $(a_\alpha, a_\alpha) = 2(\alpha, \alpha)$ we immediately reach an inequality contradicting the dominance of $A_\alpha + \rho$ in the remaining situation it follows that for any $\mu_i \in \Sigma_\rho$ for which $r_i \neq 0$ (and there must be at least one such),

$$2(A_\alpha + \rho, \mu_i) < (\alpha, \alpha)$$

and, since $\alpha$ is short, this case must also be dismissed. 

**Remark.** Under the assumptions of Lemma 4.6, $\alpha$ must in fact be quite a distance away from in particular the lower portions of $\mathcal{C}_{\alpha_0}$. The separation increases as $\lambda$ decreases.

For $\alpha \in \mathcal{A}_\alpha^+$ let $\lambda_\alpha \in \mathbb{N}$ be determined by the equation

$$(A_\alpha, \lambda_\alpha + \rho)(H_\alpha) = 1.$$ (4.2)

Among those $\lambda_\alpha$'s for which $A_\alpha - \alpha$ is a highest weight for the $\mathcal{H}(t_1)$-module $\mathcal{V}_\lambda$, let $\lambda_\alpha$ denote the smallest, and let $a_\alpha$ denote the corresponding element of $\mathcal{A}_\alpha^+$. (There are no multiplicities in $\mathcal{V}_\lambda$.)

**Corollary 4.7.** Let $\tilde{\alpha} \in \mathcal{A}_\alpha^+$, $\tilde{\alpha} \neq a_\alpha$. If $\tilde{\alpha} \in \mathcal{C}_{a_\alpha}$, $A_\alpha - \tilde{\alpha}$ is not a highest weight for the module $\mathcal{V}_\lambda$.

**Proof.** If $A_\alpha - \tilde{\alpha}$ is a highest weight for the module $\mathcal{V}_\lambda$, $\lambda_\alpha < \lambda_{\tilde{\alpha}}$. By Lemma 4.6 this implies that $a_\alpha \notin \mathcal{C}_{\tilde{\alpha}}$.

**Lemma 4.8.** Consider an $a_i$ from above; $i = 1, \ldots, r$, and assume $\lambda < \lambda_0$. Then $A_\alpha - a_i$ is not a highest weight for the module $\mathcal{V}_\lambda$. Moreover, $a_i \notin \mathcal{C}_{a_\alpha}$.

**Proof.** $a_i$ satisfies

$$2((A_\alpha, \lambda) + \rho - n_i a_i \cdots - n_{i-1} a_i, a_i) = n_i > 0.$$

By Lemma 3.4, inner products between elements of $\mathcal{A}_\alpha^+$ are nonnegative and thus

$$\frac{2(A_\alpha + \rho, a_i)}{(a_i, a_i)} + \lambda \left( \frac{\gamma_{i_1} \cdots \gamma_{i_r}}{(a_i, a_i)} \right) = n_i > 0.$$

It follows that $\lambda_\alpha \leq \lambda$ and the minimality of $\lambda_\alpha$ then gives the first part of the lemma. The second statement follows from Lemma 4.6.

**Lemma 4.9.** Assume that $\lambda < \lambda_0$ and that $A_\alpha - \omega$ is a highest weight for the $\mathcal{H}(t_1)$-module $\mathcal{V}_\lambda$. Then at least one $a_i$, $i = 1, \ldots, r$, belongs to $\mathcal{C}_{a_\alpha}$.

**Proof.** Suppose not. Then, by Lemma 4.8, $a_i \notin \mathcal{C}_{a_\alpha}$ for all $i = 1, \ldots, r$. Consider an arbitrary one of these, $a_i$. It is clear that there exists a $\gamma \in \mathcal{A}_\alpha^+$ such that (a) $|a_i| \subset \mathcal{C}_{a_\alpha}$, and (b) for no $\gamma_i \in \mathcal{A}_\alpha^+$ is $|a_i| \subset \mathcal{C}_{a_\alpha}$ ($\gamma_i \neq \gamma$). It follows that there must exist two distinct elements, $\mu_{a_i}$ and $\mu_{b_i}$, of $\Sigma_\rho$ such that $\gamma + \mu_{a_i} \in \mathcal{A}_\alpha^+$ and $\gamma + \mu_{b_i} \in \mathcal{A}_\alpha^+$. In fact, by the way the diagrams build up, it follows that there are elements $\mu_{a_i}$ and $\mu_{b_i}$ of $\mathcal{A}_\alpha^+$ such that (i) $a_i = \gamma + \mu_{a_i} + \cdots + \mu_{a_i}$, and (ii) for all $j \leq s$ and $k \leq t$, $\gamma + \mu_{a_i} + \cdots + \mu_{a_i} + \mu_{b_i} + \cdots + \mu_{b_i}$ is $\mathcal{A}_\alpha^+$, and $\mu_{a_i} + \cdots + \mu_{a_i} \cap (\mu_{a_i} + \cdots + \mu_{b_i} \cdots + \mu_{b_i}) = 0$ (cf. Fig. 2). Let $\mu_i = \gamma + \mu_{a_i} + \cdots + \mu_{a_i}$. By (ii) $\mu_i \mu_{b_i} \leq 0$ for $k \leq t$ and $\mu_i \leq \mu_{b_i}$ for $j \leq s$.

If $(a_i, \mu_{b_i}) \geq 0$, $a_i + \mu_{b_i} + \mu_{b_i} \in \mathcal{A}_\alpha^+$ and this is easily seen to contradict the proof of Lemma 4.1(ii). Thus, $(a_i, \mu_{b_i}) > 0$, and analogously, $(a_i, \mu_{a_i}) > 0$. If we had $(a_i - a_i, \mu_{b_i}) = 0$ this would imply that

$$\langle a_i, \mu_{b_i} \rangle \geq \langle a_i + \mu_{a_i}, \mu_{b_i} \rangle.$$
Since, by the above and by Lemma 4.4, we also have
\[ \langle A_0, \mu_{\alpha} \rangle \geq \max \{ 1, \langle a_0 + \mu_{b}, \mu_{\alpha} \rangle \}, \]
this would further imply that \( \alpha_i + \mu_\alpha = \alpha_0 + \mu_b \) is a highest weight which, since it clearly belongs to \( C_{a_0}^+ \), is impossible. Thus
\[ \langle A_0 - \alpha_i, \mu_{b} \rangle < 0. \] (4.3)

We now invoke the assumption that \( A_0 - \omega \) is a highest weight; in particular that \( \langle A_0 - \omega, \mu_{b} \rangle \geq 0 \). Combined with (4.3) this implies that there exists an \( \alpha_i \in \{ \alpha_1, ..., \alpha_r \} \) such that \( \langle \alpha_i, \mu_b \rangle < 0 \). We claim that \( \alpha_i \) has to be located on the line through \( \alpha_i - \mu_b \), parallel to \( \mu_{\alpha_i} \) (cf. Fig. 2). Recall that \( \alpha_i \not\in C_{a_0}^+ \cup C_{a_0}^- \). This may be proved rigorously by eliminating all other possibilities. We mention briefly that Lemma 4.1 and the fact that Dynkin diagrams can have no closed loops immediately rule out all other situations but the one indicated by an \( \delta_b \) in Fig. 2 (and its symmetrical counterpart). This, however, by a straightforward reduction, can be ruled out by showing that Fig. 3, in which \( \gamma_1, ..., \gamma_5 \) are distinct elements of \( A_0^+ \), can have no existence. But this follows easily from the nature of Dynkin diagrams, Lemma 3.4, or, equivalently and at the same time justifying our brevity; see the Appendix.

The proof is completed by consecutively applying the arguments that led from \( \alpha_i \) to \( \alpha_{i+1} \) to each, from the previously thus obtained element of \( \{ \alpha_1, ..., \alpha_r \} \). To wit, it follows that there are elements \( \alpha_i, ..., \alpha_r \) of \( \{ \alpha_1, ..., \alpha_r \} \) such that for each \( s = 3, ..., \alpha_i \) is located on the line through \( \alpha_i - \mu_{b} - \cdots - \mu_{b_{s-1}} \), and parallel to the previously described line through \( \alpha_i \). Clearly, for \( s = t + 1 \), this is a contradiction. \( \blacksquare \)

5. K-TYPES

As a \( \mathcal{H}(\mathfrak{t}^{t^2}) \)-module, \( \mathcal{H}(\mathfrak{q}^+ \otimes_{\mathfrak{sl}(\mathfrak{p})} V_{\lambda}) \) is equal to \( \mathcal{H}(\mathfrak{p}^-) \otimes V_{\lambda} \). The restriction of \( B_A \) to each \( t \)-irreducible subspace is, because it is \( t \)-invariant, either zero, strictly positive definite, or strictly negative definite. The problem to which we address ourselves is, for \( A_0 \) fixed, that of determining the set of \( \lambda \)'s for which \( W_{\lambda}, A = (A_0, \lambda) \), is infinitesimally unitary. This is the case exactly when there are no subspaces on which \( B_A \) is strictly negative definite.

To begin with consider an irreducible subspace of \( \mathcal{H}(\mathfrak{p}^-) \otimes V_{\lambda} \) and let \( q \neq 0 \) be the highest weight vector. Observe that the degree of \( q \) is well defined; assume that it is \( d \). Observe moreover that since \( \lambda \) only makes its presence felt through the action of the center of \( t \), \( q \) is, for \( A_0 \) fixed, a highest weight vector for all \( \lambda \). Let \( A_0 \) and \( q \) be fixed and consider the function
\[ f_{\lambda}(q) = B_A(q, q). \] (5.1)

The following is straightforward, and in part well known.
Lemma 5.2. \( f_q(\lambda) = (-1)^d C_q \lambda^d + \text{lower order terms in } \lambda \), \( C_q > 0 \).

The zeros of \( f_q(\lambda) \) are the only places where the restriction of \( B_\lambda \) to the irreducible subspace in question can change signature. If \( q \) is a first order polynomial and if \( f_q(\lambda_0) = 0 \) it is clear that \( W_{\lambda_0} \) cannot be unitary for \( \lambda < \lambda_0 \).

The smallest \( \lambda_0 \) determined by a highest weight vector of degree 1: \( \lambda_0 \), was named "the last possible place of unitarity," and was explicitly determined for an arbitrary \( A_q \) in [6].

Proposition 5.3. Let \( A_0 \in \mathbb{C} \), ..., \( A_t - \alpha_i \) be the set of highest weights in the \( \mathcal{H}(t_1) \) module \( p \otimes V \mathcal{A}_1 \); \( a_1 \), ..., \( a_t \in A_{-1}^+ \). Let, for \( i = 1, ..., t \), \( \lambda_i \) be determined by the equation \((-A_0, \lambda_i) + \rho + (H_{-1}) = 1 \). Then \( \lambda_i = \min \{ \lambda_1, ..., \lambda_t \} \).

We shall see below that \( W_{\lambda_0} \), \( A = (A_0, \lambda_0) \), in fact is unitary.

The determination of the zeros for all \( f_q \)'s as \( q \) varies in the set of highest weight vectors in \( \mathcal{H}(p) \otimes V \mathcal{A}_1 \) would of course yield a complete solution to our original problem. However, as we shall see, it is sufficient to determine \( N_\lambda \) for \( \lambda < \lambda_0 \), and to describe this ideal it is sufficient to determine a set of generators. This set is finite, as is intuitively clear, and as follows from, e.g., the theory of the category \( \mathcal{C} \) [1].

Consider now a highest weight vector \( q \) in \( \mathcal{H}(p) \otimes V \mathcal{A}_1 \) of weight \( (A_0, \lambda_0) = n_0, a_1 = ..., a_n, \); \( a_1 \), ..., \( a_n \in A_{-1}^+ \), and \( n_0, n_1 \in \mathbb{N} \). Let \( \omega_q = n_0 a_1 + ... + n_q a_q \) and observe that the degree of \( q \) is \( d = n_0 + ... + n_q \). Let \( I_d \) denote the ideal generated by the elements of degree less than \( d \) in \( N_\lambda \), \( A = (A_0, \lambda_0) \). Assume that \( q \in N_\lambda \) and that \( q \notin I_d \).

Lemma 5.4. The module \( \mathcal{H}(p^1) \cdot q + I_d)I_d \) is a standard cyclic module of highest weight \( (A_0, \lambda_0) - \omega_q \).

Proof. By assumption \( q \) is a highest weight vector for \( f^1 \). Moreover, for any \( z^* \in \mathcal{P}^* \), \( z^* q \) is in \( N_\lambda \) and is of strictly lower degree than \( q \).

In the following proposition the assumptions on \( q \) are maintained.

Proposition 5.5. \( L_{A_0, \alpha_0} \subseteq \mathcal{J}(M_{A_0, A}), A = (A_0, \lambda_0) \).

Proof. It follows from the various identifications of Section 2 that there exist two \( \mathcal{H}(p^1) \) invariant subspaces, \( A \) and \( B \), of \( M_{A_0, A} \) such that \( B \subseteq A \), and such that \( A/B \) is isomorphic to the unique irreducible quotient of the standard cyclic module in Lemma 5.4. \( A \), \( B \), and \( M_{A_0, A} \) belong to the category \( \mathcal{C} \) and thus have finite Jordan–Hölder series.

6. \( SO^*(2n) \)

For later reference we list here a set of unitary representations of \( G_0 = SO^*(2n) \). They are obtained along the lines of [4]. Specifically, the Hermitian symmetric space corresponding to \( G = SU(n, n) \) may be represented as \( \mathcal{S} = \{ z \in M(n, \mathbb{C}) \mid z^*z < 1 \} \) and \( G_0 \) is isomorphic to a subgroup of \( G \) that leaves invariant the intersection \( \mathcal{S}_0 \) of \( \mathcal{S} \) with the space of skew-symmetric elements of \( M(n, \mathbb{C}) \). One easily checks that the results of [4] may be applied, and thus the restriction to \( G_0 \) of any unitary holomorphic representation of \( G \) yields a series of unitary representations of \( G_0 \). The representations of \( G \) in question may be taken to live in Hilbert spaces of vector-valued holomorphic functions on \( \mathcal{S} \) and hence, in particular, the representation of \( G_0 \) obtained from a unitary holomorphic representation of \( G \) by restricting the functions to \( \mathcal{S}_0 \), is unitary.

Let \( e_1, ..., e_n \) denote the standard orthonormal basis of \( \mathbb{H}^n \). Then, for \( G_0 \),

\[ A_{e_i} = (e_i - e_j) \quad 1 \leq i < j \leq n \],

and

\[ A_{e_i} = (e_i + e_j) \quad 1 \leq i < j \leq n \].

\( A = (\lambda_1, ..., \lambda_n) \) is \( t_1 \)-integral and dominant if and only if \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n \) and \( \lambda_i - \lambda_j \in \mathbb{Z} \). \( p = (n - 1, n - 2, ..., 0, 0) \), and \( \lambda = \lambda_i - \lambda_j \).

By letting \( \tau \) be the trivial representation in Proposition 2.3 of [5], and taking \( j = k \), it follows from the above that the representations

\[ (0, ..., 0, -m_1, ..., -m_l) - (j, j, ..., j) \quad (6.1) \]

of \( G_0 \) are unitary.

Consider a \( A_0 \) for which \( (A_0, e_i - e_j) = 0 \); i.e., \( \lambda_i = \lambda_j \), and let \( i \) be determined by \( \lambda_i = \lambda_j = \cdots = \lambda_j + 1 \). It is clear that the \( \lambda_0 \) of Proposition 5.3 is attained at \( a_0 = (e_{i-1} \quad e_i) \), and it follows that

\[ \lambda_0 = \lambda_{i+1} - \lambda_i = \lambda_{i-1} + \lambda_i \]

\[ = 1 - (n - (i - 1)) - (n - i) \]

\[ = -2n + 2i \]

and thus, \( \lambda_i = \cdots = \lambda_i = -(n - 2i) \). A comparison with (6.1) then gives the following:

Proposition 6.1. Let \( (A_0, e_i - e_j) = 0 \). Then \( W_{\lambda_i}, A = (A_0, \lambda_0) \), is unitarizable.

We observe that the case \( (A_0, e_i - e_j) \neq 0 \) cannot be treated by this method.
7. Unitarity at the Last Possible Place

Let \( A_0 \) be \( f_1 \)-integral and dominant. Suppose that for a given \( \lambda' \), \( W_{\lambda'} \), \( A = (A, \lambda') \), is not unitarizable. Observe that the irreducible subspaces of different highest weights are perpendicular, independently of \( \lambda \). It follows that the Hermitian form, restricted to the space spanned by the \( K \)-types of some highest weight, is not positive semidefinite. On the other hand, for \( \lambda \) sufficiently negative, \( W_{\lambda} \) is unitarizable. Moreover, the form is a smooth function of \( \lambda \) (polynomial) and changes in the signature can only happen at points at which it is degenerate. It follows easily that for some \( \lambda_0 < \lambda' \) and some \( \omega_0 \), we must be in the situation of Proposition 5.5. It thus follows that there exists a sequence \( a_1, \ldots, a_r \in A \) which satisfies condition (A) for the pair \( (A_0, \lambda_0 - \omega_0 + \rho, (A_0, \lambda_0) + \rho) \). According to Proposition 3.6 we may assume that \( a_i \in A_0^+ \), \( i = 1, \ldots, r \).

Let \( V_{a_0} \) denote the \( f \)-invariant subspace of \( p^- \otimes V_{\lambda_0} \) of highest weight \( A_0 - a_0 \).

**Lemma 7.1.** If \( q \) is a highest weight vector in \( \mathcal{H}(p^-) \otimes V_{\lambda_0} \) of weight \( A_0 - a_0 \), \( q \in \mathcal{H}(p^-) \cdot V_{a_0} \), and if for all \( i \), \( a_i \in C_0^- \), then \( q \) does not belong to the \( \mathcal{H}(p^-) \)-ideal generated by any \( f \)-invariant subspace of \( p^- \otimes V_{a_0} \).

**Proof.** Since \( q \) evidently must belong to one of the \( \mathcal{H}(p^-) \)-ideals generated by the \( f \)-irreducible subspaces of \( p^- \otimes V_{\lambda_0} \), it is sufficient to prove that \( q \) does not belong to any one different from \( \mathcal{H}(p^-) \cdot V_{a_0} \). Let \( V_{a_0} \) denote a \( f \)-irreducible subspace of \( p^- \otimes V_{\lambda_0} \) of weight \( A_0 - a_0 \) with \( a_0 \). To begin with, observe that \( a_0 \) is a highest weight in the sense that there is an argument similar to that which led to (4.3) in the proof of Lemma 4.9. Let \( \Sigma_i \) denote the smallest subset of \( \Sigma \) for which \( a_0 \) is a linear combination of \( a_i \) and the elements of the subset (coefficient \( 0 \) allowed). Let \( \mu \in \Sigma_i \) and assume that \( a_0 - \mu \in A_0^+ \). Our next observation is that \( \mu \in \Sigma_i \). To wit, if it did, by passing to the "boundary" of \( C_0^- \), one would, through Lemma 4.4, obtain an \( a' \in C_0^- \), \( a' \neq a_0 \), for which \( A_0 - a' \) is a highest weight of the module \( p^- \otimes V_{a_0} \) and this is impossible by Corollary 4.7. Now assume that \( q \in \mathcal{H}(p^-) \cdot V_{a_0} \). By the same arguments that gave that a highest weight of \( p^- \otimes V_{a_0} \) is of the form \( A_0 - a \) for some \( a \in A_0^+ \) it follows that there are \( a_1, \ldots, a_r \) such that \( q = a_1 \). Observe that \( s = n_1 + \cdots + n_r - 1 \). Now, \( a_1 \in C_0^- \) and \( a_0 \neq a_1 \), and hence there exists a \( \mu \in \Sigma_i \) such that \( a_0 - \mu \in A_0^+ \) and such that, moreover, the coefficient of \( \mu \) in \( a \) is strictly less than the coefficient of \( \mu \) in \( a_0 \). Thus one of the \( a_i \)'s must have a \( \mu \)-coefficient which is strictly larger than that of \( a_0 \). However, by the observation that \( \mu \in \Sigma_i \) and by the way the diagrams are built up, this is easily seen to be impossible.

We now begin to examine \( A = (A_0, \lambda_0) \). Even though in general there will be some \( \lambda_0 \)'s less than \( \lambda_0 \) at which \( L_{{\lambda_0} \in H(M_{\lambda_0})} \) it seems to be generally true that any \( f \)-type whose highest weight occurs in \( \mathcal{H}(p^-) \otimes V_{\lambda_0} \) belongs to \( \mathcal{H}(p^-) \cdot V_{\lambda_0} \), and thus vanishes at \( \lambda_0 \). Here we shall only investigate the situation for some groups and representations. First we wish to establish that if \( a_1, \ldots, a_r \in A_0^+ \) satisfies condition (A) for the pair \( (A_0, \lambda_0 - \omega_0 + \rho, (A_0, \lambda_0) + \rho) \), where \( A_0 - a_0 \) is the weight of a highest weight vector \( q \) in the \( \mathcal{H}(p^-) \)-module \( \mathcal{H}(p^-) \otimes V_{\lambda_0} \) and \( \lambda_0 < \lambda_0 \), then, for all \( i = 1, \ldots, r \), \( a_i \in C_0^- \). For the general case, the combinatorics has so far proved to be too complicated, but we can establish this for sufficiently many cases. To begin with, let \( n_1, \ldots, n_r \) be the integers of Definition 3.1. In particular,

\[
n_i = \langle A_0, \lambda_0 - \omega_0 + \rho - n_1 a_1 - \cdots - n_{i-1} a_{i-1}, a_i \rangle.
\]

(7.1)

Because inner products between positive noncompact roots are nonnegative, and because \( \lambda_0 < \lambda_0 \), it follows that

\[
\langle A_0, \lambda_0 - \omega_0 + \rho, a_i \rangle > \langle A_0, \lambda_0 + \rho, a_i \rangle > 0.
\]

(7.2)

(Recall that \( \langle A_0, \lambda_0 + \rho, a_i \rangle = 1 \).) Equation (7.1) implies that the \( a_i \)'s are distinct. Suppose, namely, that \( a_i = a_j \), with \( j > i \). Then

\[
n_j = \langle A_0, \lambda_0 - \omega_0 + \rho - n_1 a_1 - \cdots - n_{j-1} a_{j-1}, a_j \rangle - \langle n_{i+1} a_{i+1} + \cdots + n_{j-1} a_{j-1}, a_j \rangle \]

\[
= -n_i - \langle n_{i+1} a_{i+1} + \cdots + n_{j-1} a_{j-1}, a_j \rangle < 0.
\]

Naturally, Eq. (7.1) contains much more information, but for our limited purposes, what has been stated is sufficient.

Consider now the diagram corresponding to \( (2n - 1, 2) \). (See the Appendix.) Let \( A_0 \) be \( f_1 \)-integral and dominant, and let \( i \) denote the smallest integer such that \( \langle A_0, \mu_0 \rangle \neq 0 \). Assume that \( i < n \). In this case it is easy to see that \( a_0 = \gamma_1 - \mu_2 - \cdots - \mu_{i-1} \) if \( i = 2 \), \( \gamma_1 = \gamma_2 \) and that any root \( \alpha \in \Delta_0^+ \) that satisfies (7.2) must be in \( C_0^- \). However, it is also clear that for any such \( \alpha \) one can find a \( \mu_j > i \) such that \( \langle \alpha, \mu_j \rangle > 0 \) and such that \( \alpha \) is the only element of \( C_0^- \) with a nonzero inner product with \( \mu_j \). But this means that it is impossible to construct a highest weight of the form \( A_0 - a_1 - \cdots - a_n \) with elements \( a_j, i = 1, \ldots, r \), of \( C_0^- \). The conclusion is the same when \( i = n \).

It suffices to observe that \( \langle A_0, \mu_0 \rangle = 1 \) is the only case in which \( a_0 = \gamma_1 - \mu_2 - \cdots - \mu_{i} \), and that this \( a_0 \) satisfies \( \langle a_0, \Sigma_i \rangle = 0 \) (in fact, \( \langle a_0, \Sigma_i \rangle = 0 \)). Analogous arguments yield the same conclusion for \( (2n - 2, 2) \) and all cases for \( e_4 \) and \( e_5 \), but the ones described below:

\[
e_6.
\]

Following the notation of the Appendix, consider a \( A_0 \) for which
\[ \langle A_0, \mu_3 \rangle = \cdots = \langle A_0, \mu_5 \rangle = 0, \quad \langle A_0, \mu_6 \rangle = n > 0. \]

Clearly, \( \alpha_0 = \beta + \mu_2 + \mu_4 + \mu_3 + \mu_5 + \mu_6 \). If \( n = 1 \) the root \( \alpha' = \beta + \mu_1 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \) satisfies (7.2) but is not in \( C^+ \). However, it is impossible to have \( \alpha' \) belong to the set \( S = \{ \alpha_1, \ldots, \alpha_n \} \) because of the \( \mu_i \) pointing towards it. Suppose, namely, that \( \alpha' \in S \). Since \( A_0 - n_1 \alpha_1 - \cdots - n_n \alpha_n \) is a highest weight and \( \langle \alpha', \mu_i \rangle > 0 \), either \( \alpha' - \mu_1 + \mu_2 \) or \( \alpha' - \mu_1 + \mu_3 + \mu_4 \) (or both) must belong to \( S \). If, say, \( \alpha' - \mu_1 + \mu_2 \in S \), because of the \( \mu_i \) pointing towards this root, we must have \( \alpha' + \mu_i + \mu_j \in S \), and the coefficient of that root must be at least equal to that of \( \alpha' - \mu_1 + \mu_2 \). But there is a \( \mu_i \) pointing towards that root as well, and the net effect is that \( \langle \omega_2, \mu_3 \rangle > 0 \). By considering \( \mu_i \), the case of \( \alpha' - \mu_3 + \mu_4 + \mu_5 > 0 \) is also excluded.

\[ e_i \quad \text{In the case} \quad \langle A_0, \mu_1 \rangle = \cdots = \langle A_0, \mu_3 \rangle = 0 \quad \text{and} \quad \langle A_0, \mu_4 \rangle = n > 0, \quad \alpha_0 = \beta + \mu_1 + \mu_3 + \mu_4 + \mu_5 + \mu_6. \]

When \( n = 1 \) the roots \( \alpha_0 - \mu_1 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \) and \( \alpha_0 - \mu_1 + \mu_3 + \mu_4 + \mu_5 + \mu_6 \) must be excluded, and for \( n = 2 \) the latter must be dismissed whereas the former no longer satisfies (7.2). This is done by arguments analogous to those of \( e_i \). We stress that in the mentioned cases for \( e_i \) there are solutions to (7.1) with \( \lambda_0 < \lambda_0 \).

Finally we return to the case left open by \( e_i \). Observe that \( C^+ \) is “one-dimensional,” and hence also a smallest such \( \alpha' \). However, by considering the simple compact root \( \mu_1 \) pointing towards \( \alpha' \) it follows by an exhaustion argument along the lines of the proof of Lemma 4.9 that this is not possible. Together with Section 6, this establishes the unitarity at the last possible place for \( \mathfrak{s}o^*(2n) \).

Let us now return to the case left open for \( e_i \). Observe that \( \lambda_0 = -4 \). At \( \lambda = -4 - i, \lambda_{1+i} + \alpha_1, \ldots, \lambda_{1+i} + \alpha_n \in JH(M_{\lambda_0}, \varphi) \) is a highest weight if and only if \( \langle A_0, \mu_0 \rangle = \pm \). Let \( V_0 \) denote a subspace of \( \mathfrak{h}(\Gamma, M) \) of highest weight \( A_0 - s(\alpha_1 + \gamma) \), \( s = 1, 2, 3, \ldots \). According to Lemma 7.1, \( V_0 \subset \mathfrak{h}(\Gamma, M) \). More generally, any irreducible subspace of \( \mathfrak{h}(\Gamma, M) \) whose highest weight occurs as a such in \( \mathfrak{h}(\Gamma, M) \) actually belongs to \( \mathfrak{h}(\Gamma, M) \). This is intuitively clear since the coordinate functions of \( V_{\lambda_0} \) corresponding to the highest weight vector \( v_0 \) in \( V_{\lambda_0} \) include the terms \( z_0, \alpha_1 \in C_{\lambda_0} \). Any highest weight vector \( \bar{v} \) in \( \mathfrak{h}(\Gamma, M) \) must, because of \( \mu_1 \), have a coordinate function \( z_0 \) relative to \( v_0 \) of the form \( \bar{v}_0 = \sum_{\alpha_i} \alpha_i \), where \( \alpha_i \in C_{\lambda_0} \) and \( p_i \in \mathfrak{h}(\Gamma, M) \). This observation can easily be made rigorous since it is enough to consider the case in which none of the \( p_i \)'s contain elements corresponding to \( -C_{\lambda_0} \). \( \mathfrak{h}(\Gamma, M) \) is built up by consequtively tensoring representations with \( \Gamma \). In this coordinate function is readily seen to coincide with the coordinate function of a highest weight vector in \( \mathfrak{h}(\Gamma, M) \). The remaining case for \( e_i \) may be treated analogously but we observe that the representations in question \( \lambda_0 = -8 \) exactly are those that occur in the decomposition of the tensor product of the most singular, unitary, nontrivial representation with \( A_0 = 0 \), with itself. The \( \lambda \)-parameter of that representation is \( -4 \).

The only remaining cases now are \( \mathfrak{sp}(n, \mathbb{R}) \) and \( \mathfrak{su}(p, q) \). The unitarity at the last possible place for those groups has been established in [5, 6]. We can then state

**Proposition 7.2.** \( W \lambda \), with \( \lambda = (A_0, \lambda_0) \), is unitarizable.

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### 8. The Unitary Highest Weight Modules

With the unitarity at the last possible place thus established the description of the general situation follows by forming tensor products, along the lines of [4], of \( W \lambda \); \( \lambda = (A_0, \lambda_0) \), with the most singular, nontrivial, unitary module \( W \lambda \), corresponding to \( A_0 = 0 \) and \( \lambda = \lambda_0 \). The value of \( \lambda_0 \) has been determined in [8, 9]. \( W \lambda \) contains, of course, all first order polynomials, but the \( \mathfrak{h}(\Gamma, M) \)-ideal generated by the irreducible subspace of the second order polynomials of highest weight \(-\beta - \gamma\)'s is missing. Here, \( \gamma \) denotes the smallest element of \( A_0 \) perpendicular to \( \beta \). The restriction of \( W \lambda \otimes W \lambda \) to the diagonal (cf. [4]) is the unitarizable module \( W \lambda \), \( \lambda' = (A_0, A_0 + \lambda_0) \). If a second order polynomial is missing from \( W \lambda \), it is clear that it must be in the ideal \( \mathfrak{h}(\Gamma, M) \). \( V_{\lambda_0} \) and in no other such. Moreover, in this case it follows (cf. Lemma 5.2) that the modules \( \tilde{W} \lambda \), \( \lambda = (A_0, \lambda) \) with \( \lambda_0 + \lambda < \lambda < \lambda_0 \) are not unitarizable. From these observations it is straightforward to decide in the explicit cases at hand exactly in which cases second order polynomials are missing from \( W \lambda \), and by repeatedly forming tensor products with \( W \lambda \), the complete description of the set of \( \lambda \)'s below \( \lambda_0 \) at which certain polynomials are missing from the corresponding module \( W \lambda \) follows. Moreover, if a third order polynomial \( q \) is missing at \( \lambda_0 + 2\lambda_0 \) (this can only happen for \( q = \mathfrak{sp}(p, q), \mathfrak{sp}(n, \mathbb{R}), \) or \( \mathfrak{so}^*(2n) \)), it is in the ideal generated by the second order polynomial that is missing at \( \lambda_0 + 2\lambda_0 \). Hence \( q \) is missing at \( \lambda_0 + 2\lambda_0 \), \( \lambda_0, \lambda_0 + \lambda_0 \), and \( \lambda_0 \). Further, the Hermitian form restricted to \( q \) is positive for \( \lambda \rightarrow -\infty \), hence it is negative between \( \lambda_0 + 2\lambda_0 \) and \( \lambda_0 + \lambda_0 \). Continuing along these lines we see that if \( q \) is a fourth order polynomial is missing at \( \lambda_0 + (j - 1)\lambda_0 \), the Hermitian form cannot be positive semi-definite in the open interval from \( \lambda_0 + (j - 1)\lambda_0 \) to \( \lambda_0 + (j - 2)\lambda_0 \).

Observe that the description of the generators of the missing \( K \)-types fits nicely into the diagrammatic approach presented here. As an example with \( q = \mathfrak{sp}(p, q) \), the missing fifth order polynomial four steps below \( \lambda_0 \) is indicated in Fig. 4. The \( \lambda \)'s indicate that the roots occur exactly once in \( \omega_q \).
 Naturally, \( A_k \) as well as \((p, q)\) must satisfy certain conditions for this configuration to be possible. (The horizontal string of 1's must hit both walls of \( C_{n_0} \), cf. below.)

When \( g = sp(n, \mathbb{H}) \) or \( so^*(2n) \) the natural scene for the presentation of the missing \( K \)-types is the union of \( \Delta^+ \) with its mirror image around the line joining \( \beta \) and \( \gamma \), in the \( \Delta^+ \) of \( sp(n, \mathbb{H}) \) (cf. the remark following the proof of Lemma 4.2). In these terms the pictures are identical to those for \( g = su(n, n) \) with the exception that for \( g = so^*(2n) \) one must exclude those pictures that contain points on the above mentioned line.

Finally for \( e_6 \) and \( e_7 \), the missing second order polynomials are easily localized. Observe that \((\alpha_0, \Sigma')\) is a basis for a root system \((\Sigma'\) as in the proof of Lemma 7.1); for \( e_6 \) it corresponds to \( so(8, 2) \) and for \( e_7 \), to \( so(10, 2) \). For \( e_6 \), the corresponding \( \lambda \) is \(-7\), whereas for \( e_7 \) it is \(-12\).

To complete the description of the set of unitarizable highest weight modules we observe that by tensoring with \( W_\lambda \), a finite number of times we reach a point \( \lambda' \), beyond which there will be unitarity for all \( \lambda < \lambda' \). (\( \lambda' \) might—though unitary—be called the first possible place for nonunitarity.) This is so because of the structure of the diagrams combined with the fact that as \( \lambda \) decreases, the number of points in the diagram that can be used in a sequence satisfying condition (A), decreases. Recall that by Proposition 5.5, there must be such a sequence at the first possible place of nonunitarity. Thus it may be seen that beyond the mentioned points in the examples of \( e_k \) and \( e_7 \), there is unitarity. In all other cases for \( e_k \) and \( e_7 \), as well as all cases for \( so(n, 2) \), \( \lambda = \lambda_0 \) \((A_\lambda \neq 0) \). The cases of \( su(p, q) \), \( sp(n, \mathbb{H}) \), and \( so^*(2n) \) are straightforward. We conclude with an example for \( g = su(6, 6) \) that contains all the relevant features of this remark.

**Example.** \( g = su(6, 6) \). Assume \( \langle A_0, \mu_i \rangle = 0 \) for \( i = 1, 2, 4, ..., 10 \), and \( \langle A_0, \mu_1 \rangle = 1 \). Then \( \alpha_0 = \beta_1 + \mu_1 + \mu_2 + \mu_3 \), and \( \lambda_0 = -3 \). At \( \lambda = -4 \) a second order polynomial is missing, and at \( \lambda = -5 \) a third order polynomial is missing. The latter is indicated by the string of 1's. For \( \lambda < -5 \) one can at most use the roots indicated by circles to form a sequence satisfying condition (A). However, the result must be \( l_1 \)-dominant and so, because of \( \mu_3 \), there can be no points in such a hypothetical sequence on the line \( l_1 \). But then, because of \( \mu_1 \), there can be no points on \( l_2 \), etc. We conclude that there is unitarity for \( \lambda < -5 \).
The diagrams of $\Delta_n^+$
\[ g = so(2n - 1, 2). \]

$B_n^+$:
\[ \beta \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-2 \rightarrow n-1 \rightarrow n. \]

$A_n^+$:
\[ \beta \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-1 \rightarrow n \rightarrow n-1 \rightarrow \cdots \rightarrow 3 \rightarrow 2 \rightarrow \beta. \]

$D_n^+$:
\[ \beta \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-3 \rightarrow n-2 \rightarrow n-1 \rightarrow n. \]

$A_n^+$:
\[ \beta \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-4 \rightarrow n-3 \rightarrow n-2 \rightarrow n-1 \rightarrow 1 \rightarrow \beta. \]
\( g = \text{sp}(n, \mathbb{R}) \).

\[ C_n^1: \]

\[ D_n^1: \]

\[ g = \text{su}(p, q). \]

\[ A_{q+p-1}^1: \]

\[ D_n^2: \]
\( g = e_\alpha \).

\[ A_n^4 : \]

\( e_\gamma : \)

\[ A_n^4 : \]

\[ 1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \]

\[ 2 \]
REFERENCES


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