THE EXISTENCE AND UNIQUENESS OF THE HAAR INTEGRAL ON A LOCALLY COMPACT TOPOLOGICAL GROUP

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The discovery by Alfred Haar in 1933, cf. [**6**], of a translation invariant measure on any locally compact topological group must surely rank as one of the high moments in 20th century mathematics. Although the existence was known for all the classical groups, a result of such sweeping generality was thought improbable by most experts. John von Neumann used afterwards to tell with a wry smile how he had tried to talk Haar out of Haar measure. He made amends by giving an easy proof in [**8**] of the existence of Haar measure for compact groups.

Haar proves the existence by resorting to a choice. Since he assumes the group to be second countable this can be accomplished by a Cantor diagonal process. His argument can be adapted to the general case, but then the axiom of choice seems to be needed for the existence. (But see Remark 1.) However, already in 1935 von Neumann, [**9**] and André Weil proved (independently) that the measure was unique up to a multiplicative constant. Weil's argument is quite elementary, and can be reproduced using only a few fact about the convolution product and the partition of unit. A shorter but slightly more advanced proof is obtained by applying Fubini's theorem (though only in the "Fubinito version" for continuous functions with compact support) to the product of two Haar integrals on $G \times G$, cf. [**10**, 6.6.12]. Von Neumann's argument is the shortest, but it uses the Radon-Nikodym theorem for measures (albeit only in the case where one measure is dominated by the other, so that the existence of the Radon-Nikodym derivative is immediate from L^2-theory), and the full (Tonelli) version of Fubini's theorem. Strictly speaking the argument is therefore only valid for $\sigma-$compact groups (including all connected groups).

The point of this note is to show that Haar's theorem is not really that difficult to prove, given a certain mathematical maturity, and that it does not take an elaborate textbook to reproduce it. The proof used for the existence is due to Weil, [**11**], and builds on Haar's original ideas. It is reproduced in [**7**, §29] and [**5**, 14.1] and seems unsurpassable in elegance. For the benefit of the readers discrimination we reproduce all three proofs of the uniqueness mentioned above. Throughout the paper we have used the by now standard theory of the Daniell integral, identifying the class of inner regular Borel measures on a locally compact set X with the set of positive linear functionals on $C_c(X)$, see [**4**], [**2**], [**7**, §12] or [**10**, Chapter 6]. The notation follows that of [**10**].

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Definitions. Let G be a locally compact, Hausdorff topological group with unit element e, and consider positive functions f and g in the algebra $C_c(G)$ of continuous functions on G with compact supports. If $g \neq 0$ there is for each y in G a constant $t > 0$ and x in G, such that $f(y) < tg(x^{-1}y)$. It follows by a standard compactness argument that we can find a finite set of left translates of g, such that

$$f \leq \sum t_n g_{x_n} ,$$

where $g_x(y) = g(x^{-1}y)$. We define $(f\colon g)$ to be the infimum of all numbers $\sum t_n$ that arise in this form. The number $(f\colon g)$, which roughly measures the size of f relative to g, has some simple properties which we list below.

(i) $(f_x\colon g) = (f\colon g)$ for every x in G (left invariance)
(ii) $((f_1 + f_2)\colon g) \leq (f_1\colon g) + (f_2\colon g)$ (subadditivity)
(iii) $(tf\colon g) = (f\colon t^{-1}g) = t(f\colon g)$ for all $t > 0$ (homogeneity)
(iv) If $f_1 \leq f_2$ then $(f_1\colon g) \leq (f_2\colon g)$ (monotonicity)
(v) $(f\colon h) \leq (f\colon g)(g\colon h)$ (comparability)
(vi) $(f\colon g) \geq \|f\|_\infty \|g\|_\infty^{-1}$ (non-triviality)

Only the last two conditions are not immediate from the definition. But if $f \leq \sum t_n g_{x_n}$ and $g \leq \sum s_m h_{y_m}$, then

$$f(z) \leq \sum t_n g(x_n^{-1}z) \leq \sum t_n s_m h(y_m^{-1}x_n^{-1}z) = \sum t_n s_m h_{x_n y_m}(z) .$$

Consequently $(f\colon h) \leq \sum t_n s_m$ $(= \sum t_n \sum s_m)$, and thus (v) follows. To prove (vi) note that if $f \leq \sum t_n g_{x_n}$ and $f(y) = \|f\|_\infty$, then

$$\|f\|_\infty = f(y) \leq \sum t_n g(x_n^{-1}y) \leq \sum t_n \|g\|_\infty ;$$

whence $\|f\|_\infty \leq (f\colon g)\|g\|_\infty$, as desired.

Lemma. *For each triple (f_0, f_1, f_2) in $C_c(G)_+$ and $\varepsilon > 0$, there is a neighbourhood E of e in G such that for every non-zero g in $C_c(G)_+$ with support in E we have*

$$(f_1\colon g) + (f_2\colon g) \leq ((f_1 + f_2)\colon g) + \varepsilon(f_0\colon g) .$$

Proof. Choose a function h in $C_c(G)_+$ such that $h(y) = \|f_1 + f_2\|_\infty$ whenever $f_1(y) + f_2(y) > 0$. Then take $\delta > 0$ such that $(3\delta + 2\delta^2)(h\colon f_0) \leq \varepsilon$ and let $f = f_1 + f_2 + \delta h$. Define h_i in $C_c(G)_+$ for $i = 1, 2$ such that $h_i f = f_i$. Since both h_1 and h_2 are uniformly continuous on G, having compact supports, there is a neighbourhood E of e such that $x^{-1}y \in E$ implies that $|h_i(x) - h_i(y)| < \delta$ for $i = 1, 2$. If therefore $g \in C_c(G)_+$ with support in E, then, whenever $f \leq \sum t_n g_{x_n}$, we can estimate

$$f_i(y) = h_i(y)f(y) \leq \sum t_n h_i(y) g(x_n^{-1}y) \leq \sum t_n (h_i(x_n) + \delta) g(x_n^{-1}y) .$$

It follows that $(f_i\colon g) \leq \sum t_n(h_i(x_n) + \delta)$ and since $h_1 + h_2 \leq 1$ this implies that

$$(f_1\colon g) + (f_2\colon g) \leq \sum t_n(1 + 2\delta) .$$

Consequently, by (ii) and (iii) (note that $f_1 + f_2 \leq h$) we get

$$\begin{aligned}
&(f_1\colon g) + (f_2\colon g) \leq (1 + 2\delta)(f\colon g) \leq (1 + 2\delta)(((f_1 + f_2)\colon g) + \delta(h\colon g)) \\
&\leq ((f_1 + f_2)\colon g) + (2\delta + (1 + 2\delta)\delta)(h\colon g) \\
&\leq ((f_1 + f_2)\colon g) + (3\delta + 2\delta^2)(h\colon f_0)(f_0\colon g) \leq ((f_1 + f_2)\colon g) + \varepsilon(f_0\colon g) .
\end{aligned}$$

□

Theorem 1. *There exists a non-trivial left invariant Radon integral on G.*

Proof. Let (g_λ) denote the net of functions in $C_c(G)_+$ such that $g_\lambda(e) = 1$, where $g_\lambda \prec g_\mu$ if $g_\mu \le g_\lambda$. Fix once and for all a non-zero function f_0 in $C_c(G)_+$, and for every f in $C_c(G)_+$ define

$$I_\lambda(f) = (f\colon g_\lambda)(f_0\colon g_\lambda)^{-1}\,.$$

Evidently the function $f \to I_\lambda(f)$ is left invariant, subadditive, homogeneous and monotone, cf. (i)–(iv). Moreover, we see from condition (v) that

$$(*) \qquad 0 < (f_0\colon f)^{-1} \le I_\lambda(f) \le (f\colon f_0)\,.$$

Finally it follows from the Lemma that for every $\varepsilon > 0$ we have

(vii) $I_\lambda(f_1) + I_\lambda(f_2) \le I_\lambda(f_1 + f_2) + \varepsilon$

eventually. If we therefore choose a universal subnet of (g_λ), cf. [**10**, 1.3.8], then $I(f) = \lim I_\lambda(f)$ will exist for every f in $C_c(G)_+$ by $(*)$, and define a positive, left invariant and additive functional I on $C_c(G)_+$ by (i), (ii), (iii) and (vii). Since $I_\lambda(f_0) = I(f_0) = 1$ this functional is non-zero, and thus its linear extension to $C_c(G)$ defines a non-trivial Radon integral on G. □

Theorem 2. *The Haar integral on G is unique up to a multiplicative constant.*

First Proof. Let $\int$ denote any non-zero left invariant Radon integral on G. Without loss of generality we may assume that $\int f_0 = 1$, and must show that $\int = I$, where I is the integral found in Theorem 1.

If $f \le \sum t_n g_{x_n}$ for some f, g in $C_c(G)_+$, then evidently $\int f \le \sum t_n \int g$, from which we conclude that $\int f \le (f\colon g) \int g$. With (g_λ) the net considered in Theorem 1, put $\gamma_\lambda = (\int g_\lambda)^{-1}$. Then the estimate above shows that

$$(*) \qquad \gamma_\lambda = \gamma_\lambda \int f_0 \le (f_0\colon g_\lambda)$$

for all λ.

Now put $u_\lambda = \gamma_\lambda g_\lambda$, and note that the net (u_λ) is an approximative unit for the convolution product on $C_c(G)$ in the uniform topology. For each f in $C_c(G)_+$ and $\varepsilon > 0$ we therefore eventually have

$$f(y) \le \int f(x) u_\lambda(x^{-1}y)\,dx + \tfrac{1}{2}\varepsilon$$

for all y.

Let C be a compact subset of G supporting f, and for a fixed λ and $\varepsilon_2 \le \frac{1}{2}\varepsilon(\int f)^{-1}$ choose a finite open covering (E_n) of C and elements x_n in E_n, such that

$$u_\lambda(x^{-1}y) < u_\lambda(x_n^{-1}y) + \varepsilon_2$$

for all x in E_n and y in C. Choosing a partition of unit subordinate to (E_n) on C, cf. [**10**, 1.7.12], we can find a family of functions h_n in $C_c(G)_+$ with $\sum h_n | C = 1$, such that each h_n is supported in E_n. It follows that

$$\int f(x) u_\lambda(x^{-1}y)\,dx = \sum \int f(x) h_n(x) u_\lambda(x^{-1}y)\,dx \le$$
$$\sum \int f(x) h_n(x)\,dx\, u_\lambda(x_n^{-1}y) + \sum \int f(x) h_n(x) \varepsilon_2\,dx = \sum t_n u_\lambda(x_n^{-1}y) + \varepsilon_2 \int f\,,$$

where $t_n = \int fh_n$. If we therefore choose a function h in $C_c(G)_+$ such that $h|C = 1$, then our two estimates combine to show that

$$(**) \qquad f(y) \leq \sum t_n u_\lambda(x_n^{-1}y) + \varepsilon h(y)$$

for all y in G. Thus, $(f-\varepsilon h)_+ \leq \sum t_n (u_\lambda)_{x_n}$, whence $((f-\varepsilon h)_+ : u_\lambda) \leq \sum t_n = \int f$. Since $u_\lambda = \gamma_\lambda g_\lambda$ this, by (iii), means that

$$((f-\varepsilon h)_+) : g_\lambda) \leq \gamma_\lambda \int f \, .$$

Using that $f = (f-\varepsilon h) + \varepsilon h \leq (f-\varepsilon h)_+ + \varepsilon h$ we have by (ii) and (iv) that eventually

$$(f : g_\lambda) \leq \gamma_\lambda \int f + \varepsilon (h : g_\lambda) \, .$$

Combined with $(*)$ this shows that

$$I_\lambda(f) = (f : g_\lambda)(f_0 : g_\lambda)^{-1} \leq \gamma_\lambda (f_0 : g_\lambda)^{-1} \int f + \varepsilon I_\lambda(h) \leq \int f + \varepsilon I_\lambda(h) \, .$$

Passing to the limit along the universal net, as in Theorem 1, gives $I(f) \leq \int f + \varepsilon I(h)$, and since ε is arbitrary we conclude that $I(f) \leq \int f$ for every f in $C_c(G)_+$. But then $\int - I$ is a positive, left invariant Radon integral on G, and since we have $(\int - I)(f_0) = 0$ it must be the zero integral (by invariance), whence $\int = I$, as desired. □

Remark 1. As shown by Henri Cartan, the inequality $(**)$, above, even in the two-sided form $\|f - \sum t_n (u_\lambda)_{x_n}\|_\infty \leq \varepsilon$, can be established directly, without resorting to the convolution with any Haar integrals, cf. [**3**]. The non-trivial argument uses the almost additive functionals I_λ in lieu of the integral. This inequality then shows that the full net $(I_\lambda(f))$ actually is convergent for each f in $C_c(G)_+$, avoiding the need to choose a universal subnet. The existence of the Haar integral is therefore independent of the axiom of choice. A full discussion of these implications and a proof that simultaneaously establishes existence and uniqueness can be found in [**1**].

Second Proof. In the net (g_λ) considered in Theorem 1 we consider the subnet consisting of symmetric functions $(g(y^{-1}) = g(y))$, replacing if necessary each g_λ by the function $x \to g_\lambda(x) g_\lambda(x^{-1})$. If now $\int_x$ and $\int_y$ are left invariant Radon integrals on G we consider the numbers $\gamma_\lambda = (\int_x g_\lambda)(\int_y g_\lambda)^{-1}$. Since $\gamma_\lambda > 0$ for all λ we may assume, passing if necessary to a further subnet, that either $\gamma_\lambda \to \gamma$ or $\gamma_\lambda^{-1} \to \gamma$ for some $\gamma \geq 0$. Interchanging $\int_x$ and $\int_y$ transforms the second situation into the first, which we may therefore assume to hold.

Using Fubini's theorem, cf. [**10**, 6.6.6], see also Remark 2, applied to the product integral $\int_x \otimes \int_y$ on $G \times G$, together with the translation invariance of $\int_x$ and $\int_y$, we get for any f in $C_c(G)$ and every symmetric g_λ

$$\int_y \int_x f(x) g_\lambda(y) \, dx \, dy = \int_x \int_y f(x) g_\lambda(y) \, dy \, dx = \int_x \int_y f(x) g_\lambda(x^{-1}y) \, dy \, dx$$

$$\int_y \int_x f(x) g(x^{-1}y) \, dx \, dy = \int_y \int_x f(yx) g_\lambda(x^{-1}) \, dx \, dy = \int_y \int_x f(yx) g_\lambda(x) \, dx \, dy \, .$$

For each $\varepsilon > 0$ we can find a compact neighbourhood E of e, such that for all x in E we have $\int_y |f(yx) - f(y)|\,dy \leq \varepsilon$, cf. [**10**, 6.6.11]. Since the support of g_λ is contained in E eventually, this, combined with the equality above and the definition of γ, gives

$$\Big|\int_x f(x)\,dx - \gamma\int_y f(y)\,dy\Big| = \lim\Big(\int_y g_\lambda\Big)^{-1}\Big|\int_x\int_y (f(x)g_\lambda(y) - g_\lambda(x)f(y))\,dy\,dx\Big|$$
$$= \lim\Big(\int_y g_\lambda\Big)^{-1}\Big|\int_x\int_y (f(yx) - f(y))g_\lambda(x)\,dy\,dx\Big| \leq \limsup\Big(\int_y g_\lambda\Big)^{-1}\int_x \varepsilon g_\lambda = \varepsilon\gamma\,.$$

Since ε and f were arbitrary we conclude that $\int_x = \gamma\int_y$. □

Remark 2. If $\int_x$ and $\int_y$ are Radon integrals on locally compact Hausdorff spaces X and Y, respectively, the product integral $\int_x \otimes \int_y$ on $X \times Y$ is defined so that the Fubinito theorem, mentioned in the second proof, above, is built into the construction. For any f in $C_c(X \times Y)$ choose $c = a \otimes b$ in $C_c(X) \otimes C_c(Y)$ so that $fc = f$ and $0 \leq c \leq 1$. By the Stone-Weierstrass theorem we can then for each $\varepsilon > 0$ find a finite tensor product $f_\varepsilon = \sum g_n \otimes h_n$ such that $\|f - f_\varepsilon\|_\infty \leq \varepsilon$ and $f_\varepsilon c = f_\varepsilon$. This shows at once that both functions $x \to \int_y f(x,y)\,dy$ and $y \to \int_x f(x,y)\,dx$ are continuous with compact supports on X and Y, respectively, and that $\int_x\int_y f(e,y)\,dy\,dx = \int_y\int_x f(x,y)\,dx\,dy (= \int_x \otimes \int_y (f)\,)$. Indeed, the result is trivial for f_ε, and the functionals $g \to \int_x ga$ and $h \to \int_y hb$ are uniformly continuous on $C_b(X)$ and $C_b(Y)$, respectively, so that the approximation of f by f_ε is respected.

Third Proof. Let again $\int_x$ and $\int_y$ be left invariant Radon integrals on G, which we assume to be $\sigma-$compact. In order to prove that $\int_y$ is proportional to $\int_x$ we may assume that $\int_y \leq \int_x$, replacing if necessary $\int_x$ with $\int_x + \int_y$. By the Radon-Nikodym theorem, cf. [**10**, 6.5.4], there is a function h in $L^\infty_x(G)$, with $0 \leq h(x) \leq 1$ almost everywhere, such that $\int_y f = \int_x fh$ for all f in $C_c(G)$. This implies that for each z in G we have

$$\int_x f(x)h(x)\,dx = \int_y f(y)\,dy = \int_y f(zy)\,dy = \int_x f(zx)h(x)\,dx = \int_x f(x)h(z^{-1}x)\,dx\,.$$

Consequently, $\int_x f(x)(h(x) - h(z^{-1}x))\,dx = 0$ for every f in $C_c(G)$, whence $h = h_z$ almost everywhere. Fubini's theorem, applied to the product integral $\int_x \otimes \int_x$ on $G \times G$ and the function $(x,z) \to h(z^{-1}x) - h(x)$, now shows that the function $x \to \int_x |h(z^{-1}x) - h(x)|\,dz = \int_x |h(z^{-1}) - h(x)|\,dz$ equals zero almost everywhere. It follows that h is constant almost everywhere, as desired. □

References

[1] Erik M. Alfsen, *A simplified constructive proof of the existence and uniqueness of Haar measure*, Mathematica Scandinavica **12** (1963), 106–116.

[2] Nicolas Bourbaki, *"Éléments de Mathématiques: Livre VI, Intégration"*, Herman & C^{ie}, Paris, 1952.

[3] Henri Cartan, *Sur la mesure de Haar*, Comptes Rendus de l'Académie des Sciences de Paris **211** (1940), 759–762 [Reprinted in "Œuvres" vol. **2** 1020–1022 by Springer-Verlag, Berlin 1979].
[4] P. J. Daniell, *A general form of integral*, Annals of Mathematics **19** (1917–18), 279–294.
[5] Jean Dieudonné, *" Éléments d'Analyse"* vol. **II**, Gauthier-Villars, Paris, 1968.
[6] Alfred Haar, *Der Massbegriff in der Theorie der kontinuierlichen Gruppen*, Annals of Mathematics **34** (1933), 147–169.
[7] Lynn A. Loomis, *"An Introduction to Abstract harmonic Analysis"*, University Series of Higher Mathematics, Van Nostrand, Princeton N.J. 1953.
[8] John von Neumann, *Zum Haarschen Maß in Topologischen Gruppen*, Compositio Mathematica **1** (1934), 106–114.
[9] John von Neumann, *The uniqueness of Haar's measure*, Matematičeskiĭ Sbornik **1** (1936), 721–734 [Reprinted in "Collected Works" vol. **4**, 91–104, by Pergamon Press, Oxford 1962].
[10] Gert K. Pedersen, *"Analysis Now"*, Graduate Texts in Mathematics, vol. **118**, Springer Verlag, Berlin, 1989.
[11] André Weil, *"L'Intégration dans les Groupes Topologiques et ses Applications"*, Hermann & C^{ie}, Paris, 1940.

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