

COTORSION PAIRS ASSOCIATED WITH AUSLANDER CATEGORIES

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ABSTRACT

We prove that the Auslander category determined by a semidualizing module is the left half of a perfect cotorsion pair. We also prove that the Bass class determined by a semidualizing module is preenveloping.

0. Introduction

The notion of semidualizing modules over commutative noetherian rings goes back to Foxby [11] (PG-modules of rank one), while Golod [13] (suitable modules) and Vasconcelos [18] (spherical modules) furthered their study. Christensen [3] extended this notion to semidualizing complexes.

Received August 1, 2007

A semidualizing module or complex C over a commutative noetherian ring gives rise to two full subcategories of the derived category of the category of R -modules, namely the so-called **Auslander class** and **Bass class** defined by Avramov–Foxby [1, (3.1)] and Christensen [3, def. (4.1)]. Semidualizing complexes and their Auslander/Bass classes have caught the attention of several authors; see, for example, [1, 3–5, 8, 10–12, 14, 16, 17].

Usually, one is interested in studying the modules in the Auslander/Bass classes (by definition, the objects of these categories are complexes), and in this paper we use \mathcal{A}_C and \mathcal{B}_C to denote the categories of all modules belonging to the Auslander class and Bass class, respectively.

We mention that when C itself is a (semidualizing) module, then one can describe \mathcal{A}_C and \mathcal{B}_C in terms of vanishing of certain derived module functors and invertibility of certain module homomorphisms; see Avramov–Foxby [1, prop. (3.4)] and Christensen [3, obs. (4.10)].

In the case where C is a **dualizing** module or complex, it is possible to understand \mathcal{A}_C and \mathcal{B}_C in terms of the so-called Gorenstein homological dimensions; see Enochs–Jenda–Xu [8] and Christensen–Frankild–Holm [4]. A similar description exists for other special semidualizing complexes C (see Holm–Jørgensen [14]), but for a general semidualizing complex C , not much is known about \mathcal{A}_C or \mathcal{B}_C .

In this paper we are concerned with what covering and enveloping properties \mathcal{A}_C and \mathcal{B}_C possess. Our main results are Theorems 3.11 and 3.12 which state, respectively, that:

THEOREM A: *Let C be a semidualizing module over a commutative noetherian ring R . Then $(\mathcal{A}_C, (\mathcal{A}_C)^\perp)$ is a perfect cotorsion pair; in particular, \mathcal{A}_C is covering. Furthermore, \mathcal{A}_C is preenveloping.*

THEOREM B: *Let C be a semidualizing module over a commutative noetherian ring R . Then \mathcal{B}_C is preenveloping.*

As Corollary 3.13 we get:

COROLLARY C: *Let (R, \mathfrak{m}, k) be a commutative, noetherian and local Cohen–Macaulay ring admitting a dualizing module. Then the following conclusions hold:*

- (a) *The class of all R -modules of finite Gorenstein projective/flat dimension is covering and preenveloping.*
- (b) *The class of all R -modules of finite Gorenstein injective dimension is preenveloping.*

1. Preliminaries

In this section we briefly recall a number of definitions relevant to this paper, namely the definitions of semidualizing modules, Auslander categories, (pre)covers, (pre)envelopes, cotorsion pairs, and Kaplansky classes. These notions will be used throughout the paper without further mention.

Setup 1.1: Throughout, R is a fixed commutative noetherian ring with identity, and C is a fixed semidualizing module for R ; cf. Definition 1.3 below. We write $\text{Mod } R$ for the category of R -modules.

Remark 1.2: Actually, we only need R to be commutative and noetherian when we deal with semidualizing modules over R and their Auslander/Bass classes. But in all of Section 2 for example, R could be any ring.

The next definition goes back to [11] (where the more general PG-modules are studied) and [13], but a more recent reference is [3, def. (2.1)].

Definition 1.3: A **semidualizing** module for R is a finitely generated R -module C such that:

- (1) $\text{Ext}_R^j(C, C) = 0$ for all $j > 0$, and
- (2) The natural homothety morphism $\chi_C: R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism.

The following lemma is straightforward.

LEMMA 1.4: *Let M be any R -module. If either $\text{Hom}_R(C, M) = 0$ or $C \otimes_R M = 0$, then $M = 0$. ■*

Next we recall the definitions, cf. [11, sec. 1], [1, prop. (3.4)], and [3, def. (4.1) and obs. (4.10)] of the “module versions” of the Auslander categories with respect to the semidualizing module C .

Definition 1.5: The **Auslander categories** $\mathcal{A}_C = \mathcal{A}_C(R)$ and $\mathcal{B}_C = \mathcal{B}_C(R)$ are the full subcategories of $\text{Mod } R$ whose objects are specified as follows: An R -module M belongs to \mathcal{A}_C if

- (A1) $\text{Tor}_j^R(C, M) = 0$ for all $j > 0$;
- (A2) $\text{Ext}_R^j(C, C \otimes_R M) = 0$ for all $j > 0$;
- (A3) $\eta_M : M \longrightarrow \text{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

An R -module N belongs to \mathcal{B}_C if

- (B1) $\text{Ext}_R^j(C, N) = 0$ for all $j > 0$;
- (B2) $\text{Tor}_j^R(C, \text{Hom}_R(C, N)) = 0$ for all $j > 0$;
- (B3) $\varepsilon_N : C \otimes_R \text{Hom}_R(C, N) \longrightarrow N$ is an isomorphism.

We often refer to \mathcal{A}_C as the **Auslander class**, and to \mathcal{B}_C as the **Bass class**.

Remark 1.6: In the notation of [1, 3], the categories from Definition 1.5 should have a subscript “0”. However, since this paper only deals with modules (as opposed to complexes), and in order to keep notation as simple as possible, we have chosen to omit this “0”.

The next definition — which is important for our main results: Theorems 3.11, 3.12, and 3.13 — is taken directly from Enochs–Jenda [7, def. 5.1.1 and 6.1.1].

Definition 1.7: Let \mathcal{F} be a class of modules. An **\mathcal{F} -precover** of a module M is a homomorphism $\varphi : F \longrightarrow M$ with $F \in \mathcal{F}$ such that every homomorphism $\varphi' : F' \longrightarrow M$ with $F' \in \mathcal{F}$ factors as

$$\begin{array}{ccc}
 & & F' \\
 & \swarrow \text{---} & \downarrow \varphi' \\
 F & \xrightarrow{\varphi} & M.
 \end{array}$$

An \mathcal{F} -precover $\varphi : F \longrightarrow M$ is called an **\mathcal{F} -cover** if every endomorphism $f : F \longrightarrow F$ with $\varphi f = \varphi$ is an automorphism. The class \mathcal{F} is called **(pre)covering** if every module has an \mathcal{F} -(pre)cover. There is a similar definition of **\mathcal{F} -(pre)envelopes**.

Our main Theorem 3.11 uses the notion of a perfect cotorsion pair. The definition, which is given below, is taken directly from [7, def. 7.1.2] and Enochs–López-Ramos [9, def. 2.2].

Definition 1.8: A pair of module classes $(\mathcal{F}, \mathcal{G})$ is a **cotorsion pair** if $\mathcal{F}^\perp = \mathcal{G}$ and $\mathcal{F} = {}^\perp\mathcal{G}$, where

$$\begin{aligned} \mathcal{F}^\perp &= \{N \in \text{Mod } R \mid \text{Ext}_R^1(F, N) = 0 \text{ for all } F \in \mathcal{F}\}, \quad \text{and} \\ {}^\perp\mathcal{G} &= \{M \in \text{Mod } R \mid \text{Ext}_R^1(M, G) = 0 \text{ for all } G \in \mathcal{G}\}. \end{aligned}$$

A cotorsion pair $(\mathcal{F}, \mathcal{G})$ is called **perfect** if \mathcal{F} is covering and \mathcal{G} is enveloping.

Finally, we need for Proposition 3.10 the notion of a Kaplansky class. For the convenience of the reader we restate [9, def. 2.1] below.

Definition 1.9: A class of modules \mathcal{F} is called a **Kaplansky class** if there exists a cardinal number λ such that for every $x \in F \in \mathcal{F}$ there is a submodule $x \in F' \subseteq F$ with $|F'| \leq \lambda$ and $F', F/F' \in \mathcal{F}$.

2. Exact subcomplexes of an exact complex

The main result of this section is Proposition 2.5, which shows how to find desirable exact subcomplexes of a given exact complex. This result is the cornerstone in the proof of Proposition 3.10 in Section 3.

LEMMA 2.1: *Let λ be any infinite cardinal number with $\lambda \geq |R|$, and let M be any R -module. The following conclusions hold:*

- (a) *If M is generated by a subset $X \subseteq M$ satisfying $|X| \leq \lambda$, then also $|M| \leq \lambda$.*
- (b) *If $S \subseteq M$ is a submodule and with $|S|, |M/S| \leq \lambda$, then $|M| \leq \lambda$.*

Proof. Part (a) is clear as there is an epimorphism $R^{(X)} \rightarrow M$, and $|R^{(X)}| \leq \lambda$. For part (b) we pick a set of representatives $X \subseteq M$ for the cosets of S in M (which has $|X| = |M/S| \leq \lambda$). Then $X \cup S \subseteq M$ generates M and satisfies $|X \cup S| \leq \lambda$, so part (a) gives the desired conclusion. ■

LEMMA 2.2: *Let λ be any infinite cardinal number. If Q is a finitely generated module and P is a module with $|P| \leq \lambda$, then the functors*

$$\text{Hom}_R(Q, -), P \otimes_R - : \text{Mod } R \rightarrow \text{Ab}$$

have the property that the image of any module A with $|A| \leq \lambda$ has again cardinality $\leq \lambda$.

Proof. Pick an integer $n > 0$ and exact sequences

$$R^n \rightarrow Q \rightarrow 0 \quad \text{and} \quad R^{(P)} \rightarrow P \rightarrow 0.$$

Applying $\text{Hom}_R(-, A)$ to the first of these sequences and $-\otimes_R A$ to the second one, we get exactness of

$$0 \longrightarrow \text{Hom}_R(Q, A) \longrightarrow \text{Hom}_R(R^n, A) \cong A^n \quad \text{and}$$

$$A^{(P)} \cong R^{(P)} \otimes_R A \longrightarrow P \otimes_R A \longrightarrow 0.$$

If $|A|, |P| \leq \lambda$ then we have $|A^n|, |A^{(P)}| \leq \lambda$, and therefore also the desired conclusions, $|\text{Hom}_R(Q, A)| \leq \lambda$ and $|P \otimes_R A| \leq \lambda$. ■

LEMMA 2.3: *Let $F: \text{Mod } R \longrightarrow \text{Ab}$ be an additive covariant functor which preserves direct limits, and assume that there exists an infinite cardinal number $\lambda \geq |R|$ such that $|FA| \leq \lambda$ for all modules A with $|A| \leq \lambda$. Let*

$$\mathcal{E} = E' \xrightarrow{d'} E \xrightarrow{d} E''$$

be a complex of R -modules such that $F\mathcal{E}$ is exact. Suppose $S' \subseteq E', S \subseteq E, S'' \subseteq E''$ are submodules such that $|S'|, |S|, |S''| \leq \lambda$. Then there exists a subcomplex,

$$\mathcal{T} = T' \xrightarrow{d'} T \xrightarrow{d} T''$$

of \mathcal{E} such that $F\mathcal{T}$ is exact, such that $S' \subseteq T', S \subseteq T, S'' \subseteq T''$, and such that $|T'|, |T|, |T''| \leq \lambda$.

If, in addition, \mathcal{E} is exact, then we can choose \mathcal{T} to be exact as well.

Proof. Replacing S with $S + d'(S')$ and S'' with $S'' + d(S)$ we see that we can assume that

$$S' \xrightarrow{d'} S \xrightarrow{d} S''$$

is a subcomplex of \mathcal{E} . Note that

$$(\dagger) \quad \mathcal{E} = E' \xrightarrow{d'} E \xrightarrow{d} E'' = \varinjlim (U' \xrightarrow{d'} U \xrightarrow{d} U'')$$

where the direct limit is taken over the family \mathcal{U} of all subcomplexes $U' \longrightarrow U \longrightarrow U''$ of \mathcal{E} which contain $S' \longrightarrow S \longrightarrow S''$ and satisfy that $U'/S', U/S$ and U''/S'' are finitely generated. For each such subcomplex we also have $|U'|, |U|, |U''| \leq \lambda$ by Lemma 2.1(b).

Now, suppose that $z \in \text{Ker}(FS \longrightarrow FS'') \subseteq FS$. By (\dagger) and the assumptions on the functor F it follows that

$$(\ddagger) \quad FE' \longrightarrow FE \longrightarrow FE'' = \varinjlim (FU' \longrightarrow FU \longrightarrow FU''),$$

which is exact by assumption. As the image of z in FE belongs to

$$\text{Ker}(FE \longrightarrow FE'') = \text{Im}(FE' \longrightarrow FE),$$

the identity (‡) implies the existence of some $U'_z \longrightarrow U_z \longrightarrow U''_z$ in \mathcal{U} such that the image of z in FU_z belongs to

$$\text{Im}(FU'_z \longrightarrow FU_z).$$

Then we define

$$T'_0 \longrightarrow T_0 \longrightarrow T''_0 = \sum_z (U'_z \longrightarrow U_z \longrightarrow U''_z),$$

where the sum is taken over all $z \in \text{Ker}(FS \longrightarrow FS'')$. By construction, there is an inclusion

$$(b_0) \quad \text{Im}(\text{Ker}(FS \longrightarrow FS'') \longrightarrow FT_0) \subseteq \text{Im}(FT'_0 \longrightarrow FT_0).$$

We also note that the assumptions on F imply that

$$(‡) \quad |\text{Ker}(FS \longrightarrow FS'')| \leq |FS| \leq \lambda,$$

since $|S| \leq \lambda$. Consequently,

$$|T_0/S| = \left| \sum_z U_z/S \right| \leq \sum_z |U_z/S| \leq \sum_z \lambda \leq \lambda^2 = \lambda,$$

where the penultimate inequality follows Lemma 2.1(a), as U_z/S is finitely generated, and the last inequality comes from (‡). Thus, we have $|S|, |T_0/S| \leq \lambda$, so Lemma 2.1(b) implies that $|T_0| \leq \lambda$. Similarly,

$$|T'_0|, |T_0|, |T''_0| \leq \lambda.$$

Now, going through the same procedure as above, but using the complex $T'_0 \longrightarrow T_0 \longrightarrow T''_0$ instead of $S' \longrightarrow S \longrightarrow S''$, we get a subcomplex

$$T'_1 \longrightarrow T_1 \longrightarrow T''_1$$

of \mathcal{E} containing $T'_0 \longrightarrow T_0 \longrightarrow T''_0$, and such that $|T'_1|, |T_1|, |T''_1| \leq \lambda$ and

$$(b_1) \quad \text{Im}(\text{Ker}(FT_0 \longrightarrow FT''_0) \longrightarrow FT_1) \subseteq \text{Im}(FT'_1 \longrightarrow FT_1).$$

In this fashion we construct an increasing sequence

$$T'_n \longrightarrow T_n \longrightarrow T''_n, \quad n = 0, 1, 2, \dots$$

of subcomplexes of \mathcal{E} such that $|T'_n|, |T_n|, |T''_n| \leq \lambda$ and

$$(b_n) \quad \text{Im}(\text{Ker}(FT_{n-1} \longrightarrow FT''_{n-1}) \longrightarrow FT_n) \subseteq \text{Im}(FT'_n \longrightarrow FT_n).$$

Finally, we define a subcomplex of \mathcal{E} by

$$\begin{aligned} T' \longrightarrow T \longrightarrow T'' &= \bigcup_{n=0}^{\infty} (T'_n \longrightarrow T_n \longrightarrow T''_n) \\ &= \varinjlim_{n \geq 0} (T'_n \longrightarrow T_n \longrightarrow T''_n). \end{aligned}$$

Note that $|T| \leq \sum_{n \geq 0} |T_n| \leq \lambda + \lambda + \lambda + \dots = \lambda$, and similarly one gets $|T'|, |T''| \leq \lambda$. As F commutes with direct limits we have

$$FT' \longrightarrow FT \longrightarrow FT'' = \varinjlim (FT'_n \longrightarrow FT_n \longrightarrow FT''_n).$$

It is straightforward to verify that the conditions (b_n) ensure exactness of the complex above.

Concerning the last claim of the lemma we argue as follows: If, in addition, \mathcal{E} is exact, then we have exactness of $G\mathcal{E}$, where G is the functor $G = F \oplus \text{id}$, and $\text{id}: \text{Mod } R \longrightarrow \text{Ab}$ is the forgetful functor. Hence, applying the part of the lemma which has already been established, but with F replaced by G , we get the exactness of $G\mathcal{T}$. Consequently, both $F\mathcal{T}$ and $\text{id}\mathcal{T} = \mathcal{T}$ are exact. ■

LEMMA 2.4: *Let λ be any infinite cardinal number with $\lambda \geq |R|$. If $S \subseteq E$ are R -modules such that $|S| \leq \lambda$, then there is a pure submodule $T \subseteq E$ of E with $S \subseteq T$ and $|T| \leq \lambda$.*

Proof. A slight modification of the proof of [7, Lemma 5.3.12] gives this result. ■

PROPOSITION 2.5: *Let $F: \text{Mod } R \longrightarrow \text{Ab}$ be an additive covariant functor which preserves direct limits, and assume that there exists an infinite cardinal number $\lambda \geq |R|$ such that $|FA| \leq \lambda$ for all modules A with $|A| \leq \lambda$. Let*

$$E = \dots \longrightarrow E_{n+1} \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \dots$$

be an exact complex of R -modules such that also FE is exact. If $S_n \subseteq E_n$ is a submodule such that $|S_n| \leq \lambda$ for each $n \in \mathbb{Z}$, then there is an exact subcomplex

$$D = \dots \longrightarrow D_{n+1} \longrightarrow D_n \longrightarrow D_{n-1} \longrightarrow \dots$$

of E such that also FD is exact. Furthermore, $D_n \subseteq E_n$ is a pure submodule, $S_n \subseteq D_n$, and $|D_n| \leq \lambda$ for each $n \in \mathbb{Z}$.

Proof. For each n we construct a chain $S_n = S_n^0 \subseteq S_n^1 \subseteq S_n^2 \subseteq \dots$ of submodules of E_n with $|S_n^i| \leq \lambda$ for all n and i as follows:

First pick a function $f: \mathbb{N}_0 \rightarrow \mathbb{Z}$ with the property that for each $n \in \mathbb{Z}$ the set $f^{-1}(\{n\}) = \{k \in \mathbb{N}_0 \mid f(k) = n\}$ is infinite. Of course, we are given that $S_n^0 = S_n$ for all n .

Assume that we for some $k \geq 0$ have constructed S_n^{2k} for all n with the property that $|S_n^{2k}| \leq \lambda$. By Lemma 2.4 we get a pure submodule $S_n^{2k+1} \subseteq E_n$ which contains S_n^{2k} and which satisfies $|S_n^{2k+1}| \leq \lambda$.

Now assume that, for some $k \geq 0$, we have constructed S_n^{2k+1} for all n with the property that $|S_n^{2k+1}| \leq \lambda$. Then we construct S_n^{2k+2} by the following procedure: Consider the complex

$$\mathcal{E}_k = E_{f(k)+1} \rightarrow E_{f(k)} \rightarrow E_{f(k)-1}.$$

By hypothesis both \mathcal{E}_k and $F\mathcal{E}_k$ are exact. Applying Lemma 2.3 to this complex \mathcal{E}_k and to the submodules

$$S_{f(k)+1}^{2k+1} \subseteq E_{f(k)+1}, \quad S_{f(k)}^{2k+1} \subseteq E_{f(k)} \quad \text{and} \quad S_{f(k)-1}^{2k+1} \subseteq E_{f(k)-1}$$

we get an exact subcomplex

$$\mathcal{T}_k = T' \rightarrow T \rightarrow T''$$

of \mathcal{E}_k where also $F\mathcal{T}_k$ is exact, and furthermore,

$$S_{f(k)+1}^{2k+1} \subseteq T', \quad S_{f(k)}^{2k+1} \subseteq T \quad \text{and} \quad S_{f(k)-1}^{2k+1} \subseteq T'',$$

and $|T'|, |T|, |T''| \leq \lambda$. We then define

$$S_{f(k)+1}^{2k+2} = T', \quad S_{f(k)}^{2k+2} = T \quad \text{and} \quad S_{f(k)-1}^{2k+2} = T'',$$

and let $S_n^{2k+2} = S_n^{2k+1}$ for all $n \notin \{f(k) + 1, f(k), f(k) - 1\}$.

Having constructed the sequences $S_n = S_n^0 \subseteq S_n^1 \subseteq S_n^2 \subseteq \dots \subseteq E_n$ for $n \in \mathbb{Z}$ as above, we set $D_n = \bigcup_{i=0}^\infty S_n^i$. Clearly, $S_n \subseteq D_n$ and $|D_n| \leq \lambda$. Since $D_n = \bigcup_{k=0}^\infty S_n^{2k+1}$ and since $S_n^{2k+1} \subseteq E_n$ is a pure submodule for each k , it follows that $D_n \subseteq E_n$ is a pure submodule.

For each n , the differential $E_n \rightarrow E_{n-1}$ restricts to a homomorphism $D_n \rightarrow D_{n-1}$: If $x \in D_n$ then there exists an i_0 such that $x \in S_n^i$ for all $i \geq i_0$. Since $f^{-1}(\{n\})$ is infinite, there exists $k \geq 0$ satisfying both $f(k) = n$ and $2k + 1 \geq i_0$. By our construction,

$$S_{n+1}^{2k+2} \rightarrow S_n^{2k+2} \rightarrow S_{n-1}^{2k+2}$$

is a subcomplex of $E_{n+1} \rightarrow E_n \rightarrow E_{n-1}$, so in particular the differential $E_n \rightarrow E_{n-1}$ maps $x \in S_n^{2k+1} \subseteq S_n^{2k+2}$ into $S_{n-1}^{2k+2} \subseteq D_{n-1}$. Hence

$$D = \cdots \rightarrow D_{n+1} \rightarrow D_n \rightarrow D_{n-1} \rightarrow \cdots$$

is a subcomplex of E . In fact, for the n 'th segment of D we have the expression

$$D_{n+1} \rightarrow D_n \rightarrow D_{n-1} = \varinjlim_{k \in f^{-1}(\{n\})} (S_{n+1}^{2k+2} \rightarrow S_n^{2k+2} \rightarrow S_{n-1}^{2k+2}),$$

and since each of the complexes

$$S_{n+1}^{2k+2} \rightarrow S_n^{2k+2} \rightarrow S_{n-1}^{2k+2}, \quad k \in f^{-1}(\{n\})$$

are exact and stay exact when we apply F to them, the same is true for $D_{n+1} \rightarrow D_n \rightarrow D_{n-1}$, as F commutes with direct limits. ■

3. Covers and envelopes by Auslander categories

This last section is concerned with covering and enveloping properties of the Auslander categories. Our main results are Theorems 3.11 and 3.12.

To prove our main theorems, we need the alternative descriptions of the modules in the Auslander categories given in Propositions 3.6 and 3.7. To this end, we introduce two new classes of modules:

Definition 3.1: The classes of C -**injective** and C -**flat** modules are defined as

$$\begin{aligned} \mathcal{I}_C &= \mathcal{I}_C(R) = \{\text{Hom}_R(C, I) \mid I \text{ injective } R\text{-module}\}, \\ \mathcal{F}_C &= \mathcal{F}_C(R) = \{C \otimes_R F \mid F \text{ flat } R\text{-module}\}. \end{aligned}$$

Observation 3.2: R is a semidualizing module for itself, and by setting $C = R$ in the definition above we see that \mathcal{I}_R and \mathcal{F}_R are the classes of (ordinary) injective and flat R -modules, respectively.

The proof of the next lemma is straightforward.

LEMMA 3.3: *For R -modules U and V one has the implications:*

- (a) $U \in \mathcal{I}_C \iff U \in \mathcal{A}_C \text{ and } C \otimes_R U \in \mathcal{I}_R.$
- (b) $V \in \mathcal{F}_C \iff V \in \mathcal{B}_C \text{ and } \text{Hom}_R(C, V) \in \mathcal{F}_R. \quad \blacksquare$

Remark 3.4: The classes \mathcal{I}_C , \mathcal{F}_C and also

$$\mathcal{P}_C = \mathcal{P}_C(R) = \{C \otimes_R P \mid P \text{ projective module}\}$$

were used in [14], and it was proved in [14, lem.2.14] (compared with [9, thm. 2.5]) that \mathcal{F}_C is preenveloping. When R is a Cohen–Macaulay local ring and C is a dualizing module for R , it was proved in [8, prop. 1.5] that \mathcal{I}_C is preenveloping and \mathcal{P}_C is precovering.

We will make use of the following:

PROPOSITION 3.5: *\mathcal{I}_C is enveloping and \mathcal{F}_C is covering. In particular, for any R -module M there exist complexes*

$$\mathcal{U} = 0 \longrightarrow M \longrightarrow U^0 \longrightarrow U^1 \longrightarrow U^2 \longrightarrow \dots$$

with $U^0, U^1, U^2, \dots \in \mathcal{I}_C$, and

$$\mathcal{V} = \dots \longrightarrow V_2 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow M \longrightarrow 0$$

with $V_0, V_1, V_2 \in \mathcal{F}_C$ such that $C \otimes_R \mathcal{U}$ and $\text{Hom}_R(C, \mathcal{V})$ are exact.

Proof. Even if \mathcal{I}_C is just preenveloping and \mathcal{F}_C is just precovering there will exist complexes \mathcal{U} and \mathcal{V} with $U^i \in \mathcal{I}_C$ and $V_j \in \mathcal{F}_C$ such that

$$\begin{aligned} \text{Hom}(\mathcal{U}, \text{Hom}(C, I)) &\cong \text{Hom}(C \otimes \mathcal{U}, I), \quad \text{and} \\ \text{Hom}(C \otimes F, \mathcal{V}) & \end{aligned}$$

are exact for all injective modules I and all flat modules F . Taking I to be faithfully injective and $F = R$, we see that $C \otimes \mathcal{U}$ and $\text{Hom}(C, \mathcal{V})$ are exact.

Thus, the proposition is proved when we have argued that \mathcal{I}_C is enveloping and \mathcal{F}_C is covering. The class \mathcal{I}_R is known to be enveloping by Xu [19, thm. 1.2.11], since injective hulls in the sense of Eckmann–Schopf [6] always exists. The class \mathcal{F}_R is known to be covering by Bican–Bashir–Enochs [2]. Now, it is easy to see that for any module M , the composition

$$M \longrightarrow \text{Hom}(C, C \otimes M) \longrightarrow \text{Hom}(C, E(C \otimes M))$$

is an \mathcal{I}_C -envelope of M , where $E(-)$ denotes the injective envelope. Likewise, the composition

$$C \otimes F(\text{Hom}(C, M)) \longrightarrow C \otimes \text{Hom}(C, M) \longrightarrow M$$

is a \mathcal{F}_C -cover of M , where $F(-)$ denotes the flat cover. ■

Having established Lemma 3.3 and Proposition 3.5, the proof of the next result is similar to that of [19, prop. 5.5.4].

PROPOSITION 3.6: *A module M belongs to \mathcal{A}_C if and only if there exists an exact sequence*

$$(\dagger) \quad \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow U^2 \longrightarrow \cdots$$

satisfying the following conditions:

- (1) $F_0, F_1, F_2, \dots \in \mathcal{F}_R$ and $U^0, U^1, U^2, \dots \in \mathcal{I}_C$;
- (2) $M = \text{Ker}(U^0 \longrightarrow U^1)$;
- (3) $C \otimes_R (\dagger)$ is exact.

Dually one proves the next result, which is similar to [19, prop. 5.5.5].

PROPOSITION 3.7: *A module M belongs to \mathcal{B}_C if and only if there exists an exact sequence*

$$(\ddagger) \quad \cdots \longrightarrow V_2 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

satisfying the following conditions:

- (1) $V_0, V_1, V_2, \dots \in \mathcal{F}_C$ and $I^0, I^1, I^2, \dots \in \mathcal{I}_R$;
- (2) $M = \text{Ker}(I^0 \longrightarrow I^1)$;
- (3) $\text{Hom}_R(C, (\ddagger))$ is exact.

Remark 3.8: Taking the necessary precautions, one can study Auslander categories over non-noetherian rings. In this generality one can also prove versions of, for example, Propositions 3.6 and 3.7; see [15].

In addition to the fact that [19, chap. 5.5] assumes C to be dualizing (and not just semidualizing), there is another important difference between [19, prop. 5.5.5] and Proposition 3.7: Namely, we work with \mathcal{F}_C whereas Xu works with \mathcal{P}_C (which he denotes \mathcal{W}); see Remark 3.4.

From our point of view \mathcal{F}_C is more flexible than \mathcal{P}_C . For example, \mathcal{F}_C is closed under pure submodules and pure quotients; see Proposition 3.9 below, whereas \mathcal{P}_C in general does not have these properties.

PROPOSITION 3.9: *The classes \mathcal{I}_C and \mathcal{F}_C are closed under pure submodules and pure quotients.*

Proof. First we prove that \mathcal{I}_C is closed under pure submodules and pure quotients. To this end let

$$0 \longrightarrow M \longrightarrow \text{Hom}(C, I) \longrightarrow N \longrightarrow 0$$

be a pure exact sequence with I injective. Applying $C \otimes -$ to this sequence we get another pure exact sequence,

$$(*) \quad 0 \longrightarrow C \otimes M \longrightarrow I \longrightarrow C \otimes N \longrightarrow 0.$$

As I is injective, and as \mathcal{I}_R is closed under pure submodules and pure quotients, we conclude that the modules $C \otimes M$ and $C \otimes N$ are injective. Applying $\text{Hom}(C, -)$ to the pure exact sequence $(*)$ we get exactness of the lower row in the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & \text{Hom}(C, I) & \longrightarrow & N \longrightarrow 0 \\ & & \downarrow \eta_M & & \parallel & & \downarrow \eta_N \\ 0 & \longrightarrow & \text{Hom}(C, C \otimes M) & \longrightarrow & \text{Hom}(C, I) & \longrightarrow & \text{Hom}(C, C \otimes N) \longrightarrow 0. \end{array}$$

Since $C \otimes M$ and $C \otimes N$ are injective by Lemma 3.3(a), we are done if we can prove that η_M and η_N are isomorphisms. By the snake-lemma, η_M is injective, η_N is surjective, and $\text{Ker } \eta_N \cong \text{Coker } \eta_M$. Hence, it suffices to argue that $\text{Coker } \eta_M = 0$, and by Lemma 1.4 it is enough to show that

$$C \otimes \text{Coker } \eta_M = 0.$$

Right exactness of $C \otimes -$ gives exactness of

$$C \otimes M \xrightarrow{C \otimes \eta_M} C \otimes \text{Hom}(C, C \otimes M) \longrightarrow C \otimes \text{Coker } \eta_M \longrightarrow 0.$$

Since $C \otimes M$ is injective, $C \otimes \eta_M$ is an isomorphism with inverse $\varepsilon_{(C \otimes M)}$. In particular, $C \otimes \eta_M$ is surjective, and hence $C \otimes \text{Coker } \eta_M = 0$, as desired.

Similarly, as the class of flat modules is closed under pure submodules and pure quotients, one proves that \mathcal{F}_C also has these properties. ■

PROPOSITION 3.10: \mathcal{A}_C and \mathcal{B}_C are Kaplansky classes.

Proof. We only prove that \mathcal{A}_C is Kaplansky, as the proof for \mathcal{B}_C is similar. We claim that any infinite cardinal number $\lambda \geq |R|$ implements the Kaplansky property for \mathcal{A}_C ; cf. Definition 1.9:

Let $x \in M \in \mathcal{A}_C$. By Proposition 3.6 there is an exact sequence

$$E = \dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow U^2 \longrightarrow \dots$$

satisfying the conditions (1), (2), and (3) of that result. Now consider the submodules:

- $Rx \subseteq U^0$ (which has $|Rx| \leq \lambda$);

- $0 \subseteq U^n$ for $n \geq 1$;
- $0 \subseteq F_n$ for $n \geq 0$.

Applying Proposition 2.5 with $F = C \otimes -$ (cf. Lemma 2.2) to this situation we get an exact subcomplex

$$D = \dots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow W^2 \longrightarrow \dots$$

of E with $Rx \subseteq W^0$ and such that $C \otimes D$ is exact, and furthermore, $G_n \subseteq F_n$ and $W^n \subseteq U^n$ are pure submodules, and $|G_n|, |W^n| \leq \lambda$.

By the condition Proposition 3.6(1), $F_n \in \mathcal{F}_R$ and $U^n \in \mathcal{I}_C$, and thus Proposition 3.9 implies that

$$(*) \quad G_n, F_n/G_n \in \mathcal{F}_R \quad \text{and} \quad W^n, U^n/W^n \in \mathcal{I}_C.$$

Hence, Proposition 3.6 implies that $M' = \text{Ker}(W^0 \longrightarrow W^1)$ belongs to \mathcal{A}_C . As $x \in M = \text{Ker}(U^0 \longrightarrow U^1)$ we also have $x \in M'$, and of course $|M'| \leq |W^0| \leq \lambda$. Thus, it remains to argue that $M/M' \in \mathcal{A}_C$. By construction we have a pure exact sequence of complexes

$$(\#) \quad 0 \longrightarrow D \longrightarrow E \longrightarrow E/D \longrightarrow 0.$$

As D and E are exact, then so is E/D ; in particular it follows that $M/M' \cong \text{Ker}(U^0/W^0 \longrightarrow U^1/W^1)$. Purity of $(\#)$ gives exactness of

$$0 \longrightarrow C \otimes D \longrightarrow C \otimes E \longrightarrow C \otimes (E/D) \longrightarrow 0,$$

and since $C \otimes D$ and $C \otimes E$ are exact then so is $C \otimes (E/D)$. By $(*)$ the complex E/D consists of modules of the form Proposition 3.6(1), and these arguments show that $M/M' \in \mathcal{A}_C$, as desired. ■

THEOREM 3.11: $(\mathcal{A}_C, (\mathcal{A}_C)^\perp)$ is a perfect cotorsion pair; in particular, \mathcal{A}_C is covering. Furthermore, \mathcal{A}_C is preenveloping.

Proof. By Proposition 3.10, the class \mathcal{A}_C is Kaplansky. Clearly, \mathcal{A}_C contains the projective modules, and is closed under extensions and direct limits. Hence [9, thm. 2.9] implies that $(\mathcal{A}_C, (\mathcal{A}_C)^\perp)$ is a perfect cotorsion pair. As \mathcal{A}_C is also closed under products, [9, thm. 2.5] gives that \mathcal{A}_C is preenveloping. ■

THEOREM 3.12: \mathcal{B}_C is preenveloping.

Proof. By Proposition 3.10, the class \mathcal{B}_C is Kaplansky. Since \mathcal{B}_C is closed under direct limits and products, [9, thm. 2.5] implies that \mathcal{B}_C is preenveloping. ■

COROLLARY 3.13: *Let (R, \mathfrak{m}, k) be a commutative, noetherian and local Cohen–Macaulay ring admitting a dualizing module. Then the following conclusions hold:*

- (a) *The class of R -modules of finite Gorenstein projective/flat dimension is covering and preenveloping.*
- (b) *The class of R -modules of finite Gorenstein injective dimension is preenveloping.*

Proof. Taking C to be the dualizing module for R , the assertions follow immediately from comparing Theorems 3.11 and 3.12 with [8, cor. 2.4 and 2.6]. ■

ACKNOWLEDGEMENT. We thank Diana White for useful comments.

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