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The direct limit closure of perfect complexes ☆

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In loving memory of Hans-Bjørn Foxby—our teacher, colleague, and friend

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ABSTRACT

Every projective module is flat. Conversely, every flat module is a direct limit of finitely generated free modules; this was proved independently by Govorov and Lazard in the 1960s. In this paper we prove an analogous result for complexes of modules, and as applications we reprove some results due to Enochs and García Rozas and to Neeman.

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1. Introduction

Let R be a ring. In contrast to the projective objects in the category of R-modules, i.e. the projective R-modules, the projective objects in the category of R-complexes are not of much utility; indeed, they are nothing but contractible (split) complexes of projective R-modules. In the category of complexes, the relevant alternative to projectivity—from the homological point of view, at least—is semi-projectivity. A complex P is called *semi-projective* (or *DG-projective*) if the total Hom functor $\operatorname{Hom}(P,-)$ preserves surjective quasi-isomorphisms, i.e. surjective morphisms that induce isomorphisms in homology. The semi-projective complexes are exactly the cofibrant objects in the standard model structure on the category of complexes; see Hovey [11, §2.3]. Alternatively, a complex is semi-projective if and only if it consists of projective modules and it is K-projective in the sense of Spaltenstein [18]. The notion of semi-projectivity in the category of complexes extends the notion of projectivity in the category of modules in a natural and useful way: A module is projective if and only if it is semi-projective when viewed as a complex.

Similarly, a complex F is *semi-flat* if the total tensor product functor $-\otimes F$ preserves injective quasi-isomorphisms; equivalently, F is a complex of flat modules and K-flat in the sense of [18]. A module is flat if and only if it is semi-flat when viewed as a complex. Every semi-projective complex is semi-flat, and simple examples of semi-projective complexes are bounded complexes of finitely generated projective modules, also

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known as *perfect* complexes. The class of semi-flat complexes is closed under direct limits, and our main result, Theorem 1.1 below, shows that every semi-flat complex is a direct limit of perfect complexes. For modules, the theorem specializes to a classic result, proved independently by Govorov [9] and Lazard [13]: Every flat module is a direct limit of finitely generated free modules.

1.1. Theorem. *For an R-complex F the following conditions are equivalent.*

- (i) *F is semi-flat.*
- (ii) *Every morphism of R-complexes $\varphi\colon N \to F$ with N bounded and degreewise finitely presented admits a factorization,*

$$\begin{array}{ccc} N & \xrightarrow{\ \varphi\ } & F \\ & {}_{\kappa}\searrow \quad \nearrow_{\lambda} & \\ & L, & \end{array}$$

  where L is a bounded complex of finitely generated free R-modules.
- (iii) *There exists a set $\{L^u\}_{u\in U}$ of bounded complexes of finitely generated free R-modules and a pure epimorphism $\coprod_{u\in U} L^u \to F$.*
- (iv) *F is isomorphic to a filtered colimit of bounded complexes of finitely generated free R-modules.*
- (v) *F is isomorphic to a direct limit of bounded complexes of finitely generated free R-modules.*

The theorem is proved in Section 5. The terminology used in the statement is clarified in the sections leading up to the proof. In Section 4 we show that the finitely presented objects in the category of complexes are exactly the bounded complexes of finitely presented modules. Results of Breitsprecher [5] and Crawley-Boevey [6] show that the category of complexes is locally finitely presented, see Remark 4.7. Therefore, the equivalence of (ii), (iii), and (iv) follows from [6, (4.1)]. Furthermore, a result by Adámek and Rosický [1, Thm. 1.5] shows that (iv) and (v) are equivalent for quite general reasons; thus our task is to prove that the equivalent conditions (ii)–(v) are also equivalent to (i).

The characterization of semi-flat complexes in Theorem 1.1 opens to a study of the interplay between semi-flatness and purity in the category of complexes; this is the topic of Section 6. We show, for example, that a complex F is semi-flat if and only if every surjective quasi-isomorphism $M \to F$ is a pure epimorphism. This compares to Lazard's [13, Cor. 1.3] which states that a module F is flat if and only if every surjective homomorphism $M \to F$ is a pure epimorphism.

In the final Section 7, we use Theorem 1.1 to reprove a few results due to Enochs and García Rozas [7] and to Neeman [17]; our proofs are substantially different from the originals. In Theorem 7.3 we show that an acyclic semi-flat complex is a direct limit of contractible perfect complexes. Combined with a result of Benson and Goodearl [4] this enables us to show in Theorem 7.8 that a semi-flat complex of projective modules is semi-projective.

2. Complexes

In this paper R is a ring, and the default action on modules is on the left. Thus, R-modules are left R-modules, while right R-modules are considered to be (left) modules over the opposite ring R°. The definitions and results listed in this section are standard and more details can be found in textbooks, such as Weibel's [19], and in the paper [2] by Avramov and Foxby.

An R-complex M is a graded R-module $M = \coprod_{v\in\mathbb{Z}} M_v$ equipped with a differential, that is, an R-linear map $\partial^M\colon M \to M$ that satisfies $\partial^M\partial^M = 0$ and $\partial^M(M_v) \subseteq M_{v-1}$ for every $v \in \mathbb{Z}$. The homomorphism $M_v \to M_{v-1}$ induced by ∂^M is denoted ∂^M_v. Thus, an R-complex M can be visualized as follows,

$$M = \cdots \longrightarrow M_{v+1} \xrightarrow{\partial^M_{v+1}} M_v \xrightarrow{\partial^M_v} M_{v-1} \longrightarrow \cdots.$$

The category of R-complexes is denoted $\mathcal{C}(R)$. We identify the category of graded R-modules with the full subcategory of $\mathcal{C}(R)$ whose objects are R-complexes with zero differential.

For an R-complex M with differential ∂^M, set $\mathrm{Z}(M) = \operatorname{Ker}\partial^M$, $\mathrm{B}(M) = \operatorname{Im}\partial^M$, $\mathrm{C}(M) = \operatorname{Coker}\partial^M$, and $\mathrm{H}(M) = \mathrm{Z}(M)/\mathrm{B}(M)$; they are sub-, quotient, and subquotient complexes of M. Furthermore, $\mathrm{Z}(-)$, $\mathrm{B}(-)$, $\mathrm{H}(-)$ and $\mathrm{C}(-)$ are additive endofunctors on $\mathcal{C}(R)$.

A complex M with $\mathrm{H}(M) = 0$ is called *acyclic*. The *shift* of M is the complex ΣM with $(\Sigma M)_v = M_{v-1}$ and $\partial^{\Sigma M}_v = -\partial^M_{v-1}$. A morphism $\alpha\colon M \to N$ of complexes is called a *quasi-isomorphism* if $\mathrm{H}(\alpha)\colon \mathrm{H}(M) \to \mathrm{H}(N)$ is an isomorphism.

2.1. To a morphism $\alpha\colon M \to N$ of R-complexes one associates a complex $\operatorname{Cone}\alpha$, called the *mapping cone* of α; it fits into a degreewise split exact sequence,

$$0 \longrightarrow N \longrightarrow \operatorname{Cone}\alpha \longrightarrow \Sigma M \longrightarrow 0.$$

The morphism α is a quasi-isomorphism if and only if $\operatorname{Cone}\alpha$ is acyclic.

2.2. Let M and N be R-complexes. The total Hom complex, written $\operatorname{Hom}_R(M,N)$, yields a functor

$$\operatorname{Hom}_R(-,-)\colon \mathcal{C}(R)^{\mathrm{op}} \times \mathcal{C}(R) \longrightarrow \mathcal{C}(\mathbb{Z}).$$

The functor $\operatorname{Hom}_R(M,-)$ commutes with mapping cones, that is, for every morphism α of R-complexes there is an isomorphism of $\mathbb{Z}$-complexes,

$$\operatorname{Cone}\operatorname{Hom}_R(M,\alpha) \cong \operatorname{Hom}_R(M,\operatorname{Cone}\alpha).$$

There is an equality of abelian groups,

$$\mathrm{Z}_0\big(\operatorname{Hom}_R(M,N)\big) = \mathcal{C}(R)(M,N),$$

where the right-hand side is the hom-set in the category $\mathcal{C}(R)$.

2.3. Let M be an R°-complex and let N be an R-complex. The total tensor product complex, written $M \otimes_R N$, yields a functor

$$- \otimes_R -\colon \mathcal{C}\big(R^\circ\big) \times \mathcal{C}(R) \longrightarrow \mathcal{C}(\mathbb{Z}).$$

The functor $M \otimes_R -$ commutes with mapping cones, that is, for every morphism α of R-complexes there is an isomorphism of $\mathbb{Z}$-complexes,

$$\operatorname{Cone}(M \otimes_R \alpha) \cong M \otimes_R \operatorname{Cone}\alpha.$$

For a homogeneous element m in a graded module (or a complex) M, we write $|m|$ for its degree.

2.4. Let M be an R-complex. The *biduality* morphism

$$\delta^M\colon M \longrightarrow \operatorname{Hom}_{R^\circ}\big(\operatorname{Hom}_R(M,R),R\big)$$

is given by

$$\delta^M(m)(\psi) = (-1)^{|\psi||m|}\psi(m)$$

for homogeneous elements $m \in M$ and $\psi \in \operatorname{Hom}_R(M, R)$.

The morphism δ^M of R-complexes is an isomorphism if M is a complex of finitely generated projective R-modules.

2.5. Let M and N be R-complexes and let X be a complex of R–R°-bimodules. The *tensor evaluation* morphism

$$\omega^{MXN}\colon \operatorname{Hom}_R(M, X) \otimes_R N \longrightarrow \operatorname{Hom}_R(M, X \otimes_R N)$$

is given by

$$\omega^{MXN}(\psi \otimes n)(m) = (-1)^{|m||n|}\psi(m) \otimes n$$

for homogeneous elements $\psi \in \operatorname{Hom}_R(M, X)$, $n \in N$, and $m \in M$.

The morphism ω^{MXN} of $\mathbb{Z}$-complexes is an isomorphism if M is a bounded complex of finitely generated projective R-modules and $X = R$.

2.6. Let M be an R-complex, let N be an R°-complex, and let X be a complex of R–R°-bimodules. The *homomorphism evaluation* morphism

$$\theta^{XNM}\colon \operatorname{Hom}_{R^\circ}(X, N) \otimes_R M \longrightarrow \operatorname{Hom}_{R^\circ}\big(\operatorname{Hom}_R(M, X), N\big)$$

is given by

$$\theta^{XNM}(\psi \otimes m)(\varphi) = (-1)^{|\varphi||m|}\psi\varphi(m)$$

for homogeneous elements $\psi \in \operatorname{Hom}_{R^\circ}(X, N)$, $m \in M$, and $\varphi \in \operatorname{Hom}_R(M, X)$.

The morphism θ^{XNM} of $\mathbb{Z}$-complexes is an isomorphism if M is a bounded complex of finitely generated projective R-modules and $X = R$.

3. Filtered colimits

We refer to MacLane [14, Sec. IX.1] for background on colimits.

3.1. Definition. Let $\mathcal{A}$ be a category. By a *filtered colimit* in $\mathcal{A}$ we mean the colimit of a functor $\mathrm{F}\colon \mathcal{J} \to \mathcal{A}$, which is denoted $\operatorname{colim}_{J \in \mathcal{J}} \mathrm{F}(J)$, where $\mathcal{J}$ is a skeletally small filtered category. We reserve the term *direct limit* for the colimit of a direct system, i.e. of a functor $\mathcal{J} \to \mathcal{A}$ where $\mathcal{J}$ is the filtered category associated to a directed set, i.e. a filtered preordered set.

Notice that some authors, including Crawley-Boevey [6], use the term "direct limit" for any filtered colimit. For a direct system $\{A^u \to A^v\}_{u \leqslant v}$ it is customary to write $\varinjlim A^u$ for its direct limit, i.e. its colimit, however, we shall stick to the notation $\operatorname{colim} A^u$.

Let $\mathcal{A}$ and $\mathcal{B}$ be categories that have (all) filtered colimits. Recall that a functor $\mathrm{T}\colon \mathcal{A} \to \mathcal{B}$ is said to *preserve (filtered) colimits* if the canonical morphism in $\mathcal{B}$,

$$\operatorname{colim}_{J \in \mathcal{J}} \mathrm{T}\big(\mathrm{F}(J)\big) \longrightarrow \mathrm{T}\big(\operatorname{colim}_{J \in \mathcal{J}} \mathrm{F}(J)\big),$$

is an isomorphism for every (filtered) colimit $\operatorname{colim}_{J \in \mathcal{J}} \mathrm{F}(J)$ in $\mathcal{A}$.

We need a couple of facts about filtered colimits of complexes; the arguments are given in [19, Lem. 2.6.14 and Thm. 2.6.15].

3.2. The following assertions hold.

(a) Every homogeneous element in a filtered colimit, $\operatorname{colim}_{J\in\mathcal{J}} \mathrm{F}(J)$, in $\mathcal{C}(R)$ is in the image of the canonical morphism $\mathrm{F}(J) \to \operatorname{colim}_{J\in\mathcal{J}} \mathrm{F}(J)$ for some $J \in \mathcal{J}$.
(b) Filtered colimits in $\mathcal{C}(R)$ are exact (colimits are always right exact).

3.3. Lemma. *Let $\mathcal{A}$ be a category with filtered colimits and let $\mathrm{T}', \mathrm{T}, \mathrm{T}'' \colon \mathcal{A} \to \mathcal{C}(R)$ be functors. The following assertions hold.*

(a) *If $0 \to \mathrm{T}' \to \mathrm{T} \to \mathrm{T}''$ is an exact sequence and if T and T'' preserve filtered colimits, then T' preserves filtered colimits.*
(b) *If $\mathrm{T}' \to \mathrm{T} \to \mathrm{T}'' \to 0$ is an exact sequence and if T' and T preserve filtered colimits, then T'' preserves filtered colimits.*
(c) *If $0 \to \mathrm{T}' \to \mathrm{T} \to \mathrm{T}'' \to 0$ is an exact sequence and if T' and T'' preserve filtered colimits, then T preserves filtered colimits.*

Proof. Let $\mathcal{J}$ be a skeletally small filtered category and let $\mathrm{F}\colon \mathcal{J} \to \mathcal{A}$ be a functor.

(a) Exactness of the sequence $0 \to \mathrm{T}' \to \mathrm{T} \to \mathrm{T}''$ and left exactness of filtered colimits in $\mathcal{C}(R)$ yield the following commutative diagram,

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{colim}_{J\in\mathcal{J}} \mathrm{T}'(\mathrm{F}(J)) & \longrightarrow & \operatorname{colim}_{J\in\mathcal{J}} \mathrm{T}(\mathrm{F}(J)) & \longrightarrow & \operatorname{colim}_{J\in\mathcal{J}} \mathrm{T}''(\mathrm{F}(J)) \\
& \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu''} \\
0 \longrightarrow & \mathrm{T}'(\operatorname{colim}_{J\in\mathcal{J}} \mathrm{F}(J)) & \longrightarrow & \mathrm{T}(\operatorname{colim}_{J\in\mathcal{J}} \mathrm{F}(J)) & \longrightarrow & \mathrm{T}''(\operatorname{colim}_{J\in\mathcal{J}} \mathrm{F}(J)),
\end{array}$$

where μ', μ, and μ'' are the canonical morphisms. If μ and μ'' are isomorphisms, then so is μ' by the Five Lemma.

Parts (b) and (c) have similar proofs. □

3.4. Proposition. *The functors $\mathrm{Z}, \mathrm{C}, \mathrm{B}, \mathrm{H} \colon \mathcal{C}(R) \to \mathcal{C}(R)$ preserve filtered colimits.*

Proof. A graded R-module is considered as an R-complex with zero differential; let $\mathcal{GM}(R)$ be the full subcategory of $\mathcal{C}(R)$ whose objects are all graded R-modules. Since the inclusion functor $\mathrm{i}\colon \mathcal{GM}(R) \to \mathcal{C}(R)$ preserves filtered colimits, it suffices to argue that C preserves filtered colimits when viewed as a functor from $\mathcal{C}(R)$ to $\mathcal{GM}(R)$. However, this functor C has a right adjoint, namely the inclusion functor i, so it follows from (the dual of) [14, V§5, Thm. 1] that C preserves colimits.

Denote by I the identity functor on $\mathcal{C}(R)$. Since I and C preserve filtered colimits, Lemma 3.3 applies to the short exact sequence $0 \to \mathrm{B} \to \mathrm{I} \to \mathrm{C} \to 0$ to show that B preserves filtered colimits. From the short exact sequences,

$$0 \longrightarrow \mathrm{H} \longrightarrow \mathrm{C} \longrightarrow \Sigma\mathrm{B} \longrightarrow 0 \qquad \text{and} \qquad 0 \longrightarrow \mathrm{B} \longrightarrow \mathrm{Z} \longrightarrow \mathrm{H} \longrightarrow 0$$

we now conclude that H and Z preserve filtered colimits as well. □

3.5. Proposition. *For every R-complex N the functor $-\otimes_R N\colon \mathcal{C}(R^\circ) \to \mathcal{C}(\mathbb{Z})$ preserves colimits. For every bounded R-complex P of finitely generated projective modules the functor* $\operatorname{Hom}_R(P,-)\colon \mathcal{C}(R) \to \mathcal{C}(\mathbb{Z})$ *preserves colimits.*

Proof. The functor $-\otimes_R N$ has a right adjoint, namely $\operatorname{Hom}_{\mathbb{Z}}(N,-)$, so it follows from (the dual of) [14, V§5, Thm. 1] that $-\otimes_R N$ preserves colimits.

If P is a bounded complex of finitely generated projective R-modules, then $\operatorname{Hom}_R(P,R)$ is a bounded complex of finitely generated projective R°-modules. By 2.4 and 2.6 there are natural isomorphisms of functors from $\mathcal{C}(R)$ to $\mathcal{C}(\mathbb{Z})$,

$$\begin{aligned}\operatorname{Hom}_R(P,-) &\cong \operatorname{Hom}_R\big(\operatorname{Hom}_{R^\circ}\big(\operatorname{Hom}_R(P,R),R\big),-\big)\\ &\cong \operatorname{Hom}_R(R,-)\otimes_{R^\circ}\operatorname{Hom}_R(P,R)\\ &\cong -\otimes_{R^\circ}\operatorname{Hom}_R(P,R),\end{aligned}$$

and the desired conclusion follows from the first assertion. □

4. Finitely presented objects in the category of complexes

Let $\mathcal{A}$ be an additive category. Following Crawley-Boevey [6], an object A in $\mathcal{A}$ is called *finitely presented* if the functor $\mathcal{A}(A,-)$ preserves filtered colimits. The category $\mathcal{A}$ is called *locally finitely presented* if the category of finitely presented objects in $\mathcal{A}$ is skeletally small and if every object in $\mathcal{A}$ is a filtered colimit of finitely presented objects; see [6].

4.1. Definition. For an R-module F and $v \in \mathbb{Z}$ denote by $\mathrm{D}^v(F)$ the R-complex $0 \longrightarrow F \xrightarrow{=} F \longrightarrow 0$ concentrated in degrees v and $v-1$.

4.2. Construction. Let M be an R-complex. For a homomorphism $\pi\colon F \to M_v$ of R-modules, there is a morphism of R-complexes, $\tilde{\pi}\colon \mathrm{D}^v(F) \to M$, given by

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & F & \xrightarrow{1^F} & F & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \partial^M_v \pi} & & \\ \cdots & \longrightarrow & M_{v+1} & \xrightarrow{\partial^M_{v+1}} & M_v & \xrightarrow{\partial^M_v} & M_{v-1} & \xrightarrow{\partial^M_{v-1}} & M_{v-2} \longrightarrow \cdots . \end{array}$$

4.3. Proposition. *An R-complex M is bounded and degreewise finitely presented if and only if there exists an exact sequence of R-complexes $L^1 \to L^0 \to M \to 0$ where L^0 and L^1 are bounded complexes of finitely generated free modules.*

Proof. The "if" part is trivial. To show "only if", assume that M is a bounded and degreewise finitely presented complex. Since M is, in particular, degreewise finitely generated, we can for every $v \in \mathbb{Z}$ choose a surjective homomorphism $\pi^v\colon F^v \to M_v$ where F^v is finitely generated free, and such that F^v is zero if M_v is zero. Set $L^0 = \coprod_{v\in\mathbb{Z}} \mathrm{D}^v(F^v)$ and let $\pi\colon L^0 \to M$ be the unique morphism whose composite, $\pi\varepsilon^v$, with the embedding $\varepsilon^v\colon \mathrm{D}^v(F^v) \hookrightarrow L^0$ equals the morphism $\tilde{\pi}^v$ from 4.2. Evidently, π is surjective and L^0 is a bounded complex of finitely generated free modules. Consider the kernel $M' = \operatorname{Ker}\pi$. Since L^0 is bounded, so is M'. Furthermore, as M is degreewise finitely presented, M' is degreewise finitely generated. Hence the argument above shows that there exists a surjective morphism $L^1 \to M'$ where L^1 is a bounded complex of finitely generated free modules. The composite $L^1 \twoheadrightarrow M' \hookrightarrow L^0$ now yields the left-hand morphism in an exact sequence $L^1 \to L^0 \to M \to 0$. □

It is a well-known fact that every module is isomorphic to a direct limit of finitely presented modules. The following generalization to complexes can be found in García Rozas's [8, Lemmas 4.1.1(ii) and 5.1.1].

4.4. Every R-complex is isomorphic to a direct limit of bounded and degreewise finitely presented R-complexes. □

The next theorem identifies the finitely presented objects in the category $\mathcal{C}(R)$, and combined with 4.4 it shows that this category is locally finitely presented; see Corollary 4.6 and Remark 4.7.

4.5. Theorem. *For an R-complex M the following conditions are equivalent.*

- (i) *M is bounded and degreewise finitely presented.*
- (ii) *The functor $\operatorname{Hom}_R(M,-)$ preserves filtered colimits.*
- (iii) *The functor $\operatorname{Hom}_R(M,-)$ preserves direct limits.*
- (iv) *The functor $\mathcal{C}(R)(M,-)$ preserves filtered colimits.*
- (v) *The functor $\mathcal{C}(R)(M,-)$ preserves direct limits.*

Proof. The implications $(ii) \Longrightarrow (iii)$ and $(iv) \Longrightarrow (v)$ are trivial. By 2.2 there is for every R-complex M an identity of functors from $\mathcal{C}(R)$ to $\mathbb{Z}$-modules,

$$\mathcal{C}(R)(M,-) = \mathrm{Z}_0\big(\operatorname{Hom}_R(M,-)\big).$$

By Proposition 3.4 the functor Z_0 preserves filtered colimits, and hence the implications $(ii) \Longrightarrow (iv)$ and $(iii) \Longrightarrow (v)$ follow. It remains to show that the implications $(i) \Longrightarrow (ii)$ and $(v) \Longrightarrow (i)$ hold.

$(i) \Longrightarrow (ii)$ By Proposition 4.3 there is an exact sequence $L^1 \to L^0 \to M \to 0$ where L^0 and L^1 are bounded complexes of finitely generated free R-modules. Thus there is an exact sequence,

$$0 \longrightarrow \operatorname{Hom}_R(M,-) \longrightarrow \operatorname{Hom}_R\big(L^0,-\big) \longrightarrow \operatorname{Hom}_R\big(L^1,-\big),$$

of functors from $\mathcal{C}(R)$ to $\mathcal{C}(\mathbb{Z})$. By Proposition 3.5 the functors $\operatorname{Hom}_R(L^0,-)$ and $\operatorname{Hom}_R(L^1,-)$ preserve filtered colimits, and the conclusion follows from Lemma 3.3.

$(v) \Longrightarrow (i)$ By 4.4 there is a direct system $\{\mu^{vu}\colon M^u \to M^v\}_{u\leqslant v}$ of bounded and degreewise finitely presented R-complexes with $\operatorname{colim} M^u \cong M$. By assumption, the canonical morphism

$$\alpha\colon \operatorname{colim} \mathcal{C}(R)\big(M,M^u\big) \longrightarrow \mathcal{C}(R)\big(M,\operatorname{colim} M^u\big) \cong \mathcal{C}(R)(M,M)$$

is an isomorphism. Write

$$\mu^u\colon M^u \longrightarrow \operatorname{colim} M^u \cong M \quad \text{and} \quad \lambda^u\colon \mathcal{C}(R)\big(M,M^u\big) \longrightarrow \operatorname{colim} \mathcal{C}(R)\big(M,M^u\big)$$

for the canonical morphisms, and note that $\alpha\lambda^u = \mathcal{C}(R)(M,\mu^u)$ holds for all u. Surjectivity of α yields an element $\chi \in \operatorname{colim} \mathcal{C}(R)(M,M^u)$ with $\alpha(\chi) = 1^M$. By 3.2 one has $\chi = \lambda^u(\psi^u)$ for some $\psi^u \in \mathcal{C}(R)(M,M^u)$. Hence, there are equalities $\mu^u\psi^u = \mathcal{C}(R)(M,\mu^u)(\psi^u) = \alpha\lambda^u(\psi^u) = \alpha(\chi) = 1^M$. Thus, M is a direct summand of M^u, and since M^u is bounded and degreewise finitely presented, so is M. □

The equivalences above of (ii) and (iii) and of (iv) and (v) also follow from general principles; see [1, Cor. to Thm. 1.5].

4.6. Corollary. *The category $\mathcal{C}(R)$ is locally finitely presented, and the finitely presented objects in $\mathcal{C}(R)$ are exactly the bounded and degreewise finitely presented R-complexes.*

Proof. By the equivalence of (i) and (iv) in Theorem 4.5, the finitely presented objects in the category $\mathcal{C}(R)$ are exactly the bounded and degreewise finitely presented R-complexes. Evidently, the category of such complexes is skeletally small. By 4.4 every object in $\mathcal{C}(R)$ is a filtered colimit (even a direct limit) of finitely presented objects. □

4.7. Remark. The fact that $\mathcal{C}(R)$ is locally finitely presented also follows from [5, Satz 1.5] and [6, (2.4)]; indeed, $\mathcal{C}(R)$ is a Grothendieck category and $\{\mathrm{D}^u(R) \mid u \in \mathbb{Z}\}$ is a generating set of finitely presented objects.

5. Proof of the main theorem

The notion of semi-flatness, and the related notions of semi-projectivity and semi-freeness, originate in the treatise [3] by Avramov, Foxby, and Halperin.

A graded R-module L is called *graded-free* if it has a *graded basis*, that is, a basis consisting of homogeneous elements. It is easily seen that L is graded-free if and only if every component L_v is a free R-module.

5.1. An R-complex L is called *semi-free* if the underlying graded R-module has a graded basis E that can be written as a disjoint union $E = \biguplus_{n \geqslant 0} E^n$ such that one has $E^0 \subseteq \mathrm{Z}(L)$ and $\partial^L(E^n) \subseteq R\langle \bigcup_{i=0}^{n-1} E^i \rangle$ for every $n \geqslant 1$. Such a basis is called a *semi-basis* of L.

5.2. Example. A bounded below complex of free modules is semi-free.

5.3. Every R-complex M has a *semi-free resolution*, that is, a quasi-isomorphism of R-complexes $\pi\colon L \to M$ where L is semi-free. Moreover, π can be chosen surjective and with $L_v = 0$ for all $v < \inf\{n \in \mathbb{Z} \mid M_n \neq 0\}$. See [3, Thm. 2.2].

5.4. For an R-complex P the following conditions are equivalent.

- (i) The functor $\mathrm{Hom}_R(P, -)$ is exact and preserves quasi-isomorphisms.
- (ii) For every morphism $\alpha\colon P \to N$ and for every surjective quasi-isomorphism $\beta\colon M \to N$ there exists a morphism $\gamma\colon P \to M$ such that $\alpha = \beta\gamma$ holds.
- (iii) P is a complex of projective R-modules, and the functor $\mathrm{Hom}_R(P, -)$ preserves acyclicity.

A complex that satisfies these equivalent conditions is called *semi-projective*; see [3, Thm. 3.5].

5.5. Example. A bounded below complex of projective modules is semi-projective.

By [3, Thm. 3.5] a semi-free complex is semi-projective.

5.6. For an R-complex F the following conditions are equivalent.

- (i) The functor $- \otimes_R F$ is exact and preserves quasi-isomorphisms.
- (ii) F is a complex of flat R-modules and the functor $- \otimes_R F$ preserves acyclicity.

A complex that satisfies these equivalent conditions is called *semi-flat*; see [3, Thm. 6.5].

5.7. Example. A bounded below complex of flat modules is semi-flat.

By [3, Lem. 7.1] a semi-projective complex is semi-flat.

As noted in 2.4, the biduality morphism δ^P is an isomorphism for every complex P of finitely generated projective modules. For the proof of Theorem 1.1 we need an explicit description of the inverse.

5.8. Lemma. *For a complex P of finitely generated projective R-modules, the inverse of the isomorphism $\delta^{\operatorname{Hom}_R(P,R)}$ is $\operatorname{Hom}_R(\delta^P, R)$.*

Proof. As P is a complex of finitely generated projective R-modules, δ^P and hence $\operatorname{Hom}_R(\delta^P, R)$ are isomorphisms by 2.4. For φ in $\operatorname{Hom}_R(P, R)$ and x in P one has

$$\begin{aligned}\big(\operatorname{Hom}_R(\delta^P, R)\delta^{\operatorname{Hom}_R(P,R)}\big)(\psi)(x) &= \big(\delta^{\operatorname{Hom}_R(P,R)}(\psi)\delta^P\big)(x)\\ &= \delta^{\operatorname{Hom}_R(P,R)}(\psi)\big(\delta^P(x)\big)\\ &= \delta^P(x)(\psi)\\ &= \psi(x),\end{aligned}$$

so $\operatorname{Hom}_R(\delta^P, R)\delta^{\operatorname{Hom}_R(P,R)}$ is the identity on $\operatorname{Hom}_R(P, R)$. □

Condition (*iii*) in Theorem 1.1 asserts the existence of a certain pure epimorphism in $\mathcal{C}(R)$. In the proof below, we use that the equivalence of conditions (*ii*)–(*v*) has been established elsewhere and we do not directly address (*iii*). However, in the next section we study the relationship between purity and semi-flatness; in particular, we recall the definition of a pure epimorphism from [6, §3] in the first paragraph of Section 6.

Proof of Theorem 1.1. By Corollary 4.6 the category $\mathcal{C}(R)$ is locally finitely presented and its finitely presented objects are exactly the bounded and degreewise finitely presented complexes. It now follows from [6, (4.1)] that (*ii*), (*iii*), and (*iv*) are equivalent. Furthermore, [1, Thm. 1.5] shows that (*iv*) and (*v*) are equivalent. The remaining implications (*i*) $\Longrightarrow$ (*ii*) and (*v*) $\Longrightarrow$ (*i*) are proved below.

(*i*) $\Longrightarrow$ (*ii*) Let $\varphi\colon N \to F$ be a morphism of R-complexes where N is bounded and degreewise finitely presented. By Proposition 4.3 there is an exact sequence,

$$L^1 \xrightarrow{\psi^1} L^0 \xrightarrow{\psi^0} N \longrightarrow 0,$$

of R-complexes, where L^0 and L^1 are bounded complexes of finitely generated free modules. Consider the exact sequence of R°-complexes,

$$0 \longrightarrow K \xrightarrow{\iota} \operatorname{Hom}_R\big(L^0, R\big) \xrightarrow{\operatorname{Hom}(\psi^1,R)} \operatorname{Hom}_R\big(L^1, R\big), \tag{1}$$

where K is the kernel of $\operatorname{Hom}_R(\psi^1, R)$ and ι is the embedding. The functor $\mathrm{Z}_0(-)$ is left exact, and the functor $- \otimes_R F$ is exact by definition, so it follows that the functor $\mathrm{Z}_0(- \otimes_R F)$ leaves the sequence (1) exact. As L^0 is bounded, so is K; set $u = \inf\{n \in \mathbb{Z} \mid K_n \neq 0\}$. By 5.3 there is an exact sequence,

$$P \xrightarrow{\pi} K \longrightarrow 0, \tag{2}$$

where π is a quasi-isomorphism and P is a semi-free R°-complex with $P_v = 0$ for all $v < u$. As F is semi-flat, $\pi \otimes_R F$ is a surjective quasi-isomorphism by 5.6. A simple diagram chase shows that every surjective quasi-isomorphism is surjective on cycles, so the functor $\mathrm{Z}_0(- \otimes_R F)$ leaves the sequence (2) exact. Consequently, there is an exact sequence,

$$\mathrm{Z}_0(P \otimes_R F) \xrightarrow{(\iota\pi)\otimes F} \mathrm{Z}_0\big(\operatorname{Hom}_R\big(L^0, R\big) \otimes_R F\big) \xrightarrow{\operatorname{Hom}(\psi^1,R)\otimes F} \mathrm{Z}_0\big(\operatorname{Hom}_R\big(L^1, R\big) \otimes_R F\big).$$

For every R-complex M, denote by ξ^M the composite morphism

$$\mathrm{Hom}_R(M,R)\otimes_R F \xrightarrow{\omega^{MRF}} \mathrm{Hom}_R(M,R\otimes_R F) \xrightarrow{\cong} \mathrm{Hom}_R(M,F),$$

where ω^{MRF} is the tensor evaluation morphism 2.5 and the isomorphism is induced by the canonical one $R\otimes_R F\cong F$. The morphism ξ^M is natural in M, and by 2.5 it is an isomorphism if M is a bounded complex of finitely generated projective modules. The exact sequence above now yields another exact sequence,

$$\mathrm{Z}_0(P\otimes_R F) \xrightarrow{\xi^{L^0}\circ((\iota\pi)\otimes F)} \mathrm{Z}_0\big(\mathrm{Hom}_R\big(L^0,F\big)\big) \xrightarrow{\mathrm{Hom}(\psi^1,F)} \mathrm{Z}_0\big(\mathrm{Hom}_R\big(L^1,F\big)\big). \tag{3}$$

As $\varphi\psi^0\colon L^0\to F$ is a morphism, it is an element in $\mathrm{Z}_0(\mathrm{Hom}_R(L^0,F))$; see 2.2. Since one has $\mathrm{Hom}_R(\psi^1,F)(\varphi\psi^0)=\varphi\psi^0\psi^1=0$, exactness of (3) yields an element x in $\mathrm{Z}_0(P\otimes_R F)$ with

$$\big(\xi^{L^0}\circ\big((\iota\pi)\otimes_R F\big)\big)(x)=\varphi\psi^0. \tag{4}$$

The graded module underlying P has a graded basis E, and x has the form $x=\sum_{i=1}^n e_i\otimes f_i$ with $e_i\in E$ and $f_i\in F$. Set $w=\max\{|e_1|,\dots,|e_n|\}$; as one has $P_v=0$ for all $v<u$, each basis element e_i satisfies $u\leqslant|e_i|\leqslant w$. For $v\in\mathbb{Z}$ set $E_v=\{e\in E\mid |e|=v\}$. Next we define a bounded subcomplex P' of P such that each module P'_v is finitely generated and free. For $v\notin\{u,\dots,w\}$ set $P'_v=0$; for $v\in\{u,\dots,w\}$ the modules P'_v are constructed inductively. Let P'_w be the finitely generated free submodule of P_w generated by the set $E'_w=\{e_1,\dots,e_n\}\cap E_w$. For $v\leqslant w$ assume that a finitely generated free submodule P'_v of P_v with finite basis E'_v has been constructed. As the subset $B'_{v-1}=\{\partial^P(e)\mid e\in E'_v\}$ of P_{v-1} is finite, there is a finite subset G'_{v-1} of E_{v-1} with $B'_{v-1}\subseteq R^\circ\langle G'_{v-1}\rangle$. Now let P'_{v-1} be the submodule of P_{v-1} generated by the following finite set of basis elements,

$$E'_{v-1}=G'_{v-1}\cup\big(\{e_1,\dots,e_n\}\cap E_{v-1}\big).$$

By construction, one has $\partial^P(P'_v)\subseteq P'_{v-1}$ for all $v\in\mathbb{Z}$, so P' is a subcomplex of P. The construction shows that $x=\sum_{i=1}^n e_i\otimes f_i$ belongs to $P'\otimes_R F$. As F is a complex of flat R-modules, $P'\otimes_R F$ is a subcomplex of $P\otimes_R F$, and as the element x is in $\mathrm{Z}_0(P\otimes_R F)$ it also belongs to $\mathrm{Z}_0(P'\otimes_R F)$.

Set $L=\mathrm{Hom}_{R^\circ}(P',R)$. As P' is a bounded complex of finitely generated free R°-modules, L is a bounded complex of finitely generated free R-modules. Let $\varepsilon\colon P'\hookrightarrow P$ be the embedding and let $\kappa'\colon L^0\to L$ be the composite morphism

$$L^0 \xrightarrow{\delta^{L^0}} \mathrm{Hom}_{R^\circ}\big(\mathrm{Hom}_R\big(L^0,R\big),R\big) \xrightarrow{\mathrm{Hom}(\iota\pi\varepsilon,R)} \mathrm{Hom}_{R^\circ}\big(P',R\big)=L.$$

In the commutative diagram

$$\begin{array}{ccc} P' & \xrightarrow{\quad\iota\pi\varepsilon\quad} & \mathrm{Hom}_R(L^0,R) \\ \cong\big\downarrow \delta^{P'} & & \cong\big\downarrow \delta^{\mathrm{Hom}(L^0,R)} \\ \mathrm{Hom}_R(\mathrm{Hom}_{R^\circ}(P',R),R) & \xrightarrow{\mathrm{Hom}(\mathrm{Hom}(\iota\pi\varepsilon,R),R)} & \mathrm{Hom}_R(\mathrm{Hom}_{R^\circ}(\mathrm{Hom}_R(L^0,R),R),R) \end{array}$$

the vertical morphisms are isomorphisms by 2.4, and $\delta^{\mathrm{Hom}(L^0,R)}$ is by Lemma 5.8 the inverse of $\mathrm{Hom}_R(\delta^{L^0},R)$. One now has

$$\mathrm{Hom}_R\big(\kappa',R\big)\delta^{P'}=\iota\pi\varepsilon. \tag{5}$$

It follows that there are equalities,

$$\operatorname{Hom}_R(\kappa'\psi^1, R)\delta^{P'} = \operatorname{Hom}_R(\psi^1, R)\iota\pi\varepsilon = 0\pi\varepsilon = 0,$$

and since $\delta^{P'}$ is an isomorphism, the morphism $\operatorname{Hom}_R(\kappa'\psi^1, R)$ is zero. Thus, $\operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(\kappa'\psi^1, R), R)$ is zero, and hence the commutative diagram

$$\begin{array}{ccc} L^1 & \xrightarrow{\kappa'\psi^1} & L \\ \cong \big\downarrow \delta^{L^1} & & \cong \big\downarrow \delta^{L} \\ \operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(L^1, R), R) & \xrightarrow{\operatorname{Hom}(\operatorname{Hom}(\kappa'\psi^1, R), R)} & \operatorname{Hom}_{R^\circ}(\operatorname{Hom}_R(L, R), R) \end{array}$$

shows that $\kappa'\psi^1 = 0$ holds. Again the vertical morphisms are isomorphisms by 2.4. Since κ' vanishes on $\operatorname{Im}\psi^1 = \operatorname{Ker}\psi^0$ there is a unique morphism $\kappa : N \to L$ with $\kappa\psi^0 = \kappa'$. Finally, consider the diagram,

$$\begin{array}{ccccc} P' \otimes_R F & \xrightarrow{\delta^{P'} \otimes F} & \operatorname{Hom}_R(L, R) \otimes_R F & \xrightarrow{\xi^L} & \operatorname{Hom}_R(L, F) \\ \big\downarrow \varepsilon\otimes F & & \big\downarrow \operatorname{Hom}(\kappa', R)\otimes F & & \big\downarrow \operatorname{Hom}_R(\kappa', F) \\ P \otimes_R F & \xrightarrow{(\iota\pi)\otimes F} & \operatorname{Hom}_R(L^0, R) \otimes_R F & \xrightarrow{\xi^{L^0}} & \operatorname{Hom}_R(L^0, F), \end{array} \tag{6}$$

where the left-hand square is commutative by (5) and the right-hand square is commutative by naturality of ξ. Set

$$\lambda = \big(\xi^L \circ \big(\delta^{P'} \otimes F\big)\big)(x);$$

it is an element in $\operatorname{Hom}_R(L, F)$, and as x belongs to $\mathrm{Z}_0(P' \otimes_R F)$, also λ is a cycle; i.e. $\lambda : L \to F$ is a morphism. From (6), from the definition of λ, and from (4) one gets $\lambda\kappa' = \varphi\psi^0$. The identity $\kappa' = \kappa\psi^0$ and surjectivity of ψ^0 now yield $\lambda\kappa = \varphi$.

$(v) \implies (i)$ Every bounded complex of finitely generated free R-modules is semi-flat, see Example 5.7, and as mentioned in the introduction a direct limit of semi-flat complexes is semi-flat. A proof of this fact can be found in [3, Prop. 6.9]; for completeness we include the argument. Let $\{F^u \to F^v\}_{u \leqslant v}$ be a direct system of semi-flat R-complexes and set $F = \operatorname{colim} F^u$. By Proposition 3.5 there is a natural isomorphism of functors, $- \otimes_R F \cong \operatorname{colim}(- \otimes_R F^u)$. By assumption, each functor $- \otimes_R F^u$ is exact and preserves acyclicity. Since direct limits in $\mathcal{C}(\mathbb{Z})$ are exact, see 3.2, and since the homology functor preserves direct limits, see Proposition 3.4, it follows that the functor $- \otimes_R F$ is exact and preserves acyclicity; that is, F is semi-flat. □

6. Purity

Let $\mathcal{A}$ be a locally finitely presented category. Following [6, §3] a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$ is called *pure* if

$$0 \longrightarrow \mathcal{A}\big(A, M'\big) \longrightarrow \mathcal{A}(A, M) \longrightarrow \mathcal{A}\big(A, M''\big) \longrightarrow 0$$

is exact for every finitely presented object A in $\mathcal{A}$. In this case, the morphism $M' \to M$ is called a *pure monomorphism* and $M \to M''$ is called *pure epimorphism*.

In view of Corollary 4.6, a morphism $\alpha : X \to Y$ in $\mathcal{C}(R)$ is a pure epimorphism if and only if for every morphism $\varphi : N \to Y$ with N bounded and degreewise finitely presented there exists a morphism $\beta : N \to X$ with $\varphi = \alpha\beta$.

Semi-flat complexes have the following two-out-of-three property; see the proof of [3, Prop. 6.7].

6.1. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of R-complexes. If F'' is semi-flat, then F' is semi-flat if and only if F is semi-flat.

The next result supplements 6.1; it shows that the class of semi-flat complexes is closed under pure subcomplexes and pure quotient complexes.

6.2. Proposition. *Let $0 \to F' \to F \to F'' \to 0$ be a pure exact sequence of R-complexes. If the complex F is semi-flat, then F' and F'' are semi-flat.*

Proof. Assume that F is semi-flat and denote the given morphism from F to F'' by α. Let $\varphi: N \to F''$ be a morphism where N is a bounded and degreewise finitely presented R-complex. Since α is a pure epimorphism one has $\varphi = \alpha\beta$ for some morphism $\beta: N \to F$. Since F is semi-flat, the morphism β, and hence also φ, factors through a bounded complex of finitely generated free R-modules. Thus F'' is semi-flat by Theorem 1.1. It now follows from 6.1 that F' is semi-flat as well. $\square$

Every surjective homomorphism $M \to F$ of R-modules with F flat is a pure epimorphism; see [13, Cor. 1.3]. The next example shows that a surjective morphism $M \to F$ of R-complexes with F semi-flat need not be a pure epimorphism.

6.3. Example. Consider the $\mathbb{Z}$-complexes, $\mathrm{D}^0(\mathbb{Z})$ and $\mathbb{Z}$. As a $\mathbb{Z}$-complex, $\mathbb{Z}$ is semi-flat by Example 5.7. The surjective morphism $\pi: \mathrm{D}^0(\mathbb{Z}) \to \mathbb{Z}$, given by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1^{\mathbb{Z}}} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1^{\mathbb{Z}}} & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0, \end{array}$$

is not a pure epimorphism. Indeed, the complex $\mathbb{Z}$ is finitely presented but the identity morphism $\mathbb{Z} \to \mathbb{Z}$ does not factor through π; in other words, π is not a split surjection.

What can be salvaged is captured in the next proposition.

6.4. Proposition. *For an R-complex F the following conditions are equivalent.*

- (i) *F is semi-flat.*
- (ii) *Every surjective quasi-isomorphism $M \to F$ is a pure epimorphism.*
- (iii) *There exists a semi-free complex L and a quasi-isomorphism $L \to F$ which is also a pure epimorphism.*

Proof. $(i) \Longrightarrow (ii)$ Let $\alpha: M \to F$ be a surjective quasi-isomorphism and $\varphi: N \to F$ be a morphism with N bounded and degreewise finitely presented. Since F is semi-flat there exists by Theorem 1.1 a bounded complex L of finitely generated free R-modules and morphisms $\kappa: N \to L$ and $\lambda: L \to F$ with $\varphi = \lambda\kappa$. As L is semi-projective, see Examples 5.2 and 5.5, there exists by 5.4 a morphism $\gamma: L \to M$ with $\lambda = \alpha\gamma$, so with $\beta = \gamma\kappa$ one has $\varphi = \alpha\beta$.

$(ii) \Longrightarrow (iii)$ Immediate from 5.3.

$(iii) \Longrightarrow (i)$ Follows from Proposition 6.2 as a semi-free complex is semi-flat. $\square$

7. Semi-flat complexes of projective modules

A semi-free complex is semi-projective, see Example 5.5, but a semi-projective complex of free modules need not be semi-free. Indeed, the $\mathbb{Z}/6\mathbb{Z}$-complex,

$$\cdots \longrightarrow \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{3} \mathbb{Z}/6\mathbb{Z} \xrightarrow{2} \mathbb{Z}/6\mathbb{Z} \xrightarrow{3} \mathbb{Z}/6\mathbb{Z} \longrightarrow \cdots$$

serves as a counterexample; see [3, Ex. 7.10]. It turns out that a semi-flat complex of projective modules is, in fact, semi-projective. As Murfet notes in his thesis [15, Cor. 5.14], this follows from work of Neeman [17] on the homotopy category of flat modules. The purpose of this section is to provide an alternative proof of this fact.

7.1. For an R-complex C, the following conditions are equivalent.

(i) The identity 1^C is null-homotopic; that is, it is a boundary in $\operatorname{Hom}_R(C, C)$.
(ii) The exists a degree 1 homomorphism $\sigma: C \to C$ with $\partial^C = \partial^C \sigma \partial^C$.
(iii) There exists a graded R-module B with $\operatorname{Cone} 1^B \cong C$.

A complex that satisfies these conditions is called *contractible*; see [19, Sec. 1.4].

If C is contractible, then so are all complexes $\operatorname{Hom}_R(C, X)$, $\operatorname{Hom}_R(X, C)$, and $Y \otimes_R C$. Every contractible complex is acyclic.

7.2. Lemma. *Let N be a bounded and degreewise finitely generated R-complex, and let C be a contractible complex of projective R-modules. Every morphism $N \to C$ factors as $N \to L \to C$, where L is a bounded and contractible complex of finitely generated free R-modules.*

Proof. There is a graded R-module P with $C \cong \operatorname{Cone} 1^P = \coprod_{v \in \mathbb{Z}} \mathrm{D}^{v+1}(P_v)$. In particular, C is a coproduct of bounded contractible complexes of projective R-modules. For each module P_v there is a complementary module Q_v and a set E_v such that there is an isomorphism $P_v \oplus Q_v \cong R^{(E_v)}$. Set

$$L' = C \oplus \Bigl(\coprod_{v \in \mathbb{Z}} \mathrm{D}^{v+1}(P_v)\Bigr) \cong \coprod_{v \in \mathbb{Z}} \bigl(\mathrm{D}^{v+1}(R)\bigr)^{(E_v)}.$$

A morphism $\alpha: N \to C$ factors through L', and since N is bounded and degreewise finitely generated, it factors through a finite coproduct $L = \bigoplus_{i=1}^{n} \mathrm{D}^{v_i+1}(R)$. Evidently, this is a bounded and contractible complex of finitely generated free R-modules. □

The next result shows that the complexes characterized in [7, Thm. 2.4], in [8, Thm. 4.1.3], and in [17, Fact 2.14] are precisely the acyclic semi-flat complexes. In [3] such complexes are called *categorically flat*, in [8] they are called *flat*, and in [16] they are called *pure acyclic*.

7.3. Theorem. *For an R-complex F the following conditions are equivalent.*

(i) *F is semi-flat and acyclic.*
(ii) *F is a filtered colimit of bounded and contractible complexes of finitely generated free R-modules.*
(iii) *F is a direct limit of bounded and contractible complexes of finitely generated free R-modules.*
(iv) *F is acyclic and $\mathrm{B}(F)$ is a complex of flat R-modules.*

Proof. (i) $\Longrightarrow$ (ii) By Theorem 4.5 and [6, (4.1)] it is sufficient to prove that every morphism $\varphi: N \to F$ with N bounded and degreewise finitely presented factors through a bounded and contractible complex of finitely generated free R-modules. Fix such a morphism φ. Let $\pi: P \xrightarrow{\simeq} F$ be a surjective semi-free resolution; cf. 5.3. As P is acyclic and semi-projective, the complex $\operatorname{Hom}_R(P, P)$ is acyclic; in particular the morphism 1^P is null-homotopic so P is contractible. The morphism π is by Proposition 6.4 a pure epimorphism, so φ factors through P and hence, by Lemma 7.2, through a bounded and contractible complex L of finitely generated free R-modules.

$(ii) \Longrightarrow (iii)$ This follows from [1, Thm. 1.5].

$(iii) \Longrightarrow (iv)$ A direct limit of contractible (acyclic) complexes is acyclic, so F is acyclic. In a contractible complex L of free R-modules, the subcomplex $\mathrm{B}(L)$ consists of projective R-modules. The functor $\mathrm{B}(-)$ preserves direct limits by Proposition 3.4, and a direct limit of projective modules is a flat module, so $\mathrm{B}(F)$ is a complex of flat R-modules.

$(iv) \Longrightarrow (i)$ Each sequence $0 \to \mathrm{B}_v(F) \to F_v \to \mathrm{B}_{v-1}(F) \to 0$ is exact, so each module F_v is flat; that is, F is an acyclic complex of flat R-modules. For an acyclic R°-complex M (actually for any R°-complex) it follows from the Künneth formula [19, Thm. 3.6.3] that $M \otimes_R F$ is acyclic. Thus, F is semi-flat by 5.6. □

In the terminology of [3] the equivalence of (i) and (iii) above says that a complex is categorically flat if and only if it is a direct limit of categorically projective complexes of finitely generated free modules.

7.4. Corollary. *Let $\alpha\colon F \to F'$ be a quasi-isomorphism between semi-flat R-complexes. For every bounded and degreewise finitely presented R-complex N the morphism $\mathrm{Hom}_R(N,\alpha)\colon \mathrm{Hom}_R(N,F) \to \mathrm{Hom}_R(N,F')$ is a quasi-isomorphism.*

Proof. By 2.1 and 6.1 the complex $\mathrm{Cone}\,\alpha$ is acyclic and semi-flat. The functor $\mathrm{Hom}_R(N,-)$ preserves filtered colimits by Theorem 4.5 and maps contractible complexes to contractible complexes. Now it follows from Theorem 7.3 and Proposition 3.4 that $\mathrm{Hom}_R(N,\mathrm{Cone}\,\alpha)$ is acyclic. Since the functor $\mathrm{Hom}_R(N,-)$ commutes with mapping cone, see 2.2, it follows that the complex $\mathrm{Cone}\,\mathrm{Hom}_R(N,\alpha)$ is acyclic, and thus $\mathrm{Hom}_R(N,\alpha)$ is a quasi-isomorphism by 2.1. □

The next corollary can be proved similarly; a different proof is given in [3, 6.4].

7.5. Corollary. *Let $\alpha\colon F \to F'$ be a quasi-isomorphism between semi-flat R-complexes. For every R°-complex M the morphism $M \otimes_R \alpha\colon M \otimes_R F \to M \otimes_R F'$ is a quasi-isomorphism.* □

The next result was proved by Neeman [17, Rmk. 2.15 and Thm. 8.6] in 2008. He notes, "I do not know an elementary proof, a proof which avoids homotopy categories". We show that it follows from a theorem of Benson and Goodearl [4, Thm. 2.5][1] from 2000, which asserts that if $0 \to F \to P \to F \to 0$ is a short exact sequence of R-modules with F flat and P projective, then F is projective as well. Notice that if R has finite finitistic projective dimension, then this assertion follows from Jensen's [12, Prop. 6], and if R has cardinality $\leqslant \aleph_n$ for some $n \in \mathbb{N}$, then it follows from a theorem of Gruson and Jensen [10, Thm. 7.10].

7.6. Proposition. *If P is an acyclic complex of projective R-modules such that the subcomplex $\mathrm{B}(P)$ consists of flat R-modules, then P is contractible.*

Proof. For each $v \in \mathbb{Z}$ the sequence $0 \to \mathrm{B}_v(P) \to P_v \to \mathrm{B}_{v-1}(P) \to 0$ is exact. The coproduct of all these exact sequences yields the exact sequence

$$0 \longrightarrow \coprod_{v\in\mathbb{Z}} \mathrm{B}_v(P) \longrightarrow \coprod_{v\in\mathbb{Z}} P_v \longrightarrow \coprod_{v\in\mathbb{Z}} \mathrm{B}_v(P) \longrightarrow 0.$$

By assumption, the module $\coprod_{v\in\mathbb{Z}} \mathrm{B}_v(P)$ is flat and $\coprod_{v\in\mathbb{Z}} P_v$ is projective, so it follows from [4, Thm. 2.5] that $\coprod_{v\in\mathbb{Z}} \mathrm{B}_v(P)$ is projective. Consequently, every module $\mathrm{B}_v(P)$ is projective, and therefore P is contractible. □

[1] Benson and Goodearl's proof uses only classical results from homological algebra.

Semi-projective complexes, just like semi-flat complexes, have a two-out-of-three property; see the proof of [3, Prop. 3.7].

7.7. Let $0 \to P' \to P \to P'' \to 0$ be an exact sequence of R-complexes. If P'' is semi-projective, then P' is semi-projective if and only if P is semi-projective.

7.8. Theorem. *A semi-flat complex of projective R-modules is semi-projective.*

Proof. Let $\pi\colon L \xrightarrow{\simeq} F$ be a semi-free resolution, see 5.3, and consider the mapping cone sequence $0 \to F \to \operatorname{Cone}\pi \to \Sigma L \to 0$; see 2.1. Since ΣL is semi-projective, see Example 5.5, it suffices by 7.7 to argue that $\operatorname{Cone}\pi$ is semi-projective. As the complexes F and ΣL are semi-flat and consist of projective modules, it follows from 6.1 and 2.1 that $\operatorname{Cone}\pi$ is an acyclic semi-flat complex of projective modules. Thus Theorem 7.3 and Proposition 7.6 apply to show that $\operatorname{Cone}\pi$ is contractible. It remains to note that every contractible complex of projective modules is semi-projective; this is immediate from 5.4. □

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