

THE STRUCTURE OF BALANCED BIG COHEN–MACAULAY MODULES OVER COHEN–MACAULAY RINGS

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(Received 24 September 2015; accepted 2 January 2016; first published online 10 June 2016)

Abstract. Over a Cohen–Macaulay (CM) local ring, we characterize those modules that can be obtained as a direct limit of finitely generated maximal CM modules. We point out two consequences of this characterization: (1) Every balanced big CM module, in the sense of Hochster, can be written as a direct limit of small CM modules. In analogy with Govorov and Lazard’s characterization of flat modules as direct limits of finitely generated free modules, one can view this as a “structure theorem” for balanced big CM modules. (2) Every finitely generated module has a pre-envelope with respect to the class of finitely generated maximal CM modules. This result is, in some sense, dual to the existence of maximal CM approximations, which has been proved by Auslander and Buchweitz.

2010 *Mathematics Subject Classification.* Primary 13C14. Secondary 13D05, 13D07.

1. Introduction. Let R be a local ring. Hochster [18] defines an R -module M to be *big Cohen–Macaulay* (big CM) if some system of parameters (s.o.p.) of R is an M -regular sequence. If every s.o.p. of R is an M -regular sequence, then M is called *balanced big CM*. The term “big” refers to the fact that M need not be finitely generated; and a finitely generated (balanced) big CM module is called a *small CM module*. It is conjectured by Hochster, see (2.1) in *loc. cit.*, that every local ring has a big CM module. This conjecture is still open, however, it has been settled affirmatively by Hochster [16, 17] in the equicharacteristic case, that is, for local rings containing a field. In fact, such a ring always has a balanced big CM module. Since it is known that a positive answer to Hochster’s conjecture will imply several other classic homological conjectures, the existence of (balanced) big CM modules remains to be an important question in commutative algebra.

Although this paper makes no direct contribution to Hochster’s conjecture, it is concerned with balanced big CM modules. We study such modules over a CM ring R with a dualizing module Ω . In this setting, the conjecture is of course trivially true since small, and hence also balanced big, CM modules abound (examples are R and Ω); and the natural question is therefore if one can describe all these big modules. As it turns out, all balanced big CM R -modules do share a common “structure”: they can always be built from small ones. The following result is proved in Section 4.

THEOREM A. *Every balanced big CM R -module is a direct limit of small CM R -modules.*

Example 4.7 exhibits a (non-balanced) big CM module that is *not* a direct limit of small CM modules.

A *finitely generated maximal CM module* is a module which is either small CM or zero; a convention used by Auslander and Buchweitz [2] and others. Theorem A is a consequence of the next result—also proved in Section 4—which gives two equivalent characterizations of the direct limit closure of the class of finitely generated maximal CM modules.

THEOREM B. *For every R -module M , the following conditions are equivalent.*

- (i) M is a direct limit of finitely generated maximal CM R -modules.
- (ii) Every system of parameters of R is a weak M -regular sequence.
- (iii) M is Gorenstein flat¹ viewed as a module over the trivial extension $R \ltimes \Omega$.

This theorem is analogous to a classic result, due to Govorov [14] and Lazard [22], which shows that the direct limit closure of the class of finitely generated free modules is precisely the class of flat modules. Following Hochster's terminology, it is reasonable to call a module *weak balanced big CM* if it satisfies condition (ii) in Theorem B. This class of modules is denoted by **wbbCM**, and it is a natural extension of the class of finitely generated maximal CM modules to the realm of all modules. Indeed, a finitely generated module belongs to **wbbCM** if and only if it is maximal CM.

In Section 5 we give applications of Theorem B in relative homological algebra. It follows from works of Ischebeck [20] and Auslander and Buchweitz [2] that the class **MCM** of finitely generated maximal CM R -modules is part of a complete hereditary cotorsion pair $(\mathbf{MCM}, \mathbf{MCM}^\perp)$ on the category of finitely generated R -modules². In particular, every finitely generated R -module has an **MCM**-precover and an \mathbf{MCM}^\perp -preenvelope. We show:

THEOREM C. *Every finitely generated R -module has an **MCM**-preenvelope.*

We also extend the cotorsion pair $(\mathbf{MCM}, \mathbf{MCM}^\perp)$ and the existence of **MCM**-preenvelopes to the realm of all—not necessarily finitely generated—modules:

THEOREM D. *On the category of all R -modules, $(\mathbf{wbbCM}, \mathbf{wbbCM}^\perp)$ is a perfect hereditary cotorsion pair, in particular, every R -module has a **wbbCM**-cover and a \mathbf{wbbCM}^\perp -envelope. Furthermore, every R -module has a **wbbCM**-preenvelope.*

As a consequence of Theorem D, we prove the existence of (non-weak) balanced big CM covers for certain types of modules; see Proposition 5.2 and Example 5.3.

2. Regular sequences, depth, and CM modules

Setup 2.1. Throughout, (R, \mathfrak{m}, k) is a commutative noetherian local CM ring with Krull dimension d . It is assumed that R has a dualizing (or canonical) module Ω .

Let (A, \mathfrak{n}, ℓ) be any commutative noetherian local ring and let M be any A -module. A sequence of elements $\bar{x} = x_1, \dots, x_n \in \mathfrak{n}$ is called a *weak M -regular sequence* if x_i is a non-zero-divisor on $M/(x_1, \dots, x_{i-1})M$ for every $1 \leq i \leq n$ (for $i = 1$ this means that x_1 is a non-zero-divisor on M). If, in addition, $(x_1, \dots, x_n)M \neq M$, then \bar{x} is an

¹ In the sense of Enochs, Jenda, and Torrecillas [10]; see also Definition 4.2.

² Actually, \mathbf{MCM}^\perp is the class of finitely generated R -modules with finite injective dimension; cf. Theorem 5.1.

M-regular sequence. If $M \neq 0$ is finitely generated, then by Nakayama’s lemma every weak *M*-regular sequence is automatically *M*-regular; this is not the case in general.

The *depth* of a finitely generated *A*-module $M \neq 0$, denoted by $\text{depth}_A M$, is the supremum of the lengths of all *M*-regular sequences (alternatively, the common length of all maximal *M*-regular sequences). This invariant can be computed homologically as follows:

$$\text{depth}_A M = \inf\{i \in \mathbb{Z} \mid \text{Ext}_A^i(\ell, M) \neq 0\}. \tag{1}$$

For an arbitrary *A*-module *M*, we *define* its *depth*³ by the formula (1), with the convention that $\inf \emptyset = \infty$. So, for example, the zero module has infinite *depth*⁴.

For a finitely generated *A*-module $M \neq 0$ one always has $\text{depth}_A M \leq \dim A$, and the following conditions are equivalent; see Eisenbud [8, Prop.-Def. 21.9].

- (i) $\text{depth}_A M = \dim A$.
- (ii) Every system of parameters of *A* is an *M*-regular sequence.
- (iii) Some system of parameters of *A* is an *M*-regular sequence.

A finitely generated module *M* that satisfies these equivalent conditions is called *small CM*. A *finitely generated maximal CM module* is a module which is either small CM or zero, and the category of all such modules is denoted by **MCM**. Unlike the category of small CM modules, the category **MCM** is additive and closed under direct summands. Some authors, such as Yoshino [31], use the simpler terminology “CM module” for what we have called a “maximal CM module”.

It is well known that for an arbitrary *A*-module *M*, the conditions (i)–(iii) above are no longer equivalent, and hence there is more than one way to extend the notion of “(maximal) CM” to the realm of non-finitely generated modules. The next definition is due to Hochster [18] (the term “balanced” appears in Sharp [27]).

DEFINITION 2.2. An *A*-module *M* is called (*balanced*) *big CM* if (every) some system of parameters of *A* is an *M*-regular sequence.

It is well known that a big CM module need not be balanced; cf. Example 4.7. As noted above, a finitely generated module is big CM if and only if it is balanced big CM.

LEMMA 2.3. *Let M be any (not necessarily finitely generated) R -module and assume that $\bar{x} = x_1, \dots, x_n$ is both an R -regular and a weak M -regular sequence. Then one has*

$$\text{Tor}_i^R(R/(\bar{x}), M) = 0 \quad \text{for all } i > 0.$$

Proof. By induction on *n*. For $n = 1$, we have a single element x_1 which is a non-zerodivisor on both *R* and *M*. The assertion follows from inspection of the long exact Tor-sequence that arises from application of $-\otimes_R M$ to the short exact sequence $0 \rightarrow R \xrightarrow{x_1} R \rightarrow R/(x_1) \rightarrow 0$.

³ Some authors refer to the number in (1) as the “Ext-depth” of *M*. If $nM \neq M$, then $\text{depth}_A M$ (i.e. the “Ext-depth” of *M*) is always an upper bound for the length of any *M*-regular sequence; see Strooker [29, Prop. 5.3.7(ii)]. However, if *M* is not finitely generated, then there does not necessarily exist an *M*-regular sequence of length $\text{depth}_A M$; see p. 91 in *loc. cit.* for a counterexample.

⁴ It is also possible for a non-zero module to have infinite *depth*; see Observation 4.8.

Next, we assume that $n > 1$. Consider the ring $\bar{R} = R/(x_1, \dots, x_{n-1})$ and the \bar{R} -module $\bar{M} = \bar{R} \otimes_R M = M/(x_1, \dots, x_{n-1})M$. By the induction hypothesis,

$$\text{Tor}_i^R(\bar{R}, M) = \text{Tor}_i^R(R/(x_1, \dots, x_{n-1}), M) = 0 \quad \text{for all } i > 0.$$

Thus, in the derived category over R , one has $\bar{R} \otimes_R^L M \cong \bar{R} \otimes_R M = \bar{M}$, and consequently

$$R/(x_1, \dots, x_n) \otimes_R^L M = \bar{R}/(\bar{x}_n) \otimes_R^L M \cong \bar{R}/(\bar{x}_n) \otimes_R^L (\bar{R} \otimes_R^L M) \cong \bar{R}/(\bar{x}_n) \otimes_R^L \bar{M},$$

where \bar{x}_n denotes the image of x_n in \bar{R} . Hence $\text{Tor}_i^R(R/(x_1, \dots, x_n), M) \cong \text{Tor}_i^{\bar{R}}(\bar{R}/(\bar{x}_n), \bar{M})$ for every $i > 0$. The latter Tor is zero; this follows from the induction start as $\bar{x}_n \in \bar{R}$ is a non-zero-divisor on both \bar{R} and \bar{M} . \square

PROPOSITION 2.4. *For every (not necessarily finitely generated) R -module M , the following conditions are equivalent:*

- (i) *Every system of parameters of R is a weak M -regular sequence.*
- (ii) *For every R -regular sequence \bar{x} one has $\text{Tor}_i^R(R/(\bar{x}), M) = 0$ for all $i > 0$.*
- (iii) *For every R -regular sequence \bar{x} one has $\text{Tor}_1^R(R/(\bar{x}), M) = 0$.*

Proof. (i) \implies (ii): Let \bar{x} be any R -regular sequence. As R is CM, \bar{x} is part of a s.o.p. of R ; see [4, Thm. 2.1.2(d)]. By the assumption (i), this s.o.p. is a weak M -regular sequence, and hence so is the subsequence \bar{x} . Lemma 2.3 gives the desired conclusion.

(ii) \implies (iii): Clear.

(iii) \implies (i): Let $\bar{x} = x_1, \dots, x_d$ be any s.o.p. of R . Since R is CM, the sequence \bar{x} is R -regular; see Section 2. Thus, for every $i = 1, \dots, d$ there is an exact sequence,

$$0 \longrightarrow R/(x_1, \dots, x_{i-1}) \xrightarrow{x_i} R/(x_1, \dots, x_{i-1}) \longrightarrow R/(x_1, \dots, x_i) \longrightarrow 0.$$

Application of the functor $- \otimes_R M$ to this sequence yields the long exact sequence

$$\text{Tor}_1^R(R/(x_1, \dots, x_i), M) \longrightarrow M/(x_1, \dots, x_{i-1})M \xrightarrow{x_i} M/(x_1, \dots, x_{i-1})M.$$

The sequence x_1, \dots, x_i is R -regular, as it is a subsequence of the R -regular sequence \bar{x} . Hence $\text{Tor}_1^R(R/(x_1, \dots, x_i), M) = 0$ by the assumption (iii), and thus x_i is a non-zero-divisor on $M/(x_1, \dots, x_{i-1})M$. Therefore \bar{x} is a weak M -regular sequence. \square

3. The trivial extension. Let A be any commutative ring and let C be any A -module. The *trivial extension* of A by C (also called the *idealization* of C in A) is the ring $A \times C$ whose underlying abelian group is $A \oplus C$ and where multiplication is given by

$$(a, c)(a', c') = (aa', ac' + a'c) \quad \text{for } (a, c), (a', c') \in A \oplus C.$$

We refer to Fossum, Griffith, and Reiten [12, §5] for basic properties of this construction. The ring homomorphisms $\varphi: A \rightarrow A \times C$ given by $a \mapsto (a, 0)$, and $\psi: A \times C \rightarrow A$ given by $(a, c) \mapsto a$ allow us to consider any A -module as an $(A \times C)$ -module, and vice versa, and we shall do so freely. Note that the composition $\psi\varphi$ is the identity on A .

LEMMA 3.1. *Let A be any commutative ring. For any A -module C and any set of elements x_1, \dots, x_n in A there is an isomorphism of rings,*

$$(A \times C)/((x_1, 0), \dots, (x_n, 0)) \cong A/(x_1, \dots, x_n) \times C/(x_1, \dots, x_n)C.$$

Proof. There is a surjective homomorphism $\varphi: A \times C \rightarrow A/(x_1, \dots, x_n) \times C/(x_1, \dots, x_n)C$ given by $(a, c) \mapsto ([a]_{(x_1, \dots, x_n)}, [c]_{(x_1, \dots, x_n)C})$. Clearly, we have $(x_i, 0) \in \text{Ker } \varphi$. Conversely, if $(a, c) \in \text{Ker } \varphi$, then $a = \sum_{i=1}^n x_i a_i$ and $c = \sum_{i=1}^n x_i c_i$ where $a_i \in A$ and $c_i \in C$. It follows that $(a, c) = \sum_{i=1}^n (x_i, 0)(a_i, c_i)$, so (a, c) is in the ideal $((x_1, 0), \dots, (x_n, 0))$ in $A \times C$. □

While the previous lemma was quite general, the next one is more specific.

LEMMA 3.2. *Let $C \neq 0$ be any finitely generated maximal CM R -module. If $x_1, \dots, x_n \in \mathfrak{m}$ is an R -regular sequence, then $(x_1, 0), \dots, (x_n, 0)$ is an $(R \times C)$ -regular sequence.*

Proof. Fix $i \in \{1, \dots, n\}$. As $C \neq 0$ is a maximal CM R -module and the sequence x_1, \dots, x_{i-1} is R -regular, $\bar{C} = C/(x_1, \dots, x_{i-1})C \neq 0$ is a maximal CM module over $\bar{R} = R/(x_1, \dots, x_{i-1})$. By assumption, x_i is a non-zero-divisor on \bar{R} . We must argue that $(x_i, 0)$ is a non-zero-divisor on $(R \times C)/((x_1, 0), \dots, (x_{i-1}, 0)) \cong \bar{R} \times \bar{C}$, where the isomorphism is by Lemma 3.1. Thus, let $(r, c) \in \bar{R} \times \bar{C}$ be any element such that $(x_i, 0)(r, c) = (x_i r, x_i c)$ is $(0, 0)$; that is, we have $x_i r = 0$ and $x_i c = 0$. By assumption, x_i is a non-zero-divisor on \bar{R} , and since the \bar{R} -module \bar{C} is maximal CM, the element x_i is also a non-zero-divisor on \bar{C} . Thus the equations $x_i r = 0$ and $x_i c = 0$ imply that $r = 0$ and $c = 0$, that is, $(r, c) = (0, 0)$ as desired. □

Let A be a commutative ring and let $x \in A$ be an element. Recall that the Koszul complex on x is the complex $K_{\bullet}^A(x) = 0 \rightarrow A \xrightarrow{x} A \rightarrow 0$ concentrated in homological degrees 0, 1. For a sequence $\bar{x} = x_1, \dots, x_n \in A$ the Koszul complex is $K_{\bullet}^A(\bar{x}) = K_{\bullet}^A(x_1) \otimes_A \dots \otimes_A K_{\bullet}^A(x_n)$.

LEMMA 3.3. *Let A be any commutative ring, let C be an A -module, and let $\bar{x} = x_1, \dots, x_n$ be a sequence of elements in A . Consider the elements $y_i = (x_i, 0)$ and the sequence $\bar{y} = y_1, \dots, y_n$ in $A \times C$. For every A -module M there is the following isomorphism of both A - and $(A \times C)$ -complexes,*

$$K_{\bullet}^{A \times C}(\bar{y}) \otimes_{A \times C} M \cong K_{\bullet}^A(\bar{x}) \otimes_A M.$$

Proof. It suffices to consider the case $n = 1$. We have

$$K_{\bullet}^{A \times C}(y_1) \otimes_{A \times C} M \cong 0 \rightarrow M \xrightarrow{y_1} M \rightarrow 0 \quad (M \text{ is viewed as an } (A \times C)\text{-module})$$

$$K_{\bullet}^A(x_1) \otimes_A M \cong 0 \rightarrow M \xrightarrow{x_1} M \rightarrow 0 \quad (M \text{ is viewed as an } A\text{-module})$$

By definition of the $(A \times C)$ -action on M , multiplication by the element $y_1 = (x_1, 0)$ on M is nothing but multiplication by x_1 on M . □

4. The direct limit closure of maximal CM modules. By a *filtered colimit* of maximal CM R -modules we mean the colimit of a functor F from a skeletally small filtered category \mathcal{J} to the category of R -modules such that $F(J)$ is maximal CM for every J in \mathcal{J} . We reserve the term *direct limit* for the special situation where \mathcal{J} is the filtered category associated to a *directed set*, i.e. a filtered preordered set.

REMARK 4.1. It follows from general principles that a module is a filtered colimit of maximal CM modules if and only if it is a direct limit of maximal CM modules; see Adámek and Rosický [1, Thm. 1.5]. Thus in Theorem B condition (i), one can freely replace “direct limit” with “filtered colimit”.

In Theorem B condition (iii), we encounter the notion of Gorenstein flat modules. These modules were defined by Enochs, Jenda, and Torrecillas [10] as follows:

DEFINITION 4.2. Let A be any commutative ring. An A -module M is called *Gorenstein flat* if there exists an exact sequence of flat A -modules, $\mathbb{S} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$, with the property that $E \otimes_A \mathbb{S}$ is exact for every injective A -module E , such that $M \cong \text{Im}(F_0 \rightarrow F_{-1})$.

We are now in a position to prove our main result: Theorem B from the Introduction.

Proof of Theorem B. (i) \implies (ii): Every finitely generated maximal CM R -module satisfies condition (ii); see Section 2. And Proposition 2.4 shows that the class of R -modules that satisfy condition (ii) is closed under direct limits.

(ii) \implies (iii): As Ω is a dualizing R -module, $R \times \Omega$ is a Gorenstein ring by [12, Thm. 5.6]. So at least the $(R \times \Omega)$ -module M has finite Gorenstein flat dimension; see [11, Cor. 2.4] or [5, Thm. 5.2.10]. It therefore follows from [5, Cor. 5.4.9] that this dimension, $\text{Gfd}_{R \times \Omega} M$, can be computed by the “Chouinard formula”:

$$\text{Gfd}_{R \times \Omega} M = \sup\{\text{depth}(R \times \Omega)_{\mathfrak{q}} - \text{depth}_{(R \times \Omega)_{\mathfrak{q}}} M_{\mathfrak{q}} \mid \mathfrak{q} \in \text{Spec}(R \times \Omega)\}.$$

Thus, to prove that M is Gorenstein flat over $R \times \Omega$, equivalently, that $\text{Gfd}_{R \times \Omega} M \leq 0$ (the Gorenstein flat dimension of the zero module is $-\infty$), we must argue that the inequality

$$\text{depth}(R \times \Omega)_{\mathfrak{q}} - \text{depth}_{(R \times \Omega)_{\mathfrak{q}}} M_{\mathfrak{q}} \leq 0 \tag{2}$$

holds for every prime ideal \mathfrak{q} in $R \times \Omega$. By [12, Lem. 5.1(i)] every such \mathfrak{q} has the form $\mathfrak{q} = \mathfrak{p} \times \Omega$ for a prime ideal \mathfrak{p} in R . The rings R and $R \times \Omega$ are CM, and hence so are their localizations $R_{\mathfrak{p}}$ and $(R \times \Omega)_{\mathfrak{q}}$. By Lemma 5.1(ii) in *loc. cit.* the rings $R_{\mathfrak{p}}$ and $(R \times \Omega)_{\mathfrak{q}}$ have the same Krull dimension, and this number we denote by e .

Recall that all maximal R -regular sequences contained in \mathfrak{p} have the same length; this number is called the *grade of \mathfrak{p} on R* and it is denoted by $\text{grade}_R(\mathfrak{p}, R)$. Since R is CM, we have $\text{grade}_R(\mathfrak{p}, R) = \text{depth } R_{\mathfrak{p}} = e$ by [4, Thm. 2.1.3(b)]. Now, let $\bar{x} = x_1, \dots, x_e \in \mathfrak{p}$ be a maximal R -regular sequence in \mathfrak{p} . Set $y_i = (x_i, 0)$ and $\bar{y} = y_1, \dots, y_e$. The sequence \bar{y} is evidently contained in $\mathfrak{q} = \mathfrak{p} \times \Omega$, and it is $(R \times \Omega)$ -regular by Lemma 3.2. Hence \bar{y} (or more precisely, the sequence $y_1/1, \dots, y_e/1$) is also $(R \times \Omega)_{\mathfrak{q}}$ -regular, see [4, Cor. 1.1.3(a)]. As noted above, the ring $(R \times \Omega)_{\mathfrak{q}}$ has depth (and Krull dimension) equal to e , and thus the $(R \times \Omega)_{\mathfrak{q}}$ -module $(R \times \Omega)_{\mathfrak{q}}/(\bar{y})_{\mathfrak{q}} \cong ((R \times \Omega)/(\bar{y}))_{\mathfrak{q}}$ has depth 0, which means that the maximal ideal $\mathfrak{q}_{\mathfrak{q}}$ in $(R \times \Omega)_{\mathfrak{q}}$ is an associated prime of this module. It follows that \mathfrak{q} is an associated prime ideal of the $(R \times \Omega)$ -module

$T := (R \times \Omega)/(\bar{y})$; see [24, Thm. 6.2]. Note that the $(R \times \Omega)$ -module $T = (R \times \Omega)/(\bar{y})$ has finite projective dimension (equal to e); this follows from [4, Exerc. 1.3.6] and the fact that \bar{y} is $(R \times \Omega)$ -regular. We are therefore in a position to apply [5, Lem. 5.3.5(b)], which gives an inequality,

$$\text{depth}(R \times \Omega)_{\mathfrak{q}} - \text{depth}_{(R \times \Omega)_{\mathfrak{q}}} M_{\mathfrak{q}} \leq \sup\{i \mid \text{Tor}_i^{R \times \Omega}((R \times \Omega)/(\bar{y}), M) \neq 0\}. \quad (3)$$

Since \bar{y} is an $(R \times \Omega)$ -regular sequence, the Koszul complex $\mathbf{K}_{\bullet}^{R \times \Omega}(\bar{y})$ is a projective resolution of the $(R \times \Omega)$ -module $(R \times \Omega)/(\bar{y})$; see [24, Thm. 16.5(i)]. Similarly, $\mathbf{K}_{\bullet}^R(\bar{x})$ is a projective resolution of the R -module $R/(\bar{x})$. This explains the first and last isomorphism below; the middle isomorphism follows from Lemma 3.3:

$$\begin{aligned} \text{Tor}_i^{R \times \Omega}((R \times \Omega)/(\bar{y}), M) &\cong H_i(\mathbf{K}_{\bullet}^{R \times \Omega}(\bar{y}) \otimes_{R \times \Omega} M) \\ &\cong H_i(\mathbf{K}_{\bullet}^R(\bar{x}) \otimes_R M) \\ &\cong \text{Tor}_i^R(R/(\bar{x}), M). \end{aligned}$$

The assumption (ii) and Proposition 2.4 shows that $\text{Tor}_i^R(R/(\bar{x}), M) = 0$ for all $i > 0$. This fact, combined with (3), gives the desired conclusion (2).

(iii) \implies (i): Recall from Section 2 that the category of finitely generated maximal CM R -modules is an additive category closed under direct summands. To prove (i) we apply Lenzing [23, Prop. 2.1] (see also Remark 4.1). That is, we must show that every homomorphism of R -modules $\varphi: N \rightarrow M$, where N is finitely generated, factors through a maximal CM R -module. If we view N and M as modules over $R \times \Omega$, then N is still finitely generated and M is Gorenstein flat by assumption (iii). As $R \times \Omega$ is Gorenstein, [9, Lem. 10.3.6] yields that φ , as a homomorphism of $(R \times \Omega)$ -modules, factors through a finitely generated Gorenstein projective $(R \times \Omega)$ -module G . By viewing the hereby obtained factorization $N \rightarrow G \rightarrow M$ of φ , in the category of $(R \times \Omega)$ -modules, through the ring homomorphism $R \rightarrow R \times \Omega$, we get a factorization of the original φ in the category of R -modules. Thus, it remains to argue that G is maximal CM over R , i.e. that $\text{depth}_R G = d$. By Iyengar and Sather–Wagstaff [21, Lem. 2.8] applied to the local ring homomorphism $R \rightarrow R \times \Omega$, we get that $\text{depth}_R G = \text{depth}_{R \times \Omega} G$. As G is Gorenstein projective over $R \times \Omega$, it is also maximal CM over $R \times \Omega$, see e.g. [9, Cor. 11.5.4], so $\text{depth}_{R \times \Omega} G = \dim(R \times \Omega) = d$. \square

In view of Definition 2.2 (due to Hochster), we suggest the following:

DEFINITION 4.3. Let A be a commutative noetherian local ring. An A -module M is said to be *weak balanced big CM* if every system of parameters of A is a weak M -regular sequence. The category of such A -modules is denoted by **wbbCM** (where the ring A is understood).

REMARK 4.4. Let (A, \mathfrak{n}, ℓ) be any commutative noetherian local ring. If an A -module M satisfies $\mathfrak{n}M \neq M$, then M is balanced big CM if (and only if) it is weak balanced big CM. Indeed, under the assumption $\mathfrak{n}M \neq M$, a sequence of elements in \mathfrak{n} is M -regular if (and only if) it is weak M -regular.

With this terminology, the equivalence of conditions (i) and (ii) in Theorem B can be expressed as follows: *Over a CM ring R with a dualizing module, a module is weak balanced big CM if and only if it is a direct limit of finitely generated maximal CM*

modules. In symbols, the result can be written as

$$\text{wbbCM} = \varinjlim(\text{MCM}). \tag{4}$$

QUESTION 4.5. Over a general commutative noetherian local ring (not assumed to be CM with a dualizing module), how are the classes wbbCM and MCM related?

EXAMPLE 4.6. As always in this paper, R denotes the ring from Setup 2.1.

- (a) It follows from Proposition 2.4 that every flat R -module is weak balanced big CM.
- (b) Recall that an R -module M is said to be *torsion-free* if every non-zero-divisor on R is also a non-zero divisor on M . If R has dimension $d = 1$, then a system of parameters of R is nothing but a non-zero-divisor on R , so in this case “weak balanced big CM” just means “torsion-free”.

Next we prove Theorem A from the Introduction.

Proof of Theorem A. Let M be a balanced big CM R -module. As M is, in particular, weak balanced big CM, it can by Theorem B be written as $M = \varinjlim_{i \in I} M_i$ for some direct system $\varphi_{ji}: M_i \rightarrow M_j$ (for $i, j \in I$ with $i \leq j$) of finitely generated maximal CM R -modules. *A priori*, some of the M_i ’s could be zero, and hence they are not necessarily small CM modules. However, since $M \neq 0$ the subset $I' = \{i \in I \mid M_i \neq 0\}$ is cofinal in I . (Otherwise there would exist $i_0 \in I$ such that $M_i = 0$ for all $i \geq i_0$, in which case the subset $J = \{i \in I \mid M_i = 0\}$ would be cofinal in I , and hence $0 = \varinjlim_{i \in J} M_i = \varinjlim_{i \in I} M_i = M$; which is a contradiction.) It follows that $M = \varinjlim_{i \in I'} M_i$, and hence M can also be written as a direct limit of non-zero finitely generated maximal CM (i.e. small CM) modules. □

The following example is due to Griffith [15, Rem. 3.3].

EXAMPLE 4.7. Let $R = k[[x, y]]$ be the ring of formal power series in two variables x, y with coefficients in a field k . It is a regular, and hence a CM, local ring of dimension $d = 2$. Set $E = E_R(R/(y))$ and $M = R \oplus E$. Multiplication by x is an automorphism on E since $x \notin (y)$, see e.g. [9, Thm. 3.3.8(1)], so x is a non-zero-divisor on M with $M/xM \cong R/(x)$. It follows that y is a non-zero-divisor on M/xM and that $M/(x, y)M \cong R/(x, y) \neq 0$. Hence the system of parameters x, y of R is an M -regular sequence, so M is a big CM R -module.

However, M is not a balanced big CM module since the sequence y, x is not M -regular. Indeed, multiplication by y is not a monomorphism on E (the entire submodule $R/(y)$ of E is mapped to zero), and hence y is a zerodivisor on M .

As M is not balanced big CM, and since $(x, y)M \neq M$, it follows from Remark 4.4 that M is not even weak balanced big CM. Theorem B now shows that M can not be written as a direct limit of finitely generated maximal CM R -modules.

OBSERVATION 4.8. Recall from Section 2 that a finitely generated R -module X is maximal CM if and only if $\text{depth}_R X \geq d$ (and equality holds if $X \neq 0$). As the functors $\text{Ext}'_R(k, -)$ commute with direct limits, it follows from the definition of depth and from Theorem B that for every R -module M , the following implication holds:

$$M \text{ is weak balanced big CM} \implies \text{depth}_R M \geq d.$$

The converse is not true, as the R -module M from Example 4.7 is not weak balanced big CM, but it does have $\text{depth}_R M = 2 = d$. Here is one way to see why:

Since E is injective one has $\text{Ext}_R^i(k, E) = 0$ for all $i > 0$. We also have $\text{Hom}_R(k, E) = 0$. Indeed, since $k = R/(x, y)$ there is an isomorphism $\text{Hom}_R(k, E) \cong \{e \in E \mid (x, y)e = 0\}$. And if $e \in E$ satisfies $(x, y)e = 0$ then, in particular, $xe = 0$ which implies that $e = 0$ since x is a non-zero-divisor on E . Thus $\text{depth}_R E = \infty$ and $\text{depth}_R M = \text{depth}_R(R \oplus E) = \text{depth } R = 2$.

A more explicit way to formulate these considerations are as follows. The R -module M from Example 4.7 has the property that $\text{Ext}_R^i(k, M) = 0$ for $i = 0, 1$; but M is not a direct limit of finitely generated R -modules with this property (of course, M is the direct limit of *some* direct system of finitely generated R -modules).

We mention another easy consequence of Theorem B.

PROPOSITION 4.9. *The following conditions are equivalent:*

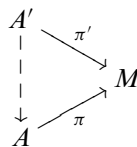
- (i) R is regular.
- (ii) Every weak balanced big CM R -module is flat.⁵

Proof. By Lazard [22, Thm. 1.2] an R -module is flat if and only if it is a direct limit of finitely generated projective R -modules. And by Cor. 1.4 in *loc. cit.* every finitely generated flat R -module is projective. By Theorem B an R -module is weak balanced big CM if and only if it is a direct limit of finitely generated maximal CM modules. And every finitely generated weak balanced big CM R -module is maximal CM. It follows that condition (ii) holds if and only if every finitely generated maximal CM module is projective, and this is tantamount to R being regular. □

In view of Example 4.6(b), we have the following special case of the implication (i) \Rightarrow (ii) in the result above: *Over a principal ideal domain every torsion-free R -module is flat.* This classic result can of course be found in most textbooks on homological algebra; see for example Rotman [26, Cor. 3.50].

5. Applications in relative homological algebra. In this final section, we give applications of Theorem B from the Introduction in relative homological algebra. Special attention is paid to preenvelopes and covers by maximal CM modules and by (weak) balanced big CM modules. First we recall some relevant notions.

Let \mathcal{A} be a class of objects in a category \mathcal{M} , and let M be an object in \mathcal{M} . Following Enochs and Jenda [9, Def. 5.1.1], a morphism $\pi: A \rightarrow M$ with $A \in \mathcal{A}$ is an \mathcal{A} -precover of M if every other morphism $\pi': A' \rightarrow M$ with $A' \in \mathcal{A}$ factors through π , as illustrated below.



⁵ Recall from Example 4.6(a) that a flat R -module is always weak balanced big CM.

An \mathcal{A} -precover $\pi : A \rightarrow M$ is called an \mathcal{A} -cover if every endomorphism φ of A that satisfies $\pi\varphi = \pi$ is an automorphism. The notion of \mathcal{A} -(pre)envelopes is categorically dual to the notion of \mathcal{A} -(pre)covers.⁶

Let \mathcal{M} be an abelian category (in our case, \mathcal{M} will be the category of all modules over some ring or the category of finitely generated modules over a noetherian ring). For a class of objects \mathcal{A} in \mathcal{M} , we define

$$\begin{aligned} {}^\perp\mathcal{A} &= \{M \in \mathcal{M} \mid \text{Ext}_{\mathcal{M}}^1(M, A) = 0 \text{ for all } A \in \mathcal{A}\}, \quad \text{and} \\ \mathcal{A}^\perp &= \{M \in \mathcal{M} \mid \text{Ext}_{\mathcal{M}}^1(A, M) = 0 \text{ for all } A \in \mathcal{A}\}. \end{aligned}$$

A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects in \mathcal{M} is called a *cotorsion pair* if $\mathcal{A}^\perp = \mathcal{B}$ and $\mathcal{A} = {}^\perp\mathcal{B}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called:

- *Hereditary* if the class \mathcal{A} is *resolving*; this means that \mathcal{A} contains all projective objects in \mathcal{M} and that \mathcal{A} is closed under extensions and kernels of epimorphisms. See [13, Def. 2.2.8(i) and Lem. 2.2.10] for further details.
 - *Complete* if the class \mathcal{A} has *enough projectives*; this means that for every $M \in \mathcal{M}$ there exists an exact sequence $0 \rightarrow B \rightarrow A \rightarrow M \rightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Equivalently, \mathcal{B} has *enough injectives*, that is for every $M \in \mathcal{M}$ there is an exact sequence $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. See [9, §7.1] and [13, Lem. 2.2.6].
- Note that if $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair, then every $M \in \mathcal{M}$ has a special \mathcal{A} -precover and a special \mathcal{B} -preenvelope in the sense of Xu [30, Props. 2.1.3 and 2.1.4]. That is, every $M \in \mathcal{M}$ has a surjective \mathcal{A} -precover, respectively, an injective \mathcal{B} -preenvelope, whose kernel, respectively, cokernel, is in \mathcal{A}^\perp , respectively, in ${}^\perp\mathcal{B}$.
- *Perfect* if every $M \in \mathcal{M}$ has an \mathcal{A} -cover and a \mathcal{B} -envelope. See [13, Def. 2.3.1].

Recall that MCM denotes the category of finitely generated maximal CM modules; see Section 2. We denote by \mathfrak{l} the category of finitely generated modules with finite injective dimension. As always, R is the ring from Setup 2.1.

A classic result of Ischebeck [20] (see also [4, Exerc. 3.1.24]) shows that $\text{Ext}_R^i(M, I) = 0$ for all $M \in \text{MCM}$, all $I \in \mathfrak{l}$, and all $i > 0$. In particular, one has $\text{MCM}^\perp \supseteq \mathfrak{l}$ and $\text{MCM} \subseteq {}^\perp\mathfrak{l}$. Combining this fact with the existence of *maximal CM approximations* and *hulls of finite injective dimension*, proved by Auslander and Buchweitz in [2, Thm. A], one gets:

THEOREM 5.1. *On the abelian category of finitely generated R -modules, the pair $(\text{MCM}, \mathfrak{l})$ is a complete hereditary cotorsion pair. In particular, every finitely generated R -module has a MCM -precover and an \mathfrak{l} -preenvelope.*

We now prove Theorems C and D from the Introduction.

Proof of Theorem C. By Crawley–Boevey [6, Thm. (4.2)], the assertion is equivalent to $\varinjlim(\text{MCM})$ being closed under products. By Theorem B, the class $\varinjlim(\text{MCM})$ is exactly the class of weak balanced big CM R -modules, and it follows from Proposition 2.4 and [9, Thm. 3.2.26] that this class is closed under products. □

⁶ Let $\mathcal{M} = \text{Mod}(A)$ be the category of (left) modules over a ring A . If $\mathcal{A} = \text{Prj}(A)$ is the class of projective A -modules, then an \mathcal{A} -cover is exactly the same as a *projective cover* in the sense of Bass [3]. If $\mathcal{A} = \text{Inj}(A)$ is the class of injective A -modules, then an \mathcal{A} -envelope is exactly the same as a *injective hull* in the sense of Eckmann and Schopf [7]. Proofs of these facts can be found in Xu [30, Thms. 1.2.11 and 1.2.12].

Proof of Theorem D. It follows from Proposition 2.4 that $wbbCM$ is resolving, and that it is closed under products, coproducts, and direct summands. Since $wbbCM = \varinjlim(MCM)$, see (4), Lenzing [23, Prop. 2.2] shows that $wbbCM$ is closed under pure submodules and pure quotient modules. Thus [19, Thm. 3.4] yields that $(wbbCM, wbbCM^\perp)$ is a perfect cotorsion pair, which is hereditary as $wbbCM$ is resolving. It follows from Rada and Saorin [25, Cor. 3.5(c)] that every R -module has a $wbbCM$ -preenvelope. \square

We end this paper by proving the existence of (non-weak) balanced big CM covers for certain types of modules.

PROPOSITION 5.2. *Let M be any R -module. If $mM \neq M$, then M has a surjective cover with respect to the class of balanced big CM R -modules.*

Proof. It follows from Theorem D that M has a cover with respect to the class of weak balanced big CM R -modules, say, $\pi: W \rightarrow M$ where W is in $wbbCM$. The homomorphism π must be surjective since every projective R -module belongs to $wbbCM$. As π is surjective, so is the induced map $W/mW \rightarrow M/mM$, and it follows that $mW \neq W$. Hence W is balanced big CM R -modules, see Remark 4.4, and π is the desired cover. \square

Proposition 5.2 is related to the main result in [28, Thm. 5.6] by Simon. This result asserts that, over a CM ring with a dualizing module, every complete module has a surjective cover w.r.t. the class \mathcal{X} of complete big CM modules including the zero module.

In Proposition 5.2, the assumption $mM \neq M$ is essential; here is a pathological example.

EXAMPLE 5.3. The zero module $M = 0$ does not have a cover with respect to the class of balanced big CM R -modules. Indeed, suppose that $W \rightarrow 0$ is such a cover. Since the zero endomorphism $0: W \rightarrow W$ makes the diagram

$$\begin{array}{ccc}
 W & & \\
 \downarrow 0 & \searrow & \\
 & & 0 \\
 \uparrow & \swarrow & \\
 W & &
 \end{array}$$

commutative, it follows from the definition of a cover that $0: W \rightarrow W$ is an automorphism. This means that $W = 0$, which is impossible as W is a big CM module.

ACKNOWLEDGEMENTS.

It is a pleasure to thank the anonymous referee for a careful reading of this manuscript and for thoughtful comments, suggestions, and corrections.

REFERENCES

1. J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189 (Cambridge University Press, Cambridge, 1994).

2. M. Auslander and R.-O. Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.), vol. 38 (Gauthier-Villars, Paris, 1989), 5–37, Colloque en l'honneur de Pierre Samuel (Orsay, 1987).
3. H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, *Trans. Amer. Math. Soc.* **95** (1960), 466–488.
4. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39 (Cambridge University Press, Cambridge, 1993).
5. L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747 (Springer-Verlag, Berlin, 2000).
6. W. Crawley-Boevey, Locally finitely presented additive categories, *Comm. Algebra* **22**(5) (1994), 1641–1674.
7. B. Eckmann and A. H. Schopf, Über injektive Moduln, *Arch. Math. (Basel)* **4** (1953), 75–78.
8. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150 (Springer-Verlag, New York, 1995).
9. E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30 (Walter de Gruyter & Co., Berlin, 2000).
10. E. E. Enochs, O. M. G. Jenda and B. Torrecillas, Gorenstein flat modules, *Nanjing Daxue Xuebao Shuxue Bannian Kan* **10**(3) (1993), 1–9.
11. E. E. Enochs, O. M. G. Jenda and J. Xu, Foxby duality and gorenstein injective and projective modules, *Trans. Amer. Math. Soc.* **348**(8) (1996), 3223–3234.
12. R. M. Fossum, P. A. Griffith and I. Reiten, *Trivial extensions of abelian categories* Homological algebra of trivial extensions of abelian categories with applications to ring theory, Lecture Notes in Mathematics, vol. 456 (Springer-Verlag, Berlin, 1975).
13. R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules*, de Gruyter Expositions in Mathematics, vol. 41 (Walter de Gruyter GmbH & Co. KG, Berlin, 2006).
14. V. E. Govorov, On flat modules, *Sibirsk. Mat. Ž.* **6** (1965), 300–304.
15. P. Griffith, A representation theorem for complete local rings, *J. Pure Appl. Algebra* **7**(3) (1976), 303–315.
16. M. Hochster, *Big Cohen-Macaulay modules and algebras and embeddability in rings of Witt vectors*, Conference on Commutative Algebra–1975, Queen's Univ., Kingston, Ont., 1975, Queen's Papers on Pure and Applied Math., No. 42 (Queen's Univ., Kingston, Ont., 1975), pp. 106–195.
17. M. Hochster, *Topics in the homological theory of modules over commutative rings*, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975, Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24–28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24.
18. M. Hochster, *Big and small Cohen-Macaulay modules*, Module theory (Proc. Special Session, Amer. Math. Soc., Univ. Washington, Seattle, Wash., 1977), Lecture Notes in Math., vol. 700 (Springer, Berlin, 1979), 119–142.
19. H. Holm and P. Jørgensen, Covers, precovers, and purity, *Illinois J. Math.* **52**(2) (2008), 691–703.
20. F. Ischebeck, Eine dualität zwischen den funktoren ext und tor, *J. Algebra* **11** (1969), 510–531.
21. S. Iyengar and S. Sather-Wagstaff, G-dimension over local homomorphisms. Applications to the Frobenius endomorphism, *Illinois J. Math.* **48**(1) (2004), 241–272.
22. D. Lazard, Autour de la platitude, *Bull. Soc. Math. France* **97** (1969), 81–128.
23. H. Lenzing, Homological transfer from finitely presented to infinite modules, Abelian group theory (Honolulu, Hawaii, 1983), Lecture Notes in Math., vol. 1006 (Springer, Berlin, 1983), 734–761.
24. H. Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Translated from the Japanese by M. Reid (Cambridge University Press, Cambridge, 1989).
25. J. Rada and M. Saorin, Rings characterized by (pre)envelopes and (pre)covers of their modules, *Comm. Algebra* **26**(3) (1998), 899–912.

26. J. J. Rotman, *An introduction to homological algebra*, second ed., (Universitext, Springer, New York, 2009). 2455920 (2009i:18011)
27. R. Y. Sharp, Cohen-Macaulay properties for balanced big Cohen-Macaulay modules, *Math. Proc. Cambridge Philos. Soc.* **90**(2) (1981), 229–238.
28. A.-M. Simon, *Approximations of complete modules by complete big Cohen-Macaulay modules over a Cohen-Macaulay local ring*, *Algebr. Represent. Theory* **12**(2–5) (2009), 385–400.
29. J. R. Strooker, *Homological questions in local algebra*, London Mathematical Society Lecture Note Series, vol. 145 (Cambridge University Press, Cambridge, 1990).
30. J. Xu, *Flat covers of modules*, Lecture Notes in Mathematics, vol. 1634 (Springer-Verlag, Berlin, 1996).
31. Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146 (Cambridge University Press, Cambridge, 1990).