Beyond totally reflexive modules and back
A survey on gorenstein dimensions

Lars Winther Christensen, Hans-Bjørn Foxby, and Henrik Holm

Abstract Starting from the notion of totally reflexive modules, we survey the theory of Gorenstein homological dimensions for modules over commutative rings. The account includes the theory's connections with relative homological algebra and with studies of local ring homomorphisms. It ends close to the starting point: with a characterization of Gorenstein rings in terms of total acyclicity of complexes.

Introduction

An important motivation for the study of homological dimensions dates back to 1956, when Auslander and Buchsbaum [7] and Serre [98] proved:

Theorem A. Let R be a commutative Noetherian local ring with residue field k. Then the following conditions are equivalent.

(*i*) R is regular.
(*ii*) k has finite projective dimension.
(*iii*) Every R-module has finite projective dimension.

This result opened to the solution of two long-standing conjectures of Krull. Moreover, it introduced the theme that finiteness of a homological dimension for all

L. W. Christensen

Department of Mathematics and Statistics, Texas Tech University, Mail Stop 1042, Lubbock, TX 79409, USA, e-mail: `lars.w.christensen@ttu.edu`

H.-B. Foxby

Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, DK-2100 København Ø, Denmark, e-mail: `foxby@math.ku.dk`

H. Holm

Department of Basic Sciences and Environment, University of Copenhagen, Thorvaldsensvej 40, DK-1871 Frederiksberg C, Denmark, e-mail: `hholm@life.ku.dk`

modules characterizes rings with special properties. Later work has shown that modules of finite projective dimension over a general ring share many properties with modules over a regular ring. This is an incitement to study homological dimensions of individual modules.

In line with these ideas, Auslander and Bridger [6] introduced in 1969 the G-dimension. It is a homological dimension for finitely generated modules over a Noetherian ring, and it gives a characterization of Gorenstein local rings (Theorem 1.27), which is similar to Theorem A. Indeed, R is Gorenstein if k has finite G-dimension, and only if every finitely generated R-module has finite G-dimension.

In the early 1990s, the G-dimension was extended beyond the realm of finitely generated modules over a Noetherian ring. This was done by Enochs and Jenda who introduced the notion of Gorenstein projective modules [41]. With the Gorenstein projective dimension at hand, a perfect parallel to Theorem A becomes available (Theorem 2.19). Subsequent work has shown that modules of finite Gorenstein projective dimension over a general ring share many properties with modules over a Gorenstein ring.

Classical homological algebra as precedent

The notions of injective dimension and flat dimension for modules also have Gorenstein counterparts. It was Enochs and Jenda who introduced Gorenstein injective modules [41] and, in collaboration with Torrecillas, Gorenstein flat modules [47]. The study of Gorenstein dimensions is often called *Gorenstein homological algebra*; it has taken directions from the following:

Meta Question. Given a result in classical homological algebra, does it have a counterpart in Gorenstein homological algebra?

To make this concrete, we review some classical results on homological dimensions and point to their Gorenstein counterparts within the main text. In the balance of this introduction, R is assumed to be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

DEPTH AND FINITELY GENERATED MODULES

The projective dimension of a finitely generated R-module is closely related to its depth. This is captured by the *Auslander–Buchsbaum Formula* [8]:

Theorem B. For every finitely generated R-module M of finite projective dimension there is an equality $\mathrm{pd}_R M = \mathrm{depth}\, R - \mathrm{depth}_R M$.

The Gorenstein counterpart (Theorem 1.25) actually strengthens the classical result; this is a recurring theme in Gorenstein homological algebra.

The injective dimension of a non-zero finitely generated R-module is either infinite or it takes a fixed value:

Theorem C. For every non-zero finitely generated R-module M of finite injective dimension there is an equality $\operatorname{id}_R M = \operatorname{depth} R$.

This result of Bass [20] has its Gorenstein counterpart in Theorem 3.24.

CHARACTERIZATIONS OF COHEN–MACAULAY RINGS

Existence of special modules of finite homological dimension characterizes Cohen–Macaulay rings. The equivalence of (i) and (iii) in the theorem below is still referred to as *Bass' conjecture,* even though it was settled more than 20 years ago. Indeed, Peskine and Szpiro proved in [86] that it follows from the New Intersection Theorem, which they proved *ibid.* for equicharacteristic rings. In 1987, Roberts [87] settled the New Intersection Theorem completely.

Theorem D. The following conditions on R are equivalent.

(i) R is Cohen–Macaulay.
(ii) There is a non-zero R-module of finite length and finite projective dimension.
(iii) There is a non-zero finitely generated R-module of finite injective dimension.

A Gorenstein counterpart to this characterization is yet to be established; see Questions 1.31 and 3.26.

Gorenstein rings are also characterized by existence of special modules of finite homological dimension. The equivalence of (i) and (ii) below is due to Peskine and Szpiro [86]. The equivalence of (i) and (iii) was conjectured by Vasconcelos [108] and proved by Foxby [56]. The Gorenstein counterparts are given in Theorems 3.22 and 4.28; see also Question 4.29.

Theorem E. The following conditions on R are equivalent.

(i) R is Gorenstein.
(ii) There is a non-zero cyclic R-module of finite injective dimension.
(iii) There is a non-zero finitely generated R-module of finite projective dimension and finite injective dimension.

LOCAL RING HOMOMORPHISMS

The Frobenius endomorphism detects regularity of a local ring of positive prime characteristic. The next theorem collects results of Avramov, Iyengar and Miller [17], Kunz [82], and Rodicio [89]. The counterparts in Gorenstein homological algebra to these results are given in Theorems 6.4 and 6.5.

Theorem F. Let R be of positive prime characteristic, and let ϕ denote its Frobenius endomorphism. Then the following conditions are equivalent.

(i) R is regular.
(ii) R has finite flat dimension as an R-module via ϕ^n for some $n \geqslant 1$.
(iii) R is flat as an R-module via ϕ^n for every $n \geqslant 1$.
(iv) R has finite injective dimension as an R-module via ϕ^n for some $n \geqslant 1$.
(v) R has injective dimension equal to $\dim R$ as an R-module via ϕ^n for every $n \geqslant 1$.

Let $(S, \mathfrak{n})$ be yet a commutative Noetherian local ring. A ring homomorphism $\varphi \colon R \to S$ is called *local* if there is an inclusion $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. A classical chapter of local algebra, initiated by Grothendieck, studies transfer of ring theoretic properties along such homomorphisms. If φ is flat, then it is called *Cohen–Macaulay* or *Gorenstein* if its closed fiber $S/\mathfrak{m}S$ is, respectively, a Cohen–Macaulay ring or a Gorenstein ring. These definitions have been extended to homomorphisms of finite flat dimension. The theorem below collects results of Avramov and Foxby from [12] and [14]; the Gorenstein counterparts are given in Theorems 7.8 and 7.11.

Theorem G. Let $\varphi \colon R \to S$ be a local homomorphism and assume that S has finite flat dimension as an R-module via φ. Then the following hold:

(a) S is Cohen–Macaulay if and only if R and φ are Cohen–Macaulay.
(b) S is Gorenstein if and only if R and φ are Gorenstein.

VANISHING OF COHOMOLOGY

The projective dimension of a module M is at most n if and only if the absolute cohomology functor $\mathrm{Ext}^{n+1}(M, -)$ vanishes. Similarly (Theorem 5.25), M has Gorenstein projective dimension at most n if and only if the relative cohomology functor $\mathrm{Ext}_{\mathrm{GP}}^{n+1}(M, -)$ vanishes. Unfortunately, the similarity between the two situations does not run too deep. We give a couple of examples:

The absolute Ext is balanced, that is, it can be computed from a projective resolution of M or from an injective resolution of the second argument. In general, however, the only known way to compute the relative Ext is from a (so-called) proper Gorenstein projective resolution of M.

Secondly, if M is finitely generated, then the absolute Ext commutes with localization, but the relative Ext is not known to do so, unless M has finite Gorenstein projective dimension.

Such considerations motivate the search for an alternative characterization of modules of finite Gorenstein projective dimension, and this has been a driving force behind much research on Gorenstein dimensions within the past 15 years. What follows is a brief review.

EQUIVALENCE OF MODULE CATEGORIES

For a finitely generated R-module, Foxby [57] gave a "resolution-free" criterion for finiteness of the Gorenstein projective dimension; that is, one that does not

involve construction of a Gorenstein projective resolution. This result from 1994 is Theorem 8.2. In 1996, Enochs, Jenda, and Xu [49] extended Foxby's criterion to non-finitely generated R-modules, provided that R is Cohen–Macaulay with a dualizing module D. Their work is related to a 1972 generalization by Foxby [54] of a theorem of Sharp [100]. Foxby's version reads:

Theorem H. Let R be Cohen–Macaulay with a dualizing module D. Then the horizontal arrows below are equivalences of categories of R-modules.

$$
\begin{array}{ccc}
\mathcal{A}_D(R) & \underset{\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R-}{\rightleftarrows}} & \mathcal{B}_D(R) \\[2pt]
\big\uparrow & & \big\uparrow \\[2pt]
\{A \mid \mathrm{pd}_R A \text{ is finite}\} & \underset{\mathrm{Hom}_R(D,-)}{\overset{D\otimes_R-}{\rightleftarrows}} & \{B \mid \mathrm{id}_R B \text{ is finite}\}
\end{array}
$$

Here, $\mathcal{A}_D(R)$ is the *Auslander class* (Definition 9.1) with respect to D and $\mathcal{B}_D(R)$ is the *Bass class* (Definition 9.4). What Enochs, Jenda, and Xu prove in [49] is that the R-modules of finite Gorenstein projective dimension are exactly those in $\mathcal{A}_D(R)$, and the modules in $\mathcal{B}_D(R)$ are exactly those of finite Gorenstein injective dimension. Thus, the upper level equivalence in Theorem H is the Gorenstein counterpart of the lower level equivalence.

A commutative Noetherian ring has a dualizing complex $\boldsymbol{D}$ if and only if it is a homomorphic image of a Gorenstein ring of finite Krull dimension; see Kawasaki [79]. For such rings, a result similar to Theorem H was proved by Avramov and Foxby [13] in 1997. An interpretation in terms of Gorenstein dimensions (Theorems 9.2 and 9.5) of the objects in $\mathcal{A}_{\boldsymbol{D}}(R)$ and $\mathcal{B}_{\boldsymbol{D}}(R)$ was established by Christensen, Frankild, and Holm [31] in 2006. Testimony to the utility of these results is the frequent occurrence—e.g., in Theorems 3.16, 4.13, 4.25, 4.30, 6.5, 6.8, 7.3, and 7.7—of the assumption that the ground ring is a homomorphic image of a Gorenstein ring of finite Krull dimension. Recall that every complete local ring satisfies this assumption.

Recent results, Theorems 2.20 and 4.27, by Esmkhani and Tousi [52] and Theorem 9.11 by Christensen and Sather-Wagstaff [35] combine with Theorems 9.2 and 9.5 to provide resolution-free criteria for finiteness of Gorenstein dimensions over general local rings; see Remarks 9.3 and 9.12.

Scope and organization

A survey of this modest length is a portrait painted with broad pen strokes. Inevitably, many details are omitted, and some generality has been traded in for simplicity. We have chosen to focus on modules over commutative, and often Noetherian, rings. Much of Gorenstein homological algebra, though, works flaw-lessly over non-commutative rings, and there are statements in this survey about

Noetherian rings that remain valid for coherent rings. Furthermore, most statements about modules remain valid for complexes of modules. The reader will have to consult the references to qualify these claims.

In most sections, the opening paragraph introduces the main references on the topic. We strive to cite the strongest results available and, outside of this introduction, we do not attempt to trace the history of individual results. In notes, placed at the end of sections, we give pointers to the literature on directions of research—often new ones—that are not included in the survey. Even within the scope of this paper, there are open ends, and more than a dozen questions and problems are found throughout the text.

From this point on, R denotes a commutative ring. Any extra assumptions on R are explicitly stated. We say that R is *local* if it is Noetherian and has a unique maximal ideal. We use the shorthand $(R, \mathfrak{m}, k)$ for a local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

Acknowledgments

1 G-dimension of finitely generated modules

The topic of this section is Auslander and Bridger's notion of G-dimension for finitely generated modules over a Noetherian ring. The notes [5] from a seminar by Auslander outline the theory of G-dimension over commutative Noetherian rings. In [6] Auslander and Bridger treat the G-dimension within a more abstract framework. Later expositions are given by Christensen [28] and by Maşek [84].

A *complex M* of modules is (in homological notation) an infinite sequence of homomorphisms of R-modules

$$M = \cdots \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_i \partial_{i+1} = 0$ for every i in $\mathbb{Z}$. The ith *homology module* of $\boldsymbol{M}$ is $\mathrm{H}_i(\boldsymbol{M}) = \mathrm{Ker}\,\partial_i/\mathrm{Im}\,\partial_{i+1}$. We call $\boldsymbol{M}$ *acyclic* if $\mathrm{H}_i(\boldsymbol{M}) = 0$ for all $i \in \mathbb{Z}$.

Lemma 1.1. *Let $\boldsymbol{L}$ be an acyclic complex of finitely generated projective R-modules. The following conditions on $\boldsymbol{L}$ are equivalent.*

(i)   *The complex $\mathrm{Hom}_R(\boldsymbol{L}, R)$ is acyclic.*
(ii) *The complex $\mathrm{Hom}_R(\boldsymbol{L}, F)$ is acyclic for every flat R-module F.*
(iii) *The complex $E \otimes_R \boldsymbol{L}$ is acyclic for every injective R-module E.*

Proof. The Lemma is proved in [28], but here is a cleaner argument: Let F be a flat module and E be an injective module. As $\boldsymbol{L}$ consists of finitely generated projective modules, there is an isomorphism of complexes

$$\mathrm{Hom}_R(\mathrm{Hom}_R(\boldsymbol{L}, F), E) \cong \mathrm{Hom}_R(F, E) \otimes_R \boldsymbol{L}.$$

It follows from this isomorphism, applied to $F = R$, that (i) implies (iii). Applied to a faithfully injective module E, it shows that (iii) implies (ii), as $\mathrm{Hom}_R(F, E)$ is an injective module. It is evident that (ii) implies (i).   □

The following nomenclature is due to Avramov and Martsinkovsky [19]; Lemma 1.6 clarifies the rationale behind it.

Definition 1.2. A complex $\boldsymbol{L}$ that satisfies the conditions in Lemma 1.1 is called *totally acyclic*. An R-module M is called *totally reflexive* if there exists a totally acyclic complex $\boldsymbol{L}$ such that M is isomorphic to $\mathrm{Coker}(L_1 \to L_0)$.

Note that every finitely generated projective module L is totally reflexive; indeed, the complex $0 \to L \xrightarrow{=} L \to 0$, with L in homological degrees 0 and -1, is totally acyclic.

Example 1.3. If there exist elements x and y in R such that $\mathrm{Ann}_R(x) = (y)$ and $\mathrm{Ann}_R(y) = (x)$, then the complex

$$\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} \cdots$$

is totally acyclic. Thus, (x) and (y) are totally reflexive R-modules. For instance, if X and Y are non-zero non-units in an integral domain D, then their residue classes x and y in $R = D/(XY)$ generate totally reflexive R-modules.

An elementary construction of rings of this kind—Example 1.4 below—shows that non-projective totally reflexive modules may exist over a variety of rings; see also Problem 1.30.

Example 1.4. Let S be a commutative ring, and let $m > 1$ be an integer. Set $R = S[X]/(X^m)$, and denote by x the residue class of X in R. Then for every integer n between 1 and $m - 1$, the module (x^n) is totally reflexive.

From Lemma 1.1 it is straightforward to deduce:

Proposition 1.5. *Let S be an R-algebra of finite flat dimension. For every totally reflexive R-module G, the module $S \otimes_R G$ is totally reflexive over S.*

Proposition 1.5 applies to $S = R/(\boldsymbol{x})$, where $\boldsymbol{x}$ is an R-regular sequence. If $(R, \mathfrak{m})$ is local, then it also applies to the $\mathfrak{m}$-adic completion $S = \widehat{R}$.

Noetherian rings

Recall that a finitely generated R-module M is called *reflexive* if the canonical map from M to $\operatorname{Hom}_R(\operatorname{Hom}_R(M,R),R)$ is an isomorphism. The following characterization of totally reflexive modules goes back to [6, 4.11].

Lemma 1.6. *Let R be Noetherian. A finitely generated R-module G is totally reflexive if and only if it is reflexive and for every $i \geqslant 1$ one has*

$$\operatorname{Ext}_R^i(G,R) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(G,R),R).$$

Definition 1.7. An (augmented) *G-resolution* of a finitely generated module M is an exact sequence $\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$, where each module G_i is totally reflexive.

Note that if R is Noetherian, then every finitely generated R-module has a G-resolution, indeed it has a resolution by finitely generated free modules.

Definition 1.8. Let R be Noetherian. For a finitely generated R-module $M \neq 0$ the *G-dimension*, denoted by G-dim$_R M$, is the least integer $n \geqslant 0$ such that there exists a G-resolution of M with $G_i = 0$ for all $i > n$. If no such n exists, then G-dim$_R M$ is infinite. By convention, set G-dim$_R 0 = -\infty$.

The 'G' in the definition above is short for Gorenstein.

In [6, Chap. 3] one finds the next theorem and its corollary; see also [28, 1.2.7].

Theorem 1.9. *Let R be Noetherian and M be a finitely generated R-module of finite G-dimension. For every $n \geqslant 0$ the next conditions are equivalent.*

(i)   G-dim$_R M \leqslant n$.
(ii) $\operatorname{Ext}_R^i(M,R) = 0$ *for all $i > n$.*
(iii) $\operatorname{Ext}_R^i(M,N) = 0$ *for all $i > n$ and all R-modules N with fd$_R N$ finite.*
(iv)   *In every augmented G-resolution*

$$\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$$

the module $\operatorname{Coker}(G_{n+1} \to G_n)$ is totally reflexive.

Corollary 1.10. *Let R be Noetherian. For every finitely generated R-module M of finite G-dimension there is an equality*

$$\text{G-dim}_R M = \sup\{\, i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(M,R) \neq 0 \,\}.$$

Remark 1.11. Examples due to Jorgensen and Şega [77] show that in Corollary 1.10 one cannot avoid the *a priori* condition that G-dim$_R M$ is finite.

Remark 1.12. For a module M as in Corollary 1.10, the small finitistic projective dimension of R is an upper bound for G-dim$_R M$; cf. Christensen and Iyengar [33, 3.1(a)].

A standard argument, see [6, 3.16] or [19, 3.4], yields:

Proposition 1.13. *Let R be Noetherian. If any two of the modules in an exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated R-modules have finite G-dimension, then so has the third.*

The following quantitative comparison establishes the G-dimension as a refinement of the projective dimension for finitely generated modules. It is easily deduced from Corollary 1.10; see [28, 1.2.10].

Proposition 1.14. *Let R be Noetherian. For every finitely generated R-module M one has $\text{G-dim}_R M \leqslant \text{pd}_R M$, and equality holds if $\text{pd}_R M$ is finite.*

By [6, 4.15] the G-dimension of a module can be measured locally:

Proposition 1.15. *Let R be Noetherian. For every finitely generated R-module M there is an equality $\text{G-dim}_R M = \sup\{\,\text{G-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}\,R\,\}$.*

For the projective dimension even more is known: Bass and Murthy [21, 4.5] prove that if a finitely generated module over a Noetherian ring has finite projective dimension locally, then it has finite projective dimension globally—even if the ring has infinite Krull dimension. A Gorenstein counterpart has recently been established by Avramov, Iyengar, and Lipman [18, 6.3.4].

Theorem 1.16. *Let R be Noetherian and let M be a finitely generated R-module. If $\text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m}$ is finite for every maximal ideal $\mathfrak{m}$ in R, then $\text{G-dim}_R M$ is finite.*

Recall that a local ring is called *Gorenstein* if it has finite self-injective dimension. A Noetherian ring is Gorenstein if its localization at each prime ideal is a Gorenstein local ring, that is, $\text{id}_{R_\mathfrak{p}} R_\mathfrak{p}$ is finite for every prime ideal $\mathfrak{p}$ in R. Consequently, the self-injective dimension of a Gorenstein ring equals its Krull dimension; that is $\text{id}_R R = \dim R$. The next result follows from [6, 4.20] in combination with Proposition 1.15.

Theorem 1.17. *Let R be Noetherian and $n \geqslant 0$ be an integer. Then R is Gorenstein with $\dim R \leqslant n$ if and only if one has $\text{G-dim}_R M \leqslant n$ for every finitely generated R-module M.*

A corollary to Theorem 1.16 was established by Goto [63] already in 1982; it asserts that also Gorenstein rings of infinite Krull dimension are characterized by finiteness of G-dimension.

Theorem 1.18. *Let R be Noetherian. Then R is Gorenstein if and only if every finitely generated R-module has finite G-dimension.*

Recall that the *grade* of a finitely generated module M over a Noetherian ring R can be defined as follows:

$$\text{grade}_R M = \inf\{\, i \in \mathbb{Z} \mid \text{Ext}^i_R(M, R) \neq 0 \,\}.$$

Foxby [55] makes the following:

Definition 1.19. Let R be Noetherian. A finitely generated R-module M is called *quasi-perfect* if it has finite G-dimension equal to grade$_R M$.

The next theorem applies to $S = R/(\boldsymbol{x})$, where $\boldsymbol{x}$ is an R-regular sequence. Special (local) cases of the theorem are due to Golod [62] and to Avramov and Foxby [13, 7.11]. Christensen's proof [29, 6.5] establishes the general case.

Theorem 1.20. *Let R be Noetherian and S be a commutative Noetherian module-finite R-algebra. If S is a quasi-perfect R-module of grade g such that $\mathrm{Ext}_R^g(S,R) \cong S$, then the next equality holds for every finitely generated S-module N,*

$$\mathrm{G\text{-}dim}_R N = \mathrm{G\text{-}dim}_S N + \mathrm{G\text{-}dim}_R S.$$

Note that an S-module has finite G-dimension over R if and only if it has finite G-dimension over S; see also Theorem 7.7. The next question is raised in [13]; it asks if the assumption of quasi-perfectness in Theorem 1.20 is necessary.

Question 1.21. Let R be Noetherian, let S be a commutative Noetherian module-finite R-algebra, and let N be a finitely generated S-module. If G-dim$_S N$ and G-dim$_R S$ are finite, is then G-dim$_R N$ finite?

This is known as the *Transitivity Question.* By [13, 4.7] and [29, 3.15 and 6.5] it has an affirmative answer if pd$_S N$ is finite; see also Theorem 7.4.

Local rings

Before we proceed with results on G-dimension of modules over local rings, we make a qualitative comparison to the projective dimension. Theorem 1.20 reveals a remarkable property of the G-dimension, one that has almost no counterpart for the projective dimension. Here is an example:

Example 1.22. Let $(R, \mathfrak{m}, k)$ be local of positive depth. Pick a regular element x in $\mathfrak{m}$ and set $S = R/(x)$. Then one has grade$_R S = 1 = \mathrm{pd}_R S$ and $\mathrm{Ext}_R^1(S,R) \cong S$, but pd$_S N$ is infinite for every S-module N such that x is in $\mathfrak{m} \, \mathrm{Ann}_R N$; see Shamash [99, Section 3]. In particular, if R is regular and x is in $\mathfrak{m}^2$, then S is not regular, so pd$_S k$ is infinite while pd$_R k$ is finite; see Theorem A.

If G is a totally reflexive R-module, then every R-regular element is G-regular. A strong converse holds for modules of finite projective dimension; it is (still) referred to as *Auslander's zero-divisor conjecture:* let R be local and $M \neq 0$ be a finitely generated R-module with pd$_R M$ finite. Then every M-regular element is R-regular; for a proof see Roberts [88, 6.2.3]. An instance of Example 1.3 shows that one can *not* relax the condition on M to finite G-dimension:

Example 1.23. Let k be a field and consider the local ring $R = k[\![X,Y]\!]/(XY)$. Then the residue class x of X generates a totally reflexive module. The element x is (x)-regular but nevertheless a zero-divisor in R.

While a tensor product of projective modules is projective, the next example shows that totally reflexive modules do not have an analogous property.

Example 1.24. Let R be as in Example 1.23. The R-modules (x) and (y) are totally reflexive, but $(x) \otimes_R (y) \cong k$ is not. Indeed, k is not a submodule of a free R-module.

The next result [6, 4.13] is parallel to Theorem B in the Introduction; it is known as the *Auslander–Bridger Formula.*

Theorem 1.25. *Let R be local. For every finitely generated R-module M of finite G-dimension there is an equality*

$$\text{G-dim}_R M = \text{depth}\,R - \text{depth}_R M.$$

In [84] Maşek corrects the proof of [6, 4.13]. Proofs can also be found in [5] and [28].

By Lemma 1.6 the G-dimension is preserved under completion:

Proposition 1.26. *Let R be local. For every finitely generated R-module M there is an equality*
$$\text{G-dim}_R M = \text{G-dim}_{\widehat{R}}(\widehat{R} \otimes_R M).$$

The following main result from [5, Section 3.2] is akin to Theorem A, but it differs in that it only deals with finitely generated modules.

Theorem 1.27. *For a local ring $(R, \mathfrak{m}, k)$ the next conditions are equivalent.*

(*i*) R is Gorenstein.
(*ii*) G-dim$_R k$ is finite.
(*iii*) G-dim$_R M$ is finite for every finitely generated R-module M.

It follows that non-projective totally reflexive modules exist over any non-regular Gorenstein local ring. On the other hand, Example 1.4 shows that existence of such modules does not identify the ground ring as a member of one of the standard classes, say, Cohen–Macaulay rings.

A theorem of Christensen, Piepmeyer, Striuli, and Takahashi [34, 4.3] shows that fewness of totally reflexive modules comes in two distinct flavors:

Theorem 1.28. *Let R be local. If there are only finitely many indecomposable totally reflexive R-modules, up to isomorphism, then R is Gorenstein or every totally reflexive R-module is free.*

This dichotomy brings two problems to light:

Problem 1.29. Let R be a local ring that is not Gorenstein and assume that there exists a non-free totally reflexive R-module. Find constructions that produce infinite families of non-isomorphic indecomposable totally reflexive modules.

Problem 1.30. Describe the local rings over which every totally reflexive module is free.

While the first problem is posed in [34], the second one was already raised by Avramov and Martsinkovsky [19], and it is proved *ibid.* that over a Golod local ring that is not Gorenstein, every totally reflexive module is free. Another partial answer to Problem 1.30 is obtained by Yoshino [116], and by Christensen and Veliche [36]. The problem is also studied by Takahashi in [105].

Finally, Theorem D in the Introduction motivates:

Question 1.31. Let R be a local ring. If there exists a non-zero R-module of finite length and finite G-dimension, is then R Cohen–Macaulay?

A partial answer to this question is obtained by Takahashi [101, 2.3].

Notes

A topic that was only treated briefly above is constructions of totally reflexive modules. Such constructions are found in [16] by Avramov, Gasharov and Peeva, in work of Takahashi and Watanabe [106], and in Yoshino's [116].

Hummel and Marley [73] extend the notion of G-dimension to finitely presented modules over coherent rings and use it to define and study coherent Gorenstein rings.

Gerko [61, Section 2] studies a dimension—the PCI-dimension or CI$_*$-dimension—based on a subclass of the totally reflexive modules. Golod [62] studies a generalized notion of G-dimension: the G$_C$-dimension, based on total reflexivity with respect to a semidualizing module C. These studies are continued by, among others, Gerko [61, Section 1] and Salarian, Sather-Wagstaff, and Yassemi [91]; see also the notes in Section 8.

An approach to homological dimensions that is not treated in this survey is based on so-called quasi-deformations. Several authors—among them Avramov, Gasharov, and Peeva [16] and Veliche [109]—take this approach to define homological dimensions that are intermediate between the projective dimension and the G-dimension for finitely generated modules. Gerko [61, Section 3] defines a Cohen–Macaulay dimension, which is a refinement of the G-dimension. Avramov [10, Section 8] surveys these dimensions.

2 Gorenstein projective dimension

To extend the G-dimension beyond the realm of finitely generated modules over Noetherian rings, Enochs and Jenda [41] introduced the notion of Gorenstein projective modules. The same authors, and their collaborators, studied these modules in several subsequent papers. The associated dimension, which is the focus of this section, was studied by Christensen [28] and Holm [66].

In organization, this section is parallel to Section 1.

Definition 2.1. An R-module A is called *Gorenstein projective* if there exists an acyclic complex P of projective R-modules such that $\mathrm{Coker}(P_1 \to P_0) \cong A$ and such that $\mathrm{Hom}_R(P, Q)$ is acyclic for every projective R-module Q.

It is evident that every projective module is Gorenstein projective.

Example 2.2. Every totally reflexive module is Gorenstein projective; this follows from Definition 1.2 and Lemma 1.1.

Basic categorical properties are recorded in [66, Section 2]:

Proposition 2.3. *The class of Gorenstein projective R-modules is closed under direct sums and summands.*

Every projective module is a direct summand of a free one. A parallel result for Gorenstein projective modules, Theorem 2.5 below, is due to Bennis and Mahdou [24, Section 2]; as substitute for free modules they define:

Definition 2.4. An R-module A is called *strongly Gorenstein projective* if there exists an acyclic complex $\boldsymbol{P}$ of projective R-modules, in which all the differentials are identical, such that $\mathrm{Coker}(P_1 \to P_0) \cong A$, and such that $\mathrm{Hom}_R(\boldsymbol{P},Q)$ is acyclic for every projective R-module Q.

Theorem 2.5. *An R-module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective R-module.*

Definition 2.6. An (augmented) *Gorenstein projective resolution* of a module M is an exact sequence $\cdots \to A_i \to A_{i-1} \to \cdots \to A_0 \to M \to 0$, where each module A_i is Gorenstein projective.

Note that every module has a Gorenstein projective resolution, as a free resolution is trivially a Gorenstein projective one.

Definition 2.7. The *Gorenstein projective dimension* of a module $M \neq 0$, denoted by $\mathrm{Gpd}_R M$, is the least integer $n \geqslant 0$ such that there exists a Gorenstein projective resolution of M with $A_i = 0$ for all $i > n$. If no such n exists, then $\mathrm{Gpd}_R M$ is infinite. By convention, set $\mathrm{Gpd}_R 0 = -\infty$.

In [66, Section 2] one finds the next standard theorem and corollary.

Theorem 2.8. *Let M be an R-module of finite Gorenstein projective dimension. For every integer $n \geqslant 0$ the following conditions are equivalent.*

(i) $\mathrm{Gpd}_R M \leqslant n$.
(ii) $\mathrm{Ext}^i_R(M,Q) = 0$ *for all $i > n$ and all projective R-modules Q.*
(iii) $\mathrm{Ext}^i_R(M,N) = 0$ *for all $i > n$ and all R-modules N with $\mathrm{pd}_R N$ finite.*
(iv)   *In every augmented Gorenstein projective resolution*

$$\cdots \to A_i \to A_{i-1} \to \cdots \to A_0 \to M \to 0$$

the module $\mathrm{Coker}(A_{n+1} \to A_n)$ is Gorenstein projective.

Corollary 2.9. *For every R-module M of finite Gorenstein projective dimension there is an equality*

$$\mathrm{Gpd}_R M = \sup\{\, i \in \mathbb{Z} \mid \mathrm{Ext}^i_R(M,Q) \neq 0 \textit{ for some projective R-module } Q \,\}.$$

Remark 2.10. For every R-module M as in the corollary, the finitistic projective dimension of R is an upper bound for $\mathrm{Gpd}_R M$; see [66, 2.28].

The next result [66, 2.24] extends Proposition 1.13.

Proposition 2.11. *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of R-modules. If any two of the modules have finite Gorenstein projective dimension, then so has the third.*

The Gorenstein projective dimension is a refinement of the projective dimension; this follows from Corollary 2.9:

Proposition 2.12. *For every R-module M one has* $\mathrm{Gpd}_R M \leqslant \mathrm{pd}_R M$, *and equality holds if M has finite projective dimension.*

Supplementary information comes from Holm [67, 2.2]:

Proposition 2.13. *If M is an R-module of finite injective dimension, then there is an equality* $\mathrm{Gpd}_R M = \mathrm{pd}_R M$.

The next result of Foxby is published in [32, Ascent table II(b)].

Proposition 2.14. *Let S be an R-algebra of finite projective dimension. For every Gorenstein projective R-module A, the module* $S \otimes_R A$ *is Gorenstein projective over S.*

Noetherian rings

Finiteness of the Gorenstein projective dimension characterizes Gorenstein rings. The next result of Enochs and Jenda [43, 12.3.1] extends Theorem 1.17.

Theorem 2.15. *Let R be Noetherian and* $n \geqslant 0$ *be an integer. Then R is Gorenstein with* $\dim R \leqslant n$ *if and only if* $\mathrm{Gpd}_R M \leqslant n$ *for every R-module M.*

The next result [28, 4.2.6] compares the Gorenstein projective dimension to the G-dimension.

Proposition 2.16. *Let R be Noetherian. For every finitely generated R-module M there is an equality* $\mathrm{Gpd}_R M = \mathrm{G\text{-}dim}_R M$.

The Gorenstein projective dimension of a module can not be measured locally; that is, Proposition 1.15 does not extend to non-finitely generated modules. As a consequence of Proposition 2.14, though, one has the following:

Proposition 2.17. *Let R be Noetherian of finite Krull dimension. For every R-module M and every prime ideal* $\mathfrak{p}$ *in R one has* $\mathrm{Gpd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \mathrm{Gpd}_R M$.

Theorem E and Proposition 2.13 yield:

Theorem 2.18. *Let R be Noetherian and M a finitely generated R-module. If* $\mathrm{Gpd}_R M$ *and* $\mathrm{id}_R M$ *are finite, then* $R_\mathfrak{p}$ *is Gorenstein for all* $\mathfrak{p} \in \mathrm{Supp}_R M$.

Local rings

The next characterization of Gorenstein local rings—akin to Theorem A in the Introduction—follows from Theorems 1.27 and 2.15 via Proposition 2.16.

Theorem 2.19. *For a local ring* $(R, \mathfrak{m}, k)$ *the next conditions are equivalent.*

(i)   *R is Gorenstein.*
(ii) $\mathrm{Gpd}_R k$ *is finite.*
(iii) $\mathrm{Gpd}_R M$ *is finite for every R-module M.*

The inequality in the next theorem is a consequence of Proposition 2.14. The second assertion is due to Esmkhani and Tousi [52, 3.5], cf. [31, 4.1]. The result should be compared to Proposition 1.26.

Theorem 2.20. *Let R be local and M be an R-module. Then one has*

$$\mathrm{Gpd}_{\widehat{R}}(\widehat{R} \otimes_R M) \leqslant \mathrm{Gpd}_R M,$$

and if $\mathrm{Gpd}_{\widehat{R}}(\widehat{R} \otimes_R M)$ *is finite, then so is* $\mathrm{Gpd}_R M$.

Notes

Holm and Jørgensen [69] extend Golod's [62] notion of G_C-dimension to non-finitely generated modules in the form of a C-Gorenstein projective dimension. Further studies of this dimension are made by White [112].

3 Gorenstein injective dimension

The notion of Gorenstein injective modules is (categorically) dual to that of Gorenstein projective modules. The two were introduced in the same paper by Enochs and Jenda [41] and investigated in subsequent works by the same authors, by Christensen and Sather-Wagstaff [35], and by Holm [66].

This section is structured parallelly to the previous ones.

Definition 3.1. An R-module B is called *Gorenstein injective* if there exists an acyclic complex I of injective R-modules such that $\mathrm{Ker}(I^0 \to I^1) \cong B$, and such that $\mathrm{Hom}_R(E, I)$ is acyclic for every injective R-module E.

It is clear that every injective module is Gorenstein injective.

Example 3.2. Let L be a totally acyclic complex of finitely generated projective R-modules, see Definition 1.2, and let I be an injective R-module. Then the acyclic complex $I = \mathrm{Hom}_R(L, I)$ consists of injective modules, and from Lemma 1.1 it follows that the complex $\mathrm{Hom}_R(E, I) \cong \mathrm{Hom}_R(E \otimes_R L, I)$ is acyclic for every injective module E. Thus, if G is a totally reflexive R-module and I is injective, then the module $\mathrm{Hom}_R(G, I)$ is Gorenstein injective.

Basic categorical properties are established in [66, 2.6]:

Proposition 3.3. *The class of Gorenstein injective R-modules is closed under direct products and summands.*

Under extra assumptions on the ring, Theorem 3.16 gives more information.

Definition 3.4. An (augmented) *Gorenstein injective resolution* of a module M is an exact sequence $0 \to M \to B^0 \to \cdots \to B^{i-1} \to B^i \to \cdots$, where each module B^i is Gorenstein injective.

Note that every module has a Gorenstein injective resolution, as an injective resolution is trivially a Gorenstein injective one.

Definition 3.5. The *Gorenstein injective dimension* of an R-module $M \neq 0$, denoted by $\mathrm{Gid}_R M$, is the least integer $n \geqslant 0$ such that there exists a Gorenstein injective resolution of M with $B^i = 0$ for all $i > n$. If no such n exists, then $\mathrm{Gid}_R M$ is infinite. By convention, set $\mathrm{Gid}_R 0 = -\infty$.

The next standard theorem is [66, 2.22].

Theorem 3.6. *Let M be an R-module of finite Gorenstein injective dimension. For every integer $n \geqslant 0$ the following conditions are equivalent.*

(i) $\mathrm{Gid}_R M \leqslant n$.
(ii) $\mathrm{Ext}_R^i(E,M) = 0$ *for all $i > n$ and all injective R-modules E.*
(iii) $\mathrm{Ext}_R^i(N,M) = 0$ *for all $i > n$ and all R-modules N with $\mathrm{id}_R N$ finite.*
(iv) *In every augmented Gorenstein injective resolution*

$$0 \to M \to B^0 \to \cdots \to B^{i-1} \to B^i \to \cdots$$

the module $\mathrm{Ker}(B^n \to B^{n+1})$ is Gorenstein injective.

Corollary 3.7. *For every R-module M of finite Gorenstein injective dimension there is an equality*

$$\mathrm{Gid}_R M = \sup\{\, i \in \mathbb{Z} \mid \mathrm{Ext}_R^i(E,M) \neq 0 \text{ for some injective R-module } E \,\}.$$

Remark 3.8. For every R-module M as in the corollary, the finitistic injective dimension of R is an upper bound for $\mathrm{Gid}_R M$; see [66, 2.29].

The next result [66, 2.25] is similar to Proposition 2.11.

Proposition 3.9. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If any two of the modules have finite Gorenstein injective dimension, then so has the third.*

The Gorenstein injective dimension is a refinement of the injective dimension; this follows from Corollary 3.7:

Proposition 3.10. *For every R-module M one has* $\operatorname{Gid}_R M \leqslant \operatorname{id}_R M$, *and equality holds if M has finite injective dimension.*

Supplementary information comes from Holm [67, 2.1]:

Proposition 3.11. *If M is an R-module of finite projective dimension, then there is an equality* $\operatorname{Gid}_R M = \operatorname{id}_R M$. *In particular, one has* $\operatorname{Gid}_R R = \operatorname{id}_R R$.

In [32] Christensen and Holm study (co)base change of modules of finite Gorenstein homological dimension. The following is elementary to verify:

Proposition 3.12. *Let S be an R-algebra of finite projective dimension. For every Gorenstein injective R-module B, the module* $\operatorname{Hom}_R(S, B)$ *is Gorenstein injective over S.*

For a conditional converse see Theorems 3.27 and 9.11.

The next result of Bennis, Mahdou, and Ouarghi [25, 2.2] should be compared to characterizations of Gorenstein rings like Theorems 2.15 and 3.14, and also to Theorems 2.18 and 3.21. It is a perfect Gorenstein counterpart to a classical result due to Faith and Walker among others; see e.g. [111, 4.2.4].

Theorem 3.13. *The following conditions on R are equivalent.*

(i)   *R is quasi-Frobenius.*
(ii) *Every R-module is Gorenstein projective.*
(iii) *Every R-module is Gorenstein injective.*
(iv) *Every Gorenstein projective R-module is Gorenstein injective.*
(v)   *Every Gorenstein injective R-module is Gorenstein projective.*

Noetherian rings

Finiteness of the Gorenstein injective dimension characterizes Gorenstein rings; this result is due to Enochs and Jenda [42, 3.1]:

Theorem 3.14. *Let R be Noetherian and* $n \geqslant 0$ *be an integer. Then R is Gorenstein with* $\dim R \leqslant n$ *if and only if* $\operatorname{Gid}_R M \leqslant n$ *for every R-module M.*

A ring is Noetherian if every countable direct sum of injective modules is injective (and only if every direct limit of injective modules is injective). The "if" part has a perfect Gorenstein counterpart:

Proposition 3.15. *If every countable direct sum of Gorenstein injective R-modules is Gorenstein injective, then R is Noetherian.*

Proof. It is sufficient to see that every countable direct sum of injective R-modules is injective. Let $\{E_n\}_{n \in \mathbb{N}}$ be a family of injective modules. By assumption, the module $\bigoplus E_n$ is Gorenstein injective; in particular, there is an epimorphism $\varphi \colon I \twoheadrightarrow \bigoplus E_n$ such that I is injective and $\operatorname{Hom}_R(E, \varphi)$ is surjective for every injective R-module E. Applying this to $E = E_n$ it is elementary to verify that φ is a split epimorphism. $\square$

Christensen, Frankild, and Holm [31, 6.9] provide a partial converse:

Theorem 3.16. *Assume that R is a homomorphic image of a Gorenstein ring of finite Krull dimension. Then the class of Gorenstein injective modules is closed under direct limits; in particular, it is closed under direct sums.*

As explained in the Introduction, the hypothesis on R in this theorem ensures the existence of a dualizing R-complex and an associated Bass class, cf. Section 9. These tools are essential to the known proof of Theorem 3.16.

Question 3.17. Let R be Noetherian. Is then every direct limit of Gorenstein injective R-modules Gorenstein injective?

Next follows a Gorenstein version of Chouinard's formula [27, 3.1]; it is proved in [35, 2.2]. Recall that the *width* of a module M over a local ring $(R, \mathfrak{m}, k)$ is defined as

$$\operatorname{width}_R M = \inf\{ i \in \mathbb{Z} \mid \operatorname{Tor}_i^R(k, M) \neq 0 \}.$$

Theorem 3.18. *Let R be Noetherian. For every R-module M of finite Gorenstein injective dimension there is an equality*

$$\operatorname{Gid}_R M = \{ \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \}.$$

Let M be an R-module, and let $\mathfrak{p}$ be a prime ideal in R. Provided that $\operatorname{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is finite, the inequality $\operatorname{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leqslant \operatorname{Gid}_R M$ follows immediately from the theorem. However, the next question remains open.

Question 3.19. Let R be Noetherian and B be a Gorenstein injective R-module. Is then $B_{\mathfrak{p}}$ Gorenstein injective over $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ in R?

A partial answer is known from [31, 5.5]:

Proposition 3.20. *Assume that R is a homomorphic image of a Gorenstein ring of finite Krull dimension. For every R-module M and every prime ideal $\mathfrak{p}$ there is an inequality* $\operatorname{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leqslant \operatorname{Gid}_R M$.

Theorem E and Proposition 3.11 yield:

Theorem 3.21. *Let R be Noetherian and M a finitely generated R-module. If $\operatorname{Gid}_R M$ and $\operatorname{pd}_R M$ are finite, then $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{Supp}_R M$.*

Local rings

The following theorem of Foxby and Frankild [58, 4.5] generalizes work of Peskine and Szpiro [86], cf. Theorem E.

Theorem 3.22. *A local ring R is Gorenstein if and only if there exists a non-zero cyclic R-module of finite Gorenstein injective dimension.*

Theorems 3.14 and 3.22 yield a parallel to Theorem1.27, akin to Theorem A.

Corollary 3.23. *For a local ring $(R, \mathfrak{m}, k)$ the next conditions are equivalent.*

(i) *R is Gorenstein.*
(ii) *$\mathrm{Gid}_R k$ is finite.*
(iii) *$\mathrm{Gid}_R M$ is finite for every R-module M.*

The first part of the next theorem is due to Christensen, Frankild, and Iyengar, and published in [58, 3.6]. The equality in Theorem 3.24—the Gorenstein analogue of Theorem C in the Introduction—is proved by Khatami, Tousi, and Yassemi [80, 2.5]; see also [35, 2.3].

Theorem 3.24. *Let R be local and $M \neq 0$ be a finitely generated R-module. Then $\mathrm{Gid}_R M$ and $\mathrm{Gid}_{\widehat{R}}(\widehat{R} \otimes_R M)$ are simultaneously finite, and when they are finite, there is an equality*

$$\mathrm{Gid}_R M = \mathrm{depth}\, R.$$

Remark 3.25. Let R be local and $M \neq 0$ be an R-module. If M has finite length and finite G-dimension, then its Matlis dual has finite Gorenstein injective dimension, cf. Example 3.2. See also Takahashi [103].

This remark and Theorem D from the Introduction motivate:

Question 3.26. Let R be local. If there exists a non-zero finitely generated R-module of finite Gorenstein injective dimension, is then R Cohen–Macaulay?

A partial answer to this question is given by Yassemi [115, 1.3].

Esmkhani and Tousi [53, 2.5] prove the following conditional converse to Proposition 3.12. Recall that an R-module M is said to be *cotorsion* if $\mathrm{Ext}_R^1(F, M) = 0$ for every flat R-module F.

Theorem 3.27. *Let R be local. An R-module M is Gorenstein injective if and only if it is cotorsion and $\mathrm{Hom}_R(\widehat{R}, M)$ is Gorenstein injective over $\widehat{R}$.*

The example below demonstrates the necessity of the cotorsion hypothesis. Working in the derived category one obtains a stronger result; see Theorem 9.11.

Example 3.28. Let $(R, \mathfrak{m})$ be a local domain which is not $\mathfrak{m}$-adically complete. Aldrich, Enochs, and López-Ramos [1, 3.3] show that the module $\mathrm{Hom}_R(\widehat{R}, R)$ is zero and hence Gorenstein injective over $\widehat{R}$. However, $\mathrm{Gid}_R R$ is infinite if R is not Gorenstein, cf. Proposition 3.11.

Notes

Dual to the notion of strongly Gorenstein projective modules, see Definition 2.4, Bennis and Mahdou [24] also study strongly Gorenstein injective modules.

Several authors—Asadollahi, Sahandi, Salarian, Sazeedeh, Sharif, and Yassemi—have studied the relationship between Gorenstein injectivity and local cohomology; see [3], [90], [96], [97], and [115].

4 Gorenstein flat dimension

Another extension of the G-dimension is based on Gorenstein flat modules—a notion due to Enochs, Jenda, and Torrecillas [47]. Christensen [28] and Holm [66] are other main references for this section.

The organization of this section follows the pattern from Sections 1–3.

Definition 4.1. An R-module A is called *Gorenstein flat* if there exists an acyclic complex $\boldsymbol{F}$ of flat R-modules such that $\mathrm{Coker}(F_1 \to F_0) \cong A$, and such that $E \otimes_R \boldsymbol{F}$ is acyclic for every injective R-module E.

It is evident that every flat module is Gorenstein flat.

Example 4.2. Every totally reflexive module is Gorenstein flat; this follows from Definition 1.2 and Lemma 1.1.

Here is a direct consequence of Definition 4.1:

Proposition 4.3. *The class of Gorenstein flat R-modules is closed under direct sums.*

See Theorems 4.13 and 4.14 for further categorical properties of Gorenstein flat modules.

Definition 4.4. An (augmented) *Gorenstein flat resolution* of a module M is an exact sequence $\cdots \to A_i \to A_{i-1} \to \cdots \to A_0 \to M \to 0$, where each module A_i is Gorenstein flat.

Note that every module has a Gorenstein flat resolution, as a free resolution is trivially a Gorenstein flat one.

Definition 4.5. The *Gorenstein flat dimension* of an R-module $M \neq 0$, denoted by $\mathrm{Gfd}_R M$, is the least integer $n \geqslant 0$ such that there exists a Gorenstein flat resolution of M with $A_i = 0$ for all $i > n$. If no such n exists, then $\mathrm{Gfd}_R M$ is infinite. By convention, set $\mathrm{Gfd}_R 0 = -\infty$.

The next duality result is an immediate consequence of the definitions.

Proposition 4.6. *Let M be an R-module. For every injective R-module E there is an inequality $\mathrm{Gid}_R \mathrm{Hom}_R(M,E) \leqslant \mathrm{Gfd}_R M$.*

Recall that an R-module E is called *faithfully* injective if it is injective and $\mathrm{Hom}_R(M,E) = 0$ only if $M = 0$. The next question is inspired by the classical situation. It has an affirmative answer for Noetherian rings; see Theorem 4.16.

Question 4.7. Let M and E be R-modules. If E is faithfully injective and the module $\mathrm{Hom}_R(M,E)$ is Gorenstein injective, is then M Gorenstein flat?

A straightforward application of Proposition 4.6 shows that the Gorenstein flat dimension is a refinement of the flat dimension; cf. Bennis [23, 2.2]:

Proposition 4.8. *For every R-module M one has* $\mathrm{Gfd}_R M \leqslant \mathrm{fd}_R M$, *and equality holds if M has finite flat dimension.*

The following result is an immediate consequence of Definition 4.1. Over a local ring a stronger result is available; see Theorem 4.27.

Proposition 4.9. *Let S be a flat R-algebra. For every R-module M there is an inequality* $\mathrm{Gfd}_S(S \otimes_R M) \leqslant \mathrm{Gfd}_R M$.

Corollary 4.10. *Let M be an R-module. For every prime ideal* $\mathfrak{p}$ *in R there is an inequality* $\mathrm{Gfd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \mathrm{Gfd}_R M$.

Noetherian rings

Finiteness of the Gorenstein flat dimension characterizes Gorenstein rings; this is a result of Enochs and Jenda [42, 3.1]:

Theorem 4.11. *Let R be Noetherian and* $n \geqslant 0$ *be an integer. Then R is Gorenstein with* $\dim R \leqslant n$ *if and only if* $\mathrm{Gfd}_R M \leqslant n$ *for every R-module M.*

A ring is coherent if and only if every direct product of flat modules is flat. We suggest the following problem:

Problem 4.12. Describe the rings over which every direct product of Gorenstein flat modules is Gorenstein flat.

Partial answers are due to Christensen, Frankild, and Holm [31, 5.7] and to Murfet and Salarian [85, 6.21].

Theorem 4.13. *Let R be Noetherian. The class of Gorenstein flat R-modules is closed under direct products under either of the following conditions:*

(a) R is homomorphic image of a Gorenstein ring of finite Krull dimension.
(b) $R_\mathfrak{p}$ *is Gorenstein for every non-maximal prime ideal* $\mathfrak{p}$ *in R.*

The next result follows from work of Enochs, Jenda, and López-Ramos [46, 2.1] and [40, 3.3].

Theorem 4.14. *Let R be Noetherian. Then the class of Gorenstein flat R-modules is closed under direct summands and direct limits.*

A result of Govorov [64] and Lazard [83, 1.2] asserts that a module is flat if and only if it is a direct limit of finitely generated projective modules. For Gorenstein flat modules, the situation is more complicated:

Remark 4.15. Let R be Noetherian. It follows from Example 4.2 and Theorem 4.14 that a direct limit of totally reflexive modules is Gorenstein flat. If R is Gorenstein of finite Krull dimension, then every Gorenstein flat R-module can be written as a direct limit of totally reflexive modules; see Enochs and Jenda [43, 10.3.8]. If R is not Gorenstein, this conclusion may fail; see Beligiannis and Krause [22, 4.2 and 4.3] and Theorem 4.30.

The next result [28, 6.4.2] gives a partial answer to Question 4.7.

Theorem 4.16. *Let R be Noetherian, and let M and E be R-modules. If E is faithfully injective, then there is an equality*

$$\operatorname{Gid}_R \operatorname{Hom}_R(M, E) = \operatorname{Gfd}_R M.$$

Theorem 4.17 is found in [66, 3.14]. It can be obtained by application of Theorem 4.16 to Theorem 3.6.

Theorem 4.17. *Let R be Noetherian and M be an R-module of finite Gorenstein flat dimension. For every integer $n \geqslant 0$ the following are equivalent.*

(i) $\operatorname{Gfd}_R M \leqslant n$.
(ii) $\operatorname{Tor}_i^R(E, M) = 0$ *for all $i > n$ and all injective R-modules E.*
(iii) $\operatorname{Tor}_i^R(N, M) = 0$ *for all $i > n$ and all R-modules N with $\operatorname{id}_R N$ finite.*
(iv) *In every augmented Gorenstein flat resolution*

$$\cdots \to A_i \to A_{i-1} \to \cdots \to A_0 \to M \to 0$$

the module $\operatorname{Coker}(A_{n+1} \to A_n)$ is Gorenstein flat.

Corollary 4.18. *Let R be Noetherian. For every R-module M of finite Gorenstein flat dimension there is an equality*

$$\operatorname{Gfd}_R M = \sup\{i \in \mathbb{Z} \mid \operatorname{Tor}_i^R(E, M) \neq 0 \text{ for some injective R-module } E\}.$$

Remark 4.19. For every R-module M as in the corollary, the finitistic flat dimension of R is an upper bound for $\operatorname{Gfd}_R M$; see [66, 3.24].

The next result [66, 3.15] follows by Theorem 4.16 and Proposition 3.9.

Proposition 4.20. *Let R be Noetherian. If any two of the modules in an exact sequence $0 \to M' \to M \to M'' \to 0$ have finite Gorenstein flat dimension, then so has the third.*

A result of Holm [67, 2.6] supplements Proposition 4.8:

Proposition 4.21. *Let R be Noetherian of finite Krull dimension. For every R-module M of finite injective dimension one has $\operatorname{Gfd}_R M = \operatorname{fd}_R M$.*

Recall that the *depth* of a module M over a local ring $(R, \mathfrak{m}, k)$ is given as

$$\operatorname{depth}_R M = \inf\{i \in \mathbb{Z} \mid \operatorname{Ext}_R^i(k, M) \neq 0\}.$$

Theorem 4.22 is a Gorenstein version of Chouinard's [27, 1.2]. It follows from [66, 3.19] and [30, 2.4(b)]; see also Iyengar and Sather-Wagstaff [76, 8.8].

Theorem 4.22. *Let R be Noetherian. For every R-module M of finite Gorenstein flat dimension there is an equality*

$$\mathrm{Gfd}_R M = \{\, \mathrm{depth}\, R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Spec}\, R \,\}.$$

The next two results compare the Gorenstein flat dimension to the Gorenstein projective dimension. The inequality in Theorem 4.23 is [66, 3.4], and the second assertion in this theorem is due to Esmkhani and Tousi [52, 3.4].

Theorem 4.23. *Let R be Noetherian of finite Krull dimension, and let M be an R-module. Then there is an inequality*

$$\mathrm{Gfd}_R M \leqslant \mathrm{Gpd}_R M,$$

and if $\mathrm{Gfd}_R M$ is finite, then so is $\mathrm{Gpd}_R M$.

It is not known whether the inequality in Theorem 4.23 holds over every commutative ring. For finitely generated modules one has [28, 4.2.6 and 5.1.11]:

Proposition 4.24. *Let R be Noetherian. For every finitely generated R-module M there is an equality $\mathrm{Gfd}_R M = \mathrm{Gpd}_R M = \mathrm{G\text{-}dim}_R M$.*

The next result [31, 5.1] is related to Theorem 4.16; the question that follows is prompted by the classical situation.

Theorem 4.25. *Assume that R is a homomorphic image of a Gorenstein ring of finite Krull dimension. For every R-module M and every injective R-module E there is an inequality*

$$\mathrm{Gfd}_R \mathrm{Hom}_R(M,E) \leqslant \mathrm{Gid}_R M,$$

and equality holds if E is faithfully injective.

Question 4.26. Let R be Noetherian and M and E be R-modules. If M is Gorenstein injective and E is injective, is then $\mathrm{Hom}_R(M,E)$ Gorenstein flat?

Local rings

Over a local ring there is a stronger version [52, 3.5] of Proposition 4.9:

Theorem 4.27. *Let R be local. For every R-module M there is an equality*

$$\mathrm{Gfd}_{\widehat{R}}(\widehat{R} \otimes_R M) = \mathrm{Gfd}_R M.$$

Combination of [67, 2.1 and 2.2] with Theorem E yields the next result. Recall that a non-zero finitely generated module has finite depth.

Theorem 4.28. *For a local ring R the following conditions are equivalent.*

(i)   *R is Gorenstein.*
(ii) *There is an R-module M with $\operatorname{depth}_R M$, $\operatorname{fd}_R M$, and $\operatorname{Gid}_R M$ finite.*
(iii) *There is an R-module M with $\operatorname{depth}_R M$, $\operatorname{id}_R M$, and $\operatorname{Gfd}_R M$ finite.*

We have for a while been interested in:

Question 4.29. Let R be local. If there exists an R-module M with $\operatorname{depth}_R M$, $\operatorname{Gfd}_R M$, and $\operatorname{Gid}_R M$ finite, is then R Gorenstein?

   A theorem of Jørgensen and Holm [71] brings perspective to Remark 4.15.

Theorem 4.30. *Assume that R is Henselian local and a homomorphic image of a Gorenstein ring. If every Gorenstein flat R-module is a direct limit of totally reflexive modules, then R is Gorenstein or every totally reflexive R-module is free.*

Notes

Parallel to the notion of strongly Gorenstein projective modules, see Definition 2.4, Bennis and Mahdou [24] also study strongly Gorenstein flat modules. A different notion of strongly Gorenstein flat modules is studied by Ding, Li, and Mao in [38].

5 Relative homological algebra

Over a Gorenstein local ring, the totally reflexive modules are exactly the maximal Cohen–Macaulay modules, and their representation theory is a classical topic. Over rings that are not Gorenstein, the representation theory of totally reflexive modules was taken up by Takahashi [102] and Yoshino [116]. Conclusive results have recently been obtained by Christensen, Piepmeyer, Striuli, and Takahashi [34] and by Holm and Jørgensen [71]. These results are cast in the language of precovers and preenvelopes; see Theorem 5.4.

   Relative homological algebra studies dimensions and (co)homology functors based on resolutions that are constructed via precovers or preenvelopes. Enochs and Jenda and their collaborators have made extensive studies of the precovering and preenveloping properties of the classes of Gorenstein flat and Gorenstein injective modules. Many of their results are collected in [43].

Terminology

Let $\mathcal{H}$ be a class of R-modules. Recall that an $\mathcal{H}$-*precover* (also called a right $\mathcal{H}$-approximation) of an R-module M is a homomorphism $\varphi \colon H \to M$ with H in $\mathcal{H}$ such that

$$\operatorname{Hom}_R(H', \varphi) \colon \operatorname{Hom}_R(H', H) \longrightarrow \operatorname{Hom}_R(H', M)$$

is surjective for every H' in $\mathcal{H}$. That is, every homomorphism from a module in $\mathcal{H}$ to M factors through φ. Dually one defines $\mathcal{H}$-preenvelopes (also called left $\mathcal{H}$-approximations).

Remark 5.1. If $\mathcal{H}$ contains all projective modules, then every $\mathcal{H}$-precover is an epimorphism. Thus, Gorenstein projective/flat precovers are epimorphisms.

If $\mathcal{H}$ contains all injective modules, then every $\mathcal{H}$-preenvelope is a monomorphism. Thus, every Gorenstein injective preenvelope is a monomorphism.

Fix an $\mathcal{H}$-precover φ. It is called *special* if one has $\mathrm{Ext}^1_R(H', \mathrm{Ker}\, \varphi) = 0$ for every module H' in $\mathcal{H}$. It is called a *cover* (or a minimal right approximation) if in every factorization $\varphi = \varphi \psi$, the map $\psi \colon H \to H$ is an automorphism. If $\mathcal{H}$ is closed under extensions, then every $\mathcal{H}$-cover is a special precover. This is known as Wakamatsu's lemma; see Xu [113, 2.1.1]. Dually one defines special $\mathcal{H}$-preenvelopes and $\mathcal{H}$-envelopes.

Remark 5.2. Let I be a complex of injective modules as in Definition 3.1. Then every differential in I is a special injective precover of its image; this fact is used in the proof of Proposition 3.15. Similarly, in a complex P of projective modules as in Definition 2.1, every differential ∂_i is a special projective preenvelope of the cokernel of the previous differential ∂_{i+1}.

Totally reflexive covers and envelopes

The next result of Avramov and Martsinkovsky [19, 3.1] corresponds over a Gorenstein local ring to the existence of maximal Cohen–Macaulay approximations in the sense of Auslander and Buchweitz [9].

Proposition 5.3. *Let R be Noetherian. For every finitely generated R-module M of finite G-dimension there is an exact sequence $0 \to K \to G \to M \to 0$ of finitely generated modules, where G is totally reflexive and $\mathrm{pd}_R K = \max\{0, \mathrm{G\text{-}dim}_R M - 1\}$. In particular, every finitely generated R-module of finite G-dimension has a special totally reflexive precover.*

An unpublished result of Auslander states that every finitely generated module over a Gorenstein local ring has a totally reflexive cover; see Enochs, Jenda, and Xu [50] for a generalization. A strong converse is contained in the next theorem, which combines Auslander's result with recent work of several authors; see [34] and [71].

Theorem 5.4. *For a local ring $(R, \mathfrak{m}, k)$ the next conditions are equivalent.*

(i)   *Every finitely generated R-module has a totally reflexive cover.*
(ii) *The residue field k has a special totally reflexive precover.*
(iii) *Every finitely generated R-module has a totally reflexive envelope.*

(*iv*) *Every finitely generated R-module has a special totally reflexive preenvelope.*
(*v*)   *R is Gorenstein or every totally reflexive R-module is free.*

If R is local and Henselian (e.g. complete), then existence of a totally reflexive precover implies existence of a totally reflexive cover; see [102, 2.5]. In that case one can drop "special" in part (*ii*) above. In general, though, the next question from [34] remains open.

Question 5.5. Let $(R, \mathfrak{m}, k)$ be local. If k has a totally reflexive precover, is then R Gorenstein or every totally reflexive R-module free?

Gorenstein projective precovers

The following result is proved by Holm in [66, 2.10].

Proposition 5.6. *For every R-module M of finite Gorenstein projective dimension there is an exact sequence $0 \to K \to A \to M \to 0$, where A is Gorenstein projective and $\mathrm{pd}_R K = \max\{0, \mathrm{Gpd}_R M - 1\}$. In particular, every R-module of finite Gorenstein projective dimension has a special Gorenstein projective precover.*

For an important class of rings, Jørgensen [78] and Murfet and Salarian [85] prove existence of Gorenstein projective precovers for all modules:

Theorem 5.7. *If R is Noetherian of finite Krull dimension, then every R-module has a Gorenstein projective precover.*

Remark 5.8. Actually, the argument in Krause's proof of [81, 7.12(1)] applies to the setup in [78] and yields existence of a *special* Gorenstein projective precover for every module over a ring as in Theorem 5.7.

Over any ring, every module has a special projective precover; hence:

Problem 5.9. Describe the rings over which every module has a (special) Gorenstein projective precover.

Gorenstein injective preenvelopes

In [66, 2.15] one finds:

Proposition 5.10. *For every R-module M of finite Gorenstein injective dimension there is an exact sequence $0 \to M \to B \to C \to 0$, where B is Gorenstein injective and $\mathrm{id}_R C = \max\{0, \mathrm{Gid}_R M - 1\}$. In particular, every R-module of finite Gorenstein injective dimension has a special Gorenstein injective preenvelope.*

Over Noetherian rings, existence of Gorenstein injective preenvelopes for all modules is proved by Enochs and López-Ramos in [51]. Krause [81, 7.12] proves a stronger result:

Theorem 5.11. *If R is Noetherian, then every R-module has a special Gorenstein injective preenvelope.*

Over Gorenstein rings, Enochs, Jenda, and Xu [48, 6.1] prove even more:

Proposition 5.12. *If R is Gorenstein of finite Krull dimension, then every R-module has a Gorenstein injective envelope.*

Over any ring, every module has an injective envelope; this suggests:

Problem 5.13. Describe the rings over which every module has a Gorenstein injective (pre)envelope.

Over a Noetherian ring, every module has an injective cover; see Enochs [39, 2.1]. A Gorenstein version of this result is recently established by Holm and Jørgensen [72, 3.3(b)]:

Proposition 5.14. *If R is a homomorphic image of a Gorenstein ring of finite Krull dimension, then every R-module has a Gorenstein injective cover.*

Gorenstein flat covers

The following existence result is due to Enochs and López-Ramos [51, 2.11].

Theorem 5.15. *If R is Noetherian, then every R-module has a Gorenstein flat cover.*

Remark 5.16. Let R be Noetherian and M be a finitely generated R-module. If M has finite G-dimension, then by Proposition 5.3 it has a finitely generated Gorenstein projective/flat precover, cf. Proposition 4.24. If M has infinite G-dimension, it still has a Gorenstein projective/flat precover by Theorems 5.7 and 5.15, but by Theorem 5.4 this need *not* be finitely generated.

Over any ring, every module has a flat cover, as proved by Bican, El Bashir, and Enochs [26]. This motivates:

Problem 5.17. Describe the rings over which every module has a Gorenstein flat (pre)cover.

Over a Noetherian ring, every module has a flat preenvelope; cf. Enochs [39, 5.1]. A Gorenstein version of this result follows from Theorem 4.13 and [51, 2.5]:

Proposition 5.18. *If R is a homomorphic image of a Gorenstein ring of finite Krull dimension, then every R-module has a Gorenstein flat preenvelope.*

Relative cohomology via Gorenstein projective modules

The notion of a proper resolution is central in relative homological algebra. Here is an instance:

Definition 5.19. An augmented Gorenstein projective resolution,

$$A^+ = \cdots \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

of an R-module M is said to be *proper* if the complex $\operatorname{Hom}_R(A', A^+)$ is acyclic for every Gorenstein projective R-module A'.

Remark 5.20. Assume that every R-module has a Gorenstein projective precover. Then every R-module has a proper Gorenstein projective resolution constructed by taking as φ_0 a Gorenstein projective precover of M and as φ_i a Gorenstein projective precover of $\operatorname{Ker}\varphi_{i-1}$ for $i > 0$.

Definition 5.21. The *relative Gorenstein projective dimension* of an R-module $M \neq 0$, denoted by rel-$\operatorname{Gpd}_R M$, is the least integer $n \geqslant 0$ such that there exists a proper Gorenstein projective resolution of M with $A_i = 0$ for all $i > n$. If no such n (or no such resolution) exists, then rel-$\operatorname{Gpd}_R M$ is infinite. By convention, set rel-$\operatorname{Gpd}_R 0 = -\infty$.

The following result is a consequence of Proposition 5.6.

Proposition 5.22. *For every R-module M one has* rel-$\operatorname{Gpd}_R M = \operatorname{Gpd}_R M$.

It is shown in [43, Section 8.2] that the next definition makes sense.

Definition 5.23. Let M and N be R-modules and assume that M has a proper Gorenstein projective resolution A. The ith relative cohomology module $\operatorname{Ext}^i_{\mathrm{GP}}(M,N)$ is $\mathrm{H}^i(\operatorname{Hom}_R(A,N))$.

Remark 5.24. Let R be Noetherian, and let M and N be finitely generated R-modules. Unless M has finite G-dimension, it is not clear whether the cohomology modules $\operatorname{Ext}^i_{\mathrm{GP}}(M,N)$ are finitely generated, cf. Remark 5.16.

Vanishing of relative cohomology $\operatorname{Ext}^i_{\mathrm{GP}}$ characterizes modules of finite Gorenstein projective dimension. The proof is standard; see [68, 3.9].

Theorem 5.25. *Let M be an R-module that has a proper Gorenstein projective resolution. For every integer $n \geqslant 0$ the next conditions are equivalent.*

(i) $\operatorname{Gpd}_R M \leqslant n$.
(ii) $\operatorname{Ext}^i_{\mathrm{GP}}(M,-) = 0$ *for all* $i > n$.
(iii) $\operatorname{Ext}^{n+1}_{\mathrm{GP}}(M,-) = 0$.

Remark 5.26. Relative cohomology based on totally reflexive modules is studied in [19]. The results that correspond to Proposition 5.22 and Theorem 5.25 in that setting are contained in [19, 4.8 and 4.2].

Notes

Based on a notion of coproper Gorenstein injective resolutions, one can define a relative Gorenstein injective dimension and cohomology functors Ext^i_{GI} with properties analogous to those of Ext^i_{GP} described above. There is, similarly, a relative Gorenstein flat dimension and a relative homology theory based on proper Gorenstein projective/flat resolutions. The relative Gorenstein injective dimension and the relative Gorenstein flat dimension were first studied by Enochs and Jenda [42] for modules over Gorenstein rings. The question of balancedness for relative (co)homology is treated by Enochs and Jenda [43], Holm [65], and Iacob [74].

In [19] Avramov and Martsinkovsky also study the connection between relative and Tate cohomology for finitely generated modules. This study is continued by Veliche [110] for arbitrary modules, and a dual theory is developed by Asadollahi and Salarian [4]. Jørgensen [78], Krause [81], and Takahashi [104] study connections between Gorenstein relative cohomology and generalized notions of Tate cohomology.

Sather-Wagstaff and White [95] use relative cohomology to define an Euler characteristic for modules of finite G-dimension. In collaboration with Sharif, they study cohomology theories related to generalized Gorenstein dimensions [93].

6 Modules over local homomorphisms

In this section, $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ is a local homomorphism, that is, there is a containment $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. The topic is Gorenstein dimensions over R of finitely generated S-modules. The utility of this point of view is illustrated by a generalization, due to Christensen and Iyengar [33, 4.1], of the Auslander–Bridger Formula (Theorem 1.25):

Theorem 6.1. *Let N be a finitely generated S-module. If N has finite Gorenstein flat dimension as an R-module via φ, then there is an equality*

$$\text{Gfd}_R N = \text{depth}\,R - \text{depth}_R N.$$

For a finitely generated S-module N of finite flat dimension over R, this equality follows from work of André [2, II.57]. For a finitely generated S-module of finite injective dimension over R, an affirmative answer to the next question is already in [107, 5.2] by Takahashi and Yoshino.

Question 6.2. Let N be a non-zero finitely generated S-module. If N has finite Gorenstein injective dimension as an R-module via φ, does then the equality $\text{Gid}_R N = \text{depth}\,R$ hold? (For $\varphi = \text{Id}_R$ this is Theorem 3.24.)

The next result of Christensen and Iyengar [33, 4.8] should be compared to Theorem 4.27.

Theorem 6.3. *Let N be a finitely generated S-module. If N has finite Gorenstein flat dimension as an R-module via φ, there is an equality*

$$\text{Gfd}_R N = \text{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N).$$

The Frobenius endomorphism

If R has positive prime characteristic, we denote by ϕ the Frobenius endomorphism on R and by ϕ^n its n-fold composition. The next two theorems are special cases of [76, 8.14 and 8.15] by Iyengar and Sather-Wagstaff and of [58, 5.5] by Foxby and Frankild.; together, they constitute the Gorenstein counterpart of Theorem F.

Theorem 6.4. *Let R be local of positive prime characteristic. The following conditions are equivalent.*

(i)   R is Gorenstein.
(ii)   R has finite Gorenstein flat dimension as an R-module via ϕ^n for some $n \geqslant 1$.
(iii) R is Gorenstein flat as an R-module via ϕ^n for every $n \geqslant 1$.

Theorem 6.5. *Let R be local of positive prime characteristic, and assume that it is a homomorphic image of a Gorenstein ring. The following conditions are equivalent.*

(i)   R is Gorenstein.
(ii)   R has finite Gorenstein injective dimension as R-module via ϕ^n for some $n \geqslant 1$.
(iii) R has Gorenstein injective dimension equal to $\dim R$ as an R-module via ϕ^n for every $n \geqslant 1$.

Part *(iii)* in Theorem 6.5 is actually not included in [58, 5.5]. Part *(iii)* follows, though, from part *(i)* by Corollary 3.23 and Theorem 3.18.

G-dimension over a local homomorphism

The homomorphism $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ fits in a commutative diagram of local homomorphisms:

$$
\begin{array}{ccc}
 & R' & \\
\dot{\phi} \nearrow & & \searrow \varphi' \\
R \xrightarrow{\;\varphi\;} S & \lhook\joinrel\xrightarrow{\;\iota\;} & \widehat{S}
\end{array}
$$

where $\dot{\phi}$ is flat with regular closed fiber $R'/\mathfrak{m}R'$, the ring R' is complete, and φ' is surjective. Set $\dot{\varphi} = \iota\varphi$; a diagram as above is called a *Cohen factorization of $\dot{\varphi}$*. This is a construction due to Avramov, Foxby, and Herzog [15, 1.1].

The next definition is due to Iyengar and Sather-Wagstaff [76, Section 3]; it is proved *ibid.* that it is independent of the choice of Cohen factorization.

Definition 6.6. Choose a Cohen factorization of $\dot{\varphi}$ as above. For a finitely generated S-module N, the *G-dimension of N over φ* is given as

$$\text{G-dim}_\varphi N = \text{G-dim}_{R'}(\widehat{S} \otimes_S N) - \text{edim}(R'/\mathfrak{m}R').$$

Example 6.7. Let k be a field and let φ be the extension from k to the power series ring $k[\![x]\!]$. Then one has $\text{G-dim}_\varphi k[\![x]\!] = -1$.

Iyengar and Sather-Wagstaff [76, 8.2] prove:

Theorem 6.8. *Assume that R is a homomorphic image of a Gorenstein ring. A finitely generated S-module N has finite G-dimension over φ if and only if it has finite Gorenstein flat dimension as an R-module via φ.*

It is clear from Example 6.7 that $\mathrm{Gfd}_R N$ and $\mathrm{G\text{-}dim}_\varphi N$ need not be equal.

7 Local homomorphisms of finite G-dimension

This section treats transfer of ring theoretic properties along a local homomorphism of finite G-dimension. Our focus is on the Gorenstein property, which was studied by Avramov and Foxby in [13], and the Cohen–Macaulay property, studied by Frankild in [59].

As in Section 6, $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local homomorphism. In view of Definition 6.6, a notion from [13, 4.3] can be defined as follows:

Definition 7.1. Set $\mathrm{G\text{-}dim}\,\varphi = \mathrm{G\text{-}dim}_\varphi S$; the homomorphism φ is said to be of *finite G-dimension* if this number is finite.

Remark 7.2. The homomorphism φ has finite G-dimension if S has finite Gorenstein flat dimension as an R-module via φ, and the converse holds if R is a homomorphic image of a Gorenstein ring. This follows from Theorems 6.3 and 6.8, in view of [76, 3.4.1],

The next descent result is [13, 4.6].

Theorem 7.3. *Let φ be of finite G-dimension, and assume that R is a homomorphic image of a Gorenstein ring. For every S-module N one has:*

(a) *If $\mathrm{fd}_S N$ is finite then $\mathrm{Gfd}_R N$ is finite.*
(b) *If $\mathrm{id}_S N$ is finite then $\mathrm{Gid}_R N$ is finite.*

It is not known if the composition of two local homomorphisms of finite G-dimension has finite G-dimension, but it would follow from an affirmative answer to Question 1.21, cf. [13, 4.8]. Some insight is provided by Theorem 7.9 and the next result, which is due to Iyengar and Sather-Wagstaff [76, 5.2].

Theorem 7.4. *Let $\psi \colon S \to T$ be a local homomorphism such that $\mathrm{fd}_S T$ is finite. Then $\mathrm{G\text{-}dim}\,\psi\varphi$ is finite if and only if $\mathrm{G\text{-}dim}\,\varphi$ is finite.*

Quasi-Gorenstein homomorphisms

Let M be a finitely generated module over a local ring $(R, \mathfrak{m}, k)$. For every integer $i \geqslant 0$ the ith *Bass number* $\mu_R^i(M)$ is the dimension of the k-vector space $\mathrm{Ext}_R^i(k, M)$.

Definition 7.5. The homomorphism φ is called *quasi-Gorenstein* if it has finite G-dimension and for every $i \geqslant 0$ there is an equality of Bass numbers

$$\mu_R^{i+\operatorname{depth}R}(R) = \mu_S^{i+\operatorname{depth}S}(S).$$

Example 7.6. If R is Gorenstein, then the natural surjection $R \twoheadrightarrow k$ is quasi-Gorenstein. More generally, if φ is surjective and S is quasi-perfect as an R-module via φ, then φ is quasi-Gorenstein if and only if there is an isomorphism $\operatorname{Ext}_R^g(S,R) \cong S$ where $g = \operatorname{G-dim}_R S$ with; see [13, 6.5, 7.1, 7.4].

Several characterizations of the quasi-Gorenstein property are given in [13, 7.4 and 7.5]. For example, it is sufficient that G-dim φ is finite and the equality of Bass numbers holds for some $i > 0$.

The next ascent–descent result is [13, 7.9].

Theorem 7.7. *Let φ be quasi-Gorenstein and assume that R is a homomorphic image of a Gorenstein ring. For every S-module N one has:*

(a) $\operatorname{Gfd}_S N$ *is finite if and only if* $\operatorname{Gfd}_R N$ *is finite.*
(b) $\operatorname{Gid}_S N$ *is finite if and only if* $\operatorname{Gid}_R N$ *is finite.*

Ascent and descent of the Gorenstein property is described by [13, 7.7.2]. It should be compared to part (b) in Theorem G.

Theorem 7.8. *The following conditions on φ are equivalent.*

(i)   *R and S are Gorenstein.*
(ii)   *R is Gorenstein and φ is quasi-Gorenstein.*
(iii) *S is Gorenstein and φ is of finite G-dimension.*

The following (de)composition result is [13, 7.10, 8.9, and 8.10]. It should be compared to Theorem 1.20.

Theorem 7.9. *Assume that φ is quasi-Gorenstein, and let $\psi\colon S \to T$ be a local homomorphism. The following assertions hold.*

(a) $\operatorname{G-dim}\psi\varphi$ *is finite if and only if* $\operatorname{G-dim}\psi$ *is finite.*
(b) $\psi\varphi$ *is quasi-Gorenstein if and only if ψ is quasi-Gorenstein.*

Quasi-Cohen–Macaulay homomorphisms

The next definition from [59, 5.8 and 6.2] uses terminology from Definition 1.19 and the remarks before Definition 6.6.

Definition 7.10. The homomorphism φ is *quasi-Cohen–Macaulay*, for short *quasi-CM*, if $\grave{\varphi}$ has a Cohen factorization where $\acute{S}$ is quasi-perfect over R'.

If φ is quasi-CM, then $\widehat{S}$ is a quasi-perfect R'-module in every Cohen factorization of $\hat{\varphi}$; see [59, 5.8]. The following theorem is part of [59, 6.7]; it should be compared to part (a) in Theorem G.

Theorem 7.11. *The following assertions hold.*

(a)   If R is Cohen–Macaulay and φ is quasi-CM, then S is Cohen–Macaulay.
(b)   If S is Cohen–Macaulay and G-dim φ is finite, then φ is quasi-CM.

   In view of Theorem 7.9, Frankild's work [59, 6.4 and 6.5] yields:

Theorem 7.12. *Assume that φ is quasi-Gorenstein, and let $\psi\colon S \to T$ be a local homomorphism. Then $\psi\varphi$ is quasi-CM if and only if ψ is quasi-CM.*

Notes

The composition question addressed in the remarks before Theorem 7.4 is investigated further by Sather-Wagstaff [92].

8   Reflexivity and finite G-dimension

In this section, R is Noetherian. Let M be a finitely generated R-module. If M is totally reflexive, then the cohomology modules $\mathrm{Ext}_R^i(M,R)$ vanish for all $i > 0$. The converse is true if M is known *a priori* to have finite G-dimension, cf. Corollary 1.10. In general, though, one can not infer from such vanishing that M is totally reflexive—explicit examples to this effect are constructed by Jorgensen and Şega in [77]—and this has motivated a search for alternative criteria for finiteness of G-dimension.

Reflexive complexes

One such criterion was given by Foxby and published in [114]. Its habitat is the derived category $\mathcal{D}(R)$ of the category of R-modules. The objects in $\mathcal{D}(R)$ are R-complexes, and there is a canonical functor F from the category of R-complexes to $\mathcal{D}(R)$. This functor is the identity on objects and it maps homology isomorphisms to isomorphisms in $\mathcal{D}(R)$. The restriction of F to modules is a full embedding of the module category into $\mathcal{D}(R)$.

   The homology $\mathrm{H}(\boldsymbol{M})$ of an R-complex $\boldsymbol{M}$ is a (graded) R-module, and $\boldsymbol{M}$ is said to have *finitely generated homology* if this module is finitely generated. That is, if every homology module $\mathrm{H}_i(\boldsymbol{M})$ is finitely generated and only finitely many of them are non-zero.

   For R-modules M and N, the (co)homology of the derived Hom and tensor product complexes gives the classical Ext and Tor modules:

$$\mathrm{Ext}_R^i(M,N) = \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(M,N)) \quad \text{and} \quad \mathrm{Tor}_i^R(M,N) = \mathrm{H}_i(M \otimes_R^{\mathbf{L}} N).$$

Definition 8.1. An R-complex M is *reflexive* if M and $\mathbf{R}\mathrm{Hom}_R(M,R)$ have finitely generated homology and the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M,R),R)$$

is an isomorphism in the derived category $\mathcal{D}(R)$. The full subcategory of $\mathcal{D}(R)$ whose objects are the reflexive R-complexes is denoted by $\mathcal{R}(R)$.

The requirement in the definition that the complex $\mathbf{R}\mathrm{Hom}_R(M,R)$ has finitely generated homology is redundant but retained for historical reasons; see [18, 3.3].

Theorem 8.2 below is Foxby's criterion for finiteness of G-dimension of a finitely generated module [114, 2.7]. It differs significantly from Definition 1.8 as it does not involve construction of a G-resolution of the module.

Theorem 8.2. *Let R be Noetherian. A finitely generated R-module has finite G-dimension if and only if it belongs to $\mathcal{R}(R)$.*

If R is local, then the next result is [28, 2.3.14]. In the generality stated below it follows from Theorems 1.18 and 8.2: the implication $(ii) \Rightarrow (iii)$ is the least obvious, it uses [28, 2.1.12].

Corollary 8.3. *Let R be Noetherian. The following conditions are equivalent.*

(i) *R is Gorenstein.*
(ii) *Every R-module is in $\mathcal{R}(R)$.*
(iii) *Every R-complex with finitely generated homology is in $\mathcal{R}(R)$.*

G-dimension of complexes

Having made the passage to the derived category, it is natural to consider G-dimension for complexes. For every R-complex M with finitely generated homology there exists a complex G of finitely generated free R-modules, which is isomorphic to M in $\mathcal{D}(R)$; see [11, 1.7(1)]. In Christensen's [28, Chap. 2] one finds the next definition and the two theorems that follow.

Definition 8.4. Let M be an R-complex with finitely generated homology. If $\mathrm{H}(M)$ is not zero, then the *G-dimension* of M is the least integer n such that there exists a complex G of totally reflexive R-modules which is isomorphic to M in $\mathcal{D}(R)$ and has $G_i = 0$ for all $i > n$. If no such integer n exists, then G-dim$_R M$ is infinite. If $\mathrm{H}(M) = 0$, then G-dim$_R M = -\infty$ by convention.

Note that this extends Definition 1.8. As a common generalization of Theorem 8.2 and Corollary 1.10 one has [28, 2.3.8]:

Theorem 8.5. *Let R be Noetherian. An R-complex M with finitely generated homology has finite G-dimension if an only if it is reflexive. Furthermore, for every reflexive R-complex M there is an equality*

$$\text{G-dim}_R M = \sup\{\, i \in \mathbb{Z} \mid \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(M,R)) \neq 0 \,\}.$$

In broad terms, the theory of G-dimension for finitely generated modules extends to complexes with finitely generated homology. One example is the next extension [28, 2.3.13] of the Auslander–Bridger Formula (Theorem 1.25).

Theorem 8.6. *Let R be local. For every complex* M *in* $\mathcal{R}(R)$ *one has*

$$\text{G-dim}_R M = \text{depth}\, R - \text{depth}_R M.$$

Here, the *depth* of a complex M over a local ring $(R, \mathfrak{m}, k)$ is defined by extension of the definition for modules, that is,

$$\text{depth}_R M = \inf\{\, i \in \mathbb{Z} \mid \text{H}^i(\mathbf{R}\text{Hom}_R(k, M)) \neq 0 \,\}.$$

Notes

In [28, Chap. 2] the theory of G-dimension for complexes is developed in detail.

Generalized notions of G-dimension—based on reflexivity with respect to semidualizing modules and complexes—are studied by Avramov, Iyengar, and Lipman [18], Christensen [29], Frankild and Sather-Wagstaff [60], Gerko [61], Golod [62], Holm and Jørgensen [70], by Salarian, Sather-Wagstaff, and Yassemi [91], and White [112]. See also the notes in Section 1.

9 Detecting finiteness of Gorenstein dimensions

In the previous section, we discussed a resolution-free characterization of modules of finite G-dimension (Theorem 8.2). The topic of this section is similar characterizations of modules of finite Gorenstein projective/injective/flat dimension. By work of Christensen, Frankild, and Holm [31], appropriate criteria are available for modules over a Noetherian ring that has a dualizing complex (Theorems 9.2 and 9.5). As mentioned in the Introduction, a Noetherian ring has a dualizing complex if and only if it is a homomorphic image of a Gorenstein ring of finite Krull dimension. For example, every complete local ring has a dualizing complex by Cohen's structure theorem.

Auslander categories

The next definition is due to Foxby; see [13, 3.1] and [54, Section 2].

Definition 9.1. Let R be Noetherian and assume that it has a dualizing complex D. The *Auslander class* $\mathcal{A}(R)$ is the full subcategory of the derived category $\mathcal{D}(R)$ whose objects M satisfy the following conditions.

(1) $\text{H}_i(M) = 0$ for $|i| \gg 0$.
(2) $\text{H}_i(D \otimes_R^{\mathbf{L}} M) = 0$ for $i \gg 0$.
(3) The natural map $M \to \mathbf{R}\text{Hom}_R(D, D \otimes_R^{\mathbf{L}} M)$ is invertible in $\mathcal{D}(R)$.

The relation to Gorenstein dimensions is given by [31, 4.1]:

Theorem 9.2. *Let R be Noetherian and assume that it has a dualizing complex. For every R-module M, the following conditions are equivalent.*

(i)   *M has finite Gorenstein projective dimension.*
(ii)   *M has finite Gorenstein flat dimension.*
(iii) *M belongs to $\mathcal{A}(R)$.*

Remark 9.3. The equivalence of $(i)/(ii)$ and (iii) in Theorem 9.2 provides a resolution-free characterization of modules of finite Gorenstein projective/flat dimension over a ring that has a dualizing complex. Every complete local ring has a dualizing complex, so in view of Theorem 2.20/4.27 there is a resolution-free characterization of modules of finite Gorenstein projective/flat dimension over any local ring.

The next definition is in [13, 3.1]; the theorem that follows is [31, 4.4].

Definition 9.4. Let R be Noetherian and assume that it has a dualizing complex D. The *Bass class* $\mathcal{B}(R)$ is the full subcategory of the derived category $\mathcal{D}(R)$ whose objects M satisfy the following conditions.

(1) $H^i(M) = 0$ for $|i| \gg 0$.
(2) $H^i(\mathbf{R}\mathrm{Hom}_R(D, M)) = 0$ for $i \gg 0$.
(3) The natural map $D \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(D, M) \to M$ is invertible in $\mathcal{D}(R)$.

Theorem 9.5. *Let R be Noetherian and assume that it has a dualizing complex. For every R-module M, the following conditions are equivalent.*

(i)   *M has finite Gorenstein injective dimension.*
(ii)   *M belongs to $\mathcal{B}(R)$.*

From the two theorems above and from Theorems 3.14 and 4.11 one gets:

Corollary 9.6. *Let R be Noetherian and assume that it has a dualizing complex. The following conditions are equivalent.*

(i)   *R is Gorenstein.*
(ii)   *Every R-complex M with $H_i(M) = 0$ for $|i| \gg 0$ belongs to $\mathcal{A}(R)$.*
(iii) *Every R-complex M with $H_i(M) = 0$ for $|i| \gg 0$ belongs to $\mathcal{B}(R)$.*

Gorenstein dimensions of complexes

It turns out to be convenient to extend the Gorenstein dimensions to complexes; this is illustrated by Theorem 9.11 below.

In the following we use the notion of a semi-projective resolution. Every complex has such a resolution, by [11, 1.6], and a projective resolution of a module is semi-projective. In view of this and Theorem 2.8, the next definition, which is due to Veliche [110, 3.1 and 3.4], extends Definition 2.7.

Definition 9.7. Let M be an R-complex. If $\mathrm{H}(M) \neq 0$, then the *Gorenstein projective dimension* of M is the least integer n such that $\mathrm{H}_i(M) = 0$ for all $i > n$ and there exists a semi-projective resolution P of M for which the module $\mathrm{Coker}(P_{n+1} \to P_n)$ is Gorenstein projective. If no such n exists, then $\mathrm{Gpd}_R M$ is infinite. If $\mathrm{H}(M) = 0$, then $\mathrm{Gpd}_R M = -\infty$ by convention.

In the next theorem, which is due to Iyengar and Krause [75, 8.1], the *unbounded Auslander class* $\widehat{\mathcal{A}}(R)$ is the full subcategory of $\mathcal{D}(R)$ whose objects satisfy conditions (2) and (3) in Definition 9.1.

Theorem 9.8. *Let R be Noetherian and assume that it has a dualizing complex. For every R-complex M, the following conditions are equivalent.*

(i) M *has finite Gorenstein projective dimension.*
(ii) M *belongs to* $\widehat{\mathcal{A}}(R)$.

One finds the next definition in [4, 2.2 and 2.3] by Asadollahi and Salarian. It uses the notion of a semi-injective resolution. Every complex has such a resolution, by [11, 1.6], and an injective resolution of a module is semi-injective. In view of Theorem 3.6, the following extends Definition 3.5.

Definition 9.9. Let M be an R-complex. If $\mathrm{H}(M) \neq 0$, then the *Gorenstein injective dimension* of M is the least integer n such that $\mathrm{H}^i(M) = 0$ for all $i > n$ and there exists a semi-injective resolution I of M for which the module $\mathrm{Ker}(I^n \to I^{n+1})$ is Gorenstein injective. If no such integer n exists, then $\mathrm{Gid}_R M$ is infinite. If $\mathrm{H}(M) = 0$, then $\mathrm{Gid}_R M = -\infty$ by convention.

In the next result, which is [75, 8.2], the *unbounded Bass class* $\widehat{\mathcal{B}}(R)$ is the full subcategory of $\mathcal{D}(R)$ whose objects satisfy (2) and (3) in Definition 9.4.

Theorem 9.10. *Let R be Noetherian and assume that it has a dualizing complex. For every R-complex M, the following conditions are equivalent.*

(i) M *has finite Gorenstein injective dimension.*
(ii) M *belongs to* $\widehat{\mathcal{B}}(R)$.

The next result is [35, 1.7]; it should be compared to Theorem 3.27.

Theorem 9.11. *Let R be local. For every R-module M there is an equality*

$$\mathrm{Gid}_R M = \mathrm{Gid}_{\widehat{R}} \mathbf{R}\mathrm{Hom}_R(\widehat{R}, M).$$

Remark 9.12. Via this result, Theorem 9.10 gives a resolution-free characterization of modules of finite Gorenstein injective dimension over any local ring.

Acyclicity versus total acyclicity

The next results characterize Gorenstein rings in terms of the complexes that define Gorenstein projective/injective/flat modules. The first one is [75, 5.5].

Theorem 9.13. *Let R be Noetherian and assume that it has a dualizing complex. Then the following conditions are equivalent.*

(*i*)   *R is Gorenstein.*
(*ii*) *For every acyclic complex **P** of projective R-modules and every projective R-module Q, the complex* $\mathrm{Hom}_R(\boldsymbol{P}, Q)$ *is acyclic.*
(*iii*) *For every acyclic complex **I** of injective R-modules and every injective R-module E, the complex* $\mathrm{Hom}_R(E, \boldsymbol{I})$ *is acyclic.*

In the terminology of [75], part (*ii*)/(*iii*) above says that every acyclic complex of projective/injective modules is totally acyclic.

The final result is due to Christensen and Veliche [37]:

Theorem 9.14. *Let R be Noetherian and assume that it has a dualizing complex. Then there exist acyclic complexes **F** and **I** of flat R-modules and injective R-modules, respectively, such that the following conditions are equivalent.*

(*i*)   *R is Gorenstein.*
(*ii*) *For every injective R-module E, the complex* $E \otimes_R \boldsymbol{F}$ *is acyclic.*
(*iii*) *For every injective R-module E, the complex* $\mathrm{Hom}_R(E, \boldsymbol{I})$ *is acyclic.*

The complexes **F** and **I** in the theorem have explicit constructions. It is not known, in general, if there is an explicit construction of an acyclic complex **P** of projective R-modules such that R is Gorenstein if $\mathrm{Hom}_R(\boldsymbol{P}, Q)$ is acyclic for every projective R-module Q.

Notes

In broad terms, the theory of Gorenstein dimensions for modules extends to complexes. It is developed in detail by Asadollahi and Salarian [4], Christensen, Frankild, and Holm [31] and [32], Christensen and Sather-Wagstaff [35], and by Veliche [110].

Objects in Auslander and Bass classes with respect to semi-dualizing complexes have interpretations in terms of generalized Gorenstein dimensions; see [44] and [45] by Enochs and Jenda, [69] by Holm and Jørgensen, and [92] by Sather-Wagstaff.

Sharif, Sather-Wagstaff, and White [94] study totally acyclic complexes of Gorenstein projective modules. They show that the cokernels of the differentials in such complexes are Gorenstein projective. That is, a "Gorenstein projective" module is Gorenstein projective.

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