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Algebras that satisfy Auslander's condition on vanishing of cohomology

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Abstract Auslander conjectured that every Artin algebra satisfies a certain condition on vanishing of cohomology of finitely generated modules. The failure of this conjecture—by a 2003 counterexample due to Jorgensen and Şega—motivates the consideration of the class of rings that *do* satisfy Auslander's condition. We call them AC rings and show that an AC Artin algebra that is left-Gorenstein is also right-Gorenstein. Furthermore, the Auslander–Reiten Conjecture is proved for AC rings, and Auslander's G-dimension is shown to be functorial for AC rings that are commutative or have a dualizing complex.

Keywords AB ring · AC ring · Conjectures of Auslander, Reiten, and Tachikawa · G-dimension · Gorenstein algebra

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In memory of Anders J. Frankild.

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0 Introduction

The studies of algebras and modules by methods of homological algebra pivot around cohomology groups and functors—in particular, their vanishing. The conjecture of Auslander we refer to in the abstract asserts that every Artin algebra satisfies the condition (AC) defined below. Auslander's conjecture is stronger than the Finitistic Dimension Conjecture and several other long-standing conjectures for finite dimensional algebras—including the Auslander–Reiten and Nakayama Conjectures; see [1, ch. V], [21,42]. In [29] Jorgensen and Şega exhibit a finite dimensional algebra that fails to satisfy (AC), thereby overturning Auslander's conjecture. This makes relevant a subtle point: one knows that if *all* finite dimensional algebras had satisfied (AC), then they would all have finite finitistic dimension, but it is *not* known if a given algebra that satisfies (AC) must have finite finitistic dimension. What *is* known, is that a finite dimensional algebra Λ over a field k has finite finitistic dimension if the enveloping algebra $\Lambda^e = \Lambda \otimes_k \Lambda^\circ$ satisfies (AC); see [21, sec. 1].

Such observations motivate the study of *AC rings*, that is, left-noetherian rings A that satisfy Auslander's condition on vanishing of cohomology:

(AC) For every finitely generated left A-module M there exists an integer $b_M \geq 0$ such that for every finitely generated left A-module N one has: $\operatorname{Ext}_A^{\gg 0}(M,N) = 0$ implies $\operatorname{Ext}_A^{>b_M}(M,N) = 0$.

For certain commutative rings this study was initiated by Huneke and Jorgensen [23]. In this paper we give special attention to problems from Auslander's work in representation theory—including the conjectures mentioned above.

Auslander and Reiten conjectured [4] that a finitely generated module M over an Artin algebra Λ is projective if $\operatorname{Ext}_\Lambda^i(M,M) = 0 = \operatorname{Ext}_\Lambda^i(M,\Lambda)$ for all $i \geq 1$. See Appendix A for a brief survey of this and related conjectures. To facilitate the discussion, we distinguish between *conjectures* (about all algebras) and *conditions* (on a single algebra). Consider the following condition on a left-noetherian ring A:

(ARC) Every finitely generated left A-module M with $\operatorname{Ext}_A^{\geq 1}(M, M \oplus A) = 0$ is projective.

The Auslander–Reiten Conjecture can now be restated as "All Artin algebras satisfy (ARC)". At the level of conjectures, Auslander's conjecture is stronger than the Finitistic Dimension Conjecture, and that one implies the Auslander–Reiten Conjecture. Thus, had *all* algebras satisfied (AC), then one would know that all algebras satisfy (ARC). Theorem A gives new insight at the level of conditions: it implies that *any* given AC ring satisfies (ARC). Our proof of Theorem A avoids considerations of finitistic dimensions, and it remains unknown if every AC Artin algebra has finite finitistic dimension.

Theorem A *Let A be a left-noetherian ring that satisfies* (AC)*, and let M be a finitely generated left A-module. If one has $\operatorname{Ext}_A^{\gg 0}(M,M) = 0$ and $\operatorname{Ext}_A^{\geq 1}(M,A) = 0$, then M is projective.*

This theorem is a special case of our main result 2.3. Notice that the vanishing conditions imposed on M in Theorem A appear to be weaker than those in the Auslander–Reiten Conjecture; we discuss this in 2.4.

It is an open question—also due to Auslander and Reiten [5]—whether an Artin algebra is left-Gorenstein if and only if it is right-Gorenstein. This is known as the Gorenstein Symmetry Question; the next partial answer is proved in 3.2 and 3.5.

Theorem B *Let A be a two-sided noetherian ring. If A and A° satisfy* (AC) *and*

(1) *A is an Artin algebra, or*

(2) *A has a dualizing complex* (*as defined in* [13]),

then $\mathrm{id}_A A < \infty$ *if and only if* $\mathrm{id}_{A^\circ} A < \infty$ (*whence,* $\mathrm{id}_A A = \mathrm{id}_{A^\circ} A$ *by* [26]).

We do not know if every Artin algebra has a dualizing complex, but every finite dimensional k-algebra does have one, cf. 3.4.

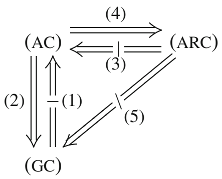
To study the module category of a Gorenstein ring, Auslander and Bridger [2] introduced the so-called G-dimension. A finitely generated left module $M \neq 0$ over a two-sided noetherian ring A is of G-dimension 0 if it is reflexive and $\mathrm{Ext}^i_A(M, A) = 0 = \mathrm{Ext}^i_{A^\circ}(\mathrm{Hom}_A(M, A), A)$ for all $i \geq 1$. Implicit in their work is the question whether all two-sided noetherian rings A satisfy the condition:

(GC) Every finitely generated left A-module $M \neq 0$ with $\mathrm{Ext}^{\geq 1}_A(M, A) = 0$ is of G-dimension 0.

By another example of Jorgensen and Şega [30], also this question has a negative answer, even for commutative local finite dimensional k-algebras. The following partial answer is part of 4.4.

Theorem C *Let A be a two-sided noetherian ring that has a dualizing complex* (*as defined in* [13]) *or is commutative. If A satisfies* (AC), *then it satisfies* (GC).

By work of Huneke et al. [25], the Auslander–Reiten Conjecture holds for commutative noetherian local rings with radical cube zero, and the counterexamples in [29, 30] show that such rings need not satisfy (AC) or (GC). Here is a summary in diagram form:



(1) [2, thm. (4.13) and (4.20)] and [29, cor. 3.3(1)];
(2) Theorem C, for two-sided noetherian rings that have a dualizing complex or are commutative;
(3) [29, cor. 3.3(2)] and [25, thm. 4.1(1)];
(4) Theorem A;
(5) [30, thm. 1.7] and [25, thm. 4.1(1)].

Theorems A, B, and C are proved in Sects. 2–4. In Sect. 5 we discuss simple procedures for generating new AC rings from existing ones.

Appendix A recapitulates certain aspects of the homological conjectures for finite dimensional k-algebras in order to place the present work in proper perspective.

Theorem C relies on a technical result, Lemma 4.1, which owes an intellectual debt to work of Huneke and Jorgensen [23]. Combined with other techniques, 4.1 yields new proofs and modest generalizations of the main result in [23] on symmetric Ext-vanishing over commutative noetherian Gorenstein AC rings; these are given in Appendix B.

Many of our proofs use the derived category over a ring. In the next section we recall the (standard) notation used throughout the paper.

1 Prerequisites

Throughout, A denotes a left-noetherian ring which is an algebra over a commutative ring $\Bbbk$ (e.g. $\Bbbk = \mathbb{Z}$), and A° is the opposite ring. The letter k denotes a field, and Λ denotes a finite dimensional k-algebra or, more generally, an Artin algebra.

1.1 Modules (over A or Λ) are left modules, unless otherwise specified. We write $\mathsf{Mod}(A)$ for the category of all A-modules and $\mathsf{mod}(A)$ for the full subcategory of finitely generated A-modules.

For M and N in $\mathsf{Mod}(A)$, the notation $\operatorname{Ext}_A^{\geq n}(M, N) = 0$ means that $\operatorname{Ext}_A^i(M, N)$ vanish for all $i \geq n$. We write $\operatorname{Ext}_A^{\gg 0}(M, N) = 0$ if $\operatorname{Ext}_A^{\geq n}(M, N) = 0$ for some n. For M in $\mathsf{mod}(A)$, a number b_M with the property required in (AC), see Sect. 0, is called an *Auslander bound* for M. We also consider rings A over which there is a uniform Auslander bound for all M in $\mathsf{mod}(A)$, i.e. rings that satisfy:

(UAC) There is a $b \geq 0$ such that for all finitely generated A-modules M and N one has: $\operatorname{Ext}_A^{\gg 0}(M, N) = 0$ implies $\operatorname{Ext}_A^{>b}(M, N) = 0$.

In [23] the smallest integer b with this property is called the Ext-index of A.

1.2 A complex of A-modules is graded homologically,

$$M = \cdots \longrightarrow M_{\nu+1} \xrightarrow{\partial^M_{\nu+1}} M_\nu \xrightarrow{\partial^M_\nu} M_{\nu-1} \longrightarrow \cdots,$$

and, for short, called an *A-complex*. The suspension of M is the complex ΣM with $(\Sigma M)_\nu = M_{\nu-1}$ and $\partial^{\Sigma M} = -\partial^M$. With the notation

$$\mathrm{C}_\nu(M) = \operatorname{Coker} \partial^M_{\nu+1} \quad \text{and} \quad \mathrm{Z}_\nu(M) = \operatorname{Ker} \partial^M_\nu,$$

soft truncations of M are defined as

$$\begin{aligned} M_{\subset u} &= 0 \to \mathrm{C}_u(M) \to M_{u-1} \to M_{u-2} \to \cdots \text{ and} \\ M_{\supset w} &= \cdots \to M_{w+2} \to M_{w+1} \to \mathrm{Z}_w(M) \to 0. \end{aligned}$$

The hard truncations of M are defined as

$$M_{\leq u} = 0 \to M_u \to M_{u-1} \to \cdots \quad \text{and} \quad M_{\geq w} = \cdots \to M_{w+1} \to M_w \to 0.$$

We say that M is *left-bounded* if $M_\nu = 0$ for $\nu \gg 0$, *right-bounded* if $M_\nu = 0$ for $\nu \ll 0$, and *bounded* if $M_\nu = 0$ for $|\nu| \gg 0$. If the homology complex $\mathrm{H}(M)$ is (left/right-) bounded, then M is said to be *homologically (left/right-)bounded*. The notation $\sup M$ and $\inf M$ is used for the supremum and infimum of the set $\{\nu \in \mathbb{Z} \mid \mathrm{H}_\nu(M) \neq 0\}$ with the conventions that $\sup \varnothing = -\infty$ and $\inf \varnothing = \infty$.

A morphism α of complexes is called a *quasiisomorphism*, and marked by the symbol $\simeq$, if it induces an isomorphism on the level of homology. The mapping cone of α is denoted $\operatorname{Cone} \alpha$. Recall that the complex $\operatorname{Cone} \alpha$ is exact if and only if α is a quasiisomorphism. Quasiisomorphisms between A-complexes are isomorphisms in the derived category $\mathsf{D}(A)$. Isomorphisms in $\mathsf{D}(A)$ are also marked by the symbol $\simeq$.

1.3 We use standard notation, $\mathbf{R}\mathrm{Hom}_A(-,-)$ and $- \otimes^{\mathbf{L}}_A -$, for the right derived Hom functor and the left derived tensor product functor; see [41, ch. 10]. Recall that for all A-modules M and N and all A°-modules K there are isomorphisms

$$\operatorname{Ext}^i_A(M, N) \cong \mathrm{H}_{-i}\, \mathbf{R}\mathrm{Hom}_A(M, N) \quad \text{and} \quad \operatorname{Tor}^A_i(K, M) \cong \mathrm{H}_i(K \otimes^{\mathbf{L}}_A M).$$

Resolutions of complexes, projective dimension (pd), and injective dimension (id) are treated in [9]. We make frequent use of the following: Every homologically left-bounded complex has a left-bounded injective resolution; every homologically right-bounded complex M has a right-bounded free resolution L, and if M has degreewise finitely generated homology, then L can be taken to be degreewise finitely generated. In particular, every homologically right-bounded complex M has a projective resolution and the projective dimension is given as:

$$\mathrm{pd}_R\, M = \inf\,\{\sup\{i \in \mathbb{Z} \mid P_i \neq 0\} \mid P \text{ is a projective resolution of } M\}\,.$$

The injective dimension of a complex is defined similarly.

Lemma 1.4 *Let X and Y be A-complexes. Assume that X is homologically right-bounded and let $P \xrightarrow{\simeq} X$ be a projective resolution; assume that Y is homologically left-bounded and let $Y \xrightarrow{\simeq} I$ be an injective resolution. If $\mathbf{R}\mathrm{Hom}_A(X, Y)$ is homologically bounded and $s \geq \sup X$, then $\mathrm{Ext}_A^{\geq 1}(\mathrm{C}_s(P), \mathrm{Z}_\nu(I)) = 0$ for all $0 \gg \nu$.*

Proof Let $s \geq \sup X$ and note that $P_{\subset s} \simeq X$ in $\mathsf{D}(A)$. Application of $\mathbf{R}\mathrm{Hom}_A(-, Y)$ to the distinguished triangle in $\mathsf{D}(A)$,

$$P_{\leq s-1} \longrightarrow P_{\subset s} \longrightarrow \Sigma^s \mathrm{C}_s(P) \longrightarrow,$$

induces a long exact sequence of homology modules, which yields isomorphisms

$$\begin{aligned}\mathrm{H}_{\nu+1}\, \mathbf{R}\mathrm{Hom}_A(P_{\leq s-1}, Y) &\cong \mathrm{H}_\nu\, \mathbf{R}\mathrm{Hom}_A(\Sigma^s \mathrm{C}_s(P), Y)\\ &\cong \mathrm{H}_{\nu+s}\, \mathbf{R}\mathrm{Hom}_A(\mathrm{C}_s(P), Y),\end{aligned}$$

for $\nu + 1 < \inf \mathbf{R}\mathrm{Hom}_A(X, Y)$. Obviously, $\mathrm{pd}_A(P_{\leq s-1}) \leq s - 1$ and, therefore,

$$\inf \mathbf{R}\mathrm{Hom}_A(P_{\leq s-1}, Y) \geq \inf Y - (s - 1);$$

see [9, thm. 2.4.P]. Set $w = \min\{\inf Y, \inf \mathbf{R}\mathrm{Hom}_A(X, Y) + s - 1\}$; it follows that

$$\mathrm{H}_\nu\, \mathbf{R}\mathrm{Hom}_A(\mathrm{C}_s(P), Y) = 0 \quad \text{for all} \quad \nu < w. \tag{1.4.1}$$

If $\nu \leq w$, then $\nu \leq \inf Y$, so there is an isomorphism of module functors

$$\mathrm{Ext}_A^i(-, \mathrm{Z}_\nu(I)) \cong \mathrm{H}_{\nu-i}\, \mathbf{R}\mathrm{Hom}_A(-, Y)$$

for every $i > 0$, cf. [12, proof of lem. (6.1.12)]. In particular,

$$\mathrm{Ext}_A^i(\mathrm{C}_s(P), \mathrm{Z}_\nu(I)) \cong \mathrm{H}_{\nu-i}\, \mathbf{R}\mathrm{Hom}_A(\mathrm{C}_s(P), Y) = 0$$

for all $i > 0$, where the last equality follows from (1.4.1). □

2 The Auslander–Reiten conjecture

In this section we prove Theorem A from Sect. 0. We open with a technical lemma.

Lemma 2.1 *Assume that A satisfies* (AC). *Let U be an exact A-complex and C be a finitely generated A-module. If*

(a) *U_ν is finitely generated for all $\nu \gg 0$,*
(b) *$\mathrm{Ext}_A^{\geq 1}(C, U_\nu) = 0$ for all $\nu \in \mathbb{Z}$, and*
(c) *there exists a $w \in \mathbb{Z}$ such that $\mathrm{Ext}_A^{\gg 0}(C, \mathrm{Z}_w(U)) = 0$,*

then $\mathrm{Ext}_A^{\geq 1}(C, \mathrm{Z}_\nu(U)) = 0$ for all $\nu \in \mathbb{Z}$. In particular, $\mathrm{Hom}_A(C, U)$ is exact.

Proof Apply $\operatorname{Hom}_A(C,-)$ to $0 \to \mathrm{Z}_{\nu+1}(U) \to U_{\nu+1} \to \mathrm{Z}_\nu(U) \to 0$, then (b) yields

$$\operatorname{Ext}^i_A(C,\mathrm{Z}_\nu(U)) \cong \operatorname{Ext}^{i+n}_A(C,\mathrm{Z}_{\nu+n}(U)) \text{ for all } \nu \in \mathbb{Z}, i>0, \text{ and } n \geq 0. \tag{2.1.1}$$

If $\nu \geq w$, then $\operatorname{Ext}^{\gg 0}_A(C,\mathrm{Z}_\nu(U))=0$. Indeed, (2.1.1) yields isomorphisms

$$\operatorname{Ext}^{i+\nu-w}_A(C,\mathrm{Z}_\nu(U)) \cong \operatorname{Ext}^{i+(\nu-w)}_A(C,\mathrm{Z}_{w+(\nu-w)}(U)) \cong \operatorname{Ext}^i_A(C,\mathrm{Z}_w(U)),$$

for $i>0$, and the right-most Ext group vanishes by (c) for $i \gg 0$. By (a) there is an integer t such that $\mathrm{Z}_\nu(U)$ is finitely generated for $\nu \geq t$. As A satisfies (AC),

$$\operatorname{Ext}^{>b}_A(C,\mathrm{Z}_\nu(U))=0 \text{ for all } \nu \geq m = \max\{t,w\}, \tag{2.1.2}$$

where b is an Auslander bound for C. To see that $\operatorname{Ext}^{\geq 1}_A(C,\mathrm{Z}_\nu(U))=0$ for all ν, consider the cases $\nu \geq m-b$ and $\nu \leq m-b$ separately. In the following, let $i>0$. If $\nu \geq m-b$, then

$$\operatorname{Ext}^i_A(C,\mathrm{Z}_\nu(U)) \cong \operatorname{Ext}^{i+b}_A(C,\mathrm{Z}_{\nu+b}(U))=0$$

by (2.1.1) and (2.1.2). If $\nu \leq m-b$ then, in particular, $m-\nu \geq b \geq 0$, and thus

$$\operatorname{Ext}^i_A(C,\mathrm{Z}_\nu(U)) \cong \operatorname{Ext}^{i+(m-\nu)}_A(C,\mathrm{Z}_{\nu+(m-\nu)}(U)) = \operatorname{Ext}^{i+m-\nu}_A(C,\mathrm{Z}_m(U))=0;$$

again by (2.1.1) and (2.1.2). □

Remark 2.2 The lemma above may fail for rings that do not satisfy (AC). Indeed, one counterexample to Auslander's conjecture is a commutative local self-injective finite dimensional k-algebra R for which there exist finitely generated modules C and Z, such that $\operatorname{Ext}^i_R(C,Z) \neq 0$ if and only if $i=0,1$; see [29, cor. 3.3.(1)]. Because R is self-injective, the modules C and Z have G-dimension 0; see [2, prop. 3.8]. Let U be a complete projective resolution of Z, see [12, thm. (4.1.4)], then U and C satisfy the requirements in Lemma 2.1, but $\operatorname{Ext}^1_R(C,Z) \neq 0$.

Theorem A in Sect. 0 is an immediate consequence of the next result.

Theorem 2.3 *Assume that A satisfies* (AC)*, and let M be an A-complex. If M has bounded and degreewise finitely generated homology, and $\mathbf{R}\operatorname{Hom}_A(M, M \oplus A)$ is homologically bounded, then M has finite projective dimension given by*

$$\operatorname{pd}_A M = -\inf \mathbf{R}\operatorname{Hom}_A(M,A) < \infty.$$

Proof We may assume that $M \not\simeq 0$ in $\mathsf{D}(A)$. We need only prove that $\operatorname{pd}_A M$ is finite, then a standard argument yields the equality displayed above; see the proof of [12, prop. (2.3.10)]. Take a right-bounded resolution $L \xrightarrow{\simeq} M$ by finitely generated free A-modules and consider the integer

$$s = \max\{-\inf \mathbf{R}\operatorname{Hom}_A(M,A),\ \sup M\}.$$

We will show that the cokernel $\mathrm{C}_s(L)$ is projective, i.e. $\operatorname{Ext}^1_A(\mathrm{C}_s(L),\mathrm{C}_{s+1}(L))=0$. To this end, take an injective resolution $M \xrightarrow{\simeq} I$ with $I_\nu = 0$ for $\nu > \sup M$; see [9, cor. 2.7.I]. Since $\mathbf{R}\operatorname{Hom}_A(M,M)$ is homologically bounded, there is by Lemma 1.4 an integer $u \leq \inf M$ such that

$$\operatorname{Ext}^{\geq 1}_A(\mathrm{C}_s(L),\mathrm{Z}_u(I))=0. \tag{2.3.1}$$

There are quasiisomorphisms

$$L \xrightarrow{\simeq} M \xrightarrow{\simeq} I \xleftarrow{\simeq} I_{\supset u},$$

so by [9, 1.4.P] there is a quasiisomorphism $\alpha\colon L \xrightarrow{\simeq} I_{\supset u}$. We claim that Lemma 2.1 applies to $U = \operatorname{Cone}\alpha$ and the finitely generated module $C = \mathrm{C}_s(L)$. Requirement 2.1(a) is clearly met, and so is 2.1(c), as $\operatorname{Cone}\alpha$ is right-bounded. To verify 2.1(b) it suffices, in view of (2.3.1), to show that $\operatorname{Ext}_A^{\geq 1}(\mathrm{C}_s(L), A) = 0$, and this follows as

$$\operatorname{Ext}_A^i(\mathrm{C}_s(L), A) \cong \mathrm{H}_{-(i+s)}\,\mathbf{R}\mathrm{Hom}_A(M, A) = 0 \text{ for all } i > 0;$$

cf. [12, prf. of (4.3.9)]. In particular, Lemma 2.1 gives $\operatorname{Ext}_A^{\geq 1}(\mathrm{C}_s(L), \mathrm{Z}_{s+1}(\operatorname{Cone}\alpha)) = 0$, and by the choice of I we have $\mathrm{Z}_{s+1}(\operatorname{Cone}\alpha) = \mathrm{C}_{s+1}(L)$. □

Remark 2.4 The condition (ARC) and Theorem A in Sect. 0 draw identical conclusions from apparently different assumptions on a finitely generated A-module M, namely:

(a) $\operatorname{Ext}_A^{\geq 1}(M, M \oplus A) = 0$; compared to

(b) $\operatorname{Ext}_A^{\gg 0}(M, M) = 0$ and $\operatorname{Ext}_A^{\geq 1}(M, A) = 0$.

Clearly, (a) implies (b). We do not know if the two are equivalent, not even if A is commutative local and Gorenstein. Theorem 2.3 shows that if A is AC, then (a) and (b) are equivalent. A much stronger result holds if A is commutative local and complete intersection, then $\operatorname{Ext}_A^{\geq 1}(M, A) = 0$ and vanishing of $\operatorname{Ext}_A^{2i}(M, M)$ for a single integer $i > 0$ implies that M is free; see [7, thm. 4.2]. If A is commutative local (AC or not) with radical cube zero, then vanishing of $\operatorname{Ext}_A^i(M, M \oplus A)$ for four consecutive values of $i \geq 2$ implies that M is free; see [25, thm. 4.1].

3 The Gorenstein symmetry question

For a two-sided noetherian ring A, we do not know if Auslander's condition is symmetric, that is, if A and A° satisfy (AC) simultaneously. For Artin algebras, however, the uniform condition (UAC), defined in 1.1, is symmetric.

Observation 3.1 Let Λ be an Artin algebra. The canonical duality functor

$$\mathrm{D}\colon \mathsf{mod}(\Lambda^{\circ}) \longrightarrow \mathsf{mod}(\Lambda),$$

see [6, thm. II.3.3], provides isomorphisms

$$\operatorname{Ext}_{\Lambda^{\circ}}^i(M, N) \cong \operatorname{Ext}_{\Lambda}^i(\mathrm{D}(N), \mathrm{D}(M))$$

for all finitely generated Λ°-modules M and N and all integers i. This shows that Λ° satisfies (UAC) if and only if Λ does.

Auslander and Reiten [5] raise the question whether an Artin algebra is left-Gorenstein if and only if it is right-Gorenstein. The next proposition contains part (1) of Theorem B from Sect. 0, and it uses [5, prop. 6.10] to establish an "algebra-wise" relation between Auslander's conjecture and the Finitistic Dimension Conjecture.

Proposition 3.2 *Let Λ be an Artin algebra that satisfies* (AC). *If* $\operatorname{id}_{\Lambda}\Lambda$ *is finite, then* $\operatorname{id}_{\Lambda^{\circ}}\Lambda$ *and the finitistic dimension of Λ (on both sides)*[1] *is finite.*

[1] In general, it is not known if the left-finitistic dimension of a finite dimensional algebra is finite if the right-finitistic dimension is, but one knows that they may differ; see [28, exa. 2.2].

Proof The finitely generated Λ-module $\mathrm{D}(\Lambda_\Lambda)$ is injective. Set $n = \mathrm{id}_\Lambda \Lambda$, then

$$\mathrm{Ext}_\Lambda^{>n}(\mathrm{D}(\Lambda_\Lambda), \mathrm{D}(\Lambda_\Lambda) \oplus {}_\Lambda\Lambda) = 0,$$

so it follows from Theorem 2.3 that $\mathrm{pd}_\Lambda \mathrm{D}(\Lambda_\Lambda)$ is at most n. For every finitely generated Λ°-module N, the isomorphism from Observation 3.1 yields

$$\mathrm{Ext}^i_{\Lambda^\circ}(N_\Lambda, \Lambda_\Lambda) \cong \mathrm{Ext}^i_\Lambda(\mathrm{D}(\Lambda_\Lambda), \mathrm{D}(N_\Lambda)) = 0 \;\text{ for }\; i > n,$$

whence $\mathrm{id}_{\Lambda^\circ} \Lambda \leq n$. Now the finitistic dimension of Λ is finite by [5, prop. 6.10]. □

Remark 3.3 For an Artin algebra Λ that satisfies (UAC), it follows from Observation 3.1 and Proposition 3.2 that $\mathrm{id}_\Lambda \Lambda$ is finite if and only if $\mathrm{id}_{\Lambda^\circ} \Lambda$ is finite.

Nagata's regular ring of infinite Krull dimension [35, ex. 1, p. 203] is an example of a commutative noetherian ring that satisfies (AC) but not (UAC). However, in the realm of Artin algebras (or local rings) we do not know of such an example.

Part (2) of Theorem B is a special case of Proposition 3.5, which addresses a natural generalization of the conditions (TC1) and (ABŞC) discussed in Appendix A.

3.4 Let B be a right-noetherian ring, which is also a $\Bbbk$-algebra; Proposition 3.5 involves a dualizing complex ${}_AD_B$ for the pair $\langle A, B\rangle$ in the sense of [13, def. 1.1]. That is,

(1) The complex D has bounded and degreewise finitely generated homology over A and over B°.

(2) There exists a quasi-isomorphism of complexes of bimodules, ${}_AP_B \xrightarrow{\simeq} {}_AD_B$, where ${}_AP_B$ is right-bounded and consists of modules that are projective over A and over B°.

(3) There exists a quasi-isomorphism of complexes of bimodules, ${}_AD_B \xrightarrow{\simeq} {}_AI_B$, where ${}_AI_B$ is bounded and consists of modules that are injective over A and over B°.

(4) The homothety morphisms

$$ {}_AA_A \longrightarrow \mathbf{R}\mathrm{Hom}_{B^\circ}({}_AD_B, {}_AD_B) \quad\text{and}\quad {}_BB_B \longrightarrow \mathbf{R}\mathrm{Hom}_A({}_AD_B, {}_AD_B),$$

are isomorphisms in homology.

If A is two-sided noetherian, then a dualizing complex for $\langle A, A\rangle$ is called a dualizing complex for A. This generalizes the definition for commutative rings in [22, V. Sect. 2].

We do not know if every Artin $\Bbbk$-algebra Λ has a dualizing complex. To be precise, we do not know if the obvious candidate $D = \mathrm{Hom}_\Bbbk({}_\Lambda\Lambda_\Lambda, \Bbbk)$ has a resolution by Λ-bimodules, as required in (2). If $\Bbbk$ is a field, however, this D is a dualizing complex for Λ; see [43, exa. 2.3(b)] and [13, app. A].

Proposition 3.5 *Let the rings A and B be as in 3.4, and let D be a dualizing complex for the pair $\langle A, B\rangle$. The complexes $\mathbf{R}\mathrm{Hom}_A(D, A)$ and $\mathbf{R}\mathrm{Hom}_{B^\circ}(D, B)$ are isomorphic in $\mathsf{D}(\Bbbk)$, and when they are homologically bounded, the following hold*:

(a) *If A satisfies* (AC)*, then $\mathrm{id}_{A^\circ} A$ and $\mathrm{id}_{B^\circ} B$ are at most $\mathrm{pd}_A D + \mathrm{id}_{B^\circ} D < \infty$.*

(b) *If B° satisfies* (AC)*, then $\mathrm{id}_A A$ and $\mathrm{id}_B B$ are at most $\mathrm{pd}_{B^\circ} D + \mathrm{id}_A D < \infty$.*

Proof The first assertion is an elementary application of swap in $\mathsf{D}(\Bbbk)$:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_A({}_AD, {}_AA) &\simeq \mathbf{R}\mathrm{Hom}_A({}_AD, \mathbf{R}\mathrm{Hom}_{B^\circ}(D_B, {}_AD_B))\\ &\simeq \mathbf{R}\mathrm{Hom}_{B^\circ}(D_B, \mathbf{R}\mathrm{Hom}_A({}_AD, {}_AD_B))\\ &\simeq \mathbf{R}\mathrm{Hom}_{B^\circ}(D_B, B_B).\end{aligned}$$

By symmetry it suffices to prove part (a). As $\mathbf{R}\mathrm{Hom}_A(D, A)$ is homologically bounded, it follows from Theorem 2.3 that $\mathrm{pd}_A D$ is finite. For every A°-module M we have

$$\begin{aligned} -\inf \mathbf{R}\mathrm{Hom}_{A^\circ}(M_A, A_A) &= -\inf \mathbf{R}\mathrm{Hom}_{A^\circ}(M_A, \mathbf{R}\mathrm{Hom}_{B^\circ}({}_AD_B, D_B)) \\ &= -\inf \mathbf{R}\mathrm{Hom}_{B^\circ}(M_A \otimes^{\mathbf{L}}_A {}_AD_B, D_B) \\ &\leq \mathrm{id}_{B^\circ} D + \sup (M_A \otimes^{\mathbf{L}}_A {}_AD_B) \\ &\leq \mathrm{id}_{B^\circ} D + \mathrm{pd}_A D, \end{aligned}$$

where the inequalities are by [9, thm. 2.4.I and 2.4.F]. Thus, $\mathrm{id}_{A^\circ} A$ is at most $\mathrm{id}_{B^\circ} D + \mathrm{pd}_A D$ by [9, thm. 2.4.I]. Similarly, for every B°-module N we have

$$\begin{aligned} -\inf \mathbf{R}\mathrm{Hom}_{B^\circ}(N_B, B_B) &= -\inf \mathbf{R}\mathrm{Hom}_{B^\circ}(N_B, \mathbf{R}\mathrm{Hom}_A({}_AD, {}_AD_B)) \\ &= -\inf \mathbf{R}\mathrm{Hom}_A({}_AD, \mathbf{R}\mathrm{Hom}_{B^\circ}(N_B, {}_AD_B)) \\ &\leq \mathrm{pd}_A D - \inf \mathbf{R}\mathrm{Hom}_{B^\circ}(N_B, {}_AD_B) \\ &\leq \mathrm{pd}_A D + \mathrm{id}_{B^\circ} D; \end{aligned}$$

this time by [9, thm. 2.4.P and 2.4.I]. □

4 Functoriality of G-dimension

Now we prove Theorem C from Sect. 0; our proof hinges on the following lemma about invertibility of the tensor evaluation morphism; cf. [9, 4.3].

Lemma 4.1 *Let M and N be A-complexes and T be an A-bimodule. Assume that M and N have bounded and degreewise finitely generated homology and that ${}_AT$ is finitely generated. Consider the tensor evaluation morphism in $\mathsf{D}(\Bbbk)$:*

$$\omega_{MTN}\colon \mathbf{R}\mathrm{Hom}_A(M, T) \otimes^{\mathbf{L}}_A N \longrightarrow \mathbf{R}\mathrm{Hom}_A(M, T \otimes^{\mathbf{L}}_A N).$$

If A satisfies (AC) *and the three complexes*

$$\mathbf{R}\mathrm{Hom}_A(M, T), \quad T \otimes^{\mathbf{L}}_A N, \quad \textit{and} \quad \mathbf{R}\mathrm{Hom}_A(M, T \otimes^{\mathbf{L}}_A N)$$

are homologically bounded, then ω_{MTN} is an isomorphism.

The lemma may fail if A does not satisfy (AC); see remarks after the proof.

Proof Take right-bounded resolutions $P \xrightarrow{\simeq} M$ and $Q \xrightarrow{\simeq} N$ by finitely generated free A-modules. The goal is to prove that the tensor evaluation morphism ω_{PTQ} is a quasiisomorphism in the category of $\Bbbk$-complexes. This is achieved as follows: As $T \otimes^{\mathbf{L}}_A N$ is homologically bounded, we may take a left-bounded injective resolution $\rho\colon T \otimes_A Q \xrightarrow{\simeq} I$. Set $s = \max\{\sup M, -\inf \mathbf{R}\mathrm{Hom}_A(M, T)\}$; it is an integer as we are free to assume $M \not\simeq 0$ in $\mathsf{D}(A)$. There is now a quasiisomorphism $\tau\colon P \xrightarrow{\simeq} P_{\subset s}$. Consider the commutative diagram in the category of $\Bbbk$-complexes

$$\begin{array}{ccc}
\operatorname{Hom}_A(P,T)\otimes_A Q & \xrightarrow{\omega_{PTQ}} & \operatorname{Hom}_A(P,T\otimes_A Q)\\
\uparrow{\scriptstyle \operatorname{Hom}_A(\tau,T)\otimes_A Q} & & \downarrow{\scriptstyle \simeq\ \operatorname{Hom}_A(P,\rho)}\\
\operatorname{Hom}_A(P_{\subset s},T)\otimes_A Q & & \operatorname{Hom}_A(P,I)\\
\downarrow{\scriptstyle \omega_{P_{\subset s}TQ}\ \cong} & & \uparrow{\scriptstyle \simeq\ \operatorname{Hom}_A(\tau,I)}\\
\operatorname{Hom}_A(P_{\subset s},T\otimes_A Q) & \xrightarrow{\operatorname{Hom}_A(P_{\subset s},\rho)} & \operatorname{Hom}_A(P_{\subset s},I).
\end{array}$$

The vertical morphisms on the right are clearly quasiisomorphisms, and the tensor evaluation morphism $\omega_{P_{\subset s}TQ}$ is easily seen to be invertible, cf. [14, prop. 2.1(v)]. It remains to prove that $\operatorname{Hom}_A(\tau,T)\otimes_A Q$ and $\operatorname{Hom}_A(P_{\subset s},\rho)$ are quasiisomorphisms.

For the first one, it is sufficient to demonstrate exactness of

$$\operatorname{Cone}\operatorname{Hom}_A(\tau,T)\cong \Sigma\operatorname{Hom}_A(\operatorname{Cone}\tau,T).$$

Since the complex $\operatorname{Cone}\tau$ is exact and right-bounded, it is enough to argue that

$$\operatorname{Ext}_A^{\geq 1}((\operatorname{Cone}\tau)_\nu,T)=0 \text{ for all } \nu\in\mathbb{Z}.$$

For $\nu\neq s$ this is clear, as the module $(\operatorname{Cone}\tau)_\nu$ is projective. Since $(\operatorname{Cone}\tau)_s=\mathrm{C}_s(P)\oplus P_{s-1}$, the case $\nu=s$ follows from the isomorphisms

$$\operatorname{Ext}_A^i(\mathrm{C}_s(P),T)\cong \mathrm{H}_{-(s+i)}\,\mathbf{R}\operatorname{Hom}_A(M,T)=0 \text{ for all } i>0, \tag{4.1.1}$$

which are immediate by the choice of s; cf. [12, proof of lem. (4.3.9)].

To see that $\operatorname{Hom}_A(P_{\subset s},\rho)$ is a quasiisomorphism, it suffices by [13, prop. 2.6(a)] to argue that $\operatorname{Hom}_A((P_{\subset s})_\nu,\rho)$ is a quasiisomorphism for all $\nu\in\mathbb{Z}$. For $\nu\neq s$ this is clear, as $(P_{\subset s})_\nu$ is projective. Since $(P_{\subset s})_s=\mathrm{C}_s(P)$, the case $\nu=s$ is equivalent to exactness of

$$\operatorname{Cone}\operatorname{Hom}_A(\mathrm{C}_s(P),\rho)\cong\operatorname{Hom}_A(\mathrm{C}_s(P),\operatorname{Cone}\rho).$$

To complete the proof we show that Lemma 2.1 applies to the complex $\operatorname{Cone}\rho$ and the finitely generated module $\mathrm{C}_s(P)$. Since $(\operatorname{Cone}\rho)_\nu=I_\nu\oplus(T\otimes_A Q)_{\nu-1}$, where $(T\otimes_A Q)_{\nu-1}$ is a finite direct sum of copies of ${}_AT$, it follows from (4.1.1) that requirement 2.1(b) is fulfilled. Furthermore, since $I_\nu=0$ for $\nu\gg 0$ also 2.1(a) is met. Finally, homological boundedness of $\mathbf{R}\operatorname{Hom}_A(M,T\otimes_A^{\mathbf{L}}N)$ implies by Lemma 1.4 that $\operatorname{Ext}_A^{\geq 1}(\mathrm{C}_s(P),\mathrm{Z}_\nu(I))=0$ for all $\nu\ll 0$. Since $\mathrm{Z}_\nu(\operatorname{Cone}\rho)=\mathrm{Z}_\nu(I)$ for $\nu\ll 0$, also requirement 2.1(c) is fulfilled. □

In [29] is given an example of a self-injective finite dimensional k-algebra that does not satisfy (AC), so it follows from the next proposition that Lemma 4.1 may fail for a ring that does not satisfy (AC).

For a Gorenstein ring—i.e. a two-sided noetherian ring with $\operatorname{id}_A A$ and $\operatorname{id}_{A^\circ} A$ finite—the equivalence of (i) and (ii) is proved by Mori [33, thm. 3.3].

Proposition 4.2 *If $\operatorname{id}_A A$ is finite, then the following conditions are equivalent*:

(i) *A satisfies* (AC).

(ii) *A satisfies* (UAC).

(iii) *For all A-complexes M and N with bounded and degreewise finitely generated homology one has: if $\mathbf{R}\operatorname{Hom}_A(M,N)$ is homologically bounded, then*

$$\omega_{MAN}\colon \mathbf{R}\operatorname{Hom}_A(M,A)\otimes_A^{\mathbf{L}} N \longrightarrow \mathbf{R}\operatorname{Hom}_A(M,N)$$

is an isomorphism in $\mathsf{D}(\Bbbk)$.

Proof Since $\mathrm{id}_A A$ is finite, the implication (i)⇒(iii) follows by Lemma 4.1. Obviously (ii) implies (i), so it remains to show the implication (iii)⇒(ii).

Let M and N be finitely generated A-modules such that $\mathrm{Ext}_A^{\gg 0}(M, N) = 0$. This means that $\mathbf{R}\mathrm{Hom}_A(M, N)$ is bounded, so by (iii) there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_A(M, A) \otimes_A^{\mathbf{L}} N \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_A(M, N)$$

in $\mathsf{D}(\Bbbk)$. Consequently,

$$\begin{aligned} -\inf \mathbf{R}\mathrm{Hom}_A(M, N) &= -\inf (\mathbf{R}\mathrm{Hom}_A(M, A) \otimes_A^{\mathbf{L}} N) \\ &\leq -\inf \mathbf{R}\mathrm{Hom}_A(M, A) \\ &\leq \mathrm{id}_A A, \end{aligned}$$

where the first inequality follows by [17, lem. 2.1.(2)] and the second by [9, 2.4.I]. This shows that $\mathrm{Ext}_A^i(M, N) = 0$ for all $i > \mathrm{id}_A A$. □

4.3 One says that the G-dimension is functorial over a two-sided noetherian ring if it satisfies the condition (GC) from Sect. 0. Examples of such rings include:

- Gorenstein rings; see [2, prop. (3.8)].
- Commutative noetherian rings that are locally Gorenstein, see [12, (1.3.2)].
- Local Artin algebras with radical square zero; see [31, prop. 2].
- Commutative noetherian Golod local rings, see [30, prop. 1.4].

The next result establishes Theorem C from Sect. 0, which adds (certain) AC rings to the list above.

Theorem 4.4 *Let A be a two-sided noetherian ring that satisfies* (AC)*, and assume that A has a dualizing complex or is commutative. For every A-complex M with bounded and degreewise finitely generated homology there is an equality*:

$$\text{G-dim}_A M = -\inf \mathbf{R}\mathrm{Hom}_A(M, A).$$

Remark 4.5 Jorgensen and Şega [30] construct a commutative local finite dimensional k-algebra R and a finitely generated R-module M with $\mathrm{Ext}_R^{\geq 1}(M, R) = 0$ but infinite G-dimension. Note that in view of Theorem 4.4, R cannot satisfy (AC). Further, it has length 8 and $\mathfrak{m}^3 = 0$, where $\mathfrak{m}$ is its radical, and thus this example is minimal: Primarily with respect to the invariant $\min\{n \mid \mathfrak{m}^n = 0\}$—as every ring with radical square zero satisfies (AC) by [29, prop. 1.1]. Secondarily with respect to length—as every commutative local artinian ring with radical cube zero and length at most 7 satisfies (AC), also by [29, prop. 1.1].

Proof of 4.4 First assume that A is commutative. It is sufficient to prove that homological boundedness of $\mathbf{R}\mathrm{Hom}_A(M, A)$ implies that the biduality morphism

$$\delta_M^A \colon M \longrightarrow \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, A), A)$$

is an isomorphism in $\mathsf{D}(A)$; see [12, cor. (2.3.8)]. This can be verified locally, as $(\delta_M^A)_{\mathfrak{p}} = \delta_{M_{\mathfrak{p}}}^{A_{\mathfrak{p}}}$ for all $\mathfrak{p}$ in Spec A, so we may assume that A is local.

Now, let K be the Koszul complex on a set of generators for the maximal ideal $\mathfrak{m}$, and let E be the injective hull of $A/\mathfrak{m}$. As the complex $\mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, A), A)$ has degreewise finitely generated homology, it follows from [19, 1.3] that δ_M^A is an isomorphism if $\delta_M^A \otimes_A^{\mathbf{L}} K$

is one. Set $J = \operatorname{Hom}_A(K, E)$, and note that this is a bounded complex of injective modules and has homology modules of finite length. By Lemma 4.1 there is an isomorphism:

$$\omega_{MAJ}\colon \mathbf{R}\mathrm{Hom}_A(M, A) \otimes^{\mathbf{L}}_A J \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_A(M, J).$$

Furthermore, as K has homology modules of finite length, the biduality morphism

$$\delta^E_K\colon K \to \operatorname{Hom}_A(\operatorname{Hom}_A(K, E), E)$$

is an isomorphism in $\mathsf{D}(A)$. The target complex is isomorphic to $\mathbf{R}\mathrm{Hom}_A(J, E)$, and there is a commutative diagram in $\mathsf{D}(A)$

$$\begin{array}{ccc}
M \otimes^{\mathbf{L}} K & \xrightarrow{\delta^A_M \otimes^{\mathbf{L}} K} & \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M, A), A) \otimes^{\mathbf{L}} K \\
{\scriptstyle M\otimes^{\mathbf{L}}\delta^E_K}\big\downarrow \simeq & & \simeq \big\downarrow {\scriptstyle \omega_{\mathbf{R}\mathrm{Hom}(M,A)AK}} \\
M \otimes^{\mathbf{L}} \mathbf{R}\mathrm{Hom}(J, E) & & \mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_A(M, A), K) \\
& & \simeq \big\downarrow {\scriptstyle \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M,A),\delta^E_K)} \\
{\scriptstyle \theta_{MJE}}\big\downarrow \simeq & & \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M, A), \mathbf{R}\mathrm{Hom}(J, E)) \\
& & \simeq \big\downarrow \\
\mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M, J), E) & \xrightarrow[\mathbf{R}\mathrm{Hom}(\omega_{MAJ}, E)]{\simeq} & \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(M, A) \otimes^{\mathbf{L}} J, E).
\end{array}$$

The unlabeled isomorphism is adjointness. The morphism $\omega_{\mathbf{R}\mathrm{Hom}(M,A)AK}$ is an isomorphism by [14, prop. 2.1(v)], and the Hom-evaluation morphism θ_{MJE} is an isomorphism by [9, lem. 4.4.(I)]. It follows that $\delta^A_M \otimes^{\mathbf{L}}_A K$ is an isomorphism. □

To prove the non-commutative part of Theorem 4.4 we need the following:

Lemma 4.6 *Let A be a two-sided noetherian ring with a dualizing complex; see 3.4. An A-complex M with bounded and degreewise finitely generated homology has finite G-dimension if and only if the complex $\mathbf{R}\mathrm{Hom}_A(M, A)$ is homologically bounded and the biduality morphism $\delta^A_M\colon M \to \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), A)$ is an isomorphism in $\mathsf{D}(A)$.*[2]

Proof By [13, prop. 3.8(b) and thm. 4.1] the complex M has finite G-dimension if and only if the complex $D \otimes^{\mathbf{L}}_A M$ is homologically bounded and the natural morphism $\eta_M\colon M \to \mathbf{R}\mathrm{Hom}_A(D, D \otimes^{\mathbf{L}}_A M)$ is an isomorphism in $\mathsf{D}(A)$. The next two isomorphisms are adjointness and Hom evaluation; see [9, lem. 4.4.(I)].

$$\mathbf{R}\mathrm{Hom}_A(M, A) \simeq \mathbf{R}\mathrm{Hom}_A(D \otimes^{\mathbf{L}}_A M, D) \quad \text{and} \tag{4.6.1}$$

$$D \otimes^{\mathbf{L}}_A M \simeq \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), D). \tag{4.6.2}$$

[2] By 3.4 the dualizing complex D has resolutions ${}_AP_A \xrightarrow{\simeq} {}_AD_A \xrightarrow{\simeq} {}_AI_A$ by A-bimodules, where each module in P is projective over both A and A°, and each module in I is injective over both A and A°. It follows that A has a resolution $A \xrightarrow{\simeq} J = \operatorname{Hom}_A(P, I)$ by A-bimodules, where each module in J is injective over both A and A°. Consequently, δ^A_M is represented by $M \to \operatorname{Hom}_{A^\circ}(\operatorname{Hom}_A(M, J), J)$.

It follows that $\mathbf{R}\mathrm{Hom}_A(M, A)$ is homologically bounded if and only if $D \otimes^{\mathbf{L}}_R M$ is so. The diagram below shows that δ^A_M is an isomorphism if and only if η_M is one.

$$\begin{array}{ccc}
\mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), A) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), \mathbf{R}\mathrm{Hom}_A(D, D)) \\
\uparrow{\scriptstyle \delta^A_M} & & \\
M & & \uparrow{\scriptstyle \simeq} \\
\downarrow{\scriptstyle \eta_M} & & \\
\mathbf{R}\mathrm{Hom}_A(D, D \otimes^{\mathbf{L}}_A M) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_A(D, \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), D))
\end{array}$$

The upper horizontal isomorphism is by definition of a dualizing complex, and the lower one is induced by (4.6.2); the right vertical isomorphism is swap. □

Proof of 4.4 continued Assume that A has a dualizing complex D; see 3.4 for the definition. By Lemma 4.6 it suffices, as in the commutative case, to show that homological boundedness of $\mathbf{R}\mathrm{Hom}_A(M, A)$ implies that $\delta^A_M : M \to \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), A)$ is an isomorphism in $\mathsf{D}(A)$. This follows from the commutative diagram below.

$$\begin{array}{ccc}
M & \xrightarrow{\delta^A_M} & \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), A) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
\mathbf{R}\mathrm{Hom}_{A^\circ}(D, D) \otimes^{\mathbf{L}}_A M & & \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A), \mathbf{R}\mathrm{Hom}_{A^\circ}(D, D)) \\
\downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} \\
\mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, D), D) & \xrightarrow{\simeq} & \mathbf{R}\mathrm{Hom}_{A^\circ}(\mathbf{R}\mathrm{Hom}_A(M, A) \otimes^{\mathbf{L}}_A D, D)
\end{array}$$

The vertical isomorphisms on the left follow by definition of a dualizing complex 3.4 and by [9, lem. 4.4.(I)]. The horizontal isomorphism is induced by ω_{MAD}, see Lemma 4.1. The vertical isomorphisms on the right follow by Hom-tensor adjointness and the definition of a dualizing complex.

5 Examples

We consider three elementary constructions that preserve the AC property.

Proposition 5.1 *Let A and B be left-noetherian and Morita equivalent rings. If A satisfies* (AC)/(UAC)*, then B satisfies* (AC)/(UAC).

Proof There exist bimodules ${}_AP_B$ and ${}_BQ_A$, which are finitely generated, projective from both sides, and provide an equivalence

$$\mathsf{mod}(A) \underset{P\otimes_B -}{\overset{Q\otimes_A -}{\rightleftarrows}} \mathsf{mod}(B).$$

Moreover, for every B-module N there is an isomorphism $N \cong \mathrm{Hom}_A(P, P \otimes_B N)$; see [41, sec. 9.5]. For finitely generated B-modules M and N it follows that

$$\begin{aligned}\operatorname{Ext}^i_B(M,N) &\cong \mathrm{H}_{-i}\,\mathbf{R}\mathrm{Hom}_B(M,\mathrm{Hom}_A(P,P\otimes_B N))\\ &\cong \mathrm{H}_{-i}\,\mathbf{R}\mathrm{Hom}_A(P\otimes^{\mathbf{L}}_B M, P\otimes^{\mathbf{L}}_B N)\\ &\cong \operatorname{Ext}^i_A(P\otimes_B M, P\otimes_B N).\end{aligned}$$

□

Example 5.2 If A satisfies (AC)/(UAC), then so does every matrix ring over A.

Proposition 5.3 *Let A and B be left-noetherian rings. The product ring $A\times B$ satisfies* (AC)/(UAC) *if and only if both A and B satisfy* (AC)/(UAC).

Proof Note that $A\times B$ is left-noetherian. There are equivalences of categories

$$\mathsf{Mod}(A)\times\mathsf{Mod}(B) \underset{\mathsf{s}}{\overset{\times}{\rightleftarrows}} \mathsf{Mod}(A\times B),$$

with the obvious definition of the functor $\times$. The functor s associates to an $A\times B$-module M the pair $\langle(1,0)M,(0,1)M\rangle$, and to an $A\times B$-linear map $\psi\colon M\longrightarrow N$ the pair of restrictions $\psi_{(1,0)}\colon (1,0)M\longrightarrow(1,0)N$ and $\psi_{(0,1)}\colon(0,1)M\longrightarrow(0,1)N$. Thus, for every pair M, N of $A\times B$-modules, s induces an isomorphism

$$\mathrm{Hom}_{A\times B}(M,N)\cong\mathrm{Hom}_A((1,0)M,(1,0)N)\oplus\mathrm{Hom}_B((0,1)M,(0,1)N).$$

The functor s is exact and preserves projectivity, indeed, $(1,0)M\cong {}_AA_{A\times B}\otimes_{A\times B}M$ and similarly $(0,1)M\cong {}_BB_{A\times B}\otimes_{A\times B}M$. Thus there are isomorphisms

$$\operatorname{Ext}^i_{A\times B}(M,N)\cong\operatorname{Ext}^i_A((1,0)M,(1,0)N)\oplus\operatorname{Ext}^i_B((0,1)M,(0,1)N),$$

for all $A\times B$-modules M and N, and all integers i. Clearly, an $A\times B$-module X is finitely generated over $A\times B$ exactly when $(1,0)X$ and $(0,1)X$ are finitely generated over A and B, respectively. Straightforward arguments finish the proof. □

The Chinese Remainder Theorem now yields:

Example 5.4 If $\mathfrak{a}$ and $\mathfrak{b}$ are proper coprime ideals in a commutative noetherian ring R, then $R/\mathfrak{a}\mathfrak{b}$ is AC if and only if both $R/\mathfrak{a}$ and $R/\mathfrak{b}$ are AC.

The results in [29] show, in particular, that the AC property does not ascend along flat ring homomorphisms. Descent, however, is straightforward:

Proposition 5.5 *Let A be commutative, and let B be a faithfully flat left-noetherian A-algebra. If B satisfies* (AC)/(UAC)*, then A satisfies* (AC)/(UAC).

Proof Note that B has a bimodule structure ${}_{A,B}B$. Let M and N be finitely generated A-modules. Because B is A-flat, one has the following chain of isomorphisms, where the second is by [9, lem. 4.4.(F)] and the third is by adjointness.

$$\begin{aligned}\operatorname{Ext}^i_A(M,N)\otimes_A B &\cong \mathrm{H}_{-i}\,\mathbf{R}\mathrm{Hom}_A(M,N)\otimes^{\mathbf{L}}_A B\\ &\cong \mathrm{H}_{-i}\,\mathbf{R}\mathrm{Hom}_A(M,N\otimes^{\mathbf{L}}_A B)\\ &\cong \mathrm{H}_{-i}\,\mathbf{R}\mathrm{Hom}_B(M\otimes^{\mathbf{L}}_A B, N\otimes^{\mathbf{L}}_A B)\\ &\cong \operatorname{Ext}^i_B(M\otimes_A B, N\otimes_A B).\end{aligned}$$

The desired conclusion now follows by faithful flatness of B over A. □

Example 5.6 A commutative noetherian ring R is AC if either $R[X]$ or $R[[x]]$ is so. Furthermore, if $(R, \mathfrak{m})$ is local and its $\mathfrak{m}$-adic completion $\widehat{R}$ is AC, then so is R.

Remark 5.7 For a commutative noetherian Cohen-Macaulay local ring R one gets stronger results [15]. Indeed, let $\mathfrak{m}$ be the maximal ideal of R, and let $x \in \mathfrak{m}$ be an R-regular element. If one of the rings R, $\widehat{R}$, $R/(x)$, $R[[X]]$, or $R[X]_{(\mathfrak{m},X)}$ satisfies (AC)/(UAC), then they all do.

Appendix A: Conjectures for rings and algebras

The Auslander–Reiten and Tachikawa Conjectures originate in representation theory of algebras, but they have recently received considerable attention in commutative algebra; see, e.g. [8,24,25,38]. This appendix provides a quick guide to these and related conjectures, and it explains, in greater detail, some of the points raised in Sect. 0.

Auslander's conjecture According to [21] and [1, intro. to ch. V], Auslander conjectured that every Artin algebra satisfies the condition (AC), defined in Sect. 0. In [29] Jorgensen and Şega showed that the conjecture fails, even for commutative local finite dimensional k-algebras: one counterexample $(R, \mathfrak{m})$ is Gorenstein with $\mathfrak{m}^4 = 0$, another is not Gorenstein and has $\mathfrak{m}^3 = 0$ and length $R = 8$. A subsequent short construction due to Smalø [39] shows that $k\langle x, y\rangle/(x^2, y^2, xy + qyx)$, where $q^n \neq 0, 1$ for all n, does not satisfy (AC). Further counterexamples are constructed by Mori in [32, sec. 6].

A.1 A commutative noetherian regular ring of infinite Krull dimension satisfies (AC) but not (UAC). We do not know of any Artin algebra or commutative noetherian local ring with that property. Rings known to satisfy (UAC) include:

- Left-noetherian rings of finite global dimension.
- Artin algebras of finite representation type; see [21, sec. 2.3].
- Group algebras of finite groups; this follows from [11, thm. 2.4].[3]
- Rings of finite global repetition index. For example quotients $\mathcal{O}/\pi$, where $\mathcal{O}$ is a classical order over a discrete valuation ring, and π is a uniformizing parameter; see [20, sec. 4].
- Exterior algebras; see [33, cor. 2.4].
- Commutative noetherian local rings that are Golod or complete intersection; see [29, prop. 1.4] and [7, thm. 4.7].
- Commutative noetherian Gorenstein local rings R of multiplicity codim $R + 2$ or with codim $R \leq 4$; see [23, thm. 3.5] and [38, thm. 3.4].
- The trivial extension of a commutative artinian local ring by its residue field; see [37, cor. 3.5].

Further examples of commutative noetherian local rings that satisfy (UAC) are given in [29, prop. 1.1][4] and in [23, thm. 3.7].

The Auslander–Reiten Conjecture The root of this is the Nakayama Conjecture posed in [36]. By work of Müller [34], it can be phrased as follows:

Every finite dimensional k-algebra Λ satisfies the following condition:
(NC) If each term in the minimal injective resolution of ${}_{\Lambda}\Lambda$ is projective, then Λ is quasi-Frobenius.

3 By the isomorphisms $\mathrm{Ext}^i_{kG}(M, N) \cong \mathrm{Hom}_k(M, \widehat{H}^i(G, N))$ for $i > 0$.

4 Where part (2) should read: edim $R -$ depth $R \leq 2$.

In [4] Auslander and Reiten propose the Generalized Nakayama Conjecture:

Every Artin algebra Λ satisfies the following condition:
(GNC) Every indecomposable injective Λ-module occurs as a summand in one of the terms in the minimal injective resolution of ${}_\Lambda\Lambda$.

A finite dimensional k-algebra is an Artin algebra, and an Artin algebra that satisfies (GNC) also satisfies (NC), cf. [6, prop. IV.3.1]. It is proved in [4] that the Generalized Nakayama Conjecture is equivalent to:

Every Artin algebra Λ satisfies the following condition:
(ARC-G) Every finitely generated Λ-generator[5] M with $\mathrm{Ext}_\Lambda^{\geq 1}(M, M) = 0$ is projective.

It is *not* known if a given finite dimensional k-algebra satisfies (GNC) if and only if it satisfies (ARC-G). What *is* known is that (GNC) holds for all k-algebras if and only if (ARC-G) does; see [42, remark after thm. 3.4.3].

In [3] the condition (ARC-G) is considered for any noetherian ring, and it is noted that a ring A satisfies (ARC-G) if and only if it satisfies (ARC); see Sect. 0. Indeed, an A-generator M with $\mathrm{Ext}_A^{\geq 1}(M, M) = 0$ also has $\mathrm{Ext}_A^{\geq 1}(M, A) = 0$, and for every A-module N the module $N \oplus A$ is an A-generator.

A.2 Rings know to satisfy (ARC) include:

- Left-noetherian rings over which every finitely generated module has an ultimately closed projective resolution;[6] see [4, prop. 1.3].
- Rings $\Lambda/(\boldsymbol{x})\Lambda$ where Λ is a noetherian algebra of finite global dimension over a commutative noetherian complete local ring $(R, \mathfrak{m})$, and $\boldsymbol{x} \in \mathfrak{m}$ is a Λ-sequence; see [3, prop. 1.9]. In particular, commutative noetherian complete intersection local rings; see also [7, thm. 4.2].
- Group algebras kG, where G is a finite group and k is a field of characteristic $p > 0$; see [10, 5.2.3].
- Commutative artinian local rings $(R, \mathfrak{m})$ with $2\,\mathrm{length}_R(\mathrm{Soc}\, R) > \mathrm{length}\, R$ or with $\mathfrak{m}^3 = 0$; see [16, 4.3] and [25, thm. 4.1].
- Commutative noetherian Golod local rings; see [29, prop. 1.4].
- Rings $R/(\boldsymbol{x})$ where $\boldsymbol{x}$ is an R-sequence, and R is commutative, noetherian, local, excellent, Cohen-Macaulay, normal, and either Gorenstein or a $\mathbb{Q}$-algebra. This is a special case of [24, thm. 0.1].
- Commutative noetherian Gorenstein local rings R with $\mathrm{codim}\, R \leq 4$; see [38, cor. 3.5].

The Tachikawa Conjectures The conditions above relate to two conjectures of Tachikawa [40, Sect. 8]:

Every finite dimensional k-algebra Λ satisfies the following condition:
(TC1) If $\mathrm{Ext}_\Lambda^{\geq 1}(\mathrm{Hom}_k(\Lambda_\Lambda, k), \Lambda) = 0$, then Λ is quasi-Frobenius.[7]

and

Every quasi-Frobenius finite dimensional k-algebra Λ satisfies:

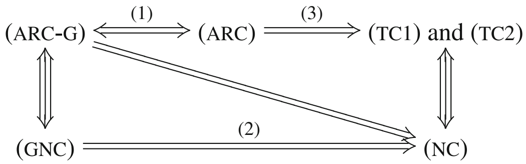
[5] Defined as follows: for every finitely generated Λ-module T there is an epimorphism $M' \twoheadrightarrow T$ such that $M' \in \mathsf{add}(M)$.

[6] Defined as a degreewise finitely generated projective resolution for which there is a $d > 0$ such that the dth syzygy has a decomposition whose factors are summands of earlier syzygies; see [27, sec. 3].

[7] The conjecture on p. 115 in [40] is equivalent to this one by the arguments on p. 114 ibid.

(TC2) Every finitely generated Λ-module M with $\operatorname{Ext}_{\Lambda}^{\geq 1}(M, M) = 0$ is projective.

It is proved in [40,42] that the Nakayama Conjecture holds if and only if both Tachikawa Conjectures hold. The diagram below depicts the known relations between conditions on finite dimensional k-algebras.



(A.3)

The notation (P)⟹(Q) means that every algebra that satisfies (P) also satisfies (Q), while (P)⟺(Q) means that *all* algebras satisfy (P) if and only *all* algebras satisfy (Q).

The implications (1) and (2) were discussed above; the implication (3) is clear; cf. the proof of Proposition 3.5. The remaining implications are proved in [42, thm. 3.4.3].[8]

In commutative algebra, Avramov et al. [8] make a conjecture related to the first of Tachikawa's conjectures mentioned above. Their conjecture is the following:

Every commutative noetherian Cohen-Macaulay local ring R satisfies:
(ABŞC) If R has a dualizing module D and $\operatorname{Ext}_{R}^{\geq 1}(D, R) = 0$, then R is Gorenstein.

It is clear that both conditions (GNC) and (NC) make sense for, and are satisfied by, every commutative noetherian local ring R. However, the conjecture of Avramov, Buchweitz, and Şega is still open, even in the case where R is a finite dimensional k-algebra. This emphasizes the point that the implication (NC)⟹(TC1) in (A.3) is not known to restrict to commutative local k-algebras. A list of rings that satisfy (ABŞC) is provided in [8, intro. and sec. 9].

A.4 We end this appendix by summarizing a couple of contributions of this paper.

Theorem 2.3 is new, even for finite dimensional k-algebras. In particular, it adds exterior algebras and rings of finite global repetition index to the list of rings known to satisfy (ARC).

Proposition 3.5 shows that (AC) implies a generalized version of (TC1) for two-sided noetherian rings with a dualizing complex.

Appendix B: AB rings

Huneke and Jorgensen [23] introduce *AB rings* as commutative noetherian Gorenstein local rings that satisfy (UAC)—equivalently (AC), cf. Proposition 4.2. Our Lemma 4.1 is inspired by ideas in [23]; in particular by [23, prop. 5.2 and 5.5]. In this appendix we apply Lemma 4.1 to reestablish two main results [23, thm. 4.1 and cor. 4.2] in the setting of complexes over a commutative noetherian ring R with $\operatorname{id}_R R$ finite.

In the following we use the term *totally reflexive* for a module that is either 0 or of G-dimension 0 in the sense of Auslander and Bridger [2], cf. Sect. 0.

Lemma B.1 *Let R be commutative and noetherian with $\operatorname{id}_R R$ finite, and let M be an R-complex. If M is isomorphic in $\mathsf{D}(R)$ to a complex of totally reflexive R-modules, then the biduality morphism δ_M^R is invertible*:

[8] The remark following [42, thm. 3.4.3] indicates that *any given algebra* satisfies (NC) if and only if it satisfies (TC1) and (TC2). However, this strong statement is *not* known to be true, cf. thm. 3.4.2 ibid. We thank Professor Yamagata for clarifying this to us.

$$M \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R).$$

In particular, M is homologically bounded if and only if $\mathbf{R}\mathrm{Hom}_R(M, R)$ *is so.*

Proof Let G be a complex of totally reflexive R-modules such that there is an isomorphism $M \simeq G$ in $\mathsf{D}(R)$; further let $\alpha\colon R \xrightarrow{\simeq} I$ be a bounded injective resolution. We start by proving that the complex $\mathrm{Hom}_R(G, R)$ is isomorphic to $\mathbf{R}\mathrm{Hom}_R(M, R)$ in $\mathsf{D}(R)$. We do so by arguing that $\mathrm{Hom}_R(G, -)$ preserves the quasiisomorphism α, that is, we show exactness of the complex

$$\mathrm{Cone}\,\mathrm{Hom}_R(G, \alpha) \cong \mathrm{Hom}_R(G, \mathrm{Cone}\,\alpha).$$

Note that $\mathrm{Cone}\,\alpha$ is a bounded and exact complex of modules of finite injective dimension. Thus, for every v the complex $\mathrm{Hom}_R(G_v, \mathrm{Cone}\,\alpha)$ is exact by [12, cor. (2.4.4)(a)], and the claim follows by [13, lem. 2.4]. The complex $\mathrm{Hom}_R(G, R)$ consists of totally reflexive R-modules, see [12, obs. (1.1.7)], so the argument above applies to show that $\mathrm{Hom}_R(\mathrm{Hom}_R(G, R), R)$ is isomorphic to $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$ in $\mathsf{D}(R)$. Consequently, the morphism

$$\delta^R_M\colon M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$$

in $\mathsf{D}(R)$ is represented by

$$\delta^R_G\colon G \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_R(G, R), R),$$

which is an isomorphism of R-complexes, as each module G_v is totally reflexive. □

In the next two results, we use the notation $(-)^\star = \mathbf{R}\mathrm{Hom}_R(-, R)$.

Theorem B.2 *Let R be commutative and noetherian with $\mathrm{id}_R R$ finite and assume that R satisfies* (AC). *For R-complexes M and N with bounded and degreewise finitely generated homology the following conditions are equivalent*:

(i) $\mathbf{R}\mathrm{Hom}_R(M, N)$ *is homologically bounded.*

(ii) $\mathbf{R}\mathrm{Hom}_R(N, M)$ *is homologically bounded.*

(iii) $M^\star \otimes^{\mathbf{L}}_R N$ *is homologically bounded.*

Proof We prove the implications (i)⇒(iii)⇒(ii), then (ii)⇒(i) by symmetry.

Homological boundedness of $\mathbf{R}\mathrm{Hom}_R(M, N)$ yields by Proposition 4.2 an isomorphism $M^\star \otimes^{\mathbf{L}}_R N \simeq \mathbf{R}\mathrm{Hom}_R(M, N)$ in $\mathsf{D}(R)$. This shows the first implication.

For the second implication, note that there are isomorphisms

$$M^\star \otimes^{\mathbf{L}}_R N \simeq N \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, R) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(N, M), R),$$

where the last one uses finiteness of $\mathrm{id}_R R$, see [18, (1.4)]. Thus, the complex $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(N, M), R)$ is homologically bounded, and Lemma B.1 finishes the proof once we show that $\mathbf{R}\mathrm{Hom}_R(N, M)$ is isomorphic in $\mathsf{D}(R)$ to a complex of totally reflexive modules. To this end, let $L \xrightarrow{\simeq} N$ be a degreewise finitely generated free resolution and choose a bounded complex G of totally reflexive modules such that $G \simeq M$; see [12, thm. (2.3.7)] and [18, (1.4)]. The complex $\mathrm{Hom}_R(L, G)$ is isomorphic to $\mathbf{R}\mathrm{Hom}_R(N, M)$ in $\mathsf{D}(R)$ and consists of totally reflexive modules. □

Corollary B.3 *Let R be commutative and noetherian with $\mathrm{id}_R\, R$ finite and assume that R satisfies* (AC). *For R-complexes M and N with bounded and degreewise finitely generated homology the following conditions are equivalent*:

(i) $M \otimes^{\mathbf{L}}_R N$ *is homologically bounded.*

(ii) $\mathbf{R}\mathrm{Hom}_R(M^\star, N)$ *is homologically bounded.*

(iii) $\mathbf{R}\mathrm{Hom}_R(N^\star, M)$ *is homologically bounded.*

Proof From the isomorphisms $N \simeq N^{\star\star}$ and $M \simeq M^{\star\star}$, see [12, thm. (2.3.14)], it follows that the complexes in (ii) and (iii) are isomorphic by swap. By Theorem B.2 condition (ii) holds if and only if the complex $M^{\star\star} \otimes^{\mathbf{L}}_R N \simeq M \otimes^{\mathbf{L}}_R N$ is homologically bounded. □

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