The Bohnenblust–Hille inequality for homogeneous polynomials

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Background: The work of Harald Bohr

The Bohr radius K is the largest number $0 \leq r < 1$ such that we have

$$\sum_{n=0}^{\infty} |c_n| r^n \leq \sup_{|z|<1} \left| \sum_{n=0}^{\infty} c_n z^n \right|,$$

whenever the series to the right represents a bounded analytic function in $|z| < 1$.

(Note: It is not at all clear at the outset that K is positive.)

Harald Bohr studied the problem of computing K in 1913, and he showed that $K \geq 1/6$.

The exact value of the Bohr radius

Bohr wrote the following in his note (or, to be more precise, Hardy wrote on Bohr's behalf):

"*I have learnt recently that Messrs. M. Riesz, Schur, and Wiener, whose attention had been drawn to the subject by my theorem, have succeeded in solving this problem completely. Their solutions show that* $K = \frac{1}{3}$*. Mr. Wiener has very kindly given me permission to reproduce here his very simple and elegant proof of this result.*"

Footnote on Wiener

Bohr is *not* referring to Norbert Wiener (19 years old at the time, already with a Ph.D.), but to the 10 years older Friedrich Wilhelm Wiener, born in 1884, and probably a casualty of World War One. See H. Boas and D. Khavinson's biography in Math. Intelligencer **22** (2000).

F. Wiener's proof

The proof is based on the inequality

$$|c_n| \le 1 - |c_0|^2,$$

which holds when $\sup_{|z|<1} |\sum_n c_n z^n| \le 1$. In fact, one only needs the slightly coarser inequality $|c_n| \le 2(1 - |c_0|)$ because it implies

$$\sum_{n=0}^{\infty} |c_n| \left(\frac{1}{3}\right)^n \le |c_0| + 2(1 - |c_0|) \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = |c_0| + 1 - |c_0| = 1.$$

This shows $K \ge 1/3$; to show that $K \le 1/3$, one may consider the function

$$\frac{a - z}{1 - az} \quad \text{for } a \nearrow 1.$$

F. Wiener's proof that $|c_n| \leq 1 - |c_0|^2$

Wiener observed that if ρ is a primitive *n*-th root of unity and $f(z) = \sum_n c_n z^n$, then

$$\varphi(z) = (f(z) + f(\rho z) + \cdots + f(\rho^{n-1} z))/n = \sum_{m=0}^{\infty} c_{mn} z^{mn},$$

which means that it suffices to show that $|c_1| \leq 1 - |c_0|^2$. To deal with this special case, we observe that

$$\psi(z) = \frac{c_0 - f(z)}{1 - \overline{c_0} f(z)}$$

satisfies $\psi(0) = 0$ and $\psi'(0) = c_1/(1 - |c_0|^2)$. Then apply the Schwarz lemma to ψ.

What happens for $1/3 < r < 1$?

Set $f(z) = \sum_n c_n z^n$ and $\|f\|_\infty = \sup_{|z|<1} |f(z)|$. Define

$$m(r) = \sup_{f \neq 0} \frac{\sum_{n=0}^{\infty} |c_n| r^n}{\|f\|_\infty}.$$

Bombieri proved (1962) that $m(r) = (3 - \sqrt{8(1-r^2)})/r$ for $1/3 \leq r \leq 1/\sqrt{2}$ and then Bombieri and Bourgain proved (2004) that when $r \to 1$,

$$\frac{1}{\sqrt{1-r^2}} - C(\varepsilon)\left(\log \frac{1}{1-r}\right)^{3/2+\varepsilon} \leq m(r) < \frac{1}{\sqrt{1-r^2}}.$$

(Note that $m(r) \leq 1/\sqrt{1-r^2}$ holds trivially.)

More on the “Bohr phenomenon”

There have in recent years been a number of other studies of what we may call the “Bohr phenomenon” or Bohr inequality, but let’s see why Bohr got interested in his radius:

Why did Bohr get interested in his radius?

Bohr says in his paper: "... *the solution of what is called the "absolute convergence problem" for Dirichlet's series of the type* $\sum a_n n^{-s}$ *must be based upon a study of the relations between the absolute value of a power-series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms on the other. It was in the course of this investigation that I was led to consider a problem concerning power-series in one variable only, which we discussed last year, and which seems to be of some interest in itself.*"

Dirichlet Series

Recall that an ordinary Dirichlet series is a series of the form $\sum_{n\geq 1} a_n n^{-s}$, where the exponentials n^{-s} are positive for positive arguments s. Bohr was, as many others before and after him, mainly interested in the distinguished case when $a_n \equiv 1$, or, to be more explicit: Bohr wanted to prove the Riemann hypothesis.

Convergence of Dirichlet series

In general, a Dirichlet series has several half-planes of convergence, as shown in the picture:

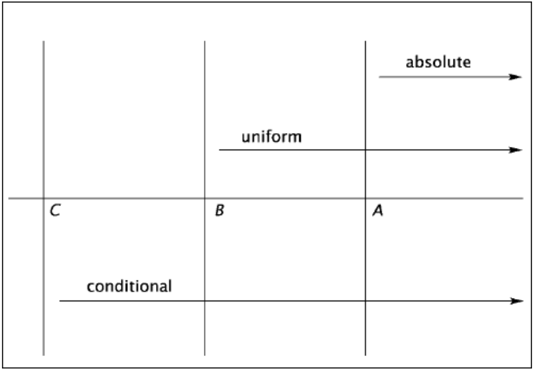

Figure: Convergence regions for Dirichlet series

Bohr's problem on absolute convergence

- It is plain that $0 \leq A - C \leq 1$, and if $a_n = e^{in^\alpha}$ with $0 \leq \alpha \leq 1$, then $C = 1 - \alpha$ and $A = 1$.
- The most interesting quantity is the difference $A - B$. In 1913, Bohr proved that it does not exceed $1/2$, but he was unable to exhibit even one example such that $A - B > 0$.
- In the same year, Toeplitz proved that we may have $A - B = 1/4$; there was no further progress on this problem until 1931, when Bohnenblust and Hille solved it completely by giving examples such that $A - B = 1/2$.
- **Alternate viewpoint**: Bohr proved that the abscissa of uniform convergence $=$ the abscissa of boundedness and regularity, i.e. the infimum of those σ_0 such that the function represented by the Dirichlet series is analytic and bounded in $\Re s = \sigma > \sigma_0$. Thus we may instead look at A for bounded analytic functions represented by Dirichlet series (and read "B for boundedness").

Bohr's insight

Let $f(s) = \sum_{n\geq 1} a_n n^{-s}$ be a Dirichlet series. We factor each integer n into a product of prime numbers $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and set $z = (p_1^{-s}, p_2^{-s}, \ldots)$. Then

$$f(s) = \sum_{n=1}^{\infty} a_n (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r} = \sum a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r}.$$

Bohr's correspondence is not just formal: thanks to a classical result of Kronecker on diophantine approximation, the supremum of $|f|$ in $\mathrm{Re}\, s = \sigma > 0$ equals the supremum of the modulus of the infinite power series in $\mathbb{D}^\infty \cap c_0$, because the vertical line $\sigma = \sigma_0$, viewed as a subset of $\mathbb{D}^\infty$ is "everywhere dense" on the infinite torus $|z_j| = p_j^{-\sigma_0}$.

Bohr's insight—and what he lacked

Bohr's correspondence is an indispensable tool for proving nontrivial results about Dirichlet series, with the question about absolute convergence as an illustrative interesting example. However, Bohr lacked two important ingredients, to be developed later:

- Polarization (first proof by Bohnenblust and Hille in 1931)
- Random Fourier series/polynomials (second proof by P. Hartman in 1939)

The Bohnenblust–Hille theorem on absolute convergence

Definition

The space $\mathscr{H}^\infty$ consists of those bounded analytic functions f in $\mathbb{C}_+ = \{s = \sigma + \mathrm{i}\, t : \ \sigma > 0\}$ such that f can be represented by an ordinary Dirichlet series $\sum_{n=1}^\infty a_n n^{-s}$ in some half-plane.

The Bohnenblust–Hille theorem can be rephrased as:

Theorem

The infimum of those $\sigma > 0$ such that

$$\sum |a_n| n^{-\sigma} < +\infty$$

for every $\sum_{n=1}^\infty a_n n^{-s}$ in $\mathscr{H}^\infty$ equals $1/2$.

Decomposition into homogeneous terms

The “natural” way to sum a power series in several (or infinitely many) variables is *not* in the order induced by the corresponding Dirichlet series; it is rather to rearrange it as a series of homogeneous polynomials (or homogeneous power series). Thus we would like to write it as

$$f(s) = \sum_{k=0}^{\infty} f_k(s), \quad \text{where} \quad f_k(s) = \sum_{\Omega(n)=k} a_n n^{-s}$$

and $\Omega(n)$ is the number of prime factors in n, counting multiplicities.

This rearrangement represents a function in $\mathscr{H}^\infty$ if $\sum_k \|f_k\|_\infty < \infty$; here we use the notation

$$\|f\|_\infty = \sup_{\sigma>0} |f(\sigma + it)|.$$

Proof using random polynomials (Hartman)

Now we would like each homogeneous term f_k to have small norm in $\mathscr{H}^\infty$, but with $\sum_{\Omega(n)=k} |a_n| n^{-\sigma}$ as large as possible. The problem is completely analogous to that originally studied by Bohr. (This is of course what Bohr had in mind!)

The proof can now be performed with what is nowadays "off-the-shelf" technology: Random polynomials or more precisely the Salem–Zygmund theorem as found in J.-P. Kahane's book "Some Random Series of Functions".

Existence of f_k with $\|f_k\|_\infty$ "small"

Let $\Pi(N,k)$ be the collection of integers n such that $\Omega(n) = k$ and each prime factor belongs to a fixed set of N primes. Then given arbitrary nonnegative numbers a_n we can find

$$f_k = \sum_{n\in\Pi(N,k)} \epsilon_n a_n n^{-s}$$

with ϵ_k taking values ± 1 and

$$\|f_k\|_\infty \le C \left(N \log k \sum_{n\in\Pi(N,k)} a_n^2 \right)^{1/2} ;$$

here C is an absolute constant independent of N and k.
The inequality is as good as we can hope for because we trivially have $\sum a_n^2 \le \|f_k\|_\infty^2$. To prove the Bohnenblust–Hille theorem, we make a careful selection of suitable k, $N = N(k)$, and the N primes that define the set $\Pi(N,k)$.

The work of Bohnenblust–Hille

Although a much simpler proof than that found in the original work of Bohnenblust and Hille is now available, the Bohnenblust–Hille paper remains a remarkable and interesting piece of work, as I'll try to convince you about in the remainder of this talk.

In addition, as it turns out, using among other things the work of Bohnenblust–Hille, we may obtain a much more precise solution to Bohr's absolute convergence problem.

What is the Bohnenblust–Hille inequality?

Let $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ be an m-homogeneous polynomial in n complex variables. Then trivially $\left(\sum_{|\alpha|=m} |a_\alpha|^2\right)^{\frac{1}{2}} \leq \|P\|_\infty$, where $\|P\|_\infty = \sup_{z\in\mathbb{D}^n} |P(z)|$.

- Is it possible to have a similar inequality

$$\left(\sum_{|\alpha|=m} |a_\alpha|^p\right)^{\frac{1}{p}} \leq C\|P\|_\infty$$

  for some $p < 2$ with C depending on m but *not* on n?

Bohnenblust–Hille

YES, and $2m/(m+1)$ is the smallest possible p.

It is of basic interest to know the asymptotic behavior of C when $p = 2m/(m+1)$ and $m \to \infty$.

A multilinear inequality

In 1930, Littlewood proved that for every bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ we have

$$\Big(\sum_{i,j} |B(e^i, e^j)|^{4/3} \Big)^{3/4} \leq \sqrt{2} \sup_{z,w \in \mathbb{D}^n} |B(z, w)|.$$

This was extended to m-linear forms by Bohnenblust and Hille in 1931:

$$\Big(\sum_{i_1,\dots,i_m} |B(e^{i_1}, \dots, e^{i_m})|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq \sqrt{2}^{m-1} \sup_{z^i \in \mathbb{D}^n} |B(z^{(1)}, \dots, z^{(m)})|.$$

The exponent $2m/(m+1)$ is best possible.

The Bohnenblust–Hille inequality

Our result is that also the polynomial Bohnenblust–Hille inequality is hypercontractive:

Theorem (Defant, Frerick, Ortega-Cerdà, Ounaïes, Seip 2011)

Let m and n be positive integers larger than 1. Then we have

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le e\sqrt{m}(\sqrt{2})^{m-1} \sup_{z\in\mathbb{D}^n}\Big|\sum_{|\alpha|=m} a_\alpha z^\alpha\Big|$$

for every m-homogeneous polynomial $\sum_{|\alpha|=m} a_\alpha z^\alpha$ on $\mathbb{C}^n$.

The novelty here is the **hypercontractivity**, i.e., the constant grows exponentially with m; known since the work of Bohnenblust–Hille that the inequality holds with constant $m^{m/2}$, modulo a factor of exponential growth.

Polarization

There is a relationship between the multilinear and polynomial inequalities, which goes via a one-to-one correspondence between symmetric multilinear forms and homogeneous polynomials.

Definition

We say that the m-linear form B is symmetric if $B(e^{i_1}, ..., e^{i_m}) = B(e^{i_{\sigma(1)}}, ..., e^{i_{\sigma(m)}})$ for every index set $(i_1, ..., i_m)$ and every permutation σ of the set $\{1, ..., m\}$.

Alternatively, B is symmetric iff

$$B(z^{(1)}, ...z^{(m)}) = B(z^{(\sigma(1))}, ..., z^{(\sigma(m))})$$

for every permutation σ of the set $\{1, ..., m\}$ and all vectors $z^{(1)}, ..., z^{(m)}$.

Polarization—continued

If we restrict a symmetric m-multilinear form to the diagonal $P(z) = B(z, \dots, z)$, then P is a homogeneous polynomial. Conversely: Given a homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ of degree m, we may define a symmetric m-multilinear form $B : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ so that $B(z, \dots, z) = P(z)$. I will show you two different ways to do it, each giving valuable complementary information.

Polarization—First approach

Just define

$$B(z^{(1)}, ..., z^{(m)}) = \frac{1}{m!} \int_{\mathbb{T}^m} P(\zeta_1 z^{(1)} + \cdots + \zeta_m z^{(m)}) \overline{\zeta_1} \cdots \overline{\zeta_m} d\mu_m(\zeta),$$

where μ_m is normalized Lebesgue measure on the m-torus $\mathbb{T}^m$. This B is clearly symmetric, and we get $P(z) = B(z, ..., z)$ because $P((\zeta_1 + \cdots + \zeta_m)z) = (\zeta_1 + \cdots + \zeta_m)^m P(z)$ and

$$(\zeta_1 + \cdots + \zeta_m)^m = m! \zeta_1 \cdots \zeta_m + R$$

with R an m-homogeneous polynomial orthogonal to $\zeta_1 \cdots \zeta_m$.

This is roughly how Bohnenblust and Hille did polarization.

Polarization lemma

Note that if $z^{(1)}$,...,$z^{(m)}$ are all in $\mathbb{D}^n$, then the point $\zeta_1 z^{(1)} + \cdots + \zeta_m z^{(m)}$ is in $m\mathbb{D}^n$. Therefore, the formula

$$B(z^{(1)}, ..., z^{(m)}) = \frac{1}{m!} \int_{\mathbb{T}^m} P(\zeta_1 z^{(1)} + \cdots + \zeta_m z^{(m)}) \overline{\zeta_1} \cdots \overline{\zeta_m} d\mu_m(\zeta),$$

gives an interesting inequality:

Lemma (Harris 1975)

We have

$$\sup_{z^{(1)},...,z^{(m)} \in \mathbb{D}^n} |B(z^{(1)}, \ldots, z^{(m)})| \leq \frac{m^m}{m!} \|P\|_\infty.$$

The inequality is hypercontractive because $m^m/m! < e^m$.

Polarization—Second approach

Write

$$P(z) = \sum_{i_1 \le \cdots \le i_m} c(i_1, \ldots, i_m) z_{i_1} \cdots z_{i_m},$$

and let B be the symmetric m-multilinear form such that $B(e^{i_1}, \cdots, e^{i_m}) = c(i_1, \ldots, i_m)/|i|$ when $i_1 \le \cdots \le i_m$ and $|i|$ is the number of different indices that can be obtained from the index $i = (i_1, \ldots, i_m)$ by permutation. Since, by this definition, the number of coefficients of B obtained from $c(i_1, ..., i_m)$ is $|i| \le m!$, a direct application of Harris's lemma and the multilinear Bohnenblust–Hille inequality gives

$$\left(\frac{1}{m!}\right)^{(m-1)/2m} \Big(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\Big)^{\frac{m+1}{2m}} \le \frac{m^m}{m!} \|P\|_\infty;$$

thus we get the constant $m^{m/2}$, modulo an exponential factor.

We need a refinement of the argument via multilinear forms!

Two lemmas

Lemma (Blei 1979)

For all sequences $(c_i)_i$ *where* $i = (i_1, \ldots, i_m)$ *and* $i_k = 1, \ldots, n$*, we have*

$$\Big(\sum_{i_1,\ldots,i_m=1}^{n} |c_i|^{\frac{2m}{m+1}} \Big)^{\frac{m+1}{2m}} \leq \prod_{1\leq k\leq m} \Big[\sum_{i_k=1}^{n} \Big(\sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} |c_i|^2 \Big)^{\frac{1}{2}} \Big]^{\frac{1}{m}}.$$

Lemma (Bayart 2002)

For any homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ *on* $\mathbb{C}^n$*:*

$$\Big(\sum_{|\alpha|=m} |a_\alpha|^2 \Big)^{\frac{1}{2}} \leq (\sqrt{2})^m \Big\| \sum_{|\alpha|=m} a_\alpha z^\alpha \Big\|_{L^1(\mathbb{T}^n)}.$$

Consequences of the hypercontractive BH inequality

The hypercontractive polynomial Bohnenblust–Hille inequality is the best one can hope for, and it has several interesting consequences: It leads to precise asymptotic results regarding certain Sidon sets, Bohr radii for polydiscs, and the moduli of the coefficients of functions in $\mathscr{H}^\infty$.

Sidon sets

Definition

If G is an Abelian compact group and Γ its dual group, a subset of the characters $S \subset \Gamma$ is called a Sidon set if

$$\sum_{\gamma \in S} |a_\gamma| \leq C \| \sum_{\gamma \in S} a_\gamma \gamma \|_\infty$$

The smallest constant $C(S)$ is called the Sidon constant of S.

We estimate the Sidon constant for homogeneous polynomials:

Definition

$S(m, n)$ is the smallest constant C such that the inequality $\sum_{|\alpha|=m} |a_\alpha| \leq C \|Q\|_\infty$ holds for every m-homogeneous polynomial in n complex variables $Q(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$.

The Sidon constant for homogeneous polynomials

Since the number of different monomials of degree m is $\binom{n+m-1}{m}$, Hölder's inequality gives:

Corollary

Let m and n be positive integers larger than 1. Then

$$S(m,n) \leq e\sqrt{m}(\sqrt{2})^{m-1}\binom{n+m-1}{m}^{\frac{m-1}{2m}}.$$

(We also have the trivial estimate

$$S(m,n) \leq \sqrt{\binom{n+m-1}{m}},$$

so the corollary is of interest only when $\log n \gg m$.)

The n-dimensional Bohr radius

Definition

The n-dimensional Bohr radius K_n is the largest $r > 0$ such that all polynomials $\sum_\alpha c_\alpha z^\alpha$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \sum_\alpha |c_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha c_\alpha z^\alpha \right|.$$

When $n > 1$, the precise value of K_n is unknown.

Problem

Determine the asymptotic behavior of K_n when $n \to \infty$.

Asymptotic behavior of K_n

The problem was studied by Boas and Khavinson in 1997. They showed that

$$\frac{1}{3}\sqrt{\frac{1}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

In 2006, Defant and Frerick showed that:

$$c\sqrt{\frac{\log n}{n \log\log n}} \leq K_n.$$

Theorem (DFOOS 2011)

The n-dimensional Bohr radius satisfies

$$c\sqrt{\frac{\log n}{n}} \leq K_n \leq 2\sqrt{\frac{\log n}{n}}.$$

F. Wiener's lemma in the polydisc

Wiener's estimate $|c_n| \leq 1 - |c_0|^2$ has the following extension to the polydisc:

Lemma

Let Q be a polynomial in n variables and $Q = \sum_{m \geq 0} Q_m$ its expansion in homogeneous polynomials. If $\|Q\|_\infty \leq 1$, then $\|Q_m\|_\infty \leq 1 - |Q_0|^2$ for every $m > 0$.

F. Wiener's proof, both of the lemma and the theorem on the Bohr radius, carries over essentially unchanged; the only new ingredient is the use of the estimate for the Sidon constant.

Remark: What is the “Bohr subset” of $\mathbb{D}^n$?

A more difficult problem would be to find the “Bohr subset” of $\mathbb{D}^n$, i.e., the set of points z in $\mathbb{D}^n$ for which

$$\sum_{\alpha} |c_\alpha z^\alpha| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_{\alpha} c_\alpha z^\alpha \right|$$

holds for all polynomials $\sum_\alpha c_\alpha z^\alpha$.

Estimates on coefficients of Dirichlet polynomials

For a Dirichlet polynomial

$$Q(s) = \sum_{n=1}^{N} a_n n^{-s},$$

we set $\|Q\|_\infty = \sup_{t\in\mathbb{R}} |Q(it)|$ and $|||Q|||_1 = \sum_{n=1}^{N} |a_n|$. Then $S(N)$ is the smallest constant C such that the inequality $|||Q|||_1 \leq C\|Q\|_\infty$ holds for every Q.

Theorem (Konyagin–Queffélec 2001, de la Bretèche 2008, DFOOS 2011)

We have

$$S(N) = \sqrt{N} \exp\Big\{ \big(-\frac{1}{\sqrt{2}} + o(1)\big) \sqrt{\log N \log\log N} \Big\}$$

when $N \to \infty$.

Historical account of the estimate for $S(N)$

The inequality

$$S(N) \geq \sqrt{N} \exp\Big\{ \big(-\frac{1}{\sqrt{2}} + o(1)\big) \sqrt{\log N \operatorname{loglog} N} \Big\}$$

was established by R. de la Bretèche, who also showed that

$$S(N) \leq \sqrt{N} \exp\Big\{ \big(-\frac{1}{2\sqrt{2}} + o(1)\big) \sqrt{\log N \operatorname{loglog} N} \Big\}$$

follows from an ingenious method developed by Konyagin and Queffélec. The same argument, using the hypercontractive BH inequality at a certain point, gives the sharp result.

A refined version of the Bohnenblust–Hille theorem

The previous result gives the following refined solution to Bohr's problem on absolute convergence:

Theorem

The supremum of the set of real numbers c such that

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\left\{c\sqrt{\log n \log\log n}\right\} < \infty$$

for every $\sum_{n=1}^{\infty} a_n n^{-s}$ *in* $\mathscr{H}^\infty$ *equals* $1/\sqrt{2}$.

This is a refinement of a theorem of R. Balasubramanian, B. Calado, and H. Queffélec (2006). (Without the hypercontractive BH inequality, one does not catch the precise bound for the constant c.)