Random Conformal Welding

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Joint work with Peter Jones, Antti Kupiainen and Eero Saksman

Conformally invariant Random Curves in the plane.

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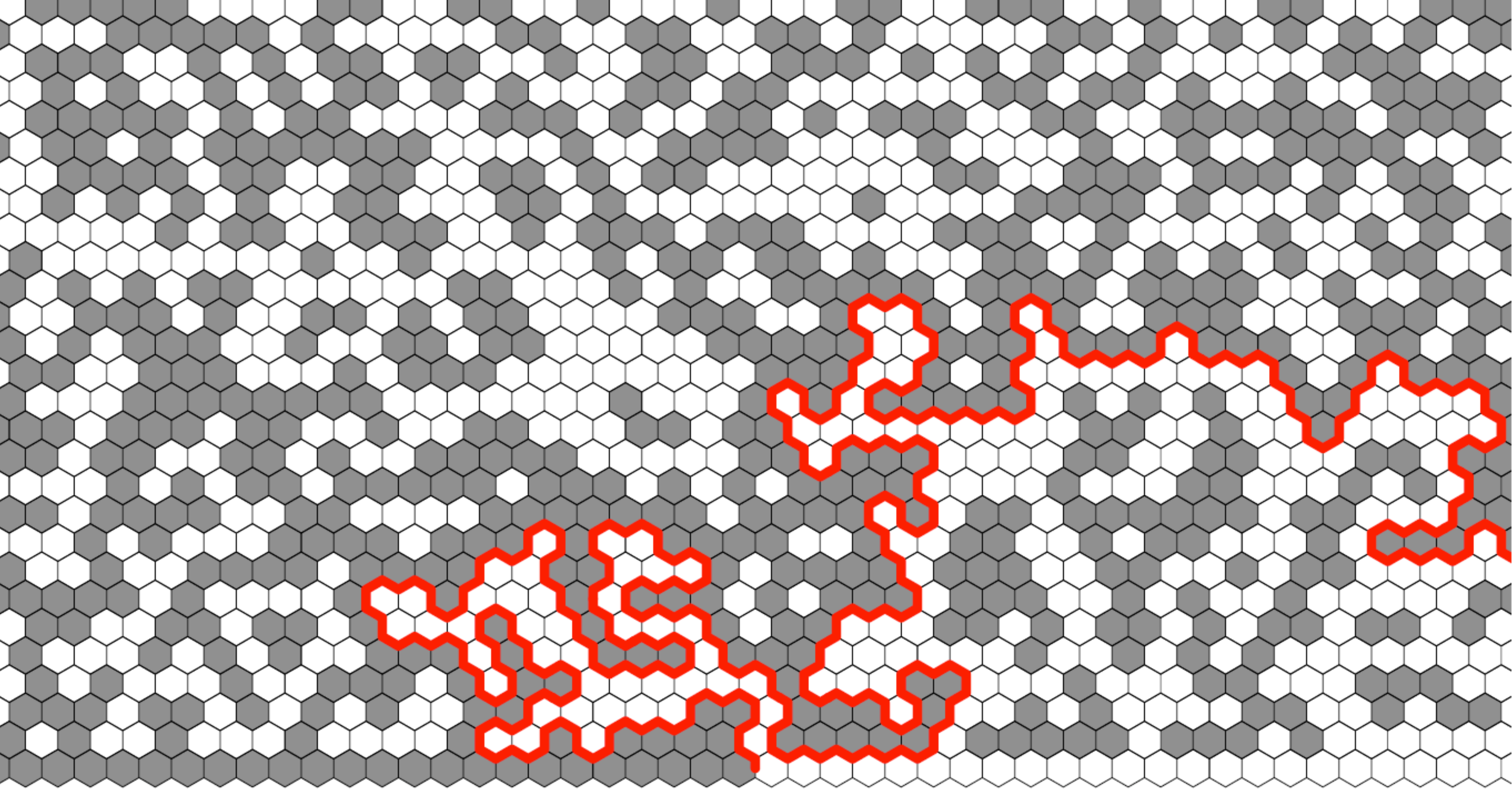
2d Statistical Mechanics:

- E.g. boundaries between phases

- Often one gets curves joining boundary points

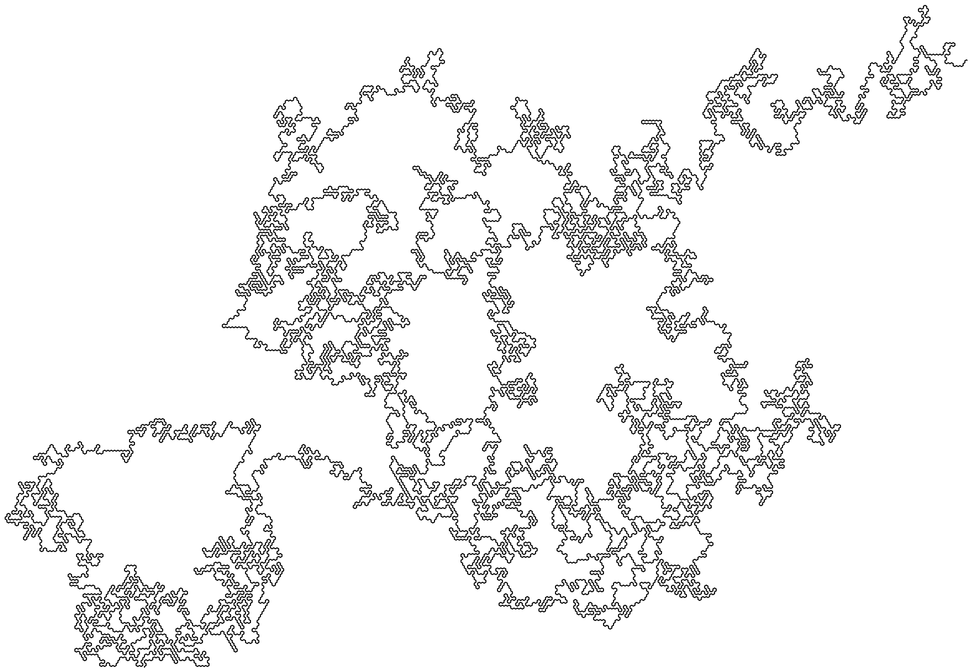
- Critical temperatures...

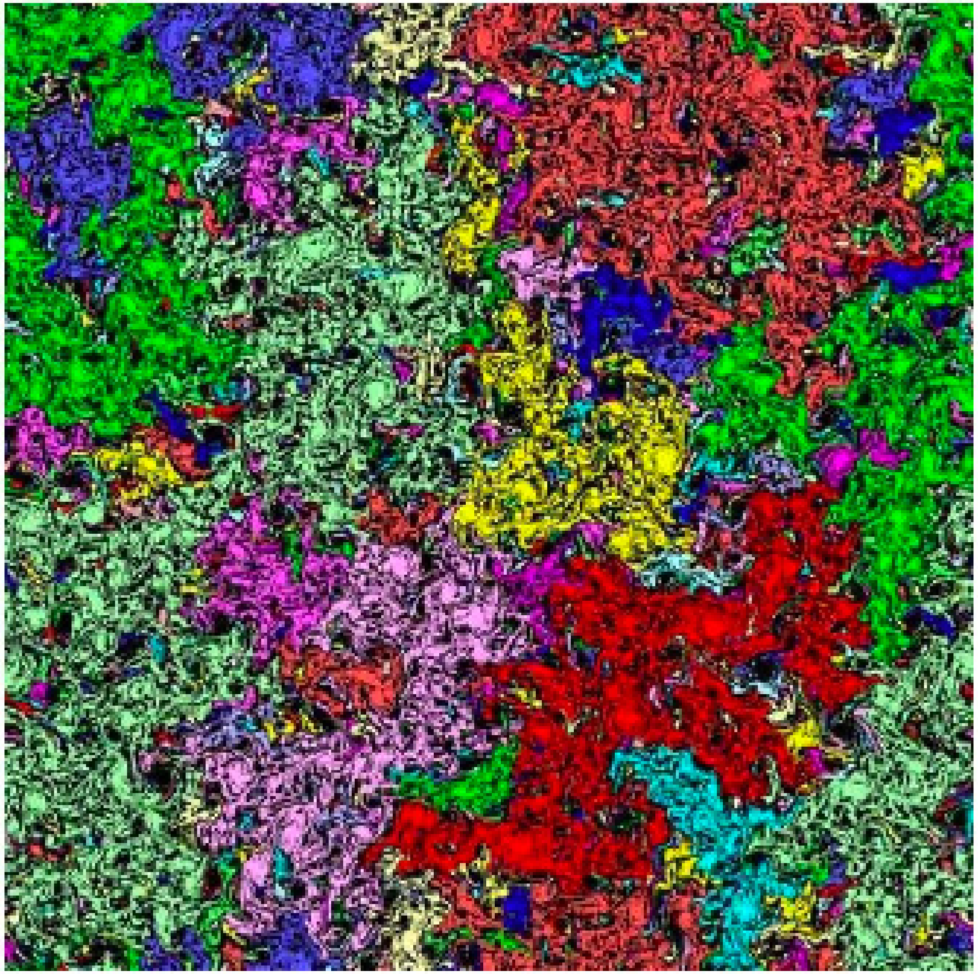
Percolation; Brownian frontier; etc., etc. ....



Pictures: Oded Schramm

Percolation interface / SLE(6) / Smirnov 2001





Scaling limit: SLE(κ)

- Curves ($\kappa < 4$) growing in fictious time, constructed with an explicit equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - B(\kappa\, t)}, \qquad g_0(z) = z.$$

- Statistics of the full curve less explicit.

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Proposal of Peter Jones: construct natural random Jordan curves by describing *statistics of homeomorphisms of line or circle*.

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Closed Jordan curves in $\widehat{\mathbb{C}} \longleftrightarrow$ Homeomorphisms $\phi : \mathbb{T} \to \mathbb{T}$.

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Jordan curve $\Gamma \subset \widehat{\mathbb{C}}$ splits $\widehat{\mathbb{C}} \setminus \Gamma = \Omega_+ \cup \Omega_-$.

Take Riemann mappings:

$$f_+ : \mathbb{D} \to \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \to \Omega_-$$

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Take Riemann mappings:

$$f_+ : \mathbb{D} \to \Omega_+ \quad \text{and} \quad f_- : \mathbb{D}_\infty \to \Omega_-$$

f_- and f_+ extend continuously to $\mathbb{T} = \partial\mathbb{D} = \partial\mathbb{D}_\infty \qquad \Rightarrow$

$$\text{get homeo :} \quad \phi = \left(f_+\right)^{-1} \circ f_- : \mathbb{T} \to \mathbb{T}$$

The Welding problem: invert this !

Given homeo $\phi : \mathbb{T} \to \mathbb{T}$, find Γ and Riemann maps $f_\pm$ so that

$$\boxed{\phi = \left(f_+\right)^{-1} \circ f_- : \mathbb{T} \to \mathbb{T}}$$

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Problem: Not possible for every homeomorphism ϕ !

Welding by QuasiConformal homeomorphisms:

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Conformally invariant structures by:

$$\partial_{\bar{z}} f = \mu(z) \partial_z f \qquad \text{with } |\mu(z)| \leq k < 1 \text{ a.e.}$$

- Has always homeomorphic solutions $F : \mathbb{C} \to \mathbb{C}$
- Any solution $f : \Omega \to \Omega'$ has form $f = h \circ F$,

  h holomorphic in $F(\Omega)$.

Suppose first that $\phi : \mathbb{T} \to \mathbb{T}$ is a restriction

$$\phi = f|_{\mathbb{T}},$$

where $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *quasiconformal* homeo:

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Solve, with a homeo F,

$$\partial_{\overline{z}} F = \begin{cases} \mu(z) \partial_z F & \text{if } x \in \mathbb{D} \\ 0 & \text{if } x \in \mathbb{D}_\infty \end{cases}$$

Then: $f_- := F|_{\mathbb{D}_\infty} : \mathbb{D}_\infty \to \Omega_-$ is conformal and $\Gamma = F(\mathbb{T}) = f_-(\mathbb{T})$ is a Jordan curve.

Beltrami equation: Now have two global solutions

$$\partial_{\bar z} f = \mu(z) \partial_z f, \quad \partial_{\bar z} F = \mu(z) \chi_{\mathbb{D}} \partial_z F$$

Uniqueness of solutions in the uniformly elliptic case $\|\mu\|_\infty < 1$:

$$\Rightarrow \quad F(z) = f_+ \circ f(z), \quad z \in \mathbb{D} \ \text{(!)}$$

Here

$$f_+ : \mathbb{D} = f(\mathbb{D}) \to F(\mathbb{D}) := \Omega_+ \quad \text{is conformal.}$$

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$f_\pm$ solve welding: Since $F|_{\mathbb{D}_\infty} := f_-$ and $f|_{\mathbb{T}} = \phi$ then

$$\phi(z) = f|_{\mathbb{T}}(z) = f_+^{-1} \circ f_-(z), \qquad z \in \mathbb{T}.$$

Have now reduction to Beltrami equation: When does this work?

When is $\phi : \mathbb{T} \to \mathbb{T}$ a restriction $\phi = f|_{\mathbb{T}}$ of a qc homeo $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$?

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In the uniformly elliptic case (i.e. $\|\mu\|_\infty < 1$), this happens $\Leftrightarrow$
ϕ is quasisymmetric:

$$\frac{|\phi(s+t) - \phi(s)|}{|\phi(s-t) - \phi(s)|} \leq K < \infty, \quad s, t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$$

These have QC extensions with $|\mu| \leq m(K) < 1$.

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In the random setting: Our $\phi : \mathbb{T} \to \mathbb{T}$ will not be quasisymmetric and our Beltrami will not be uniformly elliptic !

$\longrightarrow$ Degenerate elliptic systems/ Degenerate Beltrami equations.

$\longrightarrow$ Existence of homeomorphic solutions not obvious.

$\longrightarrow$ Uniqueness not obvious.

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We take

$$\phi(t) = \frac{\int_0^t e^{\beta X(s)}\, ds}{\int_0^1 e^{\beta X(s)}\, ds}$$

where

- $X = X(t)$ is a *Gaussian random field*, the restriction of 2D Gaussian free field on the unit circle,
- $0 \leq \beta < \beta_0$, where β_0 is a "critical value".

Recall: Gaussian random variables are determined by their expectation (take zero) and covariance.

Gaussian free field (GFF), restricted to $\mathbb{T}$:

- $X(\zeta)$ is a *Gaussian random field* with covariance

$$\mathbb{E}X(\zeta)X(\xi) = \log \frac{1}{|\zeta - \xi|}, \quad \zeta, \xi \in \mathbb{T}.$$

(Conformally invariant modulo constants !)

- X is $\mathcal{D}'(\mathbb{T})$ -valued random field: need some care!

Existence:

Set

$$X = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \Big(A_n \cos(2\pi n t) + B_n \sin(2\pi n t) \Big), \qquad t \in [0,1),$$

where

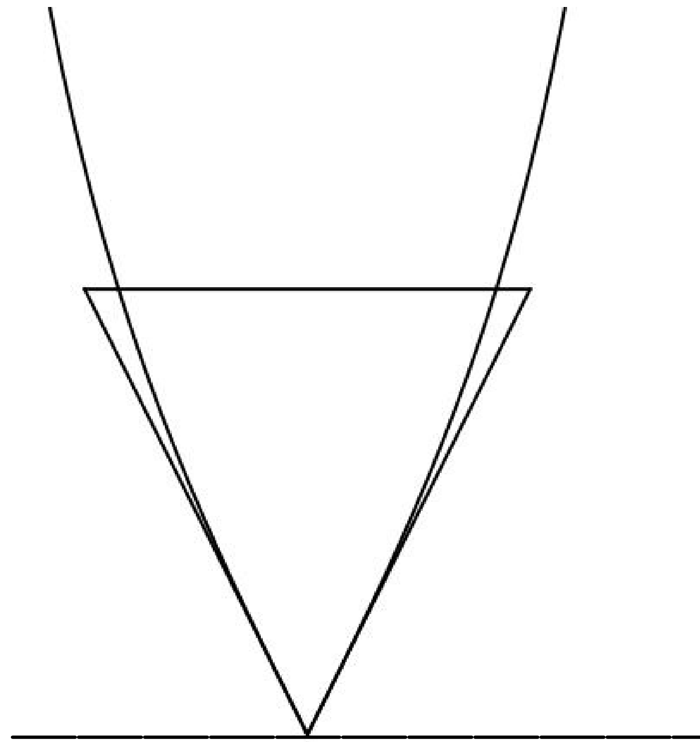
$$A_n \sim N(0,1) \sim B_n \quad (n \geq 1)$$

are independent standard Gaussians.

Bacry - Muzy; A Geometric approach:

- $X(s) = \int_{U+s} W\left(\frac{dxdy}{y^2}\right), \quad s \in \mathbb{R}.$

- W is the (periodic) white noise in $\mathbb{H}$.
  (w.r.t. hyperbolic measure in $\mathbb{H}$)



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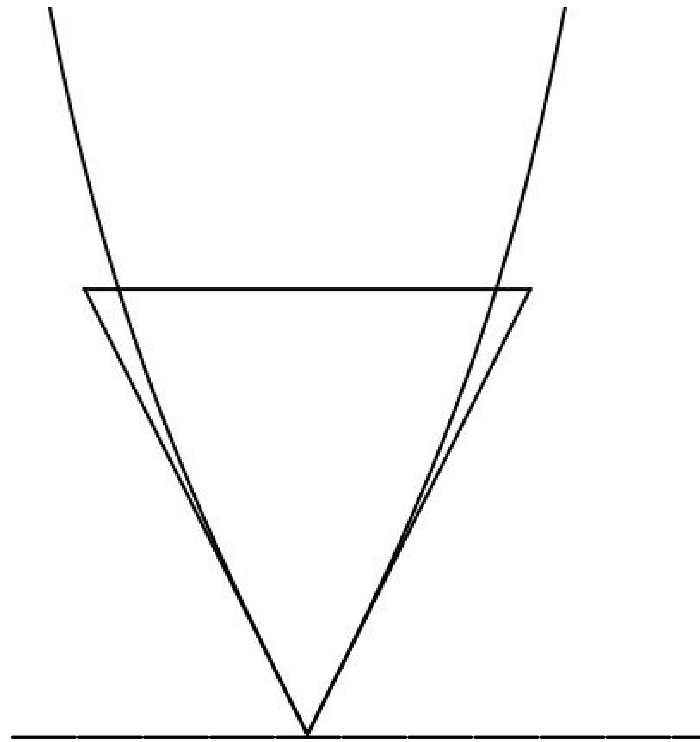

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- $U := \{(x,y) \in \mathbb{H} \ : \ -1/2 < x < 1/2, \ \ y > \frac{2}{\pi}\tan(|\pi x|)\}.$

  [Roughly, $U \simeq \{\ 2|x| < y < 1/2, \ \ |x| \le 1/4\}$]

Random homeomorphisms $\phi : \mathbb{T} \to \mathbb{T}$: We take

$$\phi(t) = \int_0^t e^{\beta X(s)} ds \Big/ \int_0^1 e^{\beta X(s)} ds, \qquad \mathbb{T} = \mathbb{R}/\mathbb{Z},$$

where $0 \le \beta < \sqrt{2} =: \beta_0$ and X is the restriction of GFF on $\mathbb{T}$.

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- X is $\mathcal{D}'(\mathbb{T})$-valued random field: How to define $\phi(t)$?!
- Regularize $:e^{\beta X_\epsilon(s)}: \; = e^{\beta X_\epsilon(s)} / \mathbb{E} e^{\beta X_\epsilon(s)}$; martingale in $\epsilon \searrow 0$.
- Almost surely $:e^{\beta X_\epsilon(s)}: ds$ converges weakly to a random Borel measure $\tau(ds) \equiv \; :e^{\beta X(s)}: ds$ on $\mathbb{T}$.

Random measure: Properties of $\tau(ds) = :e^{\beta X(s)}: ds$ for $\beta < \sqrt{2}$

- τ has no atoms
- $\mathbb{E}\,\tau(I)^p < \infty$, for $-\infty < p < 2/\beta^2$ and all intervals $I \subset \mathbb{T}$

Hence by Hölder, the distortion

$$\frac{|\phi(s+t) - \phi(s)|}{|\phi(s-t) - \phi(s)|} = \frac{\tau([s, s+t])}{\tau([s-t, s])} \in L^p(\omega), \quad p < 2/\beta^2.$$

Theorem (A-J-K-S). Let $\phi = \phi_\beta$ be the random homeomorphism

$$\phi(s) = \tau([0, s])/\tau([0, 1])$$

with $\tau(ds) = \,: e^{\beta X(s)} : ds$ and $\beta < \sqrt{2}$.

Then a.s. in ω, the random homeo ϕ admits a conformal welding

$$(\Gamma, f_+, f_-).$$

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(ii) Dependence on β is continuous 'pathwise'.

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(iii) The proof works as well for $\psi = \phi_\beta \circ (\tilde{\phi}_{\tilde{\beta}})^{-1}$,

where $\beta, \tilde{\beta} < \sqrt{2}$ and ϕ_β and $\tilde{\phi}_{\tilde{\beta}}$ are independent copies of (*).

Outline of proof.

1. Extension of ϕ to $f : \mathbb{C} \to \mathbb{C}$ by a Beurling-Ahlfors-type extension

   $\implies$ bound for $\mu = \bar{\partial} f / \partial f$ in terms of the measure τ.

2. Existence for Beltrami equation by a method of Lehto, to control moduli of annuli

3. The crucial ingredient for step 2: Probabilistic large deviation estimate for the Lehto integrals which control moduli of annuli

4. Uniqueness of welding: theorem of Jones-Smirnov on removability of Hölder curves

Extension: Many ways to extend $f : S^1 \to S^1$ to $F : \mathbb{C} \to \mathbb{C}$.

Beurling-Ahlfors: Extend $h : \mathbb{R} \to \mathbb{R}$ to upper half plane by

$$F(x+iy) = \frac{1}{2}\int_0^1 (h(x+ty) + h(x-ty) + i(h(x+ty) - h(x-ty))dt.$$

Map to disc and reflect to $|z| > 1$.

To solve Beltrami need to control $\mu = \partial_{\overline{z}} F / \partial_z F$,
i.e. look for upper bounds for distortion

$$K(z) := \frac{1+|\mu(z)|}{1-|\mu(z)|} .$$

Lehto integral. Solve degenerate Beltrami

$$\partial_{\bar{z}} F = \chi_{\mathbb{D}}(z)\mu(z)\partial_z F, \qquad F(z) = z + \mathcal{O}(1/z)$$

with the random μ to get $\Gamma = F(\partial\mathbb{D})$.

Idea by Lehto: control images of annuli under F by

$$L(\zeta, r, R) = \int_r^R \frac{1}{\int_0^{2\pi} K\left(\zeta + \rho e^{i\theta}\right) d\theta} \frac{d\rho}{\rho}, \qquad K(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

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Idea by Lehto: control images of annuli under F by

$$d(A) \leq 16\, D(A)\, e^{-2\pi^2 L(\zeta,r,R)}.$$

- $A = A(\zeta, r, R)$ annulus, center ζ, radii r, R.

- $d(A)$ inner and $D(A)$ outer diameter of $F(A)$

Existence and Hölder: Suppose for all ζ, $L(\zeta, r, 1) \geq -c \log r$.

Then

$$\text{diam}[f(B(\zeta, r))] \leq 80 r^{2\pi^2 c}$$

- Implies existence for Beltrami via equicontinuity of regularized solutions !

- Implies Hölder continuity of f.

- Need also prove $K \in L^1(\mathbb{D})$ to get $f \in W^{1,1}$ and for equicontinuity of inverse maps

Existence and Hölder: Suppose for all ζ, $L(\zeta, r, 1) \geq -c \log r$.

Then

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- Key estimate: $L(\zeta, r, 1) \geq -c \log r$ with high probability:

$$\text{Prob}\Big(L(\zeta, \rho^n, 1) < n\delta\Big) \leq \rho^{-(1+\delta)n} \qquad (\rho = 2^{-N})$$

- Suffices to consider uniform grids $\zeta_i \in \partial\mathbb{D} = \mathbb{T}$, $i = 1, \ldots \rho^{-2n}$.
- Then, by Borel-Cantelli, for a.e. ω: for $n > n(\omega)$,

$$L(\zeta_i, \rho^n, 1) > n\delta, \qquad i = 1, \ldots \rho^{-2n}.$$

$\implies$ Hölder continuity a.e. in ω

Distortion: Local and scale invariant distortion bound !

Distortion of the extension in the Whitney cube at scale 2^{-n} depends on distortion of the random homeo at the same scale and place.

For all z in Whitney cube spanned by dyadic interval $I \in \mathcal{D}_n$,

$$K(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \leq C \sum_{J,J'} \frac{\tau(J)}{\tau(J')}$$

Here $J, J' \in \mathcal{D}_{n+5}$ contained in I and its dyadic neighbors.

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Linear and local bound essential to cover all $\beta < \sqrt{2}$.

Uniqueness for welding   follows from Hölder continuity:

Suppose $f_\pm$ and $f'_\pm$ are two solutions, mapping $\mathbb{D}, \mathbb{D}_\infty$ onto $\Omega_\pm$ and $\Omega'_\pm$. Show:

$$f'_\pm = \Phi \circ f_\pm, \qquad \Phi : \widehat{C} \to \widehat{C} \text{ Möbius.}$$

Now

$$\Psi(z) := \begin{cases} f'_+ \circ \left(f_+\right)^{-1}(z) & \text{if } z \in \Omega_+ \\ f'_- \circ (f_-)^{-1}(z) & \text{if } z \in \Omega_- \end{cases}$$

is conformal outside $\Gamma = \partial\Omega_\pm$.

Result of Jones-Smirnov: Hölder curves are conformally removable i.e. Ψ extends conformally to $\widehat{C}$ i.e.it is Möbius.

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• [Sheffield 2010, November, manuscript] states that SLE_κ is indeed obtained from 'welding' of two weighted 'quantum wedges'! Locally this corresponds to welding of $(\phi_\beta)^{-1} \circ \widetilde{\phi}_\beta$, again with $\kappa = 2\beta^2$.

THANKS !