Brown Measures of Unbounded Operators Affiliated with a Finite von Neumann Algebra

Uffe Haagerup and Hanne Schultz *

Abstract

In this paper we generalize Brown's spectral distribution measure to a large class of unbounded operators affiliated with a finite von Neumann algebra. Moreover, we compute the Brown measure of all unbounded R–diagonal operators in this class. As a particular case, we determine the Brown measure $z = xy^{-1}$, where (x, y) is a circular system in the sense of Voiculescu, and we prove that for all $n \in \mathbb{N}$, $z^n \in L^p(\mathcal{M}, \tau)$ if and only if $0 < p < \frac{2}{n+1}$.

1 Introduction

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ, and let

$$\Delta(T) = \exp\Big(\int_0^\infty \log t \, \mathrm{d}\mu_{|T|}(t)\Big)$$

denote the corresponding Fuglede–Kadison determinant. L. G. Brown proved in [Br] that for every $T \in \mathcal{M}$, there exists a unique, compactly supported measure $\mu_T \in \mathrm{Prob}(\mathbb{C})$ with the property that

$$\log \Delta(T - \lambda \mathbf{1}) = \int_{\mathbb{C}} \log |z - \lambda| \, \mathrm{d}\mu_T(z), \qquad \lambda \in \mathbb{C}.$$

This measure is called Brown's spectral distribution measure (or just the Brown measure) of T. It was computed in a number of special cases in [HL], [BL], [DH], and [AH]. In particular, it was proven in [HL, Theorem 4.5] that if $T \in \mathcal{M}$ is R–diagonal in the sense of Nica and Speicher [NS], then μ_T can be determined from the S–transform of the distribution $\mu_{|T|^2}$. For simplicity, assume that $T \in \mathcal{M}$ is an R–diagonal element which is not proportional to a unitary and for which $\ker(T) = 0$. Then μ_T is the unique probability measure on $\mathbb{C}$ which is invariant under the rotations $z \mapsto \gamma z$, $\gamma \in \mathbb{T}$, and which satisfies

$$\mu_T\Big(B\big(0, \mathcal{S}_{\mu_{|T|^2}}(t-1)^{-\frac{1}{2}}\big)\Big) = t, \qquad 0 < t < 1.$$

*Supported by the Danish National Research Foundation.

In this paper we extend the Brown measure to all operators in the set $\mathcal{M}^\Delta$ of closed, densely defined operators T affiliated with $\mathcal{M}$ satisfying

$$\int_0^\infty \log^+ t \,\mathrm{d}\mu_{|T|}(t) < \infty.$$

Moreover, we extend [HL, Theorem 4.5] to all R–diagonal operators in $\mathcal{M}^\Delta$. Finally, we will study a particular example of an unbounded R–diagonal element, namely the operator $z = xy^{-1}$, where (x, y) is a circular system in the sense of Voiculescu.

The material in this paper is organized as follows: In section 2 we introduce the class $\mathcal{M}^\Delta$ and generalize the Brown measure to all $T \in \mathcal{M}^\Delta$ by proving, that for such T, there is a unique $\mu_T \in \mathrm{Prob}(\mathbb{C})$ satisfying

$$\int_{\mathbb{C}} \log^+ |z| \,\mathrm{d}\mu_T(z) < \infty$$

and

$$\log \Delta(T - \lambda \mathbf{1}) = \int_{\mathbb{C}} \log |z - \lambda| \,\mathrm{d}\mu_T(z), \qquad \lambda \in \mathbb{C}.$$

Moreover, we extend Weil's inequality

$$\int_{\mathbb{C}} |z|^p \,\mathrm{d}\mu_T(z) \le \|T\|_p^p$$

to all $T \in L^p(\mathcal{M}, \tau)$. The main results in section 2 are stated in the appendix of Brown's paper [Br] without proofs or with very sketchy proofs. Since the results of the remaining sections of this paper and of our forthcoming paper [HS] rely heavily on these statements, we have decided to include complete proofs. We will follow a different route than the one outlined in [Br]. For instance, we do not use the functions $\Lambda_t(T)$ and $s_T(t)$ from [Br, section 1].

In section 3 we introduce unbounded R–diagonal operators and we prove the following generalization of [HL, section 3]: The powers $(S^n)_{n=1}^\infty$ of an R–diagonal operator are R–diagonal, and the sum $S+T$ and the product ST of $*$–free R–diagonal operators are again R–diagonal. Moreover,

$$\begin{aligned} \mu_{|S^n|^2} &= \mu_{|S|^2}^{\boxtimes n}, \\ \tilde{\mu}_{|S+T|} &= \tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|}, \\ \mu_{|ST|^2} &= \mu_{|S|^2} \boxtimes \mu_{|T|^2}, \end{aligned}$$

where $\tilde{\mu} = \frac{1}{2}(\mu + \check{\mu})$ denotes the symmetrization of a measure $\mu \in \mathrm{Prob}(\mathbb{R})$, and $\boxplus$ ($\boxtimes$, resp.) denotes the additive (multiplicative, resp.) free convolution of measures (cf. [BV]). These results are applied in section 4 to determine the Brown measure of R–diagonal operators in $\mathcal{M}^\Delta$.

In section 5 we consider the operator $z = xy^{-1}$, where (x, y) is a circular system in the sense of Voiculescu, and we prove that the Brown measure of z is given by

$$\mathrm{d}\mu_z(s) = \frac{1}{\pi(1 + |s|^2)} \,\mathrm{dRe}s \,\mathrm{dIm}s.$$

Moreover, we show that for all $n \in \mathbb{N}$, $z^n, z^{-n} \in L^p(\mathcal{M}, \tau)$ iff $0 < p < \frac{2}{n+1}$, and when this holds,

$$\|z^n\|_p^p = \|z^{-n}\|_p^p = \frac{(n+1)\sin\left(\frac{\pi p}{2}\right)}{\sin\left(\frac{(n+1)\pi p}{2}\right)},$$

and

$$\|(z^n - \lambda \mathbf{1})^{-1}\|_p \leq \|z^{-n}\|_p, \qquad \lambda \in \mathbb{C}.$$

The last two formulas play a key role in our forthcoming paper [HS] on invariant subspaces for operators in a general II_1–factor.

2 The Brown measure of certain unbounded operators.

In [Br, Appendix] Brown described in outline how to define a Brown measure for certain *undbounded* operators affiliated with $\mathcal{M}$, where $\mathcal{M}$ is a von Neumann algebra equipped with a faithful, normal, semifinite trace.

In this section we give a more detailed exposition on the subject in the case where $\mathcal{M}$ is a finite von Neumann algebra with faithful, tracial state τ. To be more explicit, we show how one can extend the definition of the Brown measure to a class $\mathcal{M}^\Delta$ of closed, densely defined operators affiliated with $\mathcal{M}$. We also prove that many of the properties of the Brown measure for bounded operators carry over to the unbounded case.

We let $\tilde{\mathcal{M}}$ denote the set of closed, densely defined operators affiliated with $\mathcal{M}$. Recall that every operator $T \in \tilde{\mathcal{M}}$ has a polar decomposition

$$T = U|T| = U \int_0^\infty t \, \mathrm{d}E_{|T|}(t), \tag{2.1}$$

where $U \in \mathcal{M}$ is a unitary, and the spectral measure $E_{|T|}$ takes values in $\mathcal{M}$. In particular, for $T \in \tilde{\mathcal{M}}$ we may define $\mu_{|T|} \in \mathrm{Prob}(\mathbb{R})$ by

$$\mu_{|T|}(B) = \tau(E_{|T|}(B)), \qquad (B \in \mathbb{B}). \tag{2.2}$$

2.1 Definition. We denote by $\mathcal{M}^\Delta$ the set of operators $T \in \tilde{\mathcal{M}}$ fulfilling the condition

$$\tau(\log^+ |T|) = \int_0^\infty \log^+(t) \, \mathrm{d}\mu_{|T|}(t) < \infty. \tag{2.3}$$

For $T \in \mathcal{M}^\Delta$, the integral

$$\int_0^\infty \log t \, \mathrm{d}\mu_{|T|}(t) \ \in \ \mathbb{R} \cup \{-\infty\}$$

is well–defined, and we define the *Fuglede–Kadison determinant* of T, $\Delta(T) \in [0, \infty)$, by

$$\Delta(T) = \exp\Big(\int_0^\infty \log t \, \mathrm{d}\mu_{|T|}(t)\Big). \tag{2.4}$$

Note that for $T \in \mathcal{M}$, $\Delta(T)$ is the usual Fuglede–Kadison determinant of T.

2.2 Remark. If $T \in L^p(\mathcal{M}, \tau)$ for some $p \in (0, \infty)$, then

$$\int_0^\infty t^p \, \mathrm{d}\mu_{|T|}(t) < \infty,$$

implying that

$$\int_1^\infty \log t \, \mathrm{d}\mu_{|T|}(t) = \frac{1}{p} \int_1^\infty \log(t^p) \, \mathrm{d}\mu_{|T|}(t) \leq \frac{1}{p} \int_1^\infty t^p \, \mathrm{d}\mu_{|T|}(t) < \infty,$$

and hence $T \in \mathcal{M}^\Delta$.

2.3 Lemma. *If $T \in \mathcal{M}^\Delta$ and $\Delta(T) > 0$, then T is invertible in $\tilde{\mathcal{M}}$, $T^{-1} \in \mathcal{M}^\Delta$, and $\Delta(T^{-1}) = \frac{1}{\Delta(T)}$.*

Proof. If $T \in \mathcal{M}^\Delta$ and $\Delta(T) > 0$, then

$$\int_0^1 |\log t| \, \mathrm{d}\mu_{|T|}(t) < \infty.$$

Hence, $\tau(E_{|T|}(\{0\})) = \mu_{|T|}(\{0\}) = 0$, so that $\ker(T) = \{0\}$. Since $\mathcal{M}$ is finite, also $\ker(T^*) = \{0\}$, which implies that T has a closed, densely defined inverse $T^{-1} \in \tilde{\mathcal{M}}$. Take a unitary $U \in \mathcal{M}$ such that $T = U|T|$. Then

$$|T^{-1}| = U|T|^{-1}U^*.$$

Hence, $\mu_{|T^{-1}|} = \mu_{|T|^{-1}}$. Since $\mu_{|T|^{-1}}$ is the push–forward measure of $\mu_{|T|}$ via the map $t \mapsto \frac{1}{t}$, we now have that

$$\begin{aligned} \int_1^\infty \log t \, \mathrm{d}\mu_{|T^{-1}|}(t) &= \int_1^\infty \log t \, \mathrm{d}\mu_{|T|^{-1}}(t) \\ &= \int_0^1 \log\left(\frac{1}{t}\right) \mathrm{d}\mu_{|T|}(t) \\ &= -\int_0^1 \log t \, \mathrm{d}\mu_{|T|}(t) \\ &< \infty. \end{aligned}$$

Hence, $T^{-1} \in \mathcal{M}^\Delta$ and

$$\log \Delta(T^{-1}) = \int_0^\infty \log\left(\frac{1}{t}\right) \mathrm{d}\mu_{|T|}(t) = -\log \Delta(T),$$

i.e. $\Delta(T^{-1}) = \frac{1}{\Delta(T)}$. ■

2.4 Lemma. *Let $T \in \tilde{\mathcal{M}}$. Then the following are equivalent:*

(a) $T \in \mathcal{M}^\Delta$, *i.e.* $\int_0^\infty \log^+(t)\,\mathrm{d}\mu_{|T|}(t) < \infty$.

(b) $T = AB^{-1}$ *for some* $A, B \in \mathcal{M}$ *with* $\Delta(B) > 0$.

(c) $T = C^{-1}D$ *for some* $C, D \in \mathcal{M}$ *with* $\Delta(C) > 0$.

Moreover, if $T \in \mathcal{M}^\Delta$ *and* $T = AB^{-1} = C^{-1}D$ *for some* $A, B, C, D \in \mathcal{M}$ *with* $\Delta(B), \Delta(C) > 0$, *then*

$$\Delta(T) = \frac{\Delta(A)}{\Delta(B)} = \frac{\Delta(D)}{\Delta(C)}. \tag{2.5}$$

Proof. If $T \in \mathcal{M}^\Delta$, then $T = U|T|$ for some unitary $U \in \mathcal{M}$, and $T = AB^{-1}$, where

$$A = U|T|(|T|^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M} \tag{2.6}$$

and

$$B = (|T|^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M}. \tag{2.7}$$

Since $\frac{1}{2}\log(t^2+1) \le \log(2t)$ when $t \ge 1$, we get that

$$\begin{aligned} \log\Delta(B) &= -\frac{1}{2}\int_0^\infty \log(t^2+1)\,\mathrm{d}\mu_{|T|}(t) \\ &\ge -\frac{1}{2}\int_{[0,1[} \log 2\,\mathrm{d}\mu_{|T|}(t) - \int_{[1,\infty[} \log(2t)\mathrm{d}\mu_{|T|}(t) \\ &> -\infty, \end{aligned} \tag{2.8}$$

that is, $\Delta(B) > 0$.

Also, $T = U|T|U^*U$, and with

$$S = U|T|U^*, \tag{2.9}$$

$$C = (S^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M}, \tag{2.10}$$

and

$$D = S(S^2 + \mathbf{1})^{-\frac{1}{2}} \in \mathcal{M}, \tag{2.11}$$

we have that $T = C^{-1}DU$. Moreover,

$$\begin{aligned} \log\Delta(C) &= -\frac{1}{2}\int_0^\infty \log(t^2+1)\,\mathrm{d}\mu_S(t) \\ &= -\frac{1}{2}\int_0^\infty \log(t^2+1)\,\mathrm{d}\mu_{|T|}(t) \\ &> -\infty, \end{aligned}$$

i.e. $\Delta(C) > 0$.

Now we have shown that (a) implies (b) and (c). On the other hand, if $T = AB^{-1}$ for some $A, B \in \mathcal{M}$ with $\Delta(B) > 0$, then we may assume that $B \geq 0$. Then

$$\begin{aligned} \tau(\log^+ |T|) &\leq \tau(\log(\mathbf{1} + |T|^2)) \\ &= \tau(\log(\mathbf{1} + B^{-1}A^*AB^{-1})). \end{aligned}$$

Since $B^{-1}A^*AB^{-1} \leq \|A\|^2 B^{-2}$, and since $t \mapsto \log(1+t)$ is operator monotone on $[0, \infty)$, we get that

$$\begin{aligned} \tau(\log^+ |T|) &\leq \tau(\log(\mathbf{1} + \|A\|^2 B^{-2})) \\ &\leq \tau(\log((1 + \|A\|^2)(\mathbf{1} + B^{-2}))) \\ &= \log(1 + \|A\|^2) + \tau(\log(\mathbf{1} + B^{-2})). \end{aligned}$$

Since B is bounded and $\Delta(B) > 0$,

$$\begin{aligned} \tau(\log(\mathbf{1} + B^{-2})) &= \tau(\log(B^2 + \mathbf{1})) - 2\tau(\log B) \\ &\leq \log(\|B\|^2 + 1) - 2\Delta(B) \\ &< \infty. \end{aligned}$$

This shows that $T \in \mathcal{M}^\Delta$, i.e. (b) implies (a). It follows that if $T = C^{-1}D$ for some $C, D \in \mathcal{M}$ with $\Delta(C) > 0$, then $T^* \in \mathcal{M}^\Delta$. Take a unitary $U \in \mathcal{M}$ such that $T = U|T|$. Then $|T^*| = U|T|U^*$, implying that $\mu_{|T^*|} = \mu_{|T|}$. Hence T belongs to $\mathcal{M}^\Delta$ as well, and (c) implies (a).

Now, let $T \in \mathcal{M}^\Delta$. Then $T = AB^{-1} = C^{-1}D$ for some $A, B, C, D \in \mathcal{M}$ with $\Delta(B), \Delta(C) > 0$. Moreover, for all such choices of A, B, C and D,

$$CA = C(AB^{-1})B = C(C^{-1}D)B = DB.$$

Since Δ is multiplicative on $\mathcal{M}$ (cf. [FuKa]), it follows that

$$\Delta(C)\Delta(A) = \Delta(CA) = \Delta(DB) = \Delta(D)\Delta(B).$$

Hence,

$$\frac{\Delta(A)}{\Delta(B)} = \frac{\Delta(D)}{\Delta(C)}. \tag{2.12}$$

In particular, with A and B as in (2.6) and (2.7), respectively, we have that $\Delta(B) > 0$, $T = AB^{-1}$, and

$$\log \Delta(A) = \int_0^\infty \log\left(\frac{t}{\sqrt{t^2+1}}\right) \mathrm{d}\mu_{|T|}(t),$$

and

$$\log \Delta(B) = \int_0^\infty \log\left(\frac{1}{\sqrt{t^2+1}}\right) \mathrm{d}\mu_{|T|}(t),$$

so that

$$\log \Delta(T) = \log \Delta(A) - \log \Delta(B).$$

Then by (2.12), for all choices of $C, D \in \mathcal{M}$ with $\Delta(C) > 0$ and $T = C^{-1}D$,

$$\frac{\Delta(D)}{\Delta(C)} = \frac{\Delta(A)}{\Delta(B)} = \Delta(T).$$

Then finally, by (2.12), for all choices of $A, B \in \mathcal{M}$ with $\Delta(B) > 0$ and $T = AB^{-1}$, we also have that

$$\frac{\Delta(A)}{\Delta(B)} = \Delta(T). \qquad \blacksquare$$

2.5 Proposition. *If $S, T \in \mathcal{M}^\Delta$, then $ST \in \mathcal{M}^\Delta$, and*

$$\Delta(ST) = \Delta(S)\Delta(T). \tag{2.13}$$

Proof. Let $S, T \in \mathcal{M}^\Delta$. Take $A, B, C, D \in \mathcal{M}$ with $\Delta(B), \Delta(C) > 0$, such that $T = AB^{-1}$ and $S = C^{-1}D$. Then

$$ST = C^{-1}DAB^{-1},$$

where $DAB^{-1} \in \mathcal{M}^\Delta$. Hence there exist $E, F \in \mathcal{M}$ with $\Delta(E) > 0$ such that $DAB^{-1} = E^{-1}F$. It follows that

$$ST = C^{-1}E^{-1}F = (EC)^{-1}F, \tag{2.14}$$

where $EC, F \in \mathcal{M}$, and $\Delta(EC) = \Delta(E)\Delta(C) > 0$. That is, ST belongs to $\mathcal{M}^\Delta$.

To prove (2.13), we let A, B, C, D, E, F be as above. Applying (2.5) to $ST = (EC)^{-1}F$, $S = C^{-1}D$ and $T = AB^{-1}$, we get that

$$\Delta(ST) = \frac{\Delta(F)}{\Delta(EC)} = \frac{\Delta(F)}{\Delta(E)\Delta(C)} = \frac{\Delta(DA)}{\Delta(B)}\frac{1}{\Delta(C)} = \frac{\Delta(A)}{\Delta(B)}\frac{\Delta(D)}{\Delta(C)} = \Delta(S)\Delta(T). \qquad \blacksquare$$

2.6 Proposition. *$\mathcal{M}^\Delta$ is a subspace of $\tilde{\mathcal{M}}$. In particular, for $T \in \mathcal{M}^\Delta$ and $\lambda \in \mathbb{C}$, $T - \lambda\mathbf{1} \in \mathcal{M}^\Delta$.*

Proof. Clearly, if $T \in \mathcal{M}^\Delta$ and $\alpha \in \mathbb{C}$, then $\alpha T \in \mathcal{M}^\Delta$. If $S, T \in \mathcal{M}$, choose $A, B, C, D \in \mathcal{M}$ with $\Delta(B) > 0$, $\Delta(C) > 0$ and such that

$$S = C^{-1}D, \qquad T = AB^{-1}.$$

Then

$$S + T = C^{-1}(DB + CA)B^{-1},$$

where $DB + CA \in \mathcal{M}$ and $B^{-1}, C^{-1} \in \mathcal{M}^\Delta$ (cf. Lemma 2.3). Then, by Proposition 2.5, $S + T \in \mathcal{M}^\Delta$. $\blacksquare$

In the following we consider a fixed operator $T \in \mathcal{M}^\Delta$. Then we define $f : \mathbb{C} \to [-\infty, \infty)$ by

$$f(\lambda) = L(T - \lambda\mathbf{1}) := \log\Delta(T - \lambda\mathbf{1}), \qquad (\lambda \in \mathbb{C}). \tag{2.15}$$

The next thing we want to prove is:

2.7 Theorem. *f given by (2.15) is subharmonic in $\mathbb{C}$, and*

$$\mathrm{d}\mu_T = \frac{1}{2\pi}\nabla^2 f \,\mathrm{d}\lambda \tag{2.16}$$

(taken in the distribution sense) defines a probability measure on $(\mathbb{C}, \mathbb{B}_2)$. μ_T is the unique probability measure on $(\mathbb{C}, \mathbb{B}_2)$ satisfying

(i)

$$\int_{\mathbb{C}} \log^+ |z| \,\mathrm{d}\mu_T(z) < \infty,$$

(ii)

$$\forall \lambda \in \mathbb{C}: \quad L(T - \lambda \mathbf{1}) = \int_{\mathbb{C}} \log|\lambda - z| \,\mathrm{d}\mu_T(z). \tag{2.17}$$

Moreover,

(iii)

$$\int_{\mathbb{C}} \log^+ |z| \,\mathrm{d}\mu_T(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta}) \,\mathrm{d}\theta. \tag{2.18}$$

The following lemma was proven by F. Larsen in his unpublished thesis (cf. [La1, section 2]. For the convenience of the reader we include a (somewhat different) proof.

2.8 Lemma. *Let $a, b \in \mathcal{M}$ and let $\varepsilon > 0$. Define $g_\varepsilon, g : \mathbb{C} \to \mathbb{R}$ by*

$$g_\varepsilon(\lambda) = \tfrac{1}{2}\tau(\log((a - \lambda b)^*(a - \lambda b) + \varepsilon \mathbf{1})),$$

and

$$g(\lambda) = \log \Delta(a - \lambda b).$$

Then g_ε is subharmonic, and if $g(\lambda) > -\infty$ for some $\lambda \in \mathbb{C}$, then g is subharmonic as well.

Proof. Let $\lambda_1 = \mathrm{Re}(\lambda)$, $\lambda_2 = \mathrm{Im}(\lambda)$, $\lambda \in \mathbb{C}$. At first we show that $(\lambda_1, \lambda_2) \mapsto g_\varepsilon(\lambda_1 + \mathrm{i}\lambda_2)$ is a C^2–function in $\mathbb{R}^2$. Fix $\varepsilon > 0$, and define $h, k : \mathbb{C} \to \mathcal{M}$ by

$$\begin{aligned} h(\lambda) &= (a - \lambda b)^*(a - \lambda b) + \varepsilon \mathbf{1}, \\ k(\lambda) &= (a - \lambda b)(a - \lambda b)^* + \varepsilon \mathbf{1}. \end{aligned}$$

Then h and k are second order polynomials in (λ_1, λ_2) with coefficients in $\mathcal{M}$, and $h(\lambda) \geq \varepsilon \mathbf{1}$, $k(\lambda) \geq \varepsilon \mathbf{1}$ for all $\lambda \in \mathbb{C}$. Hence, by [HT, Lemma 4.6],

$$g_\varepsilon(\lambda) = \tfrac{1}{2}\tau(\log h(\lambda)), \qquad \lambda \in \mathbb{C},$$

has continuous partial derivatives given by

$$\frac{\partial g_\varepsilon}{\partial \lambda_j} = \tfrac{1}{2}\tau\Big(h^{-1}\tfrac{\partial h}{\partial \lambda_j}\Big), \qquad j = 1, 2.$$

Therefore, by [HT, Lemma 3.2], g_ε is a C^2–function with

$$\frac{\partial^2 g_\varepsilon}{\partial\lambda_i\partial\lambda_j} = \tfrac{1}{2}\tau\Big(- h^{-1}\tfrac{\partial h}{\partial\lambda_i}h^{-1}\tfrac{\partial h}{\partial\lambda_j} + h^{-1}\tfrac{\partial^2 h}{\partial\lambda_i\partial\lambda_j}\Big), \qquad i = 1,2,\ j = 1,2. \tag{2.19}$$

Since g_ε is C^2, g_ε is subharmonic if and only if its Laplacian

$$\frac{\partial^2 g_\varepsilon}{\partial\lambda_1^2} + \frac{\partial^2 g_\varepsilon}{\partial\lambda_2^2}$$

is positive. Following standard notation, we let

$$\frac{\partial}{\partial\lambda} = \frac{1}{2}\Big(\frac{\partial}{\partial\lambda_1} - \mathrm{i}\frac{\partial}{\partial\lambda_2}\Big) \quad \text{and} \quad \frac{\partial}{\partial\overline{\lambda}} = \frac{1}{2}\Big(\frac{\partial}{\partial\lambda_1} + \mathrm{i}\frac{\partial}{\partial\lambda_2}\Big).$$

Then

$$\frac{\partial^2 g_\varepsilon}{\partial\lambda_1^2} + \frac{\partial^2 g_\varepsilon}{\partial\lambda_2^2} = 4\frac{\partial^2 g_\varepsilon}{\partial\overline{\lambda}\partial\lambda}$$

By application of (2.19), we find that

$$\frac{\partial^2 g_\varepsilon}{\partial\overline{\lambda}\partial\lambda} = \tfrac{1}{2}\tau\Big(- h^{-1}\tfrac{\partial h}{\partial\overline{\lambda}}h^{-1}\tfrac{\partial h}{\partial\lambda} + h^{-1}\tfrac{\partial^2 h}{\partial\overline{\lambda}\partial\lambda}\Big). \tag{2.20}$$

Since

$$h(\lambda) = a^*a - \lambda a^*b - \overline{\lambda}b^*a + |\lambda|^2 b^*b + \varepsilon\mathbf{1},$$

we have

$$\begin{aligned}\frac{\partial h}{\partial\lambda} &= -a^*b + \overline{\lambda}b^*b = -(a-\lambda b)^*b,\\ \frac{\partial h}{\partial\overline{\lambda}} &= -b^*a + \lambda b^*b = -b^*(a-\lambda b),\end{aligned}$$

and

$$\frac{\partial^2 h}{\partial\overline{\lambda}\partial\lambda} = b^*b.$$

Applying the identity $x(x^*x+\varepsilon\mathbf{1})^{-1} = (xx^*+\varepsilon\mathbf{1})^{-1}x$ to $x = a-\lambda b$, we find that

$$\begin{aligned}\frac{\partial^2 h}{\partial\overline{\lambda}\partial\lambda} - \frac{\partial h}{\partial\overline{\lambda}}h^{-1}\frac{\partial h}{\partial\lambda} &= b^*b - b^*x(x^*x+\varepsilon\mathbf{1})^{-1}x^*b\\ &= b^*b - b^*(xx^*+\varepsilon\mathbf{1})^{-1}xx^*b\\ &= b^*b - b^*(\mathbf{1} - \varepsilon(xx^*+\varepsilon\mathbf{1})^{-1})b\\ &= \varepsilon b^*(xx^*+\varepsilon\mathbf{1})^{-1}b\\ &= \varepsilon b^*k^{-1}b.\end{aligned}$$

Then by (2.20),

$$\begin{aligned}\frac{\partial^2 g_\varepsilon}{\partial\overline{\lambda}\partial\lambda} &= \tfrac{\varepsilon}{2}\tau(h(\lambda)^{-1}b^*k(\lambda)^{-1}b)\\ &= \tfrac{\varepsilon}{2}\tau(h(\lambda)^{-\frac{1}{2}}b^*k(\lambda)^{-1}bh(\lambda)^{-\frac{1}{2}})\\ &\geq 0,\end{aligned}$$

showing that g_ε is subharmonic.

Fix $\lambda \in \mathbb{C}$, and let $x = a - \lambda b$ as above. Then

$$g_\varepsilon(\lambda) = \tfrac{1}{2}\int_0^{\|x\|} \log(t^2 + \varepsilon)\, \mathrm{d}\mu_{|x|}(t),$$

and

$$g(\lambda) = \tfrac{1}{2}\int_0^{\|x\|} \log(t^2)\, \mathrm{d}\mu_{|x|}(t).$$

Hence, g_ε is a monotonically decreasing function of $\varepsilon > 0$, and

$$g(\lambda) = \lim_{\varepsilon \to 0^+} g_\varepsilon(\lambda).$$

According to [HK], g is then either subharmonic or identically $-\infty$. ∎

2.9 Proposition. *Let $T \in \mathcal{M}^\Delta$. Then the function $f : \mathbb{C} \to [-\infty, \infty[$ given by*

$$f(\lambda) = \log \Delta(T - \lambda \mathbf{1})$$

is subharmonic in $\mathbb{C}$.

Proof. Define $T_1, T_2 \in \mathcal{M}$ by

$$T_1 = T(T^*T + \mathbf{1})^{-\frac{1}{2}} \tag{2.21}$$

and

$$T_2 = (T^*T + \mathbf{1})^{-\frac{1}{2}}. \tag{2.22}$$

Then for every $\lambda \in \mathbb{C}$,

$$T - \lambda\mathbf{1} = (T_1 - \lambda T_2)T_2^{-1},$$

where $\Delta(T_2) > 0$ (cf. (2.8)). Thus, $T - \lambda\mathbf{1} \in \mathcal{M}^\Delta$ with

$$\Delta(T - \lambda\mathbf{1}) = \Delta(T_1 - \lambda T_2)\Delta(T_2)^{-1},$$

i.e.

$$f(\lambda) = L(T - \lambda\mathbf{1}) = L(T_1 - \lambda T_2) - L(T_2). \tag{2.23}$$

Then by Lemma 2.8, f is either subharmonic or identically $-\infty$. With

$$h(\lambda) = L(T_2 - \lambda T_1) - L(T_2),$$

$h(0) = 0 > -\infty$, and it follows from Lemma 2.8 that h is subharmonic. In particular, $h(\lambda) > -\infty$ for almost every $\lambda \in \mathbb{C}$ w.r.t. Lebesgue measure. For $\lambda \in \mathbb{C} \setminus \{0\}$,

$$f(\lambda) = h\Big(\frac{1}{\lambda}\Big) + \log|\lambda|.$$

Hence, f is not identically $-\infty$. ∎

Recall from [HK, Section 3.5.4] that one can associate to every subharmonic function u the socalled *Riesz measure* μ_u, which is a positive Borel measure on $\mathbb{R}^2$ uniquely determined by

$$\forall \phi \in C_c^\infty(\mathbb{R}^2): \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} u \nabla^2 \phi \, \mathrm{d}m = \int_{\mathbb{R}^2} \phi \, \mathrm{d}\mu_u. \tag{2.24}$$

One uses the notation $\mathrm{d}\mu_u = \frac{1}{2\pi}\nabla^2 u \, \mathrm{d}\lambda$, and this is what is meant by (2.16).

In order to prove the rest of Theorem 2.7, we need some general lemmas on subharmonic functions:

2.10 Lemma. *Let $g : \mathbb{C} \to [-\infty, \infty[$ be a subharmonic function, and for $r > 0$ define*

$$m(g,r) = \frac{1}{2\pi}\int_0^{2\pi} g(r\mathrm{e}^{\mathrm{i}\theta}) \, \mathrm{d}\theta, \tag{2.25}$$

$$M(g,r) = \sup_{|z|=r} g(z). \tag{2.26}$$

Then

$$g(0) = \lim_{r\to 0} m(g,r) = \lim_{r \to 0} M(g,r). \tag{2.27}$$

Proof. Clearly, $m(g,r) \leq M(g,r)$ for every $r > 0$. Moreover, since g is subharmonic, $g(0) \leq m(g,r)$, $(r > 0)$. It follows that

$$g(0) \leq \begin{cases} \limsup_{r\to 0} m(g,r) \leq \limsup_{r \to 0} M(g,r) \\ \liminf_{r\to 0} m(g,r) \leq \liminf_{r\to 0} M(g,r) \end{cases} \tag{2.28}$$

Now, every upper semicontinuous function attains a maximum on every compact set. In particular, there exists for every $r > 0$ a complex number z_r of modulus r such that $g(z_r) = M(g,r)$. $z_r \to 0$ as $r \to 0$, and therefore

$$g(0) \geq \limsup_{r\to 0} g(z_r) = \limsup_{r\to 0} M(g,r). \tag{2.29}$$

It follows from (2.28) and (2.29) that

$$\begin{aligned} g(0) &\leq \liminf_{r\to 0} m(g,r) \\ &\leq \begin{Bmatrix} \limsup_{r\to 0} m(g,r) \\ \liminf_{r\to 0} M(g,r) \end{Bmatrix} \\ &\leq \limsup_{r\to 0} M(g,r) \\ &\leq g(0), \end{aligned}$$

so the four inequalities above are in fact identities, and this proves (2.27). ∎

2.11 Lemma. *f given by* (2.15) *satisfies*

$$\lim_{r\to\infty} (M(f,r) - \log r) = \lim_{r\to\infty} (m(f,r) - \log r) = 0. \tag{2.30}$$

Proof. Define $h : \mathbb{C} \to [-\infty, \infty[$ by

$$h(\lambda) = L(T_2 - \lambda T_1) - L(T_2), \qquad \lambda \in \mathbb{C}. \tag{2.31}$$

Then h is subharmonic with $h(0) = 0$, and it follows from Lemma 2.10 that

$$0 = \lim_{r \to 0} m(h, r) = \lim_{r \to 0} M(h, r). \tag{2.32}$$

Since

$$h(\lambda) = \log |\lambda| + f(\tfrac{1}{\lambda}), \qquad \lambda \neq 0, \tag{2.33}$$

we get that when $r > 0$,

$$\begin{aligned} M(f, r) &= M(h, \tfrac{1}{r}) + \log r, \\ m(f, r) &= m(h, \tfrac{1}{r}) + \log r, \end{aligned}$$

and combining this with (2.32) we obtain the desired result. ■

2.12 Lemma. *Let $R > r > 0$, and let g be subharmonic in $\mathbb{C}$. Then with $\mathrm{d}\mu = \frac{1}{2\pi} \nabla^2 g \, \mathrm{d}\lambda$ and*

$$\psi(z) = \begin{cases} \log\left(\frac{R}{r}\right) & , \quad |z| \le r \\ \log\left(\frac{R}{|z|}\right) & , \quad r < |z| < R \\ 0 & , \quad |z| \ge R \end{cases}$$

one has that

$$m(g, R) - m(g, r) = \int_{\mathbb{C}} \psi(z) \, \mathrm{d}\mu(z). \tag{2.34}$$

Proof. Cf. [HK, (3.5.7)]. ■

Proof of Theorem 2.7. When $R > 1 > 0$ define $\psi_R : \mathbb{C} \to \mathbb{R}$ by

$$\psi_R(z) = \begin{cases} \log R & , \quad |z| \le 1 \\ \log\left(\frac{R}{|z|}\right) & , \quad 1 < |z| < R \\ 0 & , \quad |z| \ge R \end{cases}$$

Then, according to Lemma 2.12,

$$\int_{\mathbb{C}} \psi_R(z) \, \mathrm{d}\mu_T(z) = m(f, R) - m(f, 1). \tag{2.35}$$

Now, $\frac{1}{\log R} \psi_R \nearrow 1$ as $R \to \infty$, so by the Monotone Convergence Theorem, (2.35) and Lemma 2.11,

$$\mu_T(\mathbb{C}) = \lim_{R \to \infty} \frac{m(f, R) - m(f, 1)}{\log R} = 1,$$

that is, μ_T is a probability measure.

When $R > 1$, let

$$\omega_R(z) = \log R - \psi_R(z), \qquad z \in \mathbb{C}. \tag{2.36}$$

Then $\omega_R(z) \nearrow \log^+ |z|$ as $\mathbb{R} \to \infty$, and hence by one more application of Lemma 2.11,

$$\begin{aligned}
\int_{\mathbb{C}} \log^+ |z| \, \mathrm{d}\mu_T(z) &= \lim_{R\to\infty} \int_{\mathbb{C}} \omega_R \, \mathrm{d}\mu_T \\
&= \lim_{R\to\infty} (\log R - m(f, R) + m(f, 1)) \\
&= m(f, 1),
\end{aligned}$$

proving (2.18). Note that since f is subharmonic, (2.18) imlies that $\int_{\mathbb{C}} \log^+ |z| \, \mathrm{d}\mu_T(z) < \infty$.

To see that (2.17) holds, it suffices to consider the case $\lambda = 0$. Indeed, for fixed $\lambda \in \mathbb{C}$ one easily sees that $\mu_{T-\lambda\mathbf{1}}$ is the push-forward measure of μ_T under the map $z \mapsto z - \lambda$ (cf. Lemma 2.14), and therefore

$$\int_{\mathbb{C}} \log |z - \lambda| \, \mathrm{d}\mu_T(z) = \int_{\mathbb{C}} \log |z| \, \mathrm{d}\mu_{T-\lambda\mathbf{1}}(z). \tag{2.37}$$

In the case $\lambda = 0$ one has to compute the integrals $\int_{\mathbb{C}} \log^\pm |z| \, \mathrm{d}\mu_T(z)$. We have just seen that

$$\int_{\mathbb{C}} \log^+ |z| \, \mathrm{d}\mu_T(z) = m(f, 1), \tag{2.38}$$

and with

$$\chi_r(z) = \begin{cases} \log \frac{1}{r} & , \quad |z| \le r \\ \log \frac{1}{|z|} & , \quad r < |z| \le 1 \\ 0 & , \quad |z| \ge 1 \end{cases}$$

$\chi_r(z) \nearrow \log^- |z|$ as $r \searrow 0$. Hence by Lemma 2.10 and Lemma 2.12,

$$\begin{aligned}
\int_{\mathbb{C}} \log^- |z| \, \mathrm{d}\mu_T(z) &= \lim_{r\to 0} \int_{\mathbb{C}} \chi_r \, \mathrm{d}\mu_T \\
&= \lim_{r\to 0} (m(f, 1) - m(f, r)) \\
&= m(f, 1) - f(0).
\end{aligned}$$

Combining this with (2.38) we get that

$$\int_{\mathbb{C}} \log |z| \, \mathrm{d}\mu_T(z) = f(0) = L(T),$$

as desired.

In order to prove that μ_T is uniquely determined by (i) and (ii) of Theorem 2.7, suppose $\nu \in \mathrm{Prob}(\mathbb{C})$ satisfies

$$\int_{\mathbb{C}} \log^+ |z| \, \mathrm{d}\nu(z) < \infty, \tag{2.39}$$

and

$$\forall\, \lambda \in \mathbb{C}: \qquad \int_{\mathbb{C}} \log|z-\lambda|\, \mathrm{d}\nu(z) = L(T-\lambda\mathbf{1}). \tag{2.40}$$

Note that (2.39) implies that $\int_{\mathbb{C}} \log|z-\lambda|\, \mathrm{d}\nu(z)$ is well-defined, since

$$\log|z-\lambda| \le \log(|z|+|\lambda|),$$

and

$$|z|+|\lambda| \le (|\lambda+1|)\cdot \max\{1,|z|\}.$$

Hence

$$\log|z-\lambda| \le \log(|\lambda|+1) + \log^+|z|. \tag{2.41}$$

Since μ and ν are both probability measures, it follows from a C^∞-version of Urysohn's Lemma (cf. [Fo, (8.18)]) that if

$$\int_{\mathbb{C}} \phi\, \mathrm{d}\mu_T = \int_{\mathbb{C}} \phi\, \mathrm{d}\nu$$

for every function $\phi \in C_c^\infty(\mathbb{R}^2)$, then $\mu_T = \nu$. Then consider an arbitrary function $\phi \in C_c^\infty(\mathbb{R}^2)$. Since the Laplacian of $w \mapsto \frac{1}{2\pi}\log|w-z|$ (in the distribution sense) is the Dirac measure δ_z at z, one has that

$$\begin{aligned}
\int_{\mathbb{C}} \phi(z)\, \mathrm{d}\nu(z) &= \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} \phi(\lambda)\, \delta_z(\lambda) \Big)\, \mathrm{d}\nu(z) \\
&= \frac{1}{2\pi} \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} (\nabla^2\phi)(\lambda) \log|z-\lambda|\, \mathrm{d}\lambda \Big)\, \mathrm{d}\nu(z).
\end{aligned} \tag{2.42}$$

At this place we would like to reverse the order of integration, but it is not entirely clear that this is a legal operation. Therefore we put $M = \|\nabla^2\phi\|_\infty$, and take $\chi \in C_c^\infty(\mathbb{R}^2)$ such that $0 \le \chi \le 1$ and $\chi|_{\mathrm{supp}(\nabla^2\phi)} = 1$. With

$$\psi_1 = \frac{1}{2}(M + \nabla^2\phi)\chi$$

and

$$\psi_2 = \frac{1}{2}(M - \nabla^2\phi)\chi$$

one has that $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^2)^+$, and $\nabla^2\phi = \psi_1 - \psi_2$.

Also not that, according to (2.41),

$$h(\lambda, z) := \log(|\lambda|+1) + \log^+|z| - \log|z-\lambda| \ge 0.$$

Therefore by Tonelli's Theorem

$$\int_{\mathbb{C}} \psi_i(\lambda) \int_{\mathbb{C}} h(\lambda,z)\, \mathrm{d}\nu(z)\, \mathrm{d}\lambda = \int_{\mathbb{C}}\int_{\mathbb{C}} \psi_i(\lambda) h(\lambda,z)\, \mathrm{d}\lambda\, \mathrm{d}\nu(z), \qquad i=1,2. \tag{2.43}$$

The map $\lambda \mapsto L(T - \lambda \mathbf{1})$ is subharmonic and therefore locally integrable. Since

$$\int_{\mathbb{C}} h(\lambda, z)\, \mathrm{d}\nu(z) = \log(|\lambda| + 1) + \int_{\mathbb{C}} \log^+ |z|\, \mathrm{d}\nu(z) - L(T - \lambda \mathbf{1}),$$

where $\lambda \mapsto L(T - \lambda \mathbf{1})$ is subharmonic and therefore locally integrable,

$$\int_{\mathbb{C}} \psi_i(\lambda) \int_{\mathbb{C}} h(\lambda, z)\, \mathrm{d}\nu(z)\, \mathrm{d}\lambda < \infty, \qquad i = 1, 2.$$

It now follows from (2.43) that

$$\int_{\mathbb{C}} (\nabla^2 \phi)(\lambda) \int_{\mathbb{C}} h(\lambda, z)\, \mathrm{d}\nu(z)\, \mathrm{d}\lambda = \int_{\mathbb{C}} \int_{\mathbb{C}} (\nabla^2 \phi)(\lambda) h(\lambda, z)\, \mathrm{d}\lambda\, \mathrm{d}\nu(z),$$

and since

$$\int_{\mathbb{C}} |(\nabla^2 \phi)(\lambda)| \int_{\mathbb{C}} \log(|\lambda| + 1)\, \mathrm{d}\nu(z)\, \mathrm{d}\lambda < \infty,$$

and

$$\int_{\mathbb{C}} |(\nabla^2 \phi)(\lambda)| \int_{\mathbb{C}} \log^+ |z|\, \mathrm{d}\nu(z)\, \mathrm{d}\lambda < \infty,$$

we deduce that

$$\begin{aligned}
\int_{\mathbb{C}} \phi(z)\, \mathrm{d}\nu(z) &= \frac{1}{2\pi} \int_{\mathbb{C}} \left(\int_{\mathbb{C}} (\nabla^2 \phi)(\lambda) \log |\lambda - z|\, \mathrm{d}\lambda \right) \mathrm{d}\nu(z) \\
&= \frac{1}{2\pi} \int_{\mathbb{C}} (\nabla^2 \phi)(\lambda) \int_{\mathbb{C}} \log |\lambda - z|\, \mathrm{d}\nu(z)\, \mathrm{d}\lambda \\
&= \frac{1}{2\pi} \int_{\mathbb{C}} (\nabla^2 \phi)(\lambda) L(T - \lambda \mathbf{1})\, \mathrm{d}\lambda \\
&= \int_{\mathbb{C}} \phi(z)\, \mathrm{d}\mu_T(z),
\end{aligned}$$

and this is the desired identity. ■

It follows from Theorem 2.7 that one can associate to every operator $T \in \mathcal{M}^\Delta$ a probability measure μ_T on $(\mathbb{C}, \mathbb{B}_2)$, such that in the case where $T \in \mathcal{M}$, μ_T agrees with the Brown measure of T. Therefore we make the following definition:

2.13 Definition. For $T \in \mathcal{M}^\Delta$ we shall say that the probability measure μ_T from Theorem 2.7 is the *Brown measure* of T.

In the remaining part of this section we will see that many of the properties of the Brown measure for bounded operators carry over to this more general setting.

2.14 Proposition. *Let $T \in \mathcal{M}^\Delta$. Then for every $r > 0$ and every $\lambda \in \mathbb{C}$, the Brown measure of $rT + \lambda \mathbf{1}$, $\mu_{rT + \lambda \mathbf{1}}$, is the push-forward measure of μ_T via the map $z \mapsto rz + \lambda$.*

Proof. Making use of Urysohn's Lemma for C^∞-functions on $\mathbb{R}^2$ (cf. [Fo, (8.18)]) and the fact that both of the measures considered here are probability measures, one easily sees that if

$$\int_{\mathbb{C}} \phi(z)\,\mathrm{d}\mu_{rT+\lambda\mathbf{1}}(z) = \int_{\mathbb{C}} \phi(rz+\lambda)\,\mathrm{d}\mu_T(z)$$

for every $\phi \in C_c^\infty(\mathbb{R}^2)$, then the two measures in speak agree on compact sets and hence on all of $\mathbb{B}_2$.

Let $\phi \in C_c^\infty(\mathbb{R}^2)$. Then by definition,

$$\begin{aligned}
\int_{\mathbb{C}} \phi(rz+\lambda)\,\mathrm{d}\mu_T(z) &= \frac{1}{2\pi}\int_{\mathbb{C}} \Big(\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}\Big)\phi(rz+\lambda)\,f(z)\,\mathrm{d}z \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} r^2\Big(\frac{\partial^2}{\partial w_1^2}+\frac{\partial^2}{\partial w_2^2}\Big)\phi(w)\,f\Big(\frac{1}{r}(w-\lambda)\Big)\frac{1}{r^2}\,\mathrm{d}w \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2\phi(w) f\Big(\frac{1}{r}(w-\lambda)\Big)\,\mathrm{d}w \\
&= \frac{1}{2\pi}\int_{\mathbb{C}} \nabla^2\phi(w)\,[L(rT+\lambda\mathbf{1}-w\mathbf{1})-\log r]\,\mathrm{d}w \\
&= \int_{\mathbb{C}} \phi(w)\,\mathrm{d}\mu_{rT+\lambda\mathbf{1}}(w) - \log r\cdot\int_{\mathbb{C}}\nabla^2\phi(w)\,\mathrm{d}w \\
&= \int_{\mathbb{C}} \phi(w)\,\mathrm{d}\mu_{rT+\lambda\mathbf{1}}(w),
\end{aligned}$$

where the last identity follows from Green's Theorem. ■

2.15 Proposition. *For every $T \in \mathcal{M}^\Delta$ and every $m \in \mathbb{N}$, μ_{T^m} is the push-forward measure of μ_T via the map $z \mapsto z^m$.*

Proof. Let $\nu \in \mathrm{Prob}(\mathbb{C})$ denote the push-forward measure of μ_T under the map $z \mapsto z^m$. According to Theorem 2.7 it suffices to prove that

$$\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\nu(z) < \infty,$$

and

$$\forall\,\lambda\in\mathbb{C}: \qquad \int_{\mathbb{C}} \log|\lambda - z|\,\mathrm{d}\nu(z) = L(T^m-\lambda\mathbf{1}).$$

Here

$$\begin{aligned}
\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\nu(z) &= \int_{\mathbb{C}} \log^+|z^m|\,\mathrm{d}\mu_T(z) \\
&= m\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\mu_T(z) \\
&< \infty,
\end{aligned}$$

and if we let $\theta_1, \ldots, \theta_m$ denote the m complex roots of $Q(z) = z^m - 1$, then for every $\lambda \in \mathbb{C}$,

$$|\lambda - z^m| = \prod_{k=1}^{m} |\theta_k \lambda^{\frac{1}{m}} - z|.$$

Hence

$$\begin{aligned}
\int_{\mathbb{C}} \log|\lambda - z| \,\mathrm{d}\nu(z) &= \int_{\mathbb{C}} \log|\lambda - z^m| \,\mathrm{d}\mu_T(z) \\
&= \int_{\mathbb{C}} \sum_{k=1}^{m} \log|\theta_k \lambda^{\frac{1}{m}} - z| \,\mathrm{d}\mu_T(z) \\
&= \sum_{k=1}^{m} L(T - \theta_k \lambda^{\frac{1}{m}} \mathbf{1}) \\
&= L\Big(\prod_{k=1}^{m} (T - \theta_k \lambda^{\frac{1}{m}} \mathbf{1})\Big) \\
&= L(T^m - \lambda \mathbf{1}),
\end{aligned}$$

as desired. ■

2.16 Proposition. *If $T \in \mathcal{M}^\Delta$ with*

$$\int_0^1 \log t \,\mathrm{d}\mu_{|T|}(t) > -\infty, \tag{2.44}$$

then $\mu_T(\{0\}) = \mu_{|T|}(\{0\}) = 0$, and T has an inverse $T^{-1} \in \mathcal{M}^\Delta$. Moreover, $\mu_{T^{-1}}$ is the push-forward measure of μ_T via the map $z \mapsto z^{-1}$.

Proof. According to Theorem 2.7,

$$\int_{\mathbb{C}} \log|z| \,\mathrm{d}\mu_T(z) = L(T) = \int_0^\infty \log t \,\mathrm{d}\mu_{|T|}(t). \tag{2.45}$$

Hence, if (2.44) holds, then

$$-\infty < \int_{\mathbb{C}} \log|z| \,\mathrm{d}\mu_T(z) < \infty, \tag{2.46}$$

and therefore $\mu_T(\{0\}) = \mu_{|T|}(\{0\}) = 0$. Moreover, $|T|$ has an inverse $|T|^{-1} \in \tilde{\mathcal{M}}$ with

$$\begin{aligned}
\int_0^\infty \log^+(t) \,\mathrm{d}\mu_{|T|^{-1}}(t) &= \int_0^\infty \log^+\Big(\frac{1}{t}\Big) \,\mathrm{d}\mu_{|T|}(t) \\
&= -\int_0^1 \log t \,\mathrm{d}\mu_{|T|}(t) \\
&< \infty,
\end{aligned}$$

so $|T|^{-1} \in \mathcal{M}^\Delta$. Take $U \in \mathcal{U}(\mathcal{M})$ such that $T = U|T|$. Then $T^{-1} = |T|^{-1} U^* \in \mathcal{M}^\Delta$.

Now, let ν denote the push-forward measure of μ_T under the map $z \mapsto z^{-1}$. According to Theorem 2.7, if

$$\int_{\mathbb{C}} \log^+ |z| \, \mathrm{d}\nu(z) < \infty, \tag{2.47}$$

and

$$\forall \lambda \in \mathbb{C} : \qquad \int_{\mathbb{C}} \log |z - \lambda| \, \mathrm{d}\nu(z) = L(T^{-1} - \lambda \mathbf{1}), \tag{2.48}$$

then $\nu = \mu_{T^{-1}}$. Applying (2.46) we find that

$$\begin{aligned} \int_{\mathbb{C}} \log^+ |z| \, \mathrm{d}\nu(z) &= \int_{\mathbb{C}} \log^+ \left| \frac{1}{z} \right| \mathrm{d}\mu_T(z) \\ &= -\int_{(|z| \leq 1)} \log |z| \, \mathrm{d}\mu_T(z) \\ &< \infty. \end{aligned}$$

In order to prove that (2.48) holds, let $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, then, using the multiplicativity of Δ on $\mathcal{M}^\Delta$, we find that

$$\begin{aligned} \int_{\mathbb{C}} \log |z - \lambda| \, \mathrm{d}\nu(z) &= \int_{\mathbb{C}} \log \left| \frac{1}{z} - \lambda \right| \mathrm{d}\mu_T(z) \\ &= \int_{\mathbb{C}} \log \left| \frac{1}{z} \Big(\frac{1}{\lambda} - z \Big) \lambda \right| \mathrm{d}\mu_T(z) \\ &= \int_{\mathbb{C}} \Big(\log |\lambda| + \log \left| \frac{1}{\lambda} - z \right| - \log |z| \Big) \, \mathrm{d}\mu_T(z) \\ &= L(\lambda \mathbf{1}) + L\Big(T - \frac{1}{\lambda} \mathbf{1} \Big) - L(T) \\ &= L\Big(\lambda \mathbf{1} \Big(T - \frac{1}{\lambda} \mathbf{1} \Big) T^{-1} \Big) \\ &= L(T^{-1} - \lambda \mathbf{1}). \end{aligned}$$

In the case $\lambda = 0$ we have:

$$\begin{aligned} \int_{\mathbb{C}} \log |z| \, \mathrm{d}\nu(z) &= -\int_{\mathbb{C}} \log |z| \, \mathrm{d}\mu_T(z) \\ &= -L(T) \\ &= L(T^{-1}). \end{aligned}$$

Hence (2.48) holds, and $\nu = \mu_{T^{-1}}$. ■

2.17 Proposition. *Let* $T \in \mathcal{M}^\Delta$. *Then* $\operatorname{supp}(\mu_T) \subseteq \sigma(T)$.

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma(T)$. Then $T - \lambda \mathbf{1}$ is invertible with bounded inverse. Moreover, according to Proposition 2.16, $\mu_{(T - \lambda \mathbf{1})^{-1}}$ is the push–forward measure of $\mu_{T - \lambda \mathbf{1}}$ via the map $z \mapsto z^{-1}$, $z \in \mathbb{C} \setminus \{0\}$. Since $(T - \lambda \mathbf{1})^{-1}$ is bounded, we have from [Br] that

$$\operatorname{supp}(\mu_{(T - \lambda \mathbf{1})^{-1}}) \subseteq \sigma((T - \lambda \mathbf{1})^{-1}) \subseteq \overline{B(0, r)},$$

where $r = \|(T-\lambda\mathbf{1})^{-1}\|$. Hence,

$$\operatorname{supp}(\mu_{T-\lambda\mathbf{1}}) \subseteq \{z \in \mathbb{C} \,|\, |z| \geq \tfrac{1}{r}\}.$$

In particular, $0 \notin \operatorname{supp}(\mu_{T-\lambda\mathbf{1}})$, which by Proposition 2.14 is equivalent to $\lambda \notin \operatorname{supp}(\mu_T)$. Hence, $\operatorname{supp}(\mu_T) \subseteq \sigma(T)$. ■

2.18 Lemma. *For every $p \in (0,\infty)$ and every $t \in [0,\infty[$,*

$$t^p = p^2 \int_0^\infty \log^+(at) a^{-p-1}\, \mathrm{d}a. \tag{2.49}$$

Proof. For $t = 0$ this is trivial. For $t > 0$ we find that

$$\begin{aligned}
\int_0^\infty \log^+(at)a^{-p-1}\,\mathrm{d}a &= \int_{\frac{1}{t}}^\infty \log(at)a^{-p-1}\,\mathrm{d}a \\
&= \left[-\frac{1}{p}\log(at)\, a^{-p}\right]_{\frac{1}{t}}^\infty - \int_{\frac{1}{t}}^\infty -\frac{1}{p\,a}\, a^{-p}\,\mathrm{d}a \\
&= 0 - \left[-\frac{1}{p^2}a^{-p}\right]_{\frac{1}{t}}^\infty \\
&= \frac{1}{p^2}t^p. \qquad ■
\end{aligned}$$

We will now prove Weil's inequality for operators T in $L^p(\mathcal{M})$ (cf. [Br, corollary 3.8] for the case $T \in \mathcal{M}$):

2.19 Theorem. *Let $p \in (0,\infty)$ and let $T \in L^p(\mathcal{M})$. Then*

$$\int_{\mathbb{C}} |z|^p \,\mathrm{d}\mu_T(z) \leq \|T\|_p^p. \tag{2.50}$$

In the proof of this theorem we shall need the following lemma, the proof of which we postpone for a while:

2.20 Lemma. *Let $T \in \mathcal{M}^\Delta$. Then*

$$\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\mu_T(z) \leq \tau(\log^+|T|). \tag{2.51}$$

Proof of Proposition 2.19. Let $a \geq 0$. Then, according to Lemma 2.14 and Lemma 2.20,

$$\begin{aligned}
\int_{\mathbb{C}} \log^+(a|z|)\,\mathrm{d}\mu_T(z) &= \int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\mu_{aT}(z) \\
&\leq \int_0^\infty \log^+ t\,\mathrm{d}\mu_{|aT|}(t) \\
&= \int_0^\infty \log^+(at)\mathrm{d}\mu_{|T|}(t).
\end{aligned}$$

Hence by Lemma 2.18 and Tonelli's Theorem,

$$\begin{aligned}\int_{\mathbb{C}} |z|^p \mathrm{d}\mu_T(z) &= p^2 \int_0^\infty \left(\int_{\mathbb{C}} \log^+(a|z|)\, \mathrm{d}\mu_T(z) \right) a^{-p-1}\, \mathrm{d}a \\ &\leq p^2 \int_0^\infty \left(\int_0^\infty \log^+(at)\, \mathrm{d}\mu_{|T|}(t) \right) a^{-p-1}\, \mathrm{d}a \\ &= \int_0^\infty t^p\, \mathrm{d}\mu_{|T|}(t) \\ &= \tau(|T|^p). \qquad \blacksquare\end{aligned}$$

In order to prove Lemma 2.20 we shall need some additional results:

2.21 Lemma. *Suppose $A, B, C \in \mathcal{M}^\Delta$ with A and B invertible in $\mathcal{M}^\Delta$ and*

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \geq 0.$$

Then

$$\Delta(C) \leq \Delta(A)^{\frac{1}{2}} \Delta(B)^{\frac{1}{2}}. \tag{2.52}$$

Proof. Note that $A, B \geq 0$ and that

$$\begin{pmatrix} \mathbf{1} & A^{-\frac{1}{2}} C^* B^{-\frac{1}{2}} \\ B^{-\frac{1}{2}} C A^{-\frac{1}{2}} & \mathbf{1} \end{pmatrix} = \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & B^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} A^{-\frac{1}{2}} & 0 \\ 0 & B^{-\frac{1}{2}} \end{pmatrix} \geq 0,$$

which is equivalent to saying that $\|B^{-\frac{1}{2}} C A^{-\frac{1}{2}}\| \leq 1$, and this clearly implies that

$$\Delta(B^{-\frac{1}{2}} C A^{-\frac{1}{2}}) \leq 1. \qquad \blacksquare$$

2.22 Lemma. *For every $S \in \mathcal{M}^\Delta$,*

$$\Delta(\mathbf{1} + S) \leq \Delta(\mathbf{1} + |S|). \tag{2.53}$$

Proof. Take a unitary $U \in \mathcal{M}$ such that $S = U|S|$. Then

$$\begin{pmatrix} |S| & |S| \\ |S| & |S| \end{pmatrix} \geq 0,$$

and

$$\begin{pmatrix} \mathbf{1} & U^* \\ U & \mathbf{1} \end{pmatrix} \geq 0,$$

whence

$$\begin{pmatrix} |S| + \mathbf{1} & |S| + U^* \\ |S| + U & |S| + \mathbf{1} \end{pmatrix} \geq 0.$$

Now Lemma 2.21 implies that

$$\begin{aligned}\Delta(S+\mathbf{1}) &= \Delta(U^*(S+\mathbf{1}))\\ &= \Delta(U^*(U|S|+\mathbf{1}))\\ &= \Delta(|S|+U^*)\\ &\le \Delta(|S|+\mathbf{1})^{\frac{1}{2}}\Delta(|S|+\mathbf{1})^{\frac{1}{2}}\\ &= \Delta(|S|+\mathbf{1}),\end{aligned}$$

as desired. ■

2.23 Lemma. *Every* $S \in \mathcal{M}^\Delta$ *satisfies*

$$\Delta(\mathbf{1}+|S^2|) \le \Delta(\mathbf{1}+|S|^2), \tag{2.54}$$

implying that for arbitrary $n \in \mathbb{N}$,

$$\Delta(\mathbf{1}+|S^{2^n}|) \le \Delta(\mathbf{1}+|S|^{2^n}). \tag{2.55}$$

Proof. Take a unitary $U \in \mathcal{M}$ such that $S^2 = U|S^2|$. Since

$$\begin{pmatrix} SS^* & S^2 \\ (S^*)^2 & S^*S \end{pmatrix} = \begin{pmatrix} S \\ S^* \end{pmatrix} \begin{pmatrix} S^* & S \end{pmatrix} \ge 0,$$

we find as in the foregoing proof that

$$\begin{pmatrix} \mathbf{1}+SS^* & U^*+S^2 \\ U+(S^*)^2 & \mathbf{1}+S^*S \end{pmatrix} \ge 0.$$

Again this implies that

$$\Delta(\mathbf{1}+|S^2|) = \Delta(S^2+U^*) \le \Delta(\mathbf{1}+S^*S)^{\frac{1}{2}}\Delta(\mathbf{1}+SS^*)^{\frac{1}{2}} = \Delta(\mathbf{1}+S^*S),$$

where the last identity follows from the fact that S^*S and SS^* have the same distribution w.r.t. τ. ■

Proof of Lemma 2.20. According to (2.38) we have:

$$\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\mu_T(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta, \tag{2.56}$$

where

$$f(\lambda) = \tau(\log|T-\lambda\mathbf{1}|) = \log\Delta(T-\lambda\mathbf{1}), \qquad \lambda \in \mathbb{C}. \tag{2.57}$$

For every positive integer n define f_n by

$$f_n(z) = \sum_{k=0}^{2^n-1} f\big(\mathrm{e}^{\frac{2\pi k}{2^n}\mathrm{i}}z\big), \qquad z \in \mathbb{C}. \tag{2.58}$$

Then clearly,

$$\frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \frac{1}{2\pi 2^n}\int_0^{2\pi} f_n(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta. \tag{2.59}$$

Applying Lemma 2.22 and Lemma 2.23 we obtain an estimate of $f_n(\mathrm{e}^{\mathrm{i}\theta})$:

$$\begin{aligned}
f_n(\mathrm{e}^{\mathrm{i}\theta}) &= \sum_{k=0}^{2^n-1} \log\Delta\big(\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{-\frac{2\pi k}{2^n}\mathrm{i}}T - \mathbf{1}\big) \\
&= \log\Delta\Big(\prod_{k=0}^{2^n-1}\big(\mathrm{e}^{-\mathrm{i}\theta}\mathrm{e}^{-\frac{2\pi k}{2^n}\mathrm{i}}T - \mathbf{1}\big)\Big) \\
&= \log\Delta\big(\mathbf{1} - \mathrm{e}^{-\mathrm{i}2^n\theta}T^{2^n}\big) \\
&\leq \log\Delta\big(\mathbf{1} + |T^{2^n}|\big) \\
&\leq \log\Delta\big(\mathbf{1} + |T|^{2^n}\big) \\
&= \tau\big(\log\big(\mathbf{1} + |T|^{2^n}\big)\big).
\end{aligned}$$

Combining (2.56) and (2.59) with the above estimate we see that

$$\begin{aligned}
\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\mu_T(z) &\leq \tfrac{1}{2^n}\tau\big(\log\big(\mathbf{1} + |T|^{2^n}\big)\big) \\
&= \frac{1}{2^n}\int_{[0,\infty[} \log(1+t^{2^n})\,\mathrm{d}\mu_{|T|}(t) \\
&\leq \frac{1}{2^n}\int_{[0,1[} \log 2\,\mathrm{d}\mu_{|T|}(t) + \frac{1}{2^n}\int_{[1,\infty[} (\log 2 + 2^n\log t)\,\mathrm{d}\mu_{|T|}(t) \\
&\leq \frac{2\log 2}{2^n} + \int_{[0,\infty[} \log^+ t\,\mathrm{d}\mu_{|T|}(t).
\end{aligned}$$

Finally, let $n\to\infty$, and conclude that

$$\int_{\mathbb{C}} \log^+|z|\,\mathrm{d}\mu_T(z) \leq \int_{[0,\infty[} \log^+ t\,\mathrm{d}\mu_{|T|}(t). \qquad \blacksquare$$

2.24 Proposition. *Let $T\in\mathcal{M}^\Delta$, and suppose $P\in\mathcal{M}$ is a non-trivial T-invariant projection, i.e. $PTP = TP$. Then*

$$\Delta(T) = \Delta_{P\mathcal{M}P}(PTP)^{\tau(P)}\Delta_{P^\perp\mathcal{M}P^\perp}(P^\perp TP^\perp)^{1-\tau(P)}, \tag{2.60}$$

where $\Delta_{P\mathcal{M}P}$ and $\Delta_{P^\perp\mathcal{M}P^\perp}$ refer to the Fuglede-Kadison determinant computed relative to the normalized traces $\frac{1}{\tau(P)}\tau|_{P\mathcal{M}P}$ and $\frac{1}{\tau(P^\perp)}\tau|_{P^\perp\mathcal{M}P^\perp}$ on $P\mathcal{M}P$ and $P^\perp\mathcal{M}P^\perp$, respectively.

Proof. Put $T_{11} = PTP$, $T_{12} = PTP^\perp$ and $T_{22} = P^\perp TP^\perp$. Then, w.r.t. to the decomposition $\mathcal{H} = P(\mathcal{H})\oplus P(\mathcal{H})^\perp$, we may write

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22}\end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & T_{22}\end{pmatrix}\begin{pmatrix} \mathbf{1} & T_{12} \\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix} T_{11} & 0 \\ 0 & \mathbf{1}\end{pmatrix},$$

where

$$\Delta\left(\begin{pmatrix}\mathbf{1} & 0\\ 0 & T_{22}\end{pmatrix}\right) = \Delta_{P^\perp \mathcal{M} P^\perp}(P^\perp T P^\perp)^{1-\tau(P)},$$

and

$$\Delta\left(\begin{pmatrix}T_{11} & 0\\ 0 & \mathbf{1}\end{pmatrix}\right) = \Delta_{P\mathcal{M}P}(PTP)^{\tau(P)}.$$

Thus, (2.60) holds if

$$\Delta\left(\begin{pmatrix}\mathbf{1} & T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\right) = 1. \tag{2.61}$$

To that (2.60) holds, note that

$$\begin{pmatrix}\mathbf{1} & T_{12}\\ 0 & \mathbf{1}\end{pmatrix}^{-1} = \begin{pmatrix}\mathbf{1} & -T_{12}\\ 0 & \mathbf{1}\end{pmatrix},$$

and hence

$$\Delta\left(\begin{pmatrix}\mathbf{1} & T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\right)\Delta\left(\begin{pmatrix}\mathbf{1} & -T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\right) = 1. \tag{2.62}$$

Also,

$$\begin{pmatrix}\mathbf{1} & -T_{12}\\ 0 & \mathbf{1}\end{pmatrix} = \begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix}\begin{pmatrix}\mathbf{1} & T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\begin{pmatrix}\mathbf{1} & 0\\ 0 & -\mathbf{1}\end{pmatrix},$$

so that

$$\Delta\left(\begin{pmatrix}\mathbf{1} & -T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\right) = \Delta\left(\begin{pmatrix}\mathbf{1} & T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\right),$$

and then by (2.62),

$$\Delta\left(\begin{pmatrix}\mathbf{1} & T_{12}\\ 0 & \mathbf{1}\end{pmatrix}\right) = 1,$$

as desired. ■

2.25 Lemma. *Let $p \in (0, \infty)$, and let $\varepsilon > 0$. Then the map $L_\varepsilon : L^p(\mathcal{M}, \tau) \to \mathbb{R}$ given by*

$$L_\varepsilon(T) = \tfrac{1}{2}\,\tau(\log(T^*T + \varepsilon\mathbf{1})), \qquad T \in L^p(\mathcal{M}, \tau), \tag{2.63}$$

is continuous w.r.t. $\|\cdot\|_p$.

Proof. Suppose T, $T_n \in L^p(\mathcal{M}, \tau)$ with

$$\lim_{n\to\infty} \|T - T_n\|_p = 0.$$

Then $\lim_{n\to\infty} \|T^*T - T_n^*T_n\|_{\frac{p}{2}} = 0$, implying that $T_n^*T_n \to T^*T$ in the measure topology. Consequently,

$$\mu_{T^*T} = w^* - \lim_{n\to\infty} \mu_{T_n^*T_n}. \tag{2.64}$$

Define a sequence $(\nu_n)_{n=1}^{\infty}$ of (finite) measures on $(\mathbb{R}, \mathbb{B})$ by

$$\mathrm{d}\nu_n(t) = (1 + t^{\frac{p}{2}})\mathrm{d}\mu_{T_n^* T_n}(t), \tag{2.65}$$

and note that since $\lim_{n\to\infty} \|T_n\|_p = \|T\|_p$,

$$\sup_{n\in\mathbb{N}} \nu_n(\mathbb{R}) < \infty. \tag{2.66}$$

Similarly define a finite measure ν on $(\mathbb{R}, \mathbb{B})$ by

$$\mathrm{d}\nu(t) = (1 + t^{\frac{p}{2}})\mathrm{d}\mu_{T^*T}(t). \tag{2.67}$$

Because of (2.64) we have that for every $\phi \in C_c(\mathbb{R})$,

$$\int_0^\infty \phi(t)\mathrm{d}\nu(t) = \lim_{n\to\infty} \int_0^\infty \phi(t)\mathrm{d}\nu_n(t). \tag{2.68}$$

When $\phi \in C_0(\mathbb{R})$, ϕ may be approximated (uniformly) by functions from $C_c(\mathbb{R})$. Thus, taking (2.66) and (2.68) into account, one easily sees that

$$\int_0^\infty \phi(t)\mathrm{d}\nu(t) = \lim_{n\to\infty} \int_0^\infty \phi(t)\mathrm{d}\nu_n(t). \tag{2.69}$$

In particular, with

$$\phi(t) = \frac{\log(t+\varepsilon)}{1 + t^{\frac{p}{2}}}, \qquad (t \geq 0), \tag{2.70}$$

(2.69) implies that

$$L_\varepsilon(T) = \int_0^\infty \phi(t)\mathrm{d}\nu(t) = \lim_{n\to\infty} \int_0^\infty \phi(t)\mathrm{d}\nu_n(t) = \lim_{n\to\infty} L_\varepsilon(T_n). \qquad \blacksquare$$

2.26 Corollary. *For* $p \in (0, \infty)$ *the map* $L : L^p(\mathcal{M}, \tau) \to [-\infty, \infty[$ *given by*

$$L(T) = \tau(\log|T|), \qquad T \in L^p(\mathcal{M}, \tau), \tag{2.71}$$

is upper semicontinuous w.r.t. $\|\cdot\|_p$.

Proof. Indeed, this follows from Lemma 2.25, since for every $T \in L^p(\mathcal{M}, \tau)$ we have that

$$L(T) = \inf_{\varepsilon>0} L_\varepsilon(T). \qquad \blacksquare$$

3 Unbounded R–diagonal operators

Consider a von Neumann algebra $\mathcal{M}$ equipped with a faithful, normal, tracial state τ.

3.1 Definition. For $T \in \tilde{\mathcal{M}}$ with polar decomposition $T = U|T|$, we denote by $W^*(T)$ the von Neumann algebra generated by U and all the spectral projections of $|T|$.

Note that T is affiliated with $W^*(T)$ and that $W^*(T)$ is the smallest von Neumann subalgebra of $\mathcal{M}$ with this property.

If $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite von Neumann algebras with faithful, normal, tracial states τ_1 and τ_2, respectively, then any $*$–isomorphism $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ with $\tau_1 = \tau_2 \circ \phi$ is continuous w.r.t. the measure topologies on the two von Neumann algebras and thus has a unique extension to a (surjective) $*$–isomorphism $\tilde{\phi} : \tilde{\mathcal{M}}_1 \to \tilde{\mathcal{M}}_2$.

3.2 Definition. Let $S, T \in \tilde{\mathcal{M}}$.

(a) We say that S and T have the same $*$–distribution, in symbols $S \underset{*\mathcal{D}}{\sim} T$, if there exists a trace–preserving $*$–isomporphism ϕ from $W^*(S)$ onto $W^*(T)$ with $\tilde{\phi}(S) = T$.

(b) We say that S and T are $*$–free if $W^*(S)$ and $W^*(T)$ are $*$-free.

Note that in case S and T are bounded, the two definitions (a) and (b) given above agree with the ones given in [VDN] .

Recall from [NS, p. 155 ff.] that if $U, H \in \mathcal{M}$ are $*$-free elements with U Haar unitary, then UH is R-diagonal in the sense of Nica and Speicher (cf. [NS]). Conversely, if $T \in \mathcal{M}$ is R–diagonal, then T has the same $*$–distribution as a product UH, where U and H are $*$–free elements in some tracial C^*–probability space, U is a Haar unitary, and $H \geq 0$. We therefore define R–diagonality for operators in $\tilde{\mathcal{M}}$ as follows:

3.3 Definition. $T \in \tilde{\mathcal{M}}$ is said to be *R-diagonal* if there exist a von Neumann algebra $\mathcal{N}$, with a faithful, normal, tracial state, and $*$–free elements U and H in $\tilde{\mathcal{N}}$, such that U is Haar unitary, $H \geq 0$, and such that T has the same $*$–distribution as UH.

3.4 Remark. Note that if $T \in \tilde{\mathcal{M}}$ is R–diagonal with $\ker(T) = 0$, then the partial isometry V in the polar decomposition of T, $T = V|T|$, is a unitary ($\mathcal{M}$ is finite). It follows from Definition 3.3 and Definition 3.2 that V is in fact a Haar unitary which is $*$–free from $|T|$.

In this section we will see that certain algebraic operations on (sets of $*$-free) R–diagonal operators preserve R–diagonality, exactly as in the bounded case (cf. [HL]). Our proofs are to a large extent inspired by the techniques used in [HL] and in [La1]. In particular, we will repeatedly make use of [HL, Lemma 3.7] which we state here for the convenience of the reader:

3.5 Lemma. *[HL] Let $U \in \mathcal{M}$ be a Haar unitary, and suppose $\mathcal{S} \subset \mathcal{M}$ is a set which is $*$-free from U. Then for any $n \in \mathbb{N}$,*

(i) the sets $\mathcal{S}$, $U\mathcal{S}U^*$,.... are $*$-free,

(ii) the sets $\mathcal{S}$, $U\mathcal{S}U^*$,...., $U^{n-1}\mathcal{S}(U^*)^{n-1}$, $\{U^n\}$ are $*$-free,

(iii) the sets $U\mathcal{S}U^*$,....,$U^n\mathcal{S}(U^*)^n$, $\{U^n\}$ are $*$-free.

3.6 Proposition. *If $T \in \tilde{\mathcal{M}}$ is R-diagonal with $\ker(T) = 0$, then T has an inverse $T^{-1} \in \tilde{\mathcal{M}}$, and T^{-1} is R-diagonal as well.*

Proof. Let $T = V|T|$ be the polar decomposition of T with $V \in \mathcal{M}$ Haar unitary and $*$–free from $|T|$. Since $\ker(T) = 0$, T has an inverse $T^{-1} \in \tilde{\mathcal{M}}$:

$$T^{-1} = V^*V|T|^{-1}V^* = V^*(V|T|V^*)^{-1},$$

where V^* is Haar unitary and, according to Lemma 3.5, it is $*$–free from $V|T|V^*$ and thus from $(V|T|V^*)^{-1}$. This shows that T^{-1} is R–diagonal. ■

3.7 Lemma. *Let $S, T \in \tilde{\mathcal{M}}$, and let $V \in \mathcal{M}$ be a Haar unitary. If S, T and V are $*$-free, then VS and TVS are R–diagonal.*

Proof. The case where S and T are bounded was treated by F. Larsen (cf. [La1, Lemma 3.6]). Our proof resembles the one given by F. Larsen.

Enlarging the algebra if necessary, we may assume that there are Haar unitaries $V_1, V_2 \in \mathcal{M}$, such that V_1, V_2 and S are $*$-free and $V = V_1V_2$.

Since $W^*(S) \subseteq \mathcal{M}$ is finite, there is a unitary $U_1 \in W^*(S)$ such that $S = U_1|S|$. Then $VS = V_1(V_2U_1)|S|$, where

(i) V_1 is $*$-free from $|S|$ and V_2U_1,

(ii) $\tau(V_1) = \tau(V_1^*) = 0$ and $\tau(V_2U_1) = \tau((V_2U_1)^*) = 0$,

(iii) for all $A \in W^*(|S|)$ with $\tau(A) = 0$, $\tau(V_2U_1A) = \tau(V_2)\tau(U_1A) = 0$, $\tau(AU_1^*V_2^*) = \tau(AU_1^*)\tau(V_2^*) = 0$ and $\tau(V_2U_1A(V_2U_1)^{-1}) = \tau(A) = 0$.

It follows now from [V1, Lemma 2.4] that $V_1(V_2U_1)$ is $*$-free from $|S|$. Thus, if $V_1(V_2U_1)$ is Haar unitary, then S is R–diagonal. Since V_1 is $*$-free from V_2U_1, we get from [HL, Lemma 3.7] that for every $n \in \mathbb{N}$, the operators

$$V_1^n, V_1^{n-1}(V_2U_1)V_1^{1-n}, \ldots, V_1(V_2U_1)V_1^{-1}, V_2U_1$$

are $*$-free. Consequently,

$$\begin{aligned}
\tau((V_1V_2U_1)^n) &= \tau(V_1^n[V_1^{1-n}(V_2U_1)V_1^{n-1}][V_1^{2-n}(V_2U_1)V_1^{n-2}]\cdots[V_1^{-1}(V_2U_1)V_1]V_2U_1) \\
&= \tau(V_1^n)\tau([V_1^{1-n}(V_2U_1)V_1^{n-1}][V_1^{2-n}(V_2U_1)V_1^{n-2}]\cdots[V_1^{-1}(V_2U_1)V_1]V_2U_1) \\
&= 0.
\end{aligned}$$

Then $\tau((V_1V_2U_1)^{-n}) = \overline{\tau((V_1V_2U_1)^n)} = 0$, and $V_1V_2U_1$ is Haar unitary. Therefore $VS = V_1V_2U_1|S|$ is R–diagonal.

Now, $TVS = V(V^*TVS)$. Put

$$\mathcal{B}_1 = W^*(V), \ \ \mathcal{B}_2 = W^*(T), \ \ \text{and} \ \ \mathcal{B}_3 = W^*(S).$$

Then $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are $*$-free. We may write T as $T = U_2|T|$ for a unitary $U_2 \in \mathcal{B}_2$. Then

$$V^*TV = (V^*U_2V)V^*|T|V, \tag{3.1}$$

where $V^*|T|V$ is affiliated with $V^*\mathcal{B}_2V$.

$\mathcal{B}_3$ and $V^*\mathcal{B}_2V$ are $*$-free, and according to [HL, Lemma 3.7], $\mathcal{B}_1$ and $V^*\mathcal{B}_2V$ are $*$-free. But then V is $*$-free from $\mathcal{B}_4 = \mathcal{B}_3 \vee V^*\mathcal{B}_2V$.

Since S and V^*TV are both affiliated with $\mathcal{B}_4$, their product, V^*TVS, is affiliated with $\mathcal{B}_4$, so V is $*$-free from V^*TVS. It follows now from the first part of the proof that $TVS = V(V^*TVS)$ is R-diagonal. ■

3.8 Proposition. *If $S, T \in \tilde{\mathcal{M}}$ are $*$-free R-diagonal elements, then ST is R-diagonal as well. Moreover,*

$$\mu_{(ST)^*ST} = \mu_{S^*S} \boxtimes \mu_{T^*T}. \tag{3.2}$$

Proof. Taking a free product of tracial von Neumann algebras if necessary, we can find a von Neumann algebra $\mathcal{N}$ with faithful, normal, tracial state ω and $*$–free elements $U_1, H_1, U_2, H_2 \in \tilde{\mathcal{N}}$ such that U_1, U_2 are Haar unitaries, $H_1, H_2 \geq 0$, and $S \underset{*\mathcal{D}}{\sim} U_1H_1$ and $T \underset{*\mathcal{D}}{\sim} U_2H_2$.

Choose trace–preserving $*$–isomorphisms

$$\begin{aligned} \phi_1 : W^*(S) &\rightarrow W^*(U_1H_1), \\ \phi_2 : W^*(T) &\rightarrow W^*(U_2H_2), \end{aligned}$$

with $\tilde{\phi_1}(S) = U_1H_1$ and $\tilde{\phi_2}(T) = U_2H_2$. ϕ_1 and ϕ_2 give rise to a trace–preserving $*$–isomorphism

$$\phi = \phi_1 * \phi_2 : W^*(S) * W^*(T) \rightarrow W^*(U_1H_1) * W^*(U_2H_2)$$

(the free products are taken within the category of tracial von Neumann algebras) with

$$\tilde{\phi}(ST) = \tilde{\phi_1}(S)\tilde{\phi_2}(T) = U_1H_1U_2H_2.$$

Thus, $\psi := \phi|_{W^*(ST)}$ is a trace–preserving $*$–isomorphism onto $W^*(U_1H_1U_2H_2)$ with $\tilde{\psi}(ST) = U_1H_1U_2H_2$. According to Lemma 3.7, $U_1(H_1U_2H_2)$ is R–diagonal, and hence ST is R–diagonal.

In order to prove (3.2), note that if $S = 0$, then $\mu_{S^*S} = \delta_0$, so that by the definition of multiplicative free convolution given on p. 744 in [BV],

$$\mu_{S^*S} \boxtimes \mu_{T^*T} = \delta_0 \boxtimes \mu_{T^*T} = \delta_0.$$

This shows that $\mu_{S^*S} \boxtimes \mu_{T^*T} = \mu_{(ST)^*ST}$ if $S = 0$. The same holds if $T = 0$.

Now assume that $S, T \neq 0$. Note that

$$\begin{aligned} S^*S &\underset{*\mathcal{D}}{\sim} H_1^2, \\ T^*T &\underset{*\mathcal{D}}{\sim} H_2^2, \\ (ST)^*ST &\underset{*\mathcal{D}}{\sim} H_2U_2^*H_1^2U_2H_2. \end{aligned}$$

Thus, (3.2) holds if

$$\mu_{H_2U_2^*H_1^2U_2H_2} = \mu_{H_1^2} \boxtimes \mu_{H_2^2}.$$

For every $n \in \mathbb{N}$, the bounded operators

$$S_n = U_1\, H_1\, 1_{[0,n]}(H_1)$$

and

$$T_n = U_2\, H_2\, 1_{[0,n]}(H_2)$$

are $*$–free. According to [HL, Lemma 3.9] they are both R-diagonal in the sense of Nica and Speicher (cf. [NS]). Then, by [HL, Proposition 3.6],

$$\mu_{(S_nT_n)^*S_nT_n} \;=\; \mu_{S_n^*S_n} \boxtimes \mu_{T_n^*T_n}. \tag{3.3}$$

Since $S_n \to U_1H_1$ and $T_n \to U_2H_2$ in the measure topology, $(S_nT_n)^*S_nT_n \to H_2U_2^*H_1^2U_2H_2$ in measure as well. These facts imply that $\mu_{S_n^*S_n} \xrightarrow{w^*} \mu_{H_1^2}$, $\mu_{T_n^*T_n} \xrightarrow{w^*} \mu_{H_2^2}$ and $\mu_{(S_nT_n)^*S_nT_n} \xrightarrow{w^*} \mu_{H_2U_2^*H_1^2U_2H_2}$. Moreover, $\mu_{H_1^2} \neq \delta_0$ and $\mu_{H_2^2} \neq \delta_0$, because S^*S and T^*T are non–zero. Hence, by [BV, Corollary 6.7] and by (3.3),

$$\mu_{H_2U_2^*H_1^2U_2H_2} = w^* - \lim_{n\to\infty} \mu_{S_n^*S_n} \boxtimes \mu_{T_n^*T_n} = \mu_{H_1^2} \boxtimes \mu_{H_2^2}. \qquad \blacksquare$$

3.9 Proposition. *Let $S \in \tilde{\mathcal{M}}$ be R-diagonal, and let $n \in \mathbb{N}$. Then S^n is R-diagonal. Moreover,*

$$\mu_{(S^n)^*S^n} = \mu_{S^*S}^{\boxtimes n}. \tag{3.4}$$

Proof. Choose a von Neumann algebra $\mathcal{N}$ with faithful, normal, tracial state ω and with $*$–free elements $U, H \in \tilde{\mathcal{N}}$ such that U is Haar unitary, $H \geq 0$, and $S \underset{*\mathcal{D}}{\sim} UH$. Then $S^n \underset{*\mathcal{D}}{\sim} (UH)^n$. Since

$$(UH)^n = U^n[U^{1-n}HU^{n-1}][U^{2-n}HU^{n-2}]\cdots[U^{-1}HU]H,$$

where

$$U^n,\ U^{1-n}HU^{n-1},\ U^{2-n}HU^{n-2}, \ldots,\ U^{-1}HU,\ H$$

are $*$–free (cf. Lemma 3.5 (ii)), and U^n is Haar unitary, Lemma 3.7 gives us that $(UH)^n$ is R–diagonal, and hence S^n is.

In order to prove (3.4), note that if $\mu_{S^*S} = \delta_0$, then $S = S^n = 0$ and (3.4) trivially holds. Now assume that $\mu_{S^*S} \neq \delta_0$. For $k \in \mathbb{N}$ define $S_k \in \mathcal{M}$ and $T_k \in \mathcal{N}$ by

$$S_k = S\, 1_{[0,k]}(|S|) \qquad \text{and} \qquad T_k = U\, H\, 1_{[0,k]}(H).$$

Then $T_k \underset{*\mathcal{D}}{\sim} S_k$. Moreover, by Lemma 3.7, T_k is R-diagonal in the sense of Nica and Speicher, so S_k is R–diagonal. It now follows from [HL, Proposition 3.10] that

$$\mu_{[(S_k)^n]^*(S_k)^n} = \mu_{[(T_k)^n]^*(T_k)^n} = \mu_{T_k^* T_k}^{\boxtimes n} = \mu_{S_k^* S_k}^{\boxtimes n}. \tag{3.5}$$

As k tends to infinity, $S_k^* S_k \to S^*S$ and $[(S_k)^n]^*(S_k)^n \to (S^n)^*S^n$ in the measure topology. Since $\mu_{S^*S} \neq \delta_0$, we infer from [BV, Corollary 6.7] and from (3.5) that

$$\mu_{(S^n)^*S^n} = w^* - \lim_{k\to\infty} \mu_{[(S_k)^n]^*(S_k)^n} = w^* - \lim_{k\to\infty} \mu_{S_k^* S_k}^{\boxtimes n} = \mu_{S^*S}^{\boxtimes n}. \qquad \blacksquare$$

3.10 Definition. For $\mu \in \mathrm{Prob}(\mathbb{R}, \mathbb{B})$ let $\tilde{\mu}$ denote the *symmetrization* of μ. That is, $\tilde{\mu} \in \mathrm{Prob}(\mathbb{R}, \mathbb{B})$ is given by

$$\tilde{\mu}(B) = \tfrac{1}{2}(\mu(B) + \mu(-B)), \qquad (B \in \mathbb{B}).$$

3.11 Proposition. *Let $S, T \in \tilde{\mathcal{M}}$ be $*$–free R–diagonal elements. Then*

$$\tilde{\mu}_{|S+T|} = \tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|}. \tag{3.6}$$

Proof. As in the proof of Proposition 3.8, choose $(\mathcal{N}, \omega)$ and $*$–free elements $U_1, H_1, U_2, H_2 \in \tilde{\mathcal{N}}$ such that U_1, U_2 are Haar unitaries, $H_1, H_2 \geq 0$, and $S \underset{*\mathcal{D}}{\sim} U_1 H_1$ and $T \underset{*\mathcal{D}}{\sim} U_2 H_2$.

Again, for $n \in \mathbb{N}$, let

$$S_n = U_1\, H_1\, 1_{[0,n]}(H_1)$$

and

$$T_n = U_2\, H_2\, 1_{[0,n]}(H_2).$$

Then S_n and T_n are $*$–free and R–diagonal and therefore, according to [HL, Proposition 3.5],

$$\tilde{\mu}_{|S_n+T_n|} = \tilde{\mu}_{|S_n|} \boxplus \tilde{\mu}_{|T_n|}. \tag{3.7}$$

$|S_n| \to H_1$ and $|T_n| \to H_2$ in measure, implying that $\mu_{|S_n|} \overset{w^*}{\to} \mu_{H_1} = \mu_{|S|}$ and $\mu_{|T_n|} \overset{w^*}{\to} \mu_{H_2} = \mu_{|T|}$. Then we also have weak convergence of the symmetrized measures:

$$\tilde{\mu}_{|S_n|} \overset{w^*}{\to} \tilde{\mu}_{|S|} \quad \text{and} \quad \tilde{\mu}_{|T_n|} \overset{w^*}{\to} \tilde{\mu}_{|T|}.$$

Let d denote the Lévy metric on $\mathrm{Prob}(\mathbb{R}, \mathbb{B})$ (cf. [BV, p. 743]). Then d induces the topology of weak convergence, and according to [BV, Proposition 4.13] and the above observations,

$$d(\tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|}, \tilde{\mu}_{|S_n|} \boxplus \tilde{\mu}_{|T_n|}) \leq d(\tilde{\mu}_{|S|}, \tilde{\mu}_{|S_n|}) + d(\tilde{\mu}_{|T|}, \tilde{\mu}_{|T_n|}) \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$\begin{aligned}\tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|} &= w^* - \lim_{n\to\infty} \tilde{\mu}_{|S_n|} \boxplus \tilde{\mu}_{|T_n|} \\ &= w^* - \lim_{n\to\infty} \tilde{\mu}_{|S_n+T_n|}. \end{aligned} \tag{3.8}$$

Since S and T (U_1H_1 and U_2H_2, resp.) are $*$–free with $S \underset{*\mathcal{D}}{\sim} U_1H_1$ and $T \underset{*\mathcal{D}}{\sim} U_2H_2$, it follows that $S+T \underset{*\mathcal{D}}{\sim} U_1H_1 + U_2H_2$. Moreover, $|S_n+T_n| \to |U_1H_1+U_2H_2| \underset{*\mathcal{D}}{\sim} |S+T|$ in measure, and thus $\tilde{\mu}_{|S_n+T_n|} \overset{w^*}{\to} \tilde{\mu}_{|S+T|}$. Finally, this implies that

$$\tilde{\mu}_{|S|} \boxplus \tilde{\mu}_{|T|} = \tilde{\mu}_{|S+T|}. \qquad \blacksquare$$

We close this section by proving two simple results on the S–transform of probability measures on $(0,\infty)$ (cf. [BV]).

For $\mu \in \text{Prob}((0,\infty),\mathbb{B})$ define $\psi_\mu : \mathbb{C} \setminus (0,\infty) \to \mathbb{C}$ by

$$\psi_\mu(z) = \int_0^\infty \frac{1}{1-zt}\,\mathrm{d}\mu(t) - 1, \qquad z \in \mathbb{C} \setminus (0,\infty). \tag{3.9}$$

Then ψ_μ is analytic and satisfies

(i) $\psi'_\mu(t) > 0$, $t \in (-\infty, 0)$,

(ii) $\psi_\mu(z) \to -1$ as $z \to -\infty$,

(iii) $\psi_\mu(z) \to 0$ as $z \to 0$.

Hence, ψ_μ maps a (connected) neighbourhood $\mathcal{U}_\mu$ of $(-\infty,0)$ injectively onto a neighbourhood $\mathcal{V}_\mu$ of $(-1,0)$. Define χ_μ, $\mathcal{S}_\mu : \mathcal{V}_\mu \to \mathbb{C}$ by

$$\chi_\mu(z) = \psi_\mu^{-1}(z), \qquad z \in \mathcal{V}_\mu, \tag{3.10}$$

$$\mathcal{S}_\mu(z) = \frac{z+1}{z}\chi_\mu(z), \qquad z \in \mathcal{V}_\mu. \tag{3.11}$$

3.12 Proposition. *The map* $\mu \mapsto \mathcal{S}_\mu$ *is one-to-one on* $\text{Prob}((0,\infty),\mathbb{B})$.

Proof. Suppose $\mu,\nu \in \text{Prob}((0,\infty),\mathbb{B})$ with $\mathcal{S}_\mu = \mathcal{S}_\nu$. That is, in a neighbourhood $\mathcal{V} = \mathcal{V}_\mu \cap \mathcal{V}_\nu$ of $(-1,0)$, χ_μ agrees with χ_ν. It follows that on $(-\infty,0)$, ψ_μ agrees with ψ_ν, and then, by uniqueness of analytic continuation,

$$\psi_\mu(\tfrac{1}{\lambda}) = \psi_\nu(\tfrac{1}{\lambda}), \qquad \lambda \in \mathbb{C} \setminus [0,\infty[. \tag{3.12}$$

That is, the Stieltjes-transforms G_μ and G_ν agree on $\mathbb{C} \setminus [0,\infty[$. Recall that

$$\mathrm{d}\mu(x) = -\tfrac{1}{\pi}\lim_{y\to 0^+} G_\mu(x+\mathrm{i}y)\mathrm{d}x \tag{3.13}$$

(weak convergence of measures), and similarly,

$$d\nu(x) = -\tfrac{1}{\pi}\lim_{y\to 0^+} G_\nu(x+iy)dx. \tag{3.14}$$

Thus $\mu = \nu$. ■

3.13 Proposition. *Let $\mathcal{M}$ be a II_1-factor with tracial state τ, and let $a \in \tilde{\mathcal{M}}_+$ with $\ker(a) = \{0\}$. Then for all z in a neighbourhood of $(-1,0)$,*

$$\mathcal{S}_{\mu_{a^{-1}}}(z) = \frac{1}{\mathcal{S}_{\mu_a}(-1-z)}. \tag{3.15}$$

Proof. Let $z \in \mathbb{C}\setminus [0,\infty[$. Then

$$\begin{aligned} \psi_{a^{-1}}(z) &= \int_0^\infty \frac{1}{1-zt}\, d\mu_{a^{-1}}(t) - 1 \\ &= \int_0^\infty \frac{1}{1-\frac{z}{t}}\, d\mu_a(t) - 1 \\ &= \int_0^\infty \frac{z}{t-z} d\mu_a(t), \end{aligned}$$

and hence

$$\psi_{a^{-1}}(\tfrac{1}{z}) = -\textstyle\int_0^\infty \frac{1}{1-zt}\, d\mu_a(t) = -(\psi_a(z)+1). \tag{3.16}$$

It follows that for all $z \in \mathbb{C}\setminus[0,\infty[$,

$$z = \chi_a(\psi_a(z)) = \chi_a(-1-\psi_{a^{-1}}(\tfrac{1}{z})), \tag{3.17}$$

implying that $w = \psi_{a^{-1}}(\frac{1}{z})$ satisfies

$$\chi_{a^{-1}}(w) = \frac{1}{z} = \frac{1}{\chi_a(-1-w)}, \tag{3.18}$$

and thus

$$\mathcal{S}_{\mu_{a^{-1}}}(w)\cdot \mathcal{S}_{\mu_a}(-1-w) = 1. \tag{3.19}$$

(3.19) holds for all $w \in \psi_{a^{-1}}(\mathbb{C}\setminus[0,\infty[)$ and in particular for all w in a neighbourhood of $(-1,0)$. ■

4 The Brown measure of an unbounded R–diagonal operator

The Brown measure of a general bounded R–diagonal operator was computed in [HL, Theorem 4.4]. We will genralize this result to unbounded R–diagonal elements in $\mathcal{M}^\Delta$. Our proof will take a different route than the one in [HL]. This new approach will enable us to obtain an estimate of the p–norm of the resolvent $(T-\lambda\mathbf{1})^{-1}$, $0<p<1$, for special R–diagonal elements T (cf. Section 5).

4.1 Lemma. *Let $T \in \tilde{\mathcal{M}}$ be an R–diagonal element, and let $U \in \mathcal{M}$ be a Haar unitary which is $*$–free from T. Then for every $\lambda \in \mathbb{C}$,*

$$|T - \lambda \mathbf{1}| \underset{*\mathcal{D}}{\sim} |T + |\lambda| U|. \tag{4.1}$$

Proof. By passing to a larger algebra, we may assume that $T = V|T|$ where $V \in \mathcal{M}$ is a Haar unitary and U, V and $|T|$ are $*$–free. The case $\lambda = 0$ is trivial. For $\lambda \neq 0$, let $\alpha = -\frac{\lambda}{|\lambda|}$. Then $\alpha U^* V$ is a Haar unitary which is $*$–free from T. Hence,

$$\alpha U^* V |T| \underset{*\mathcal{D}}{\sim} T.$$

Therefore,

$$\begin{aligned} |T - \lambda \mathbf{1}| &\underset{*\mathcal{D}}{\sim} |\alpha U^* V |T| - \lambda \mathbf{1}| \\ &= |T - \overline{\alpha} \lambda U| \\ &= |T + |\lambda| U|. \end{aligned}$$

4.2 Lemma. *Let $T \in \tilde{\mathcal{M}}$ be an R–diagonal operator, and define*

$$h(s) = s\, \tau\big((T^*T + s^2 \mathbf{1})^{-1}\big), \qquad s > 0.$$

Moreover, for $\lambda \in \mathbb{C} \setminus \{0\}$, set

$$h_\lambda(s) = s\, \tau\big([(T - \lambda \mathbf{1})^*(T - \lambda \mathbf{1}) + s^2 \mathbf{1}]^{-1}\big).$$

Then there exists an $s_\lambda > 0$ such that for $s > s_\lambda$,

$$h(s) = h_\lambda\left(s + \frac{\sqrt{1 - 4|\lambda|^2 h(s)^2} - 1}{2h(s)} \right).$$

Proof. By passing to a larger algebra, we may assume that there exists a Haar unitary $U \in \mathcal{M}$ which is $*$–free from T. Then, according to Lemma 4.1,

$$|T - \lambda \mathbf{1}| \underset{*\mathcal{D}}{\sim} |T + |\lambda| U|.$$

It follows now from Proposition 3.11 that

$$\tilde{\mu}_{|T - \lambda \mathbf{1}|} = \tilde{\mu}_{|T|} \boxplus \tilde{\mu}_{|\lambda| \mathbf{1}} = \tilde{\mu}_{|T|} \boxplus \nu,$$

where $\nu = \frac{1}{2}(\delta_{-|\lambda|} + \delta_{|\lambda|})$.

For $\beta > 0$ define

$$\Omega_\beta = \{ w \in \mathbb{C} \mid 0 < |w| < \beta,\ \tfrac{5\pi}{4} < \arg(w) < \tfrac{7\pi}{4} \}.$$

According to [BV, Corollary 5.8], there is a $\beta > 0$ such that for every $w \in \Omega_\beta$,

$$\mathcal{R}_{\tilde{\mu}_{|T-\lambda\mathbf{1}|}}(w) = \mathcal{R}_{\tilde{\mu}_{|T|}}(w) + \mathcal{R}_\nu(w),$$

where

$$\mathcal{R}_\nu(w) = \frac{\sqrt{1+4|\lambda|^2 w^2} - 1}{2w},$$

and

$$G_{\tilde{\mu}_{|T|}}(\mathrm{i}s) = -\mathrm{i}h(s), \qquad s > 0,$$

whence

$$\mathcal{R}_{\tilde{\mu}_{|T|}}(-\mathrm{i}h(s)) + \frac{1}{-\mathrm{i}h(s)} = G^{<-1>}_{\tilde{\mu}_{|T|}}(-\mathrm{i}h(s)) = \mathrm{i}s, \qquad s > 0.$$

Take $s_\lambda > 0$ such that for every $s > s_\lambda$, $-\mathrm{i}h(s) \in \Omega_\beta$. Then, when $s > s_\lambda$,

$$\mathcal{R}_{\tilde{\mu}_{|T-\lambda\mathbf{1}|}}(-\mathrm{i}h(s)) = \mathrm{i}s + \frac{1}{\mathrm{i}h(s)} + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{-2\mathrm{i}h(s)},$$

implying that

$$h(s) = h_\lambda\left(s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{2h(s)}\right).$$

That is, when $s > s_\lambda$ and

$$t = s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{2h(s)},$$

then $h(s) = h_\lambda(t)$. ■

Note that if

$$t = s + \frac{\sqrt{1-4|\lambda|^2 h(s)^2} - 1}{2h(s)},$$

then (s, t) satisfies the following equation:

$$(s-t)\Big(\frac{1}{h(s)} - s + t\Big) = |\lambda|^2. \tag{4.2}$$

In the following we will investigate this equation further.

4.3 Definition. Let $m, n \in \mathbb{N}$, and let U be an open set in $\mathbb{R}^m$. A map $f : U \to \mathbb{R}^n$ is said to be *analytic* if it has a power series expansion in m variables in a neighborhood of every $x \in U$.

We shall need the following two well–known lemmas about analytic functions of several variables:

4.4 Lemma. *Let U be a connected, open subset of $\mathbb{R}^m$. If $f, g : U \to \mathbb{R}^n$ are two analytic functions which coincide on a non–empty, open subset V of U, then $f = g$.*

4.5 Lemma. *Let $U \subseteq \mathbb{R}^m$ be open and let $f : U \to \mathbb{R}^m$ be an analytic function for which the Jacobian $\mathfrak{J}(x_0) = \det f'(x_0)$ is non–zero for some $x_0 \in U$. Then f is one–to–one in some neighborhood V of x_0, and the inverse of $f|_V$ is analytic in a neighborhood of $f(x_0)$.*

4.6 Lemma. *Let μ be a probability measure on $[0, \infty)$, and define*

$$h(s) = \int_0^\infty \frac{s}{s^2+u^2}\,\mathrm{d}\mu(u), \qquad s > 0. \tag{4.3}$$

Then h is analytic on $(0, \infty)$. Moreover, if μ is not a Dirac measure, then for all $s > 0$,

$$0 < h(s) < \frac{1}{s} \qquad \text{and} \qquad h'(s) < \frac{h(s)}{s} - 2h(s)^2.$$

Proof. Since

$$h(s) = \frac{1}{2}\int_0^\infty \left(\frac{1}{s+\mathrm{i}u} + \frac{1}{s-\mathrm{i}u} \right) \mathrm{d}\mu, \qquad s > 0,$$

h has a complex analytic extension

$$\tilde{h} : \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\} \to \mathbb{C}$$

given by the same formula. In particular, h is an analytic function of $s \in (0, \infty)$. If μ is not a Dirac measure, then $\mu \neq \delta_0$, and so $h(s) > 0$ for all $s > 0$. Moreover,

$$s\,h(s) = \int_0^\infty \frac{s^2}{s^2+u^2}\,\mathrm{d}\mu(u) < 1, \qquad s > 0.$$

Finally, for $s > 0$,

$$\begin{aligned}
h(s)^2 &= \int_0^\infty \int_0^\infty \frac{s}{s^2+u^2}\frac{s}{s^2+v^2}\,\mathrm{d}\mu(u)\,\mathrm{d}\mu(v) \\
&\le \int_0^\infty \int_0^\infty \frac{1}{2}\left(\left(\frac{s}{s^2+u^2}\right)^2 + \left(\frac{s}{s^2+v^2}\right)^2 \right) \mathrm{d}\mu(u)\,\mathrm{d}\mu(v) \\
&= \int_0^\infty \frac{s^2}{(s^2+u^2)^2}\,\mathrm{d}\mu(u) \\
&= \frac{1}{2}\left(\int_0^\infty \frac{s^2+u^2}{(s^2+u^2)^2}\,\mathrm{d}\mu(u) + \int_0^\infty \frac{s^2-u^2}{(s^2+u^2)^2}\,\mathrm{d}\mu(u) \right) \\
&= \frac{1}{2}\left(\frac{h(s)}{s} - h'(s) \right).
\end{aligned}$$

Hence,

$$h'(s) \le \frac{h(s)}{s} - 2h(s)^2,$$

and equality holds if and only if the product measure $\mu \otimes \mu$ is concentrated on the diagonal $\{(u,u) \,|\, u > 0\}$. But this would imply that μ is a Dirac measure. Thus, if μ is not a Dirac measure, then

$$h'(s) < \frac{h(s)}{s} - 2h(s)^2, \qquad s > 0 \qquad \blacksquare$$

4.7 Lemma. *Let μ be a probability measure on $[0,\infty)$ which is not a Dirac measure, and put*

$$\lambda_1(\mu) = \left(\int_0^\infty \frac{1}{u^2} \, \mathrm{d}\mu(u) \right)^{-\frac{1}{2}} \quad \text{and} \quad \lambda_2(\mu) = \left(\int_0^\infty u^2 \, \mathrm{d}\mu(u) \right)^{\frac{1}{2}},$$

with the convention that $\infty^{-\frac{1}{2}} = 0$. Then $0 \le \lambda_1(\mu) < \lambda_2(\mu) \le \infty$.

Proof. Clearly, $\lambda_1(\mu) < \infty$, and since $\mu \ne \delta_0$, $\lambda_2(\mu) > 0$. The lemma is then trivially true if $\lambda_1(\mu) = 0$ or $\lambda_2(\mu) = +\infty$. Thus, we can assume that $\lambda_1(\mu), \lambda_2(\mu) \in (0,\infty)$. Then, by the Schwartz inequality,

$$\begin{aligned} \frac{\lambda_2(\mu)}{\lambda_1(\mu)} &= \left(\int_0^\infty u^2 \, \mathrm{d}\mu(u) \right)^{\frac{1}{2}} \left(\int_0^\infty \frac{1}{u^2} \, \mathrm{d}\mu(u) \right)^{\frac{1}{2}} \\ &\ge \int_0^\infty u \, \frac{1}{u} \, \mathrm{d}\mu(u) \\ &= 1, \end{aligned}$$

and equality holds if and only if for some $c \in (0,\infty)$, $\frac{1}{u} = cu$ holds for μ–a.e. $u \in [0,\infty)$. However, this can not be the case when μ is not a Dirac measure. $\blacksquare$

4.8 Lemma. *Let $\mu, \lambda_1(u)$ and $\lambda_2(\mu)$ be as in Lemma 4.7, and let h be as in Lemma 4.6. Then put*

$$k(s,t) = (s-t)\left(\frac{1}{h(s)} - s + t \right), \qquad s > 0, \, t \in \mathbb{R}.$$

Then k is an analytic function on $(0,\infty) \times \mathbb{R}$. Moreover, for $t > 0$ the map $s \mapsto k(s,t)$ is a strictly increasing bijection of (t,∞) onto $(0,\infty)$, and for $t = 0$ the map $s \mapsto k(s,t)$ is a strictly increasing bijection of $(0,\infty)$ onto $(\lambda_1(\mu)^2, \lambda_2(\mu)^2)$.

Proof. Clearly, k is analytic. Moreover,

$$\frac{\partial k}{\partial s}(s,t) = \frac{1}{h(s)} - (s-t)\left(2 + \frac{h'(s)}{h(s)^2} \right). \tag{4.4}$$

For $s \in (0,\infty)$, we get from Lemma 4.6 that

$$\frac{\partial k}{\partial s}(s,0) = \frac{s}{h(s)^2} \left(\frac{h(s)}{s} - 2h(s)^2 - h'(s) \right) > 0,$$

and

$$\frac{\partial k}{\partial s}(s,s) = \frac{1}{h(s)} > s.$$

Since the right–hand side of (4.4) is an affine function of $t \in \mathbb{R}$, it follows that

$$\frac{\partial k}{\partial s}(s,t) > t, \quad s > 0, \quad t \in [0,s]. \tag{4.5}$$

Hence, $s \mapsto k(s,t)$ is a strictly increasing function of $s \in (t,\infty)$ for every $t \in [0,\infty)$. For $s > t > 0$,

$$k(s,t) = \int_t^s \frac{\partial k}{\partial s'}(s',t)\,\mathrm{d}s' > \int_t^s t\,\mathrm{d}s' = t(s-t). \tag{4.6}$$

Hence, when $t > 0$,

$$\lim_{s\to\infty} k(s,t) = \infty,$$

and

$$\lim_{s\to t+} k(s,t) = k(t,t) = 0.$$

Thus, $s \mapsto k(s,t)$ is a bijection of (t,∞) onto $(0,\infty)$.

Next, consider the case $t = 0$. We have already seen that $s \mapsto k(s,0)$ is strictly increasing on $(0,\infty)$. Note that for $s > 0$,

$$k(s,0) = \frac{1 - sh(s)}{h(s)/s} = \frac{n(s)}{d(s)}$$

where

$$n(s) = \int_0^\infty \frac{u^2}{s^2+u^2}\,\mathrm{d}\mu(u) \quad \text{and} \quad d(s) = \int_0^\infty \frac{1}{s^2+u^2}\,\mathrm{d}\mu(u).$$

By the monotone convergence theorem,

$$\lim_{s\to 0+} n(s) = 1,$$

$$\lim_{s\to 0+} d(s) = \int_0^\infty \frac{1}{u^2}\,\mathrm{d}\mu(u) = \frac{1}{\lambda_1(\mu)^2},$$

$$\lim_{s\to\infty} s^2 n(s) = \int_0^\infty u^2\,\mathrm{d}\mu(u) = \lambda_2(\mu)^2,$$

and

$$\lim_{s\to\infty} s^2 d(s) = 1.$$

Hence,

$$\lim_{s\to 0+} k(s,0) = \lambda_1(\mu)^2,$$

and

$$\lim_{s\to\infty} k(s,0) = \lambda_2(\mu)^2.$$

This shows that $s \mapsto k(s,0)$ is a bijection of $(0,\infty)$ onto $(\lambda_1(\mu)^2, \lambda_2(\mu)^2)$. ■

4.9 Definition. Let $\mu, \lambda_1(u)$ and $\lambda_2(\mu)$ be as in Lemma 4.7, let h be as in Lemma 4.6, and let k be as in Lemma 4.8. For $\lambda, t \in (0,\infty)$, let $s(\lambda,t)$ denote the unique solution $s \in (t,\infty)$ to the equation $k(s,t) = \lambda^2$ (cf. Lemma 4.8), and for $\lambda \in (\lambda_1(\mu), \lambda_2(\mu))$, let $s(\lambda,0)$ denote the unique solution $s \in (0,\infty)$ to the equation $k(s,0) = \lambda^2$.

4.10 Lemma. *The function* $(\lambda,t) \mapsto s(\lambda,t)$ *is analytic in* $(0,\infty)\times(0,\infty)$. *Moreover, for* $\lambda \in (\lambda_1(\mu), \lambda_2(\mu))$,

$$\lim_{t\to 0+} s(\lambda,t) = s(\lambda,0). \tag{4.7}$$

Proof. Let

$$\Omega = \{(s,t) \in \mathbb{R}^2 \,|\, 0 < t < s\}.$$

According to Lemma 4.8, k is a strictly positive, analytic function in Ω. Let

$$F(s,t) = (\sqrt{k(s,t)}, t), \quad (s,t) \in \Omega.$$

Then F is analytic in Ω, and by Lemma 4.8, F is a one–to–one map of Ω onto $(0,\infty)\times(0,\infty)$. Moreover, its inverse $F^{-1} : (0,\infty)\times(0,\infty) \to \Omega$ is given by

$$F^{-1}(\lambda,t) = (s(\lambda,t),t), \quad s,t > 0.$$

The Jacobian of F is

$$\mathcal{J}(F)(s,t) = \frac{\partial}{\partial s}\sqrt{k(s,t)} = \frac{1}{2\sqrt{k(s,t)}}\frac{\partial k}{\partial s}(s,t),$$

which by (4.5) is strictly positive for all $(s,t) \in \Omega$. Hence, by Lemma 4.5, F^{-1} is analytic in $(0,\infty)\times(0,\infty)$. In particular, $s(\lambda,t)$ is analytic in $(0,\infty)\times(0,\infty)$.

Now, let $\lambda_0 \in (\lambda_1(\mu), \lambda_2(\mu))$ and put $s_0 = s(\lambda_0, 0)$. Then $k(s_0,0) = \lambda_0^2$, and by the proof of Lemma 4.8, $\frac{\partial k}{\partial s}(s_0,0) > 0$. Let

$$F_0(s,t) := (\sqrt{k(s,t)}, t).$$

F_0 is then analytic in some neighborhood U_0 of $(s_0,0)$. Moreover, $\mathcal{J}(F_0)(s_0,0) \neq 0$, and therefore, by Lemma 4.5, F_0 has an analytic inverse F_0^{-1} in a neighborhood V_0 of $F_0(s_0,0) = (\lambda_0,0)$. Clearly, $F_0^{-1}(\lambda,t) = F^{-1}(\lambda,t)$, whenever $(\lambda,t) \in V_0 \cap [(0,\infty)\times(0,\infty)]$, and then and $F_0^{-1}(\lambda,t) \in \Omega$.

Note that

$$\lim_{t\to 0+} F_0^{-1}(\lambda_0,t) = F_0^{-1}(\lambda_0,0) = (s_0,0), \tag{4.8}$$

and since the second coordinate of $F_0^{-1}(\lambda_0,t)$ is t, we conclude that $F_0^{-1}(\lambda_0,t) \in \Omega$, eventually as $t \to 0+$. Hence,

$$\begin{aligned}
(s_0,0) &= \lim_{t\to 0+} F_0^{-1}(\lambda_0,t) \\
&= \lim_{t\to 0+} F^{-1}(\lambda_0,t) \\
&= \lim_{t\to 0+} (s(\lambda_0,t),t),
\end{aligned}$$

and therefore,

$$\lim_{t\to 0+} s(\lambda_0, t) = s_0 = s(\lambda_0, 0). \qquad \blacksquare$$

4.11 Remark. We get from Lemma 4.10 that

$$\lim_{t\to 0+} s(\lambda, t) = 0, \qquad 0 < \lambda \le \lambda_1(\mu), \tag{4.9}$$

and

$$\lim_{t\to 0+} s(\lambda, t) = +\infty, \qquad \lambda \ge \lambda_2(\mu). \tag{4.10}$$

Indeed, for fixed $t > 0$, $\lambda \mapsto s(\lambda, t)$ is a monotonically increasing function of λ. Hence, if $0 < \lambda \le \lambda_1(\mu)$, then

$$\limsup_{t\to 0+} s(\lambda, t) \le \limsup_{t\to 0+} s(\lambda', t) = s(\lambda', 0),$$

for all $\lambda' \in (\lambda_1(\mu), \lambda_2(\mu))$.

But $\lambda' \mapsto s(\lambda', 0)$ is the inverse function of $s \mapsto \sqrt{k(s,0)}$, and hence $\lambda' \mapsto s(\lambda', 0)$ is a bijection of $(\lambda_1(\mu), \lambda_2(\mu))$ onto $(0, \infty)$. It follows that $\limsup_{t\to 0+} s(\lambda, t) = 0$, and this proves (4.9).

For $\lambda \ge \lambda_2(\mu)$, a similar argument shows that $\liminf_{t\to 0+} s(\lambda, t) = +\infty$, and this proves (4.10).

4.12 Lemma. *Let $\lambda > 0$. Then*

(i) $\lim_{t\to\infty}(s(\lambda, t) - t) = 0$, *and*

(ii) there exists a $t_\lambda > 0$ such that when $t > t_\lambda$ and $s = s(\lambda, t)$, then

$$t = s + \frac{\sqrt{1 + 4\lambda^2 h(s)^2} - 1}{2h(s)}.$$

Proof. Fix $t > 0$, and put $s = s(\lambda, t)$. Then by Definition 4.9, $s > t$ and $k(s, t) = \lambda^2$. According to (4.6), $k(s, t) > t(s - t)$. Hence,

$$0 < s - t < \frac{\lambda^2}{t}.$$

This proves (i). With s and t as above,

$$\lambda^2 = k(s, t) = (s - t)\left(\frac{1}{h(s)} - s + t\right).$$

Solving this equation for t, we get that t is one of the two numbers

$$t_\pm = s - \frac{1}{h(s)} \pm \frac{\sqrt{1 + 4\lambda^2 h(s)^2}}{2h(s)}.$$

If $t = t_-$, then

$$s - t > \frac{1}{2h(s)},$$

and since $\frac{1}{h(s)} \to \infty$ as $s \to \infty$, this can not hold for large t because of (i). Hence, $t = t_+$ for t sufficiently large. ■

Combining the previous lemmas we get:

4.13 Proposition. *Let $T \in \tilde{\mathcal{M}}$ be an R–diagonal element, let $\lambda \in \mathbb{C} \setminus \{0\}$, and define $h(s)$ and $h_\lambda(s)$ as in Lemma 4.2. Let $\mu = \mu_{|T|}$, and let $s(|\lambda|, t)$ be as in Definition 4.9. Then*

$$h_\lambda(s(|\lambda|, t)) = h(t), \qquad t > 0.$$

Proof. According to Lemma 4.12, if $t > t_{|\lambda|}$ and $s = s(|\lambda|, t)$, then

$$t = s + \frac{\sqrt{1 + 4\lambda^2 h(s)^2} - 1}{2h(s)}.$$

Since $s(|\lambda|, t) > t$, we infer from Lemma 4.2 that for t sufficiently large,

$$h_\lambda(t) = h(s(|\lambda|, t)).$$

Hence, by Lemma 4.4 and Lemma 4.10, the same formula holds for all $t > 0$. ■

4.14 Lemma. *Let T be an unbounded R–diagonal element in $\mathcal{M}^\Delta$, let $\lambda \in \mathbb{C} \setminus \{0\}$, and let $t > 0$. With $\mu = \mu_{|T|}$ and $s(|\lambda|, t)$ as in Definition 4.9 we then have:*

$$\Delta\big((T - \lambda\mathbf{1})^*(T - \lambda\mathbf{1}) + t^2\mathbf{1}\big) = \frac{|\lambda|^2}{|\lambda|^2 + (s(|\lambda|, t) - t)^2}\Delta\big(T^*T + s(|\lambda|, t)^2\mathbf{1}\big). \tag{4.11}$$

Proof. Since T is R–diagonal, $T \underset{*\mathcal{D}}{\sim} cT$ for all $c \in \mathbb{T}$. Hence, the left–hand side of (4.11) depends only on $|\lambda|$. It therefore suffices to consider only the case $\lambda > 0$. For $\lambda, t > 0$, let

$$H(t) = \tfrac{1}{2}\,\log\Delta(T^*T + t^2\mathbf{1})$$

and

$$H_\lambda(t) = \tfrac{1}{2}\,\log\Delta\big((T - \lambda\mathbf{1})^*(T - \lambda\mathbf{1}) + t^2\mathbf{1}\big).$$

Then with $\mu_\lambda = \mu_{|T - \lambda\mathbf{1}|}$,

$$H(t) = \frac{1}{2}\int_0^\infty \log(u^2 + t^2)\,\mathrm{d}\mu(u),$$

and

$$H_\lambda(t) = \frac{1}{2}\int_0^\infty \log(u^2 + t^2)\,\mathrm{d}\mu_\lambda(u).$$

Since T and $T-\lambda\mathbf{1}$ belong to $\mathcal{M}^\Delta$, H and H_λ take values in $\mathbb{R}$. Moreover, H and H_λ are differentiable with derivatives $H'(t) = h(t)$ and $H'_\lambda(t) = h_\lambda(t)$. Also, since $T \in \mathcal{M}^\Delta$,

$$\lim_{t\to\infty} (H(t) - \log t) = \frac{1}{2} \lim_{t\to\infty} \int_0^\infty \log\left(1 + \frac{u^2}{t^2}\right) \mathrm{d}\mu(u) = 0, \tag{4.12}$$

and similarly

$$\lim_{t\to\infty} (H_\lambda(t) - \log t) = 0. \tag{4.13}$$

Fix $\lambda > 0$ and $t_0 > 0$. There is a constant C such that

$$H_\lambda(t) = \int_{t_0}^t h_\lambda(t')\,\mathrm{d}t' + C.$$

Moreover, according to Proposition 4.13,

$$h_\lambda(t) = h(s(\lambda, t)), \qquad t > 0.$$

Put $s(t) = s(\lambda, t)$ and $u(t) = t - s(t)$. Then $s(t) + u(t) = t$ and $s'(t) + u'(t) = 1$. Moreover, by Definition 4.9,

$$(s(t) - t)\left(\frac{1}{h(s(t))} - s(t) + t\right) = \lambda^2.$$

Hence,

$$u(t)\left(\frac{1}{h(s(t))} - u(t)\right) = \lambda^2,$$

implying that

$$h(s(t)) = \frac{u(t)}{\lambda^2 + u(t)^2}.$$

It follows that

$$\begin{aligned}
\int_{t_0}^t h_\lambda(v)\,\mathrm{d}v &= \int_{t_0}^t h(s(v))(s'(v) + u'(v))\,\mathrm{d}v \\
&= \int_{t_0}^t \left(h(s(v))s'(v) + \frac{u(v)}{\lambda^2 + u(v)^2} u'(v)\right) \mathrm{d}v \\
&= H(s(t)) - H(s(t_0)) + \frac{1}{2}\log\left(\frac{\lambda^2}{\lambda^2 + u(t)^2}\right) + \frac{1}{2}\log\left(\frac{\lambda^2 + u(t_0)^2}{\lambda^2}\right).
\end{aligned}$$

Hence,

$$H_\lambda(t) = H(s(t)) + \frac{1}{2}\log\left(\frac{\lambda^2}{\lambda^2 + (s(t) - t)^2}\right) + C',$$

for a constant C'. Recall that $s(t) - t \to 0$ as $t \to \infty$ (cf. Lemma 4.12). It then follows from (4.12) and (4.13) that C' must be 0. This finally shows us that

$$\exp(2H_\lambda(t)) = \frac{\lambda^2}{\lambda^2 + (s(t) - t)^2} \exp(2H(t)),$$

and this proves (4.11). ■

4.15 Theorem. *Let $T \in \mathcal{M}^\Delta$ be R–diagonal, let $\mu = \mu_{|T|}$, and let $s(|\lambda|, 0)$ be as in Definition 4.9.*

(i) If $\lambda_1(\mu) < |\lambda| < \lambda_2(\mu)$, then

$$\Delta(T - \lambda\mathbf{1}) = \left(\frac{|\lambda|^2}{|\lambda|^2 + s(|\lambda|,0)^2} \, \Delta(T^*T + s(|\lambda|,0)^2\mathbf{1}) \right)^{\frac{1}{2}}.$$

(ii) If $|\lambda| \le \lambda_1(\mu)$, then $\Delta(T - \lambda\mathbf{1}) = \Delta(T)$.

(iii) If $|\lambda| \ge \lambda_2(\mu)$, then $\Delta(T - \lambda\mathbf{1}) = |\lambda|$.

Proof. The theorem is obviously true for $\lambda = 0$. Moreover, as in the proof of Lemma 4.14, it suffices to consider the case $\lambda > 0$. Note that

$$\Delta(T - \lambda\mathbf{1})^2 = \lim_{t \to 0+} \Delta\big((T - \lambda\mathbf{1})^*(T - \lambda\mathbf{1}) + t^2\mathbf{1}\big). \tag{4.14}$$

Hence, (i) follows from Lemma 4.10 and Lemma 4.14. If $0 < \lambda \le \lambda_1(\mu)$, then by Remark 4.11, $\lim_{t\to 0+} s(\lambda, t) = 0$. Hence, (ii) also follows from Lemma 4.14. Now suppose $\lambda \ge \lambda_2(\mu)$. Then $s(\lambda, t) \to \infty$ as $t \to 0+$. The right–hand side of (4.11) is equal to

$$\frac{\lambda^2 s(\lambda,t)^2}{\lambda^2 + (s(\lambda,t) - t)^2} \, \frac{\Delta(T^*T - s(\lambda,t)^2\mathbf{1})}{s(\lambda,t)^2},$$

where the first factor converges to λ^2 as $t \to 0+$, and the second factor converges to 1 (cf. (4.12)). (iii) now follows from (4.11) and (4.14). ■

4.16 Remark. Note that

$$\lambda_2(\mu) = \left(\int_0^\infty u^2 \, \mathrm{d}\mu_{|T|}(u) \right)^{\frac{1}{2}} = \|T\|_2$$

and

$$\lambda_1(\mu) = \left(\int_0^\infty u^{-2} \, \mathrm{d}\mu_{|T|}(u) \right)^{-\frac{1}{2}} = \|T^{-1}\|_2^{-1},$$

where $\|T^{-1}\|_2 := +\infty$ in case $\ker(T) \ne 0$.

4.17 Theorem. *Let T be an R–diagonal element in $\mathcal{M}^\Delta$ with Brown measure μ_T, and suppose $\mu_{|T|}$ is not a Dirac measure.*

(a) If $\ker(T) = 0$, then

$$\operatorname{supp}(\mu_T) = \{\lambda \in \mathbb{C} \mid \|T^{-1}\|_2^{-1} \le |\lambda| \le \|T\|_2\}.$$

Moreover, the S–transform of $\mu_{|T|^2}$ *is well–defined and strictly increasing on* $(-1,0)$ *with*

$$\mathcal{S}_{\mu_{|T|^2}}((-1,0)) = (\|T\|_2^{-2}, \|T^{-1}\|_2^2),$$

and μ_T *is the unique probability measure on* $\mathbb{C}$ *which is invariant under rotations and satisfies*

$$\mu_T\big(B(0, \mathcal{S}_{\mu_{|T|^2}}(t-1)^{-\frac{1}{2}})\big) = t, \quad 0 < t < 1.$$

(b) If $\ker(T) \neq 0$*, let* P *denote the projection onto* $\ker(T)$*. Then*

$$\mathrm{supp}(\mu_T) = \{\lambda \in \mathbb{C} \,|\, |\lambda| \leq \|T\|_2\}.$$

Moreover, the S–transform of $\mu_{|T|^2}$ *is well–defined and strictly increasing on* $(\tau(P)-1,0)$ *with*

$$\mathcal{S}_{\mu_{|T|^2}}((\tau(P)-1,0)) = (\|T\|_2^{-2}, \infty),$$

and μ_T *is the unique probability measure on* $\mathbb{C}$ *which is invariant under rotations and satisfies*

$$\mu_T\big(B(0, \mathcal{S}_{\mu_{|T|^2}}(t-1)^{-\frac{1}{2}})\big) = t, \quad \tau(P) < t < 1.$$

Proof. By definition, $\mathrm{d}\mu_T(\lambda) = \frac{1}{2\pi}\nabla^2\big(\log\Delta(T-\lambda\mathbf{1})\big)\mathrm{d}\lambda$ (in the distribution sense). Hence, μ_T can be determined from Theorem 4.15 in the same way as [HL, Theorem 4.4.] is obtained from [HL, (4.5)]:

Using the same notation as in [HL], we define functions $f, g : (0,\infty) \to \mathbb{R}$ by

$$f(v) = \int_0^\infty \frac{1}{1+v^2w^2}\,\mathrm{d}\mu_{|T|}(w),$$

and

$$g(v) = \frac{1-f(v)}{v^2 f(v)}.$$

Moreover, for $\lambda \in (\|T^{-1}\|_2^{-2}, \|T\|_2^2)$, let $v(\lambda)$ denote the unique $v \in (0,\infty)$ such that $g(v) = \lambda^2$. Then, in our notation,

$$f(v) = \tau\big((\mathbf{1} + v^2T^*T)^{-1}\big) = v^{-1}h(v^{-1}),$$

and

$$g(v) = v^{-1}\left(\frac{1}{h(v^{-1})} - v^{-1}\right) = k(v^{-1}, 0).$$

Hence,

$$v(\lambda) = \frac{1}{s(\lambda,0)},$$

and it follows that the formula (4.15) in [HL],

$$\log\Delta(T-\lambda\mathbf{1}) = \frac{1}{2}\int_0^\infty \log(1+v^2w^2)\,\mathrm{d}\mu_{|T|}(w) + \frac{1}{2}\log\left(\frac{\lambda^2}{1+v^2\lambda^2}\right), \qquad \lambda \in (\|T^{-1}\|_2^{-2}, \|T\|_2^2),$$

is equivalent to the one in Theorem 4.15 (i). The rest of the proof of Theorem 4.17 is identical to the second part of the proof of [HL, Theorem 4.4], since boundedness of T is not a necessary assumption in the latter. ■

4.18 Remark. Let $T \in \mathcal{M}^\Delta$ be R–diagonal. Then $\mathrm{supp}(\mu_T) \subseteq \sigma(T)$, and according to Theorem 4.17,

$$\mathrm{supp}(\mu_T) = \{\lambda \in \mathbb{C} \,|\, \|T^{-1}\|_2^{-1} \leq |\lambda| \leq \|T\|_2\}.$$

Moreover, by arguments similar to the ones given in [HL, proof of Proposition 4.6], one can show that

(a) if $0 < |\lambda| < \|T^{-1}\|_2^{-1}$, then $\lambda \in \sigma(T)$ iff T does not have a bounded inverse, and

(b) if $|\lambda| > \|T\|_2$, then $\lambda \in \sigma(T)$ iff T is not bounded.

5 Properties of $z = xy^{-1}$

Let $\mathcal{M} = L(\mathbb{F}_4)$ be the von Neumann algebra associated with the free group on 4 generators. According to [V1] or [VDN], $\mathcal{M}$ is a II_1–factor generated by a semicircular system (s_1, s_2, s_3, s_4), i.e. the s_i's are freely independent self–adjoint elements w.r.t. the unique tracial state τ on $\mathcal{M}$, and s_i has distribution

$$\mathrm{d}\mu_{s_i}(t) = \frac{1}{2\pi}\sqrt{4-t^2}\, 1_{[-2,2]}(t)\,\mathrm{d}t, \qquad 1 \leq i \leq 4.$$

Put

$$x = \frac{s_1 + \mathrm{i}s_2}{\sqrt{2}} \quad \text{and} \quad y = \frac{s_3 + \mathrm{i}s_4}{\sqrt{2}}.$$

Then $\mathcal{M} = W^*(x,y)$, and (x,y) is a circular system in the sense of [VDN]. Also, by [VDN], $|y|$ has the distribution

$$\mathrm{d}\mu_{|y|}(t) = \frac{2}{\pi}\sqrt{4-t^2}\, 1_{[0,2]}(t)\,\mathrm{d}t.$$

In particular, $\ker(y) = 0$. In this section we will study the unbounded operator

$$z = xy^{-1}$$

as well as its powers z^n, $n = 2, 3, \ldots$ We will need the following simple observation:

5.1 Lemma. *Let $(\mu_n)_{n=1}^\infty$ and μ be probability measures on $\mathbb{R}$ with densities $(f_n)_{n=1}^\infty$ and f, respectively, w.r.t. Lebesgue measure. If $f_n \overset{n\to\infty}{\to} f$ a.e. w.r.t. Lebesgue measure, then $\mu_n \overset{n\to\infty}{\to} \mu$ weakly.*

Proof. Recall that $\mu_n \overset{n\to\infty}{\to} \mu$ weakly iff for all $\phi \in C_0(\mathbb{R})$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi \,\mathrm{d}\mu_n = \int_{\mathbb{R}} \phi\,\mathrm{d}\mu. \tag{5.1}$$

Then, let $\phi \in C_0(\mathbb{R})$ with $0 \leq \phi \leq 1$. (5.1) follows for such ϕ by application of Fatou's Lemma to each of the sequences of integrals $\left(\int_{\mathbb{R}} \phi f_n \mathrm{d}m\right)_{n=1}^{\infty}$ and $\left(\int_{\mathbb{R}} (1-\phi) f_n \mathrm{d}m\right)_{n=1}^{\infty}$.

■

5.2 Theorem. *Let* $(\mathcal{M}, \tau)$ *and* $z = xy^{-1}$ *be as above.*

(a) z *is an unbounded, R–diagonal operator.*

(b) The distribution of z *is given by*

$$\mathrm{d}\mu_{|z|}(t) = \frac{2}{\pi}\frac{1}{1+t^2}\, 1_{(0,\infty)}(t)\, \mathrm{d}t. \tag{5.2}$$

(c) For $p \in (0,1)$, $z, z^{-1} \in L^p(\mathcal{M}, \tau)$, *and*

$$\|z\|_p^p = \|z^{-1}\|_p^p = \left[\cos\left(\frac{p\pi}{2}\right)\right]^{-1} < \infty. \tag{5.3}$$

(d) $z, z^{-1} \in \mathcal{M}^{\Delta}$, *and the Brown measure of* z *is given by*

$$\mathrm{d}\mu_z(s) = \frac{1}{\pi(1+|s|^2)^2}\, \mathrm{d}s, \tag{5.4}$$

where $\mathrm{d}s = \mathrm{dRe}s\, \mathrm{dIm}s$ *is Lebesgue measure on* $\mathbb{C}$.

Proof. (a) Let $x = u|x|$ and $y = v|y|$ be the polar decompositions of x and y. Then, according to [VDN], $u, |x|, v$ and $|y|$ are $*$–free elements, and u and v are Haar unitaries. In particular, x and y are R–diagonal and so is y^{-1} (cf. Proposition 3.6). Moreover, y^{-1} has polar decomposition

$$y^{-1} = v^*(v|y|^{-1}v^*) = v^*|y^*|^{-1},$$

which implies that y^{-1} is affiliated with $W^*(y)$. Hence, x and y^{-1} are $*$–free, and it follows from Proposition 3.8 that $z = xy^{-1}$ is R–diagonal with

$$\mathcal{S}_{\mu_{|z|^2}}(t) = \mathcal{S}_{\mu_{|x|^2}}(t)\, \mathcal{S}_{\mu_{|y^{-1}|^2}}(t), \qquad t \in (-1,0).$$

The distribution of $|x|^2$ has density

$$\mathrm{d}\mu_{|x|^2}(t) = \frac{1}{2\pi}\sqrt{\frac{4-t}{t}}\, 1_{(0,4)}(t)\, \mathrm{d}t,$$

and thus $\mathcal{S}_{\mu_{|x|^2}}$ is given by

$$\mathcal{S}_{\mu_{|x|^2}}(t) = \frac{1}{1+t}$$

for all t in a neighborhood of $(-1,0)$ (cf. [HL, example 5.2]). Since $|y^{-1}| = |y^*|^{-1} \underset{*\mathcal{D}}{\sim} |y|^{-1} \underset{*\mathcal{D}}{\sim} |x|^{-1}$, we get from Proposition 3.13 that

$$\mathcal{S}_{\mu_{|y^{-1}|^2}}(t) = \frac{1}{\mathcal{S}_{\mu_{|x|^2}}(-1-t)} = -t, \qquad t \in (-1,0).$$

Then

$$\mathcal{S}_{\mu_{|z|^2}}(t) = -\frac{t}{1+t}, \qquad t \in (-1,0), \tag{5.5}$$

and

$$\chi_{\mu_{|z|^2}}(t) = \frac{t}{1+t}\mathcal{S}_{\mu_{|z|^2}}(t) = -\left(\frac{t}{1+t}\right)^2, \qquad t \in (-1,0).$$

The inverse function of $\chi_{\mu_{|z|^2}}$ is then

$$\psi_{\mu_{|z|^2}}(u) = \frac{-\sqrt{-u}}{1+\sqrt{-u}}, \qquad u \in (-\infty, 0),$$

and it follows that

$$G_{\mu_{|z|^2}}(\lambda) = \frac{1}{\lambda}\Big(1+\psi_{\mu_{|z|^2}}\Big(\tfrac{1}{\lambda}\Big)\Big) = \tfrac{1}{\lambda-\sqrt{-\lambda}}, \qquad \lambda < 0. \tag{5.6}$$

Let $\sqrt{w}$ denote the principal value of the square root of w for $w \in \mathbb{C}\setminus(\infty,0]$. Then both sides of (5.6) are analytic in $\mathbb{C}\setminus[0,\infty)$. Thus, (5.6) holds for all $\lambda \in \mathbb{C}\setminus[0,\infty)$, and it follows that for $t > 0$,

$$-\frac{1}{\pi}\lim_{u\to 0+}\mathrm{Im}G_{\mu_{|z|^2}}(t+\mathrm{i}u) = -\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{t+\mathrm{i}\sqrt{t}}\right) = \frac{1}{\pi}\frac{1}{\sqrt{t}(t+1)}. \tag{5.7}$$

For $\beta \in (0,1)$,

$$\int_0^\infty \frac{t^{\beta-1}}{1+t}\,\mathrm{d}t = \frac{\pi}{\sin(\beta\pi)}, \tag{5.8}$$

(cf. [Ha, p. 592, formula 613]). The right–hand side of (5.7) therefore defines the density of a probability measure, and then, by Lemma 5.1, the probability measures

$$\frac{1}{\pi}\,\mathrm{Im}G_{\mu_{|z|^2}}(t+\mathrm{i}u)\,\mathrm{d}t, \qquad u > 0,$$

converge weakly to

$$\frac{1}{\pi}\frac{1}{\sqrt{t}(t+1)}\,1_{(0,\infty)}(t)\,\mathrm{d}t, \tag{5.9}$$

as $u \to 0+$. Hence, by the inverse Stieltjes transform, $\mathrm{d}\mu_{|z|^2}(t)$ is given by (5.9), and then

$$\mathrm{d}\mu_{|z|}(t) = \frac{2}{\pi}\frac{1}{1+t^2}\,1_{(0,\infty)}(t)\,\mathrm{d}t.$$

This proves (a) and (b).

In order to prove (c), note that according to (5.8),

$$\begin{aligned}
\tau(|z|^p) &= \frac{2}{\pi}\int_0^\infty \frac{t^p}{1+t^2}\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_0^\infty \frac{w^{\frac{p-1}{2}}}{1+w}\,\mathrm{d}w \\
&= \Big[\sin\Big(\frac{\pi(p+1)}{2}\Big)\Big]^{-1},
\end{aligned}$$

proving (c). Since $L^p(\mathcal{M},\tau) \subseteq \mathcal{M}^\Delta$, $p > 0$, $z, z^{-1} \in \mathcal{M}^\Delta$. According to Theorem 4.17, μ_z is then the unique probability measure on $\mathbb{C}$ which is invariant under rotations and satisfies

$$\mu_z\big(B(0, \mathcal{S}_{\mu_{|z|^2}}(t-1)^{-\frac{1}{2}})\big) = t, \qquad 0 < t < 1.$$

Then by (5.5),

$$\mu_z\Big(B\Big(0, \sqrt{\tfrac{t}{1-t}}\Big)\Big) = t, \qquad 0 < t < 1,$$

that is,

$$\mu_z(B(0,r)) = \frac{r^2}{1+r^2}, \qquad r > 0.$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}r}\mu_z(B(0,r)) = \frac{2r}{(1+r^2)^2}$, and combining this with the fact that μ_z is invariant under rotations, we find that μ_z has density w.r.t. Lebesgue measure on $\mathbb{C}$ given by

$$\frac{1}{2\pi r}\frac{2r}{(1+r^2)^2} = \frac{1}{\pi}\frac{1}{(1+r^2)^2}, \qquad r > 0,$$

where $r = |s|$, $s \in \mathbb{C} \setminus \{0\}$. This proves (d). ■

5.3 Lemma. *Let μ be a probability measure on $[0,\infty)$ and, as in section 5, put*

$$h(s) = \int_0^\infty \frac{s}{s^2+u^2}\,\mathrm{d}\mu(u), \qquad s \in (0,\infty).$$

Then for $0 < p < 2$,

$$\int_0^\infty u^{-p}\,\mathrm{d}\mu(u) = \frac{2}{\pi}\sin\left(\frac{\pi p}{2}\right)\int_0^\infty s^{-p}h(s)\,\mathrm{d}s. \tag{5.10}$$

Proof. By Tonelli's theorem,

$$\int_0^\infty s^{-p}h(s)\,\mathrm{d}s = \int_0^\infty\left(\int_0^\infty \frac{s^{1-p}}{s^2+u^2}\,\mathrm{d}s\right)\mathrm{d}\mu(u).$$

Letting $s = ut^{\frac{1}{2}}$, we find (using (5.8)) that

$$\int_0^\infty \frac{s^{1-p}}{s^2+u^2}\,\mathrm{d}s = \frac{1}{2}u^{-p}\int_0^\infty \frac{t^{-\frac{p}{2}}}{1+t}\,\mathrm{d}t = \frac{\pi}{2}\left[\sin\left(\frac{\pi p}{2}\right)\right]^{-1}u^{-p}.$$

This proves (5.10). ■

5.4 Theorem. *Let $(\mathcal{M},\tau)$ and z be as in Theorem 5.2, and let $n \in \mathbb{N}$.*

(a) z^n is an unbounded R–diagonal operator.

(b)

$$\int_0^\infty \frac{s}{s^2+u^2}\,\mathrm{d}\mu_{|z|^n}(u) = \left(s + s^{\frac{n-1}{n+1}}\right)^{-1}, \qquad s > 0. \tag{5.11}$$

(c) *For* $p \in \left(0, \frac{2}{n+1}\right)$, z^n *and* z^{-n} *both belong to* $L^p(\mathcal{M}, \tau)$, *and*

$$\|z^n\|_p^p = \|z^{-n}\|_p^p = \frac{(n+1)\sin\left(\frac{\pi p}{2}\right)}{\sin\left(\frac{(n+1)\pi p}{2}\right)}. \tag{5.12}$$

(d) *If* $p \in \left(0, \frac{2}{n+1}\right)$ *and* $\lambda \in \mathbb{C}$, *then* $\ker(z^n - \lambda\mathbf{1}) \neq 0$. *Moreover,* $(z^n - \lambda\mathbf{1})^{-1} \in L^p(\mathcal{M}, \tau)$ *with*

$$\|(z^n - \lambda\mathbf{1})^{-1}\|_p \leq \|z^{-n}\|_p. \tag{5.13}$$

Proof. According to Proposition 3.9, z^n is R–diagonal. Moreover, since

$$\mathcal{S}_{\mu_{|z|^2}}(t)^n = \left(-\frac{t}{1+t}\right)^n, \qquad t \in (-1, 0),$$

$$\chi_{\mu_{|z^n|^2}}(t) = \frac{1}{1+t}\mathcal{S}_{\mu_{|z^n|^2}}(t) = -\left(-\frac{t}{1+t}\right)^{n+1}, \qquad t \in (-1, 0),$$

with inverse function

$$\psi_{\mu_{|z^n|^2}}(u) = -\frac{(-u)^{\frac{1}{n+1}}}{1 + (-u)^{\frac{1}{n+1}}}, \qquad u \in (-\infty, 0).$$

Hence, for $\lambda \in (-\infty, 0)$,

$$G_{\mu_{|z^n|^2}}(\lambda) = \frac{1}{\lambda}\Big(1 + \psi_{\mu_{|z^n|^2}}\left(\tfrac{1}{\lambda}\right)\Big) = \frac{1}{\lambda\Big(1 + (-\lambda)^{-\frac{1}{n+1}}\Big)}. \tag{5.14}$$

Let

$$h_n(s) = \int_0^\infty \frac{s}{s^2 + u^2}\, \mathrm{d}\mu_{|z^n|}(u), \qquad s \in (0, \infty).$$

Then

$$\begin{aligned} h_n(s) &= s\,\tau\big((s^2\mathbf{1} + |z^n|^2)^{-1}\big) \\ &= -s\,G_{\mu_{|z^n|^2}}(-s^2) \\ &\overset{(5.14)}{=} \Big(s + s^{\frac{n-1}{n+1}}\Big)^{-1}. \end{aligned}$$

This proves (b).

Since $z = xy^{-1}$, where (x, y) is a circular family, it is clear that $z^{-n} \underset{*\mathcal{D}}{\sim} z^n$ for all $n \in \mathbb{N}$. Hence, $\|z^n\|_p = \|z^{-n}\|_p$ for all $p > 0$. Note that for $p > 0$,

$$\|z^{-n}\|_p^p = \tau(|z^{-n}|^p) = \tau(|(z^n)^*|^{-p}) = \tau(|z^n|^{-p}).$$

Thus, by Lemma 5.3, for $p \in (0, 2)$,

$$\|z^{-n}\|_p^p = \int_0^\infty u^{-p}\, \mathrm{d}\mu_{|z^n|}(u) = \frac{2}{\pi}\sin\left(\frac{\pi p}{2}\right)\int_0^\infty s^{-p}h_n(s)\,\mathrm{d}s. \tag{5.15}$$

By application of (5.11) we find that

$$\int_0^\infty s^{-p} h_n(s)\,\mathrm{d}s = \int_0^\infty \frac{s^{-p-\frac{n-1}{n+1}}}{s^{\frac{2}{n+1}}+1}\,\mathrm{d}s = \frac{n+1}{2}\int_0^\infty \frac{t^{-\frac{(n+1)p}{2}}}{1+t}\,\mathrm{d}t.$$

Then by (5.15) and (5.8), for $0 < p < \frac{2}{n+1}$,

$$\begin{aligned}\|z^{-n}\|_p^p &= (n+1)\sin\Big(\frac{\pi p}{2}\Big)\Big[\sin\Big(\pi\Big(1-\frac{(n+1)p}{2}\Big)\Big)\Big]^{-1}\\ &= (n+1)\sin\Big(\frac{\pi p}{2}\Big)\Big[\sin\Big(\frac{(n+1)\pi p}{2}\Big)\Big]^{-1},\end{aligned} \tag{5.16}$$

and this proves (c). Note that the right–hand side of (5.16) converges to ∞ as $p \to \frac{2}{n+1}-$. Hence, $z^{-n} \notin L^{\frac{2}{n+1}}(\mathcal{M},\tau)$, and the same holds for z^n. In particular, z^n is not bounded, and this proves (a). In order to prove (d), let $\lambda \in \mathbb{C}\setminus\{0\}$, and put

$$h_{n,\lambda}(t) = \int_0^\infty \frac{t}{t^2+u^2}\,\mathrm{d}\mu_{|z^n-\lambda\mathbf{1}|}(u), \qquad t>0.$$

Then by Proposition 4.13,

$$h_{n,\lambda}(t) = h_n(s_n(|\lambda|,t)), \qquad t>0,$$

where $s_n(|\lambda|,t)$ is given by Definition 4.9 in the case $\mu = \mu_{|z^n|}$. Note that, according to Definition 4.9,

$$s_n(|\lambda|,t) > t, \qquad t>0.$$

Moreover, by (5.11), h_n is monotonically decreasing on $(0,\infty)$. Thus,

$$h_{n,\lambda}(t) \le h_n(t), \qquad t>0.$$

It now follows from Lemma 5.3 that for $p \in (0,2)$,

$$\int_0^\infty u^{-p}\,\mathrm{d}\mu_{|z^n-\lambda\mathbf{1}|}(u) \le \int_0^\infty u^{-p}\,\mathrm{d}\mu_{|z^n|}(u). \tag{5.17}$$

According to (c), the right–hand side of (5.17) is finite for $p \in \Big(0,\frac{2}{n+1}\Big)$. Hence, for such p, $\ker(z^n-\lambda\mathbf{1}) = 0$, $(z^n-\lambda\mathbf{1})^{-1} \in L^p(\mathcal{M},\tau)$, and

$$\|(z^n-\lambda\mathbf{1})^{-1}\|_p^p \le \|z^{-n}\|_p^p. \qquad \blacksquare$$

5.5 Remark. Note that Theorem 5.4 (a) and (c) generalize Theorem 5.2 (a) and (c) to all $n \in \mathbb{N}$. It is not hard to generalize Theorem 5.2 (b) and (d) as well. One finds that the distribution of $|z^n|$ is given by

$$\mathrm{d}\mu_{|z^n|}(t) = \frac{2}{\pi}\frac{\sin\Big(\frac{\pi}{n+1}\Big)}{t\Big(t^{\frac{2}{n+1}}+2\cos\Big(\frac{\pi}{n+1}\Big)+t^{-\frac{2}{n+1}}\Big)}1_{(0,\infty)(t)}\,\mathrm{d}t,$$

and the Brown measure of z^n is given by

$$\mathrm{d}\mu_{z^n}(s) = \frac{1}{n\pi}\frac{|s|^{\frac{2}{n}-2}}{(1+|s|^{\frac{2}{n}})^2}\,\mathrm{dRe}s\,\mathrm{dIm}s.$$

We leave the details of proof to the reader.

References

[AH] L. Aagaard and U. Haagerup, Moment formulas for the quasinilpotent DT–operator, *International J. Math.* **15** (2004), 581–628.

[BL] P. Biane and F. Lehner, Computation of some examples of Brown's spectral measure in free probability, *Colloq. Math.* **90** (2001), no. 2, 181–211.

[Br] L. G. Brown, Lidskii's Theorem in the Type II Case, Geometric methods in operator algebras (Kyoto 1983), H. Araki and E. Effros (Eds.) *Pitman Res. notes in Math.* Ser 123, Longman Sci. Tech. (1986), 1–35.

[BV] H. Bercovici, D. Voiculescu, Free Convolution of Measures with Unbounded Support, *Indiana University Mathematics Journal*, Vol. 42, No. 3 (1993), 733–773.

[DH] K. Dykema and U. Haagerup, DT–Operators and Decomposability of Voiculescu's Circular Operator, *Amer. J. Math.* **126** (2004), 121–189.

[FuKa] B. Fuglede, R. V. Kadison, Determinant theory in finite factors, *Ann. Math.* 55 (1952), 520–530.

[Fo] G. B. Folland, Real analysis, modern techniques and their applications, John Wiley and Sons (1984).

[HT] U. Haagerup, S. Thorbjørnsen, 'A new application of random matrices: $\mathrm{Ext}(C^*_{\mathrm{red}}(\mathbb{F}_2))$ is not a group', *Annals of Mathematics*, **162** (2005), 711–775.

[Ha] Handbook of tables for mathematics, 4th edition, *CRC–press* (1975).

[HK] W. K. Hayman, P. B. Kennedy, Subharmonic Functions, Volume I, Academic Press (London) (1976).

[HL] U. Haagerup, F. Larsen, Brown's Spectral Distribution Measure for R-diagonal Elements in Finite von Neumann Algebras, *Journ. Functional Analysis* 176, 331–367 (2000).

[HS] U. Haagerup and H. Schultz, Invariant Subspaces of Operators in a General II_1–factor. Preliminary version (2006). http://www.imada.sdu.dk/∼schultz/publications.html.

[La1] F. Larsen, Brown Measures and R-diagonal Elements in Finite von Neumann Algebras, Ph.D. Thesis, University of Southern Denmark, 1999.

[La2] F. Larsen, Powers of R-diagonal Elements, *Journ. Operator Theory* 47, 197–212 (2002).

[NS] A. Nica, R. Speicher, R-diagonal pairs – a common approach to Haar unitaries and circular elements, *Fiels Institute Communications* (1997), 149–188.

[V1] D. Voiculescu, Circular and Semicircular Systems and Free Product Factors, "Operator Algebras, Unitary Representations, Algebras, and Invariant Theory", *Progress in Math.* Vol. 92, Birkhäuser, 1990, 45-60.

[VDN] D. Voiculescu, K. Dykema and A. Nica, Free Random Variables, CMR Monograph Series 1, *American Mathematical Society* (1992).

Uffe Haagerup
Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55, 5230 Odense M
Denmark
haagerup@imada.sdu.dk

Hanne Schultz
Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55, 5230 Odense M
Denmark
schultz@imada.sdu.dk