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# ON THE BEST CONSTANTS IN NONCOMMUTATIVE KHINTCHINE-TYPE INEQUALITIES

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Abstract. We obtain new proofs with improved constants of the Khintchine-type inequality with matrix coefficients in two cases. The first case is the Pisier and Lust-Piquard noncommutative Khintchine inequality for $p = 1$, where we obtain the sharp lower bound of $\frac{1}{\sqrt{2}}$ in the complex Gaussian case and for the sequence of functions $\{e^{i2^n t}\}_{n=1}^{\infty}$. The second case is Junge's recent Khintchine-type inequality for subspaces of the operator space $R \oplus C$, which he used to construct a cb-embedding of the operator Hilbert space $OH$ into the predual of a hyperfinite factor. Also in this case, we obtain a sharp lower bound of $\frac{1}{\sqrt{2}}$. As a consequence, it follows that any subspace of a quotient of $(R \oplus C)^*$ is cb-isomorphic to a subspace of the predual of the hyperfinite factor of type $\text{III}_1$, with cb-isomorphism constant $\leq \sqrt{2}$. In particular, the operator Hilbert space $OH$ has this property.

## 1. Introduction

Let $r_n(t) = \text{sgn}(\sin(2^n t\pi))$, $n \in \mathbb{N}$ denote the Rademacher functions on $[0,1]$. The classical Khintchine inequality states that for every $0 < p < \infty$, there exist constants $A_p$ and $B_p$ such that

$$A_p \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left|\sum_{k=1}^{n} a_k r_k\right|^p dt\right)^{\frac{1}{p}} \leq B_p \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}, \tag{1.1}$$

for arbitrary $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$.

Suppose $A_p$ and $B_p$ denote the best constants for which (1.1) holds. While it is elementary to prove that $B_p = 1$ for $0 < p \leq 2$ and $A_p = 1$ for $2 \leq p < \infty$, it took the work of many mathematicians to settle all the other cases, including Szarek [20] who proved that $A_1 = \frac{1}{\sqrt{2}}$ (thus solving a long-standing conjecture of Littlewood), Young [22] who computed $B_p$ for $p \geq 3$, and the first-named author (cf. [6]) who computed $A_p$ and $B_p$ in the remaining cases.

The Khintchine inequality and its generalization to certain classes of Banach spaces are deeply connected with the study of the geometry of those Banach spaces (see [13]). Noncommutative generalizations of the classical Khintchine inequality to the case of matrix-valued coefficients were first proved by Lust-Piquard [11] in the case $1 < p < \infty$, and by Pisier and Lust-Piquard [12] for $p = 1$. Their method of proof follows the classical harmonic analysis approach of deriving Khintchine inequality for the sequence $\{e^{i2^n t}\}_{n=1}^{\infty}$ from a Paley inequality, for which they proved a noncommutative version (see Theorem II.1 in [12]). As a consequence, the following noncommutative Khintchine inequality holds (see Corollary II.2 in [12]). Given $d, n \in \mathbb{N}$ and $x_1, \ldots, x_d \in M_n(\mathbb{C})$, then

$$\frac{1}{1+\sqrt{2}} |||\{x_j\}_{j=1}^d|||^* \leq \left\|\sum_{j=1}^{d} x_j \otimes e^{i2^n t}\right\|_{L^1([0,1];S_1^n)} \leq |||\{x_j\}_{j=1}^d|||^*, \tag{1.2}$$

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where, by definition,

$$(1.3)\qquad |||\{x_i\}_{i=1}^d|||^* : \;=\; \inf\left\{ \mathrm{Tr}\left( \left( \sum_{i=1}^d y_i^* y_i \right)^{\frac{1}{2}} + \left( \sum_{i=1}^d z_i z_i^* \right)^{\frac{1}{2}} \right) ; x_i = y_i + z_i \in M_n(\mathbb{C}) \right\} .$$

Here $S_1^n$ is $M_n(\mathbb{C})$ with the norm $\|x\|_1 := \mathrm{Tr}((x^*x)^{1/2})$ , and Tr is the non-normalized trace on $M_n(\mathbb{C})$ . We should also point out that it was noted in the paper [12] (cf. p. 250) that, by using the lacunary sequence $\{3^n\}_{n\geq 1}$ instead of the sequence $\{2^n\}_{n\geq 1}$, the lower bound in the inequality (1.2) can be improved to $\frac{1}{2}$ .

By classical arguments (cf. Proposition 3.2 in [15]) , if one replaces $\{e^{i2^n t}\}_{n=1}^\infty$ by a sequence of independent complex Gaussian, respectively, Rademacher or Steinhauss random variables, the corresponding Khintchine inequality with matrix coefficients follows, as well, with possibly different constants.

Our method, leading to improved constants, was inspired by ideas of Pisier from [14], and it is based on proving first directly the dual inequality to (1.2) with constant $\sqrt{2}$ , where $\{e^{i2^n t}\}_{n=1}^\infty$ is replaced by a sequence of independent complex-valued standard Gaussian random variables on some probability space $(\Omega, \mathbb{P})$ . Based on a result from [8] , the constant $\sqrt{2}$ turns out to be optimal in this case, and for the sequence $\{e^{i2^n t}\}_{n=1}^\infty$ . We also consider the case of a sequence of Rademacher functions, and prove that the corresponding noncommutative Khintchine inequality holds with constant $\sqrt{3}$ instead of $\sqrt{2}$ , but we do not know yet whether this is sharp.

In the second part of our paper we obtain an improvement of a recent result of M. Junge (cf. [10]) concerning a Khintchine-type inequality for subspaces of $R \oplus_\infty C$ (the $l^\infty$-sum of the row and column Hilbert spaces). Recall that $R := \mathrm{Span}\{e_{1j}; j \geq 1\}$ , respectively, $C := \mathrm{Span}\{e_{j1}; j \geq 1\}$, where $e_{kl}$ is the element in $\mathcal{B}(l_2)$ corresponding to the matrix with entries equal to 1 on the $(k,l)$ position, and 0 elsewhere. This Khintchine-type inequality is intimately connected with the question of the existence of a completely isomorphic embedding of the operator space $OH$, introduced by G. Pisier (see [16]), into a noncommutative $L^1$-space, a problem that was resolved by Junge in the remarkable paper [9]. In [10] (see Section 8), Junge improved this result, by showing that $OH$ cb-embeds into the predual of a hyperfinite type $\mathrm{III}_1$ factor.

In our new approach, we first observe that given a closed subspace $H$ of $R \oplus C$ , there is a self-adjoint operator $A \in \mathcal{B}(H)$ satisfying $0 \leq A \leq I$ , where $I$ denotes the identity operator on $H$ , such that the operator space structure on $H$ is given by

$$(1.4)\qquad \left\| \sum_{i=1}^r x_i \otimes \xi_i \right\|_{M_n(H)} \;=\; \max\left\{ \left\| \sum_{i,j=1}^r \langle (I-A)\xi_i, \xi_j \rangle_H x_i x_j^* \right\|^{1/2} , \left\| \sum_{i,j=1}^r \langle A\xi_i, \xi_j \rangle_H x_i^* x_j \right\|^{1/2} \right\} ,$$

where $n, r$ are positive integers, $x_1 , \ldots , x_r \in M_n(\mathbb{C})$ and $\xi_1 , \ldots , \xi_r \in H$ .

As in Junge's approach from [10], we will use CAR algebra methods. We consider the associated quasi-free state $\omega_A$ on the CAR-algebra $\mathcal{A} = \mathcal{A}(H)$ built on the Hilbert space $H$ , and construct a linear map $F_A$ of $H^*$ into the predual $M_*$ of the von Neumann algebra $M := \overline{\pi_A(\mathcal{A})}^{\mathrm{sot}}$, which by [19] is a hyperfinite factor. Here $\pi_A$ is the unital $*$-homomorphism from the GNS representation associated to $(\mathcal{A}, \omega_A)$. Note that $M_*$ can be considered as a subspace of $\mathcal{A}^*$ . Next we let $F_A$ be the transpose of the map $E_A : \mathcal{A} \to H$ defined by

$$(1.5)\qquad \langle E_A(b), f \rangle_H \;=\; \omega_A(ba(f)^* + a(f)^* b) , \quad \forall b \in \mathcal{A} , \forall f \in H ,$$

where $f \mapsto a(f)$ is the map from $H$ to $\mathcal{A} = \mathcal{A}(H)$ in the definition of the $CAR$-algebra (cf. [4]). We then prove that $F_A$ is a cb-isomorphism of $H^*$ onto its range, satisfying the following estimates

$$(1.6) \qquad \frac{1}{\sqrt{2}} \|F_A(y)\|_{M_n(\mathcal{A})^*} \leq \|y\|_{M_n(H)^*} \leq \|F_A(y)\|_{M_n(\mathcal{A})^*}, \quad \forall n \in \mathbb{N}, y \in M_n(H)^*.$$

We do so by first proving the dual version of the inequalities (1.6), namely we show that

$$(1.7) \qquad \|\xi\|_{M_n(H)} \leq \|(\mathrm{Id}_n \otimes q_A)(\xi)\|_{M_n(\mathcal{A}/\mathrm{Ker}(E_A))} \leq \sqrt{2}\,\|\xi\|_{M_n(H)}, \quad \forall n \in \mathbb{N}, \xi \in M_n(H).$$

The estimate of the upper bound $\sqrt{2}$ in (1.7) (corresponding to the lower bound $\frac{1}{\sqrt{2}}$ in (1.6)) is obtained by methods very similar to those we used for the Pisier and Lust-Piquard noncommutative Khintchine inequality. We then prove that both constants in (1.6) are sharp.

Note that if $P$ is the unique hyperfinite factor of type $\mathrm{III}_1$ (cf. [7]), then the von Neumann algebra tensor product $M \bar{\otimes} P$ is isomorphic to $P$, and therefore $F_A$ can be considered as a completely bounded embedding of $H^*$ into the predual $P_*$ of $P$, as well. It follows that every subspace of a quotient of $(R \oplus C)^*$ is cb-isomorphic to a subspace of $P_*$ with cb-isomorphism constant $\leq \sqrt{2}$. In particular, due to results of G. Pisier (cf. [18] Proposition A1), the operator Hilbert space $OH$ has this property (cf. Corollary 3.8 in this paper). The question whether $OH$ embeds completely isometrically into a noncommutative $L^1$-space remains open.

In the case when the self-adjoint operator $A$ associated to the subspace $H$ of $R \oplus C$ has pure point spectrum and $\mathrm{Ker}(A) = \mathrm{Ker}(I - A) = 0$, our construction of the map $F_A : H^* \to M_*$ is very similar to Junge's construction from [10]. This can be seen by taking Lemma 3.3 into account.

We refer to the monographs [5, 17] for details on operator spaces. We shall briefly recall some definitions that are relevant for our paper. An operator space $V$ is a Banach space given together with an isometric embedding $V \subset \mathcal{B}(H)$, the algebra of bounded linear operators on a Hilbert space $H$. For all $n \in \mathbb{N}$, this embedding determines a norm on $M_n(V)$, the algebra of $n \times n$ matrices over $V$, induced by the space $M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$. If $W$ is a closed subspace of $V$, then both $W$ and $V/W$ are operator spaces; the matrix norms on $V/W$ are defined by $M_n(V/W) = M_n(V)/M_n(W)$. The morphisms in the category of operator spaces are *completely bounded maps.* Given a linear map $\phi : V_0 \to V_1$ between two operator spaces $V_0$ and $V_1$, define $\phi_n : M_n(V_0) \to M_n(V_1)$ by $\phi_n([v_{ij}]) = [\phi(v_{ij})]$, for all $[v_{ij}]_{i,j=1}^n \in M_n(V_0)$. Let $\|\phi\|_{cb} := \sup\{\|\phi_n\| \,; n \in \mathbb{N}\}$. The map $\phi$ is called *completely bounded* (for short, *cb*) if $\|\phi\|_{cb} < \infty$, and $\phi$ is called *completely isometric* if all $\phi_n$ are isometries. A cb map $\phi$ which is invertible with a cb inverse is called a *cb isomorphism.* The space of all completely bounded maps from $V_0$ to $V_1$, denoted by $\mathcal{CB}(V_0, V_1)$, is an operator space with matrix norms defined by $M_n(\mathcal{CB}(V_0, V_1)) = \mathcal{CB}(V_0, M_n(V_1))$. The dual of an operator space $V$ is, again, an operator space $V^* = \mathcal{CB}(V, \mathbb{C})$.

## 2. The Pisier and Lust-Piquard noncommutative Khintchine inequality

I. The complex Gaussian case

Let $\{\gamma_n\}_{n \geq 1}$ be a sequence of independent standard complex-valued Gaussian random variables on some probability space $(\Omega, \mathbb{P})$. Recall that a complex-valued random variable on $(\Omega, \mathbb{P})$ is called Gaussian *standard* if it has density $\frac{1}{\pi} e^{-|z|^2} d\mathrm{Re}z\, d\mathrm{Im}z$. Equivalently, its real and imaginary parts are real-valued, independent Gaussian random variables on $(\Omega, \mathbb{P})$, each having mean 0 and variance $\frac{1}{2}$. Therefore, for all $n \geq 1$, $\mathbb{E}(\gamma_n) = 0$ and $\mathbb{E}(|\gamma_n|^2) = 1$, where $\mathbb{E}$ denotes the usual expectation of a random variable.

**Theorem 2.1.** *Let $d$ and $n$ be positive integers, and consider $x_1, \ldots, x_d \in M_n(\mathbb{C})$. Then the following inequalities hold*

$$\frac{1}{\sqrt{2}}|||\{x_i\}_{i=1}^d|||^* \quad \leq \quad \left\| \sum_{i=1}^d x_i \otimes \gamma_i \right\|_{L^1(\Omega; S_1^n)} \leq |||\{x_i\}_{i=1}^d|||^* , \tag{2.1}$$

*where $|||\{x_i\}_{i=1}^d|||^*$ is defined by (1.3).*

We will prove Theorem 2.1 by obtaining first its dual version, namely,

**Proposition 2.2.** *Let $d$ be a positive integer, and let $\{\gamma_i\}_{i=1}^d$ be a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathbb{P})$. For $1 \leq i \leq d$ define a map $\phi_i : L^\infty(\Omega) \to \mathbb{C}$ by*

$$\phi_i(f) \quad = \quad \int_\Omega f(\omega)\overline{\gamma_i}(\omega) d\mathbb{P}(\omega) , \quad \forall\, f \in L^\infty(\Omega) ,$$

*and let $E : L^\infty(\Omega) \to \mathbb{C}^d$ be defined by*

$$E(f) \quad = \quad (\phi_1(f), \ldots, \phi_d(f)) , \quad \forall\, f \in L^\infty(\Omega) .$$

*Furthermore, let $q : L^\infty(\Omega) \to L^\infty(\Omega)/Ker(E)$ denote the quotient map. Then, for any positive integer $n$ and any $X \in M_n(L^\infty(\Omega))$,*

$$\big|\big|\big|\{x_i\}_{i=1}^d\big|\big|\big|_{M_n(\mathbb{C}^d)} \quad \leq \|(Id_n \otimes q)(X)\|_{M_n(L^\infty(\Omega)/Ker(E))} \leq \sqrt{2}\,\big|\big|\big|\{x_i\}_{i=1}^d\big|\big|\big|_{M_n(\mathbb{C}^d)} , \tag{2.2}$$

*where $x_i = (Id_n \otimes \phi_i)(X)$, $\forall 1 \leq i \leq d$, and*

$$\big|\big|\big|\{x_i\}_{i=1}^d\big|\big|\big|_{M_n(\mathbb{C}^d)} : \quad = \quad \max\left\{ \left\| \sum_{i=1}^d x_i^* x_i \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^d x_i x_i^* \right\|^{\frac{1}{2}} \right\} . \tag{2.3}$$

Note that $\mathbb{C}^d$ equipped with the sequence of matrix norms $\{|||\cdot|||_{M_n(\mathbb{C})}, n \in \mathbb{N}\})$ is an operator space.

*Proof.* Let $n \in \mathbb{N}$. We first prove the left hand side inequality in (2.2). For this, we need the following

**Lemma 2.3.** *Let $X \in M_n(L^\infty(\Omega))$, and set $x_i := (Id_n \otimes \phi_i)(X)$, $\forall 1 \leq i \leq d$. Then*

$$\|X\|_{M_n(L^\infty(\Omega))} \quad \geq \quad \max\left\{ \left\| \sum_{i=1}^d x_i^* x_i \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^d x_i x_i^* \right\|^{\frac{1}{2}} \right\} . \tag{2.4}$$

*Proof.* Since $X \in M_n(\mathbb{C}) \otimes L^\infty(\Omega)$ (algebraic tensor product), we can write

$$X = \sum_{k=1}^r y_k \otimes f_k ,$$

for some $y_k \in M_n(\mathbb{C})$, $f_k \in L^\infty(\Omega)$, $1 \leq k \leq r$.

Let $K = \mathrm{Span}\{\gamma_1, \ldots, \gamma_d, f_1, \ldots, f_r\} \subset L^2(\Omega)$. Choose an orthonormal basis $\{g_i\}_{i=1}^s$ for $K$ such that

$$g_i = \gamma_i , \quad 1 \leq i \leq d . \tag{2.5}$$

Then $X = \sum\limits_{i=1}^s z_i \otimes g_i$, for some $z_i \in M_n(\mathbb{C})$, $1 \leq i \leq s$. Note that for $1 \leq i \leq d$, we have

$$x_i = (\mathrm{Id}_n \otimes \phi_i)\left( \sum_{j=1}^s z_j \otimes g_j \right) = \sum_{j=1}^s z_i \phi_i(g_j) = z_i ,$$

because by (2.5) it follows that $\phi_i(g_j) = \langle g_j, \gamma_i \rangle_{L^2(\Omega)} = \langle g_j, g_i \rangle_{L^2(\Omega)} = \delta_{ij}$ , for all $1 \le j \le s$ .

Denote by $S(M_n(\mathbb{C}))$ the state space of $M_n(\mathbb{C})$ . Then, for $\omega \in S(M_n(\mathbb{C}))$ ,

$$\begin{aligned} \|X\|^2_{M_n(L^\infty(\Omega))} &\ge (\omega \otimes \mathbb{E})(X^*X) \\ &= (\omega \otimes \mathbb{E}) \left( \sum_{i,j=1}^{s} z_i^* z_j \otimes \bar{g}_i g_j \right) \\ &= \omega \left( \sum_{i=1}^{s} z_i^* z_i \right) \ge \omega \left( \sum_{i=1}^{d} z_i^* z_i \right) = \omega \left( \sum_{i=1}^{d} x_i^* x_i \right) . \end{aligned}$$

Take supremum over all $\omega \in S(M_n(\mathbb{C}))$ to obtain

$$\|X\|^2_{M_n(L^\infty(\Omega))} \ge \left\| \sum_{i=1}^{d} x_i^* x_i \right\| . \tag{2.6}$$

Since $\|X\|^2_{M_n(L^\infty(\Omega))} = \|XX^*\|_{M_n(L^\infty(\Omega))}$ , a similar argument shows that also

$$\|X\|^2_{M_n(L^\infty(\Omega))} \ge \left\| \sum_{i=1}^{d} x_i x_i^* \right\| . \tag{2.7}$$

This proves the lemma. □

**Remark 2.4.** As a consequence of this lemma, we deduce that for all $X \in M_n(L^\infty(\Omega))$ we have

$$\left|\left|\left| \{x_i\}_{i=1}^d \right|\right|\right|_{M_n(\mathbb{C}^d)} \le \|(\mathrm{Id}_n \otimes q)(X)\|_{M_n(L^\infty(\Omega)/\mathrm{Ker}(E))} , \tag{2.8}$$

i.e., the left hand side inequality in (2.2) holds. Indeed, for any $Y \in M_n(\mathrm{Ker}(E))$ we infer by (2.4) that

$$\|X+Y\|_{M_n(L^\infty(\Omega))} \ge |||(\mathrm{Id}_n \otimes E)(X+Y)|||_{M_n(\mathbb{C}^d)} = |||(\mathrm{Id}_n \otimes E)(X)|||_{M_n(\mathbb{C}^d)} = \left|\left|\left| \{x_i\}_{i=1}^d \right|\right|\right|_{M_n(\mathbb{C}^d)} .$$

By taking infimum over all $Y \in M_n(\mathrm{Ker}(E))$, inequality (2.8) follows by the definition of the quotient operator space norm.

It remains to prove the right hand side inequality in (2.2) . For this, let $y_1, \ldots, y_d \in M_n(\mathbb{C})$ and set

$$Y := \sum_{i=1}^{d} y_i \otimes \gamma_i \in M_n(L^4(\Omega)) .$$

We will first compute $(\mathrm{Id}_n \otimes \mathbb{E})(Y^*Y)$ , $(\mathrm{Id}_n \otimes \mathbb{E})(YY^*)$ , $(\mathrm{Id}_n \otimes \mathbb{E})((Y^*Y)^2)$ and $(\mathrm{Id}_n \otimes \mathbb{E})((YY^*)^2)$ .

Since $\mathbb{E}(\overline{\gamma_i}\gamma_j) = \delta_{ij}$ , for all $1 \le i, j \le d$ , we immediately get

$$(\mathrm{Id}_n \otimes \mathbb{E})(Y^*Y) = \sum_{i=1}^{d} y_i^* y_i , \qquad (\mathrm{Id}_n \otimes \mathbb{E})(YY^*) = \sum_{i=1}^{d} y_i y_i^* . \tag{2.9}$$

It is easily checked that the vectors $f_{ij} := \overline{\gamma_i}\gamma_j - \delta_{ij}1$ , $1 \le i, j \le d$ , together with the constant function 1 form an orthonormal set with respect to the usual $L_2(\Omega)$-inner product. We then obtain the expansion

$$Y^*Y = \sum_{i,j=1}^{d} y_i^* y_j \otimes f_{ij} + \sum_{i=1}^{d} y_i^* y_i \otimes 1 ,$$

from which we infer that

$$(2.10)\qquad (\mathrm{Id}_n \otimes \mathbb{E})((Y^*Y)^2) \;=\; \sum_{i=1}^{d} y_i^* \left(\sum_{j=1}^{d} y_j y_j^*\right) y_i + \left(\sum_{i=1}^{d} y_i^* y_i\right)^2 .$$

A similar argument shows that

$$(2.11)\qquad (\mathrm{Id}_n \otimes \mathbb{E})((YY^*)^2) \;=\; \sum_{i=1}^{d} y_i \left(\sum_{j=1}^{d} y_j^* y_j\right) y_i^* + \left(\sum_{i=1}^{d} y_i y_i^*\right)^2 .$$

By (2.10) , (2.11) and (2.9) we then obtain the following inequalities

$$(2.12)\qquad (\mathrm{Id}_n \otimes \mathbb{E})((Y^*Y)^2) \;\leq\; \left(\left\|\sum_{i=1}^{d} y_i^* y_i\right\| + \left\|\sum_{i=1}^{d} y_i y_i^*\right\|\right) (\mathrm{Id}_n \otimes \mathbb{E})(Y^*Y)\,,$$

$$(2.13)\qquad (\mathrm{Id}_n \otimes \mathbb{E})((YY^*)^2) \;\leq\; \left(\left\|\sum_{i=1}^{d} y_i^* y_i\right\| + \left\|\sum_{i=1}^{d} y_i y_i^*\right\|\right) (\mathrm{Id}_n \otimes \mathbb{E})(YY^*)\,.$$

The crucial point in proving the right hand side inequality in (2.2) is to show the following

**Lemma 2.5.** *Let $x_1, \dots, x_d \in M_n(\mathbb{C})$ . Then there exists $X \in M_n(L^\infty(\Omega))$ such that*

$$(Id_n \otimes E)(X) = \sum_{i=1}^{d} x_i \otimes e_i\,,$$

*where $\{e_i\}_{1\leq i\leq d}$ is the canonical unit vector basis in $\mathbb{C}^d$ , and*

$$\|X\|_{M_n(L^\infty(\Omega))} \;\leq\; \sqrt{2}\max\left\{\left\|\sum_{i=1}^{d} x_i^* x_i\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^{d} x_i x_i^*\right\|^{\frac{1}{2}}\right\}.$$

We first prove the following lemma:

**Lemma 2.6.** *If $y_1, \dots, y_d \in M_n(\mathbb{C})$ and*

$$(2.14)\qquad \max\left\{\left\|\sum_{i=1}^{d} y_i^* y_i\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^{d} y_i y_i^*\right\|^{\frac{1}{2}}\right\} \;=\; 1\,,$$

*then there exists $Z \in M_n(L^\infty(\Omega))$ such that*

$$\|Z\|_{M_n(L^\infty(\Omega))} \;\leq\; \frac{1}{\sqrt{2}}$$

*and, moreover, when $z_1, \dots, z_d$ are defined by $(Id_n \otimes E)(Z) = \sum_{i=1}^{d} z_i \otimes e_i$ , then*

$$\max\left\{\left\|\sum_{i=1}^{d}(y_i - z_i)^*(y_i - z_i)\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^{d}(y_i - z_i)(y_i - z_i)^*\right\|^{\frac{1}{2}}\right\} \;\leq\; \frac{1}{2}\,.$$

*Proof.* Set

$$Y \;=\; \sum_{i=1}^{d} y_i \otimes \gamma_i \in M_n(L^4(\Omega))\,.$$

Let $\widetilde{E} : L^4(\Omega) \to \mathbb{C}^d$ denote the natural extension of $E$ to $L^4(\Omega)$. Then

$$(\mathrm{Id}_n \otimes \widetilde{E})(Y) = \sum_{i=1}^{d} y_i \otimes \widetilde{E}(\gamma_i) = \sum_{i=1}^{d} y_i \otimes e_i \,.$$

Now let $C > 0$ and define $F_C : \mathbb{R} \to \mathbb{R}$ by

$$F_C(t) \quad = \quad \begin{cases} -C & \text{if} \quad t < -C \\ t & \text{if} \quad -C \leq t \leq C \\ C & \text{if} \quad t > C \end{cases} \tag{2.15}$$

Use functional calculus to define $Z \in M_n(L^\infty(\Omega))$ by

$$\begin{pmatrix} 0 & Z^* \\ Z & 0 \end{pmatrix} \quad = \quad F_C \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix} . \tag{2.16}$$

Note that this implies that $\|Z\|_{M_n(L^\infty(\Omega))} \leq C$. Further, set

$$G_C(t) \quad = \quad t - F_C(t) \,, \quad \forall t \in \mathbb{R} \,.$$

We then have

$$\begin{pmatrix} 0 & (Y-Z)^* \\ (Y-Z) & 0 \end{pmatrix} = G_C \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix}$$

and thus

$$\begin{pmatrix} (Y-Z)^*(Y-Z) & 0 \\ 0 & (Y-Z)(Y-Z)^* \end{pmatrix} = \left( G_C \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix} \right)^2 . \tag{2.17}$$

A simple calculation shows that

$$|G_C(t)| \quad \leq \quad \frac{1}{4C} t^2 \,, \quad \forall t \in \mathbb{R} \,. \tag{2.18}$$

By functional calculus it follows that

$$\left( G_C \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix} \right)^2 \quad \leq \quad \frac{1}{16C^2} \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix}^4 = \frac{1}{16C^2} \begin{pmatrix} (Y^*Y)^2 & 0 \\ 0 & (YY^*)^2 \end{pmatrix} .$$

Hence, by (2.17) and (2.18) we infer that

$$(Y-Z)^*(Y-Z) \quad \leq \quad \frac{1}{16C^2} (Y^*Y)^2 \,, \tag{2.19}$$

$$(Y-Z)(Y-Z)^* \quad \leq \quad \frac{1}{16C^2} (YY^*)^2 \,. \tag{2.20}$$

By letting $z_i = (\mathrm{Id}_n \otimes \phi_i)(Z)$, $1 \leq i \leq d$, we then have

$$(\mathrm{Id}_n \otimes E)(Z) = \sum_{i=1}^{d} z_i \otimes e_i \,,$$

and hence $(\mathrm{Id}_n \otimes \widetilde{E})(Y - Z) = \sum_{i=1}^{d} (y_i - z_i) \otimes e_i$.

By (2.19) , (2.12) and (2.14) we then obtain the estimates

$$\begin{aligned}
\sum_{i=1}^{d}(y_i - z_i)^*(y_i - z_i) &\leq (\mathrm{Id}_n \otimes \mathbb{E})((Y-Z)^*(Y-Z)) \\
&\leq \frac{1}{16C^2}(\mathrm{Id}_n \otimes \mathbb{E})((Y^*Y)^2) \\
&\leq \frac{1}{16C^2}\left(\left\|\sum_{i=1}^{d} y_i^* y_i\right\| + \left\|\sum_{i=1}^{d} y_i y_i^*\right\|\right)(\mathrm{Id}_n \otimes \mathbb{E})(Y^*Y) \\
&\leq \frac{2}{16C^2}(\mathrm{Id}_n \otimes \mathbb{E})(Y^*Y) \\
&= \frac{1}{8C^2}\sum_{i=1}^{d} y_i^* y_i \,.
\end{aligned}$$

It follows that

$$\left\|\sum_{i=1}^{d}(y_i - z_i)^*(y_i - z_i)\right\| \leq \frac{1}{8C^2}\left\|\sum_{i=1}^{d} y_i^* y_i\right\| \leq \frac{1}{8C^2}\,.$$

Similarly, we also get

$$\left\|\sum_{i=1}^{d}(y_i - z_i)(y_i - z_i)^*\right\| \quad \leq \quad \frac{1}{8C^2}\,.$$

Hence,

$$\max\left\{\left\|\sum_{i=1}^{d}(y_i - z_i)^*(y_i - z_i)\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^{d}(y_i - z_i)(y_i - z_i)^*\right\|^{\frac{1}{2}}\right\} \quad \leq \quad \frac{1}{\sqrt{8}C}\,.$$

Now take $C = \frac{1}{\sqrt{2}}$ to get the conclusion. □

We also need the following result:

**Lemma 2.7.** *Let $V$ and $W$ be Banach spaces. Consider $T : V \to W$ a bounded linear map. Further, let $\phi : W \to V$ be a non-linear map such that, for some $C > 0$ and some $0 < \delta < 1$, we have*

$$\|\phi(w)\| \quad \leq \quad C\|w\|\,, \tag{2.21}$$

$$\|w - (T \circ \phi)(w)\| \quad \leq \quad \delta\|w\|\,, \quad \forall w \in W\,. \tag{2.22}$$

*Then there exists a non-linear map $\psi : W \to V$ such that $T \circ \psi = Id_W$ and, moreover,*

$$\|\psi(w)\| \quad \leq \quad \frac{C}{1-\delta}\|w\|\,, \quad \forall w \in W\,.$$

*Proof.* Let $w \in W$. Set $w_0 = w$ and define recursively

$$w_n \quad = \quad w_{n-1} - (T \circ \phi)(w_{n-1})\,, \quad \forall n \geq 1\,.$$

Then, by (2.22) we have for all $n \geq 0$

$$\|w_{n+1}\| \leq \delta\|w_n\| \leq \ldots \leq \delta^{n+1}\|w_0\|\,. \tag{2.23}$$

Also, we deduce that

$$w = w_0 = (T\circ\phi)(w_0) + w_1 = (T\circ\phi)(w_0) + (T\circ\phi)(w_1) + w_2 = \ldots = \sum_{j=0}^{n}(T\circ\phi)(w_j) + w_{n+1}\,, \quad n \geq 0\,.$$

By (2.23) it follows that $w_n \to 0$ as $n \to \infty$, and therefore

$$w = \sum_{j=0}^{\infty}(T\circ\phi)(w_j) = T\left(\sum_{j=0}^{\infty}\phi(w_j)\right)\,. \tag{2.24}$$

Define

$$\psi(w) := \sum_{j=0}^{\infty}\phi(w_j)\,, \quad \forall w \in W\,.$$

By (2.24) it follows that $T(\psi(w)) = w$, for all $w \in W$. Moreover, by (2.21) and (2.23) we obtain that

$$\|\psi(w)\| = \left\|\sum_{j=0}^{\infty}\phi(w_j)\right\| \leq C\sum_{j=0}^{\infty}\|w_j\| \leq \frac{C}{1-\delta}\|w\|\,, \quad \forall w \in W\,,$$

which completes the proof. □

Now we are ready to prove Lemma 2.5. Indeed, Lemma 2.6 shows that if

$$y = \sum_{i=1}^{d} y_i \otimes e_i \in M_n(\mathbb{C}^d)$$

satisfies $|||y|||_{M_n(\mathbb{C}^d)} = 1$, then there exists $Z \in M_n(L^\infty(\Omega))$ so that $\|Z\|_{M_n(L^\infty(\Omega))} \leq \frac{1}{\sqrt{2}}$ and $|||(\mathrm{Id}_n \otimes E)(Z) - y|||_{M_n(\mathbb{C}^d)} \leq \frac{1}{2}$. By homogeneity we infer that for all $y \in M_n(\mathbb{C}^d)$ there exists $Z \in M_n(L^\infty(\Omega))$ so that $\|Z\|_{M_n(L^\infty(\Omega))} \leq \frac{1}{\sqrt{2}}|||y|||_{M_n(\mathbb{C}^d)}$, and, moreover, $|||(\mathrm{Id}_n \otimes E)(Z) - y|||_{M_n(\mathbb{C}^d)} \leq \frac{1}{2}|||y|||_{M_n(\mathbb{C}^d)}$. Apply now Lemma 2.7 with $V = M_n(L^\infty(\Omega))$, $W = M_n(\mathbb{C}^d)$, $T = \mathrm{Id}_n \otimes E$, the map $\phi : W \to V$ be defined by $\phi(y) = Z$, $\forall y \in W$, $C = \frac{1}{\sqrt{2}}$ and $\delta = \frac{1}{2}$. We deduce that for all $x_1, \ldots, x_d \in M_n(\mathbb{C})$, there exists $X \in M_n(L^\infty(\Omega))$ such that $(\mathrm{Id}_n \otimes E)(X) = \sum\limits_{i=1}^{d} x_i \otimes e_i$ and

$$\|X\|_{M_n(L^\infty(\Omega))} \leq \frac{C}{1-\delta}\left|\left|\left|\sum_{i=1}^{d} x_i \otimes e_i\right|\right|\right|_{M_n(\mathbb{C}^d)} = \sqrt{2}|||\{x_i\}_{i=1}^{d}|||_{M_n(\mathbb{C}^d)}\,, \tag{2.25}$$

which completes the proof of Lemma 2.5. Note that, since the norm on $M_n\left(L^\infty(\Omega)/\mathrm{Ker}(E)\right)$ is the quotient space norm on the space $M_n(L^\infty(\Omega))/M_n(\mathrm{Ker}(E))$, it follows by (2.25) that

$$\|(\mathrm{Id}_n \otimes q)(X)\|_{M_n(L^\infty(\Omega)/\mathrm{Ker}(E))} \leq \sqrt{2}|||\{x_i\}_{i=1}^{d}|||_{M_n(\mathbb{C}^d)}\,.$$

Therefore, by Lemmas 2.3 and 2.5 and Remark 2.4, there is a linear bijection $\widehat{E} : L^\infty(\Omega)/\mathrm{Ker}(E) \to \mathbb{C}^d$ such that

$$\widehat{E}(q(s_i)) \;=\; e_i\,, \quad \forall 1 \leq i \leq d\,,$$

where $s_i = \sqrt{\frac{4}{\pi}}\mathrm{sgn}(\gamma_i)$ , for $1 \le i \le d$ . Note that $s_i \in L^\infty(\Omega)$ and $\mathbb{E}(s_i\overline{\gamma_i}) = \delta_{ij}$ , $\forall 1 \le j \le d$ , so that $E(s_i) = e_i$ , $\forall 1 \le i \le d$ . For every positive integer $n$ , the following diagram is commutative,

$$\begin{array}{ccc} M_n(L^\infty(\Omega)) & \xrightarrow{\mathrm{Id}_n \otimes E} & M_n(\mathbb{C}^d) \\ & {\scriptstyle \mathrm{Id}_n \otimes q} \searrow \quad \nearrow {\scriptstyle \mathrm{Id}_n \otimes \widehat{E}} & \\ & M_n(L^\infty(\Omega)/\mathrm{Ker}(E)) & \end{array}$$

and moreover, the inequalities (2.2) hold. The proof of Proposition 2.2 is now complete. □

**Remark 2.8.** We should mention that, by the same proof with only minor modifications, Theorem 2.1 remains valid if we replace the sequence $\{\gamma_n\}_{n\ge1}$ of independent standard complex Gaussian random variables by a sequence $\{s_n\}_{n\ge1}$ of independent Steinhauss random variables (that is, a sequence of independent random variables which are uniformly distributed over the unit circle), or by the sequence $\{e_n\}_{n\ge1}$ given by $e_n(t) = e^{i2^n t}$ , $0 \le t \le 2\pi$ . Indeed, the only essential change in the proof is that the formulas (2.10) and (2.11) must be modified, because in the case of the sequences $\{s_n\}_{n\ge1}$ and $\{e_n\}_{n\ge1}$ we still have that $\{\bar{s}_i s_j ; 1 \le i,j \le d\} \cup \{1\}$ and, respectively, $\{\bar{e}_i e_j ; 1 \le i,j \le d\} \cup \{1\}$ form orthonormal sets, but in contrast to the case of the Gaussians $\{\gamma_n\}_{n\ge1}$ , one has

$$\bar{s}_j s_j = \bar{e}_j e_j = 1\,, \quad j \ge 1\,.$$

Therefore, the diagonal terms (corresponding to $i = j$) in the right hand sides of (2.10) and (2.11) should be removed from the double sums. However, since the diagonal terms are all positive, it follows that (2.12) and (2.13) remain valid in the case of the sequences $\{s_n\}_{n\ge1}$ and $\{e_n\}_{n\ge1}$ , as well.

We now discuss estimates for best constants in the noncommutative Khintchine inequalities ($p = 1$).

**Theorem 2.9.** *Denote by $c_1$ , $c_2$ the best constants in the inequalities*

$$c_1 |||\{x_i\}_{i=1}^d|||^* \quad \le \quad \left\| \sum_{i=1}^d x_i \otimes \gamma_i \right\|_{L^1(\Omega; S_1^n)} \le c_2 |||\{x_i\}_{i=1}^d|||^*\,, \tag{2.26}$$

*where $d$ and $n$ are positive integers, $x_1, \dots, x_d \in M_n(\mathbb{C})$, and $\{\gamma_i\}_{i=1}^d$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathbb{P})$ . Then*

$$c_1 = \frac{1}{\sqrt{2}}\,, \qquad c_2 = 1\,.$$

*Proof.* Let $m$ be a positive integer. Let $d = 2m+1$ and set $n = \binom{2m+1}{m}$ . Then, by Theorem 1.1 in [8], there exist partial isometries $a_1, \dots, a_d \in \mathcal{B}(H)$ , where $H$ is a Hilbert space of $\dim(H) = n$ , such that

$$\tau(a_i^* a_i) \quad = \quad \frac{m+1}{2m+1}\,, \quad \forall 1 \le i \le d\,, \tag{2.27}$$

where $\tau$ denotes the normalized trace on $\mathcal{B}(H)$ , satisfying, moreover,

$$\sum_{i=1}^d a_i^* a_i = \sum_{i=1}^d a_i a_i^* = (m+1)I\,, \tag{2.28}$$

where $I$ denotes the identity operator on $H$. First, we claim that

$$|||\{a_i\}_{i=1}^d|||_{M_n(\mathbb{C}^d)} = \sqrt{m+1}\,, \tag{2.29}$$

$$|||\{a_i\}_{i=1}^d|||^* = n\sqrt{m+1}\,. \tag{2.30}$$

Indeed, (2.29) follows immediately from the definition of the norm $|||\cdot|||_{M_n(\mathbb{C}^d)}$ and relation (2.28), while the equation (2.30) follows from the following estimates

$$\begin{aligned} |||\{a_i\}_{i=1}^d|||^* &= \sup\left\{\left|\mathrm{Tr}\left(\sum_{i=1}^d a_i b_i\right)\right| ; |||\{b_i\}_{i=1}^d|||_{M_n(\mathbb{C}^d)} \le 1\right\} \\ &\ge \left|\mathrm{Tr}\left(\sum_{i=1}^d a_i \left(\frac{a_i^*}{\sqrt{m+1}}\right)\right)\right| \\ &= \frac{1}{\sqrt{m+1}}\mathrm{Tr}\left(\sum_{i=1}^d a_i a_i^*\right) \\ &= \frac{1}{\sqrt{m+1}}\mathrm{Tr}((m+1)I) = n\sqrt{m+1}\,, \end{aligned}$$

respectively,

$$|||\{a_i\}_{i=1}^d|||^* \le \mathrm{Tr}\left(\left(\sum_{i=1}^d a_i^* a_i\right)^{1/2}\right) = \mathrm{Tr}(\sqrt{m+1}I) = n\sqrt{m+1}\,.$$

It was proved in [8] that $a_1, \ldots, a_d$ have the additional property that $\forall \beta_1, \ldots, \beta_d \in \mathbb{C}$ with $\sum_{i=1}^d |\beta_i|^2 = 1$, the operator $y := \sum_{i=1}^d \beta_i a_i \in \mathcal{B}(H)$ is also a partial isometry with $\tau(y^*y) = \frac{m+1}{2m+1}$. This implies that for all $\omega \in \Omega$, the operator

$$y_\omega := \sum_{i=1}^d \frac{\gamma_i(\omega)}{\left(\sum_{i=1}^d |\gamma_i(\omega)|^2\right)^{\frac{1}{2}}}\, a_i \in \mathcal{B}(H)$$

is a partial isometry with $\tau(y_\omega^* y_\omega) = \frac{m+1}{2m+1}$, and we deduce that

$$\begin{aligned} c_1|||\{a_i\}_{i=1}^d|||^* &\le \int_\Omega \left\|\sum_{i=1}^d \gamma_i(\omega) a_i\right\|_{L^1(M_n(\mathbb{C}),\mathrm{Tr})} d\mathbb{P}(\omega) \\ &= \int_\Omega \left(\sum_{i=1}^d |\gamma_i(\omega)|^2\right)^{\frac{1}{2}} \|y_\omega\|_{L^1(M_n(\mathbb{C}),\mathrm{Tr})} d\mathbb{P}(\omega) \\ &= n\cdot\frac{m+1}{2m+1}\int_\Omega \left(\sum_{i=1}^d |\gamma_i(\omega)|^2\right)^{\frac{1}{2}} d\mathbb{P}(\omega)\,, \end{aligned} \tag{2.31}$$

wherein we have used the fact that $y_\omega^* y_\omega$ is a projection satisfying $\tau(|y_\omega|) = \frac{m+1}{2m+1}$, for all $\omega \in \Omega$. A standard computation yields the formula

$$\int_\Omega \left(\sum_{i=1}^d |\gamma_i(\omega)|^2\right)^{\frac{1}{2}} d\mathbb{P}(\omega) = \frac{\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(d)}\,. \tag{2.32}$$

Indeed, since the distribution of $|\gamma_i|^2$ is $\Gamma(1,1)$ , $1 \leq i \leq d$ , it follows by independence that the distribution of $\sum\limits_{i=1}^{d} |\gamma_i|^2$ is $\Gamma(d,1)$, whose density is $\frac{1}{\Gamma(d)} x^{d-1} e^{-x}$ , $x > 0$ . Since $\int_0^\infty x^{\frac{1}{2}} x^{d-1} e^{-x} dx = \Gamma(d + \frac{1}{2})$ , formula (2.32) follows. Combining now (2.31) with (2.30) and (2.32) we deduce that

$$c_1 \leq \frac{1}{\sqrt{m+1}} \left( \frac{m+1}{2m+1} \right) \frac{\Gamma\left(d + \frac{1}{2}\right)}{\Gamma(d)} \leq \frac{\sqrt{m+1}}{2m+1} \sqrt{2m+1} = \frac{\sqrt{m+1}}{\sqrt{2m+1}} \,, \tag{2.33}$$

wherein we have used the inequality

$$\Gamma\left(k + \frac{1}{2}\right) < \sqrt{k}\Gamma(k) \,, \quad \forall k \in \mathbb{N} \,,$$

applied for $k = d = 2m+1$ . Since $m$ was arbitrarily chosen and $\lim\limits_{m\to\infty} \frac{\sqrt{m+1}}{\sqrt{2m+1}} = \frac{1}{\sqrt{2}}$ , we deduce by (2.33) that $c_1 \leq \frac{1}{\sqrt{2}}$ . By Theorem 2.1 we know that $c_1 \geq \frac{1}{\sqrt{2}}$ , hence we conclude that $c_1 = \frac{1}{\sqrt{2}}$ .

To estimate $c_2$ , let $d$ be a positive integer. Set $n = d$ . For all $1 \leq i \leq d$, set $x_i := e_{i1} \in M_d(\mathbb{C})$ . We then have

$$\left\| \sum_{i=1}^{d} x_i \otimes \gamma_i \right\|_{L^1(\Omega; S_1^n)} = \int_\Omega \left\| \sum_{i=1}^{d} \gamma_i(\omega) x_i \right\|_{L^1(M_n(\mathbb{C})\,, \mathrm{Tr})} d\mathbb{P}(\omega) = \int_\Omega \left( \sum_{i=1}^{d} |\gamma_i(\omega)|^2 \right)^{\frac{1}{2}} d\mathbb{P}(\omega) \,.$$

Note also that

$$|||\{x_i\}_{i=1}^d|||^* \geq \mathrm{Tr}\left( \left( \sum_{i=1}^{d} x_i^* x_i \right)^{\frac{1}{2}} \right) = \mathrm{Tr}(\sqrt{d}\; e_{11}) = 1 \,.$$

Then, using (2.32), together with the fact that $\lim\limits_{d\to\infty} \frac{1}{\sqrt{d}} \frac{\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(d)} = 1$ , we infer by (2.26) that $c_2 \geq 1$ . Since by Theorem 2.1 we get $c_2 \leq 1$ , we conclude that $c_2 = 1$ , and the proof is complete. □

**Remark 2.10.** If we replace the sequence of independent standard complex-valued Gaussian random variables $\{\gamma_n\}_{n\geq 1}$ by a sequence of independent Steinhauss random variables $\{s_n\}_{n\geq 1}$ or by the sequence $\{e^{i2^n t}\}_{n\geq 1}$ , and denote by $c_1, c_2$ the best constants in the corresponding inequalities (2.26), the same argument will give $c_1 = \frac{1}{\sqrt{2}}$ . Also, $c_2 = 1$ in both cases, as a consequence of Remark 2.8 and the fact that $\|s_1\|_{L^1(\mathbb{T})} = 1 = \|e^{i2t}\|_{L^1(\mathbb{T})}$ , where $\mathbb{T}$ is the unit circle with normalized Lebesgue measure $dt/2\pi$ .

II. The Rademacher case

Let $\{r_n\}_{n\geq 1}$ be a sequence of Rademacher functions on $[0,1]$ . Probabilistically, one can think of $\{r_n\}_{n\geq 1}$ as being a sequence of independent, identically distributed random variables on $[0,1]$, each taking value 1 with probability $\frac{1}{2}$, respectively, value $-1$ with probability $\frac{1}{2}$. It is easily seen that $\mathbb{E}(r_n) = 0$ and $\mathbb{E}(r_n r_m) = \delta_{nm}$ , for all $n, m \in \mathbb{N}$ .

**Theorem 2.11.** *Let $d$ and $n$ be positive integers and consider $x_1, \ldots, x_d \in M_n(\mathbb{C})$ . Then the following inequalities hold*

$$\frac{1}{\sqrt{3}} |||\{x_i\}_{i=1}^d|||^* \quad \leq \quad \left\| \sum_{i=1}^{d} x_i \otimes r_i \right\|_{L^1([0,1]; S_1^n)} \leq |||\{x_i\}_{i=1}^d|||^* \,. \tag{2.34}$$

As in the case of complex Gaussian random variables, we prove the dual version of Theorem 2.11, namely,

**Proposition 2.12.** *Let $d$ be a positive integer, and let $\{r_i\}_{1\leq i\leq d}$ be a sequence of Rademacher functions on $[0,1]$. For $1\leq i\leq d$ define $\phi_i : L^\infty([0,1])\to\mathbb{C}$ by*

$$\phi_i(f) \;=\; \int_0^1 f(t)r_i(t)dt\,, \quad \forall\, f\in L^\infty([0,1])\,,$$

*and let $E : L^\infty([0,1])\to\mathbb{C}^d$ be defined by*

$$E(f) \;=\; (\phi_1(f)\,,\ldots,\phi_d(f))\,, \quad \forall\, f\in L^\infty([0,1])\,.$$

*Furthermore, let $q : L^\infty([0,1]) \to L^\infty([0,1])/Ker(E)$ denote the quotient map. Then, for any positive integer $n$ and any $X\in M_n(L^\infty([0,1]))$,*

$$\text{(2.35)}\qquad \left|\left|\left|\{x_i\}_{i=1}^d\right|\right|\right|_{M_n(\mathbb{C}^d)} \leq \|(Id_n\otimes q)(X)\|_{M_n(L^\infty([0,1])/Ker(E))} \leq \sqrt{3}\left|\left|\left|\{x_i\}_{i=1}^d\right|\right|\right|_{M_n(\mathbb{C}^d)}\,,$$

*where $x_i = (Id_n\otimes\phi_i)(X)$, $\forall 1\leq i\leq d$.*

*Proof.* Let $n$ be a positive integer. The proof of the left hand side inequality in (2.35) is the same as in the complex Gaussian case. For the right hand side inequality we follow the same argument, but with appropriate modifications, which we indicate below.

Let $y_1\,,\ldots\,,y_d\in M_n(\mathbb{C})$ and set

$$Y := \sum_{i=1}^d y_i\otimes r_i \in M_n(L^\infty([0,1])\,.$$

As before we will estimate $(\mathrm{Id}_n\otimes\mathbb{E})(Y^*Y)$, $(\mathrm{Id}_n\otimes\mathbb{E})(YY^*)$, $(\mathrm{Id}_n\otimes\mathbb{E})((Y^*Y)^2)$ and $(\mathrm{Id}_n\otimes\mathbb{E})((YY^*)^2)$.

First note that $Y^*Y = \sum\limits_{i,j=1}^d y_i^*y_j\otimes r_ir_j$ and, respectively, $YY^* = \sum\limits_{i,j=1}^d y_iy_j^*\otimes r_ir_j$ to conclude that

$$\text{(2.36)}\qquad (\mathrm{Id}_n\otimes\mathbb{E})(Y^*Y) \;=\; \sum_{i=1}^d y_i^*y_i$$

$$\text{(2.37)}\qquad (\mathrm{Id}_n\otimes\mathbb{E})(YY^*) \;=\; \sum_{i=1}^d y_iy_i^*\,.$$

Furthermore, note that $(\mathrm{Id}_n\otimes\mathbb{E})((Y^*Y)^2) = \sum\limits_{i,j,k,l=1}^d y_i^*y_jy_k^*y_l\mathbb{E}(r_ir_jr_kr_l)$. Since $\mathbb{E}(r_ir_jr_kr_l)\in\{0,1\}$ with $\mathbb{E}(r_ir_jr_kr_l)=1$ if and only if $i=j=k=l$, or $i=j\neq k=l$, or $i=k\neq j=l$, or $i=l\neq j=k$, it then follows that

$$\begin{aligned}\text{(2.38)}\qquad (\mathrm{Id}_n\otimes\mathbb{E})((Y^*Y)^2) \;&=\; \sum_{i=1}^d y_i^*y_iy_i^*y_i + \sum_{i\neq j} y_i^*y_iy_j^*y_j + \sum_{i\neq j} y_i^*y_jy_i^*y_j + \sum_{i\neq j} y_i^*y_jy_j^*y_i \\ &=\; \sum_{i,j=1}^d y_i^*y_iy_j^*y_j + \sum_{i\neq j} y_i^*y_jy_i^*y_j + \sum_{i\neq j} y_i^*y_jy_j^*y_i\,.\end{aligned}$$

Note that $\sum\limits_{i,j=1}^d y_i^*y_iy_j^*y_j = \left(\sum\limits_{i=1}^d y_i^*y_i\right)^2$. Further, we have

$$\sum_{i\neq j} y_i^*y_jy_i^*y_j = \sum_{i<j} y_i^*y_jy_i^*y_j + \sum_{i>j} y_i^*y_jy_i^*y_j = \sum_{i<j} y_i^*y_jy_i^*y_j + \sum_{i<j} y_j^*y_iy_j^*y_i = \sum_{i<j}((y_i^*y_j)^2 + (y_j^*y_i)^2)\,.$$

Using the fact that $(y_i^*y_j)^2+(y_j^*y_i)^2\le y_i^*y_jy_j^*y_i+y_j^*y_iy_i^*y_j$ , $1\le i,j\le d$ , it follows that

$$
\begin{aligned}
\sum_{i<j}((y_i^*y_j)^2+(y_j^*y_i)^2) &\le \sum_{i<j}(y_i^*y_jy_j^*y_i+y_j^*y_iy_i^*y_j) \\
&= \sum_{i<j}y_i^*y_jy_j^*y_i+\sum_{i>j}y_i^*y_jy_j^*y_i \\
&= \sum_{i\ne j}y_i^*y_jy_j^*y_i\,.
\end{aligned}
\tag{2.39}
$$

Therefore, we conclude that

$$
\begin{aligned}
(\mathrm{Id}_n\otimes\mathbb{E})((Y^*Y)^2) &\le \left(\sum_{i=1}^d y_i^*y_i\right)^2+2\sum_{i\ne j}y_i^*y_jy_j^*y_i \\
&\le \left(\sum_{i=1}^d y_i^*y_i\right)^2+2\sum_{i,j=1}^d y_i^*y_jy_j^*y_i \\
&= \left(\sum_{i=1}^d y_i^*y_i\right)^2+2\sum_{i=1}^d y_i^*\left(\sum_{j=1}^d y_jy_j^*\right)y_i
\end{aligned}
$$

Recalling the definition (2.3), and using (2.36) we now obtain

$$
\begin{aligned}
(\mathrm{Id}_n\otimes\mathbb{E})((Y^*Y)^2) &\le |||\{y_i\}_{i=1}^d|||^2\left(\sum_{i=1}^d y_i^*y_i+2\sum_{i=1}^d y_i^*y_i\right) \\
&= 3|||\{y_i\}_{i=1}^d|||^2(\mathrm{Id}_n\otimes\mathbb{E})(Y^*Y)\,.
\end{aligned}
\tag{2.40}
$$

A similar proof based on (2.37) shows that

$$
(\mathrm{Id}_n\otimes\mathbb{E})((YY^*)^2) \le 3|||\{y_i\}_{i=1}^d|||^2(\mathrm{Id}_n\otimes\mathbb{E})(YY^*)\,. \tag{2.41}
$$

Next we prove the following

**Lemma 2.13.** *Let $x_1,\dots,x_d\in M_n(\mathbb{C})$. Then there exists $X\in M_n(L^\infty([0,1]))$ such that*

$$
(Id_n\otimes E)(X)=\sum_{i=1}^d x_i\otimes e_i\,,
$$

*satisfying, moreover,*

$$
\|X\|_{M_n(L^\infty([0,1]))} \le \sqrt{3}\max\left\{\left\|\sum_{i=1}^d x_i^*x_i\right\|^{\frac12},\left\|\sum_{i=1}^d x_ix_i^*\right\|^{\frac12}\right\}. \tag{2.42}
$$

As in the case of independent standard complex Gaussians, the crucial point in the argument is the following version of Lemma 2.6, whose proof carries over verbatim to this setting, except for choosing $C=\frac{\sqrt{3}}{2}$ .

**Lemma 2.14.** *If $y_1,\dots,y_d\in M_n(\mathbb{C})$ satisfy*

$$
\max\left\{\left\|\sum_{i=1}^d y_i^*y_i\right\|^{\frac12},\left\|\sum_{i=1}^d y_iy_i^*\right\|^{\frac12}\right\} = 1\,, \tag{2.43}
$$

*then there exists* $Z \in M_n(L^\infty([0,1]))$ *such that* $\|Z\|_{M_n(L^\infty([0,1]))} \le \frac{\sqrt{3}}{2}$ *, and, moreover, when* $z_1, \dots, z_d$ *are defined by* $(Id_n \otimes E)(Z) = \sum_{i=1}^d z_i \otimes e_i$ *, then*

$$\max\left\{ \left\| \sum_{i=1}^d (y_i - z_i)^*(y_i - z_i) \right\|^{\frac{1}{2}}, \left\| \sum_{i=1}^d (y_i - z_i)(y_i - z_i)^* \right\|^{\frac{1}{2}} \right\} \quad \le \quad \frac{1}{2}.$$

Hence, for all $y \in M_n(\mathbb{C}^d)$ there is $Z \in M_n(L^\infty([0,1]))$ such that $\|Z\|_{M_n(L^\infty([0,1]))} \le \frac{\sqrt{3}}{2}|||y|||_{M_n(H)}$, and $|||(\mathrm{Id}_n \otimes E)(Z) - y|||_{M_n(\mathbb{C}^d)} \le \frac{1}{2}|||y|||_{M_n(\mathbb{C}^d)}$. An application of Lemma 2.7 with $V = M_n(L^\infty([0,1]))$, $W = M_n(\mathbb{C}^d)$, $T = \mathrm{Id}_n \otimes E$, the map $\phi : W \to V$ be defined by $\phi(y) = Z$, $\forall y \in W$, $C = \frac{\sqrt{3}}{2}$ and $\delta = \frac{1}{2}$ shows that for all $x \in M_n(\mathbb{C}^d)$ there exists $X \in M_n(L^\infty([0,1]))$ such that $(\mathrm{Id}_n \otimes E)(X) = x$ and

$$\|X\|_{M_n(L^\infty([0,1]))} \le \sqrt{3}|||x|||_{M_n(\mathbb{C}^d)}. \tag{2.44}$$

This completes the proof of Lemma 2.13. As explained before, (2.44) implies that

$$\|(\mathrm{Id}_n \otimes q)(X)\|_{M_n(L^\infty([0,1])/\mathrm{Ker}(E))} \le \sqrt{3}\,\left|\left|\left|\{x_i\}_{i=1}^d\right|\right|\right|_{M_n(\mathbb{C}^d)}.$$

We conclude that there exists a linear bijection $\widehat{E} : L^\infty([0,1])/\mathrm{Ker}(E) \to \mathbb{C}^d$ such that $\widehat{E}(q(r_i)) = e_i = E(r_i)$, for all $1 \le i \le d$, and moreover, with respect to the operator space structure of the quotient space $L^\infty([0,1])/\mathrm{Ker}(E)$, the inequalities (2.35) hold. This completes the proof of Proposition 2.12. □

**Remark 2.15.** Let $c_1, c_2$ denote the best constants in the inequalities

$$c_1|||\{x_i\}_{i=1}^d|||^* \quad \le \quad \left\| \sum_{i=1}^d x_i \otimes r_i \right\|_{L^1([0,1];S_1^n)} \le c_2|||\{x_i\}_{i=1}^d|||^*, \tag{2.45}$$

where $d, n$ are positive integers, and $x_1, \dots, x_d \in M_n(\mathbb{C})$. Then the following estimates hold

$$\frac{1}{\sqrt{3}} \le c_1 \le \frac{1}{\sqrt{2}}, \quad c_2 = 1.$$

Indeed, the estimate $c_1 \le \frac{1}{\sqrt{2}}$ is a consequence of Szarek's result (see [20]) that the best constant in the classical Khintchine inequalities for Rademachers is $\frac{1}{\sqrt{2}}$, while the estimate $\frac{1}{\sqrt{3}} \le c_1$ follows by Theorem 2.11, which also shows that $c_2 \le 1$. Since $\mathbb{E}(|r_1|) = 1$, we deduce by taking $d = n = 1$ and $x_1 = 1$ in (2.45) that $c_2 \ge 1$. Hence $c_2 = 1$.

## 3. A noncommutative Khintchine-type inequality for subspaces of $R \oplus C$

Let $H \subseteq R \oplus C$ be a subspace, equipped with the Hilbert space structure induced by the usual direct sum of Hilbert spaces inner product. More precisely, given $\xi \in H$, write $\xi = (\xi_R, \xi_C) \in R \oplus C$; then

$$\langle \xi, \eta \rangle_H \quad = \quad \langle \xi_R, \eta_R \rangle_R + \langle \xi_C, \eta_C \rangle_C, \quad \forall \xi, \eta \in H. \tag{3.1}$$

Consider $R \oplus C$ equipped with the operator space structure of the $l_\infty$-direct sum $R \oplus_\infty C$. Note that the norm induced on $H$ by the inner product $\langle \cdot, \cdot \rangle_H$ is not the same as the one coming from $R \oplus_\infty C$. For all $\xi \in H$, define further

$$U_1(\xi) = \xi_R, \quad U_2(\xi) = \xi_C.$$

Then $U_1 \in \mathcal{B}(H, R)$, respectively $U_2 \in \mathcal{B}(H, C)$ and formula (3.1) becomes

$$\langle \xi, \eta \rangle_H \quad = \quad \langle U_1(\xi), U_1(\eta) \rangle_R + \langle U_2(\xi), U_2(\eta) \rangle_C, \quad \forall \xi, \eta \in H. \tag{3.2}$$

The operator $U : H \to R \oplus C$ defined by $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ is an isometry, where $H$ and $R \oplus C$ are equipped with the above Hilbert space structure. This implies that $U_1^* U_1 + U_2^* U_2 = I$, where $I$ denotes the identity operator on $H$. Let

$$A \quad = \quad U_2^* U_2 \in \mathcal{B}(H)\,. \tag{3.3}$$

Then $0 \leq A \leq I$.

We now discuss the operator space structure of $H$. Let $n$ be a positive integer. Then for all $r \in \mathbb{N}$, all $x_i \in M_n(\mathbb{C})$ and all $\xi_i \in H$, $1 \leq i \leq r$, we have

$$\left\| \sum_{i=1}^{r} x_i \otimes \xi_i \right\|_{M_n(H)} \quad = \quad \max \left\{ \left\| \sum_{i=1}^{r} x_i \otimes U_1 \xi_i \right\|_{M_n(R)}, \left\| \sum_{i=1}^{r} x_i \otimes U_2 \xi_i \right\|_{M_n(C)} \right\}. \tag{3.4}$$

We claim that

$$\left\| \sum_{i=1}^{r} x_i \otimes \xi_i \right\|_{M_n(H)} \quad = \quad \max \left\{ \left\| \sum_{i,j=1}^{r} \langle (I-A)\xi_i, \xi_j \rangle_H x_i x_j^* \right\|^{\frac{1}{2}}, \left\| \sum_{i,j=1}^{r} \langle A\xi_i, \xi_j \rangle_H x_i^* x_j \right\|^{\frac{1}{2}} \right\}. \tag{3.5}$$

Indeed, by the definition of operator space matrix norms on $R$ and $C$ we have

$$\left\| \sum_{i=1}^{r} x_i \otimes U_1 \xi_i \right\|_{M_n(R)} = \left\| \sum_{i,j=1}^{r} x_i x_j^* \langle U_1 \xi_i, U_1 \xi_j \rangle_R \right\|^{\frac{1}{2}} = \left\| \sum_{i,j=1}^{r} x_i x_j^* \langle (I-A)\xi_i, \xi_j \rangle_H \right\|^{\frac{1}{2}},$$

respectively,

$$\left\| \sum_{i=1}^{r} x_i \otimes U_2 \xi_i \right\|_{M_n(C)} = \left\| \sum_{i,j=1}^{r} x_i^* x_j \langle U_2 \xi_i, U_2 \xi_j \rangle_C \right\|^{\frac{1}{2}} = \left\| \sum_{i,j=1}^{r} x_i^* x_j \langle A\xi_i, \xi_j \rangle_H \right\|^{\frac{1}{2}},$$

and the claim is proved.

Let $\mathcal{A}$ be the CAR algebra over the Hilbert space $H$. Recall that $\mathcal{A}$ is a unital $C^*$-algebra (unique up to $*$-isomorphism) with the property that there exists a linear map

$$H \ni f \mapsto a(f) \in \mathcal{A}$$

whose range generates $\mathcal{A}$, satisfying for all $f, g \in H$ the anticommutation relations

$$\begin{aligned} a(f)a(g)^* + a(g)^* a(f) &= \langle f, g \rangle_H I \\ a(f)a(g) + a(g)a(f) &= 0\,. \end{aligned} \tag{3.6}$$

Let $\omega_A$ be the gauge-invariant quasi-free state on $\mathcal{A}$ corresponding to the operator $A$ $(0 \leq A \leq I)$ associated to the subspace $H$ of $R \oplus C$. Recall that a state $\omega$ on $\mathcal{A}$ is called gauge-invariant if it is invariant under the group of gauge transformations $\tau_\theta(a(f)) = a(e^{i\theta} f)$, $\forall \theta \in [0, 2\pi)$. It turns out (see [1] and [2]) that a gauge-invariant quasi-free state $\omega$ on $\mathcal{A}$ is completely determined by one truncated function $\omega_T$. More precisely, a functional $\omega_T(\cdot,\cdot)$ over the monomials in $a^*(f)$ and $a(g)$, $\forall f, g \in H$, which is linear in the first argument and conjugate-linear in the second determines a gauge-invariant quasi-free state $\omega$ on $\mathcal{A}$ if and only if

$$0 \leq \omega_T(a(f)^*, a(f)) \leq \|f\|^2\,, \quad \forall f \in H\,. \tag{3.7}$$

Now, given the operator $0 \le A \le I$, define

$$\omega_T^A(a(f)^*, a(g)) := \langle Ag, f\rangle_H \,.$$

The positivity condition (3.7) is clearly satisfied. Let $\omega_A$ be the gauge-invariant quasi-free state on $\mathcal{A}$ determined by the truncated function $\omega_T^A$. Then for all $n \ge 1$, the $n$-point functions of $\omega_A$ have the form

$$\omega_A(a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)) = \delta_{nm} \det(\langle Ag_i, f_j\rangle_H \,, i, j)\,, \quad \forall f_1, \dots, f_n, g_1, \dots, g_m \in H\,. \tag{3.8}$$

Given $b \in \mathcal{A}$, the map

$$H \ni f \mapsto \omega_A(a(f)b^* + b^* a(f)) \in \mathbb{C}$$

is a bounded linear functional on $\mathcal{A}$. By the Riesz representation theorem, there exists a unique element $E_A(b) \in H$ such that

$$\langle f, E_A(b)\rangle_H \quad = \quad \omega_A(a(f)b^* + b^* a(f))\,, \quad \forall f \in H\,. \tag{3.9}$$

Equivalently,

$$\langle E_A(b), f\rangle_H \quad = \quad \omega_A(ba(f)^* + a(f)^* b)\,, \quad \forall f \in H\,. \tag{3.10}$$

We obtain in this way a bounded linear map $E_A : \mathcal{A} \to H$. By uniqueness in the Riesz representation theorem and the anticommutation relations (3.6) it follows that

$$E_A(a(f)) = f\,, \quad \forall f \in H\,. \tag{3.11}$$

Consider the GNS representation $(\pi_{\omega_A}, H, \xi_{\omega_A})$ associated to $(\mathcal{A}, \omega_A)$. For simplicity of notation, write $\pi_{\omega_A} = \pi_A$ and $\xi_{\omega_A} = \xi_A$ (the cyclic unit vector for the representation). Then for all $f \in H$ and all $b \in \mathcal{A}$,

$$\omega_A(a(f)b^* + b^* a(f)) = \langle \pi_A(a(f)b^* + b^* a(f))\xi_A \,, \xi_A\rangle_H = \langle \{\pi_A(a(f)) \,, \pi_A(b^*)\}\xi_A \,, \xi_A\rangle_H \,,$$

where $\{K, L\} = KL + LK$. Equivalently,

$$\omega_A(ba(f)^* + a(f)^* b) \quad = \quad \langle \{\pi_A(a(f)^*) \,, \pi_A(b)\}\xi_A \,, \xi_A\rangle_H \,, \quad \forall f \in H \,, \forall b \in \mathcal{A}\,.$$

Note that the map

$$\mathcal{A} \ni c \mapsto \langle \{\pi_A(a(f)^*) \,, c\}\xi_A \,, \xi_A\rangle_H \in \mathbb{C}$$

extends to a normal (positive) linear functional on the von Neumann algebra $\overline{\pi_A(\mathcal{A})}^{\mathrm{sot}}$. This implies that $E_A$ extends to a bounded linear map on the von Neumann algebra generated by $\pi_A(\mathcal{A})$ and moreover the range of the dual map $E_A^*$ is contained in the predual of $\overline{\pi_A(\mathcal{A})}^{\mathrm{sot}}$.

With the notation set forth above, we prove the following

**Theorem 3.1.** *The map $E_A : \mathcal{A} \to H$ yields a complete isomorphism*

$$H \cong \mathcal{A}/Ker(E_A)$$

*with cb-isomorphism constant $\le \sqrt{2}$. More precisely, if $q_A : \mathcal{A} \to \mathcal{A}/Ker(E_A)$ denotes the quotient map, then given any positive integers $n, r$ we have for all $x_i \in M_n(\mathbb{C})$ and $b_i \in \mathcal{A}$, $1 \le i \le r$:*

$$\left\| \sum_{i=1}^r x_i \otimes E_A(b_i) \right\|_{M_n(H)} \le \left\| \sum_{i=1}^r x_i \otimes q_A(b_i) \right\|_{M_n(\mathcal{A}/Ker(E_A))} \le \sqrt{2} \left\| \sum_{i=1}^r x_i \otimes E_A(b_i) \right\|_{M_n(H)} . \tag{3.12}$$

*Furthermore, the dual map $E_A^*$ is a complete isomorphism of $H^*$ onto a subspace of the predual of $\overline{\pi_A(\mathcal{A})}^{sot}$.*

**Remark 3.2.** Note that Theorem 3.1 is equivalent to the statement that for any positive integers $n$, $r$ we have for all $x_i \in M_n(\mathbb{C})$ and $\xi_i \in H$, $1 \le i \le r$,

$$\left\| \sum_{i=1}^r x_i \otimes \xi_i \right\|_{M_n(H)} \le \left\| \sum_{i=1}^r x_i \otimes q_A(a(\xi_i)) \right\|_{M_n(\mathcal{A}/\mathrm{Ker}(E_A))} \le \sqrt{2} \left\| \sum_{i=1}^r x_i \otimes \xi_i \right\|_{M_n(H)} . \tag{3.13}$$

Indeed, to prove that (3.12) implies (3.13), put $b_i := a(\xi_i)$, $1 \le i \le r$ and use the fact that by (3.11), $E_A(a(\xi_i)) = \xi_i$, $1 \le i \le r$. To prove that, conversely, (3.13) implies (3.12), put $\xi_i := E_A(b_i)$, $1 \le i \le r$. Then $E_A(b_i - a(\xi_i)) = 0$, which implies that $q_A(b_i - a(\xi_i)) = 0$, so the middle term of (3.12) is equal to the middle term of (3.13). The equivalence of (3.12) and (3.13) will be used several times in the following.

*Proof of Theorem 3.1.* We first prove the theorem in the finite dimensional case.

Assume $\dim(H) = d < \infty$. Consider the associated operator $A$ $(0 \le A \le I)$ defined by (3.3). There exists an orthonormal basis $\{e_i\}_{1 \le i \le d}$ of $H$ with respect to which the matrix $A$ is diagonal. That is,

$$\langle Ae_i , e_j \rangle_H = \nu_i \delta_{ij} , \quad \forall 1 \le i, j \le d , \tag{3.14}$$

which implies that $0 \le \nu_i \le 1$, $\forall 1 \le i \le d$.
Let $\mathcal{A}$ be the CAR-algebra over $H$ and $\omega_A$ be the quasi-free state on $\mathcal{A}$ corresponding to the operator $A$. Further, set

$$a_i \quad := \quad a(e_i) , \quad \forall 1 \le i \le d .$$

By (3.8) it follows that

$$\omega_A(a_i^* a_j) \quad = \quad \nu_i \delta_{ij} , \quad \forall 1 \le i, j \le d , \tag{3.15}$$

and, respectively,

$$\omega_A(a_i a_j^*) \quad = \quad (1 - \nu_i) \delta_{ij} , \quad \forall 1 \le i, j \le d . \tag{3.16}$$

Let $n$ be a positive integer. Given $x_1 , \dots , x_d \in M_n(\mathbb{C})$, we have by (3.5) that

$$|||\{x_i\}_{i=1}^d|||_A := \left\| \sum_{i=1}^d x_i \otimes e_i \right\|_{M_n(H)} = \max \left\{ \left\| \sum_{i=1}^d (1 - \nu_i) x_i x_i^* \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^d \nu_i x_i^* x_i \right\|^{\frac{1}{2}} \right\} . \tag{3.17}$$

In view of Remark 3.2, we have to prove that

$$\begin{aligned} \max \left\{ \left\| \sum_{i=1}^d (1 - \nu_i) x_i x_i^* \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^d \nu_i x_i^* x_i \right\|^{\frac{1}{2}} \right\} &\le \left\| \sum_{i=1}^d x_i \otimes q_A(a_i) \right\|_{M_n(\mathcal{A}/\mathrm{Ker}(E_A))} \\ &\le \sqrt{2} \max \left\{ \left\| \sum_{i=1}^d (1 - \nu_i) x_i x_i^* \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^d \nu_i x_i^* x_i \right\|^{\frac{1}{2}} \right\} . \end{aligned} \tag{3.18}$$

We first prove the left hand side inequality in (3.18). For each $1 \le i \le d$ set

$$\phi_i^A(b) \quad := \quad \omega_A(a_i^* b + b a_i^*) , \quad \forall b \in \mathcal{A} . \tag{3.19}$$

Note that $\langle E_A(b), e_i \rangle_H = \phi_i^A(b)$, for all $b \in \mathcal{A}$ and that by the anticommutation relations (3.6),

$$\phi_i^A(a_j) = \delta_{ij} , \quad \forall 1 \le i, j \le d .$$

In particular, $E_A(a_i) = e_i$, $\forall 1 \le i \le d$.

**Lemma 3.3.** *For all* $1 \le i \le d$ *we have*

$$\omega_A(a_i^* b) = \nu_i \phi_i^A(b)\,, \qquad \omega_A(b a_i^*) = (1-\nu_i)\phi_i^A(b)\,, \quad \forall b \in \mathcal{A}\,. \tag{3.20}$$

*Proof.* We consider a special representation of the CAR algebra $\mathcal{A}$. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $I_2 = I_{M_2(\mathbb{C})}$ and set

$$a_1' := e \otimes (\otimes_{j=2}^d I_2)\,, \quad a_i' := \left(\otimes_{j=1}^{i-1} u\right) \otimes e \otimes (\otimes_{j=i+1}^d I_2)\,, \quad 2 \le i \le d\,. \tag{3.21}$$

Since $u^2 = I_2$, $ee^* + e^*e = I_2$, $eu + ue = 0$, it follows that $\{a_i'\}_{1 \le i \le d}$ satisfy the CAR relations (3.6). Thus $C^*(\{a_1', \dots, a_d'\}) = \otimes_{i=1}^d M_2(\mathbb{C})$ (see [2] and [4]), and there is a $*$-isomorphism $\psi : \mathcal{A} \to C^*(\{a_1', \dots, a_d'\})$ such that $\psi(a_i) = a_i'$, $\forall 1 \le i \le d$. From now on we identify $\mathcal{A}$ with $\otimes_{i=1}^d M_2(\mathbb{C})$, and write $a_i' = a_i$, $1 \le i \le d$. Then, by [19] (see pp. 4 and 5),

$$\omega_A(b) := \left(\otimes_{i=1}^d \psi_i\right)(b)\,, \quad \forall b \in \mathcal{A}\,, \tag{3.22}$$

where $\psi_i(h) = \mathrm{Tr}\left(\begin{pmatrix} 1-\nu_i & 0 \\ 0 & \nu_i \end{pmatrix} h\right)$, $\forall h \in M_2(\mathbb{C})$, $1 \le i \le d$.

We first show that for all $1 \le i \le d$,

$$(1-\nu_i)\omega_A((a_i)^* b) \;=\; \nu_i \omega_A(b (a_i)^*)\,, \quad \forall b \in \mathcal{A}\,. \tag{3.23}$$

To check (3.23), it is enough to look at simple tensors $b = b_1 \otimes b_2 \otimes \ldots \otimes b_d \in \mathcal{A}$. Consider first the case $i = 1$. Then

$$\omega_A((a_1)^* b) = \psi_1(e^* b_1) \prod_{i=2}^d \psi_i(b_i)\,, \quad \omega_A(b (a_1)^*) = \psi_1(e^* b_1) \prod_{i=2}^d \psi_i(b_i)\,.$$

Let $b_1 = \begin{pmatrix} b_1^{(11)} & b_1^{(12)} \\ b_1^{(21)} & b_1^{(22)} \end{pmatrix}$. Then $\psi_1(e^* b_1) = \psi_1\left(\begin{pmatrix} 0 & 0 \\ b_1^{(11)} & b_1^{(12)} \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} 0 & 0 \\ 0 & \nu_1 b_1^{(12)} \end{pmatrix}\right) = \nu_1 b_1^{(12)}$, respectively, $\psi_1(b_1 e^*) = \pi_1\left(\begin{pmatrix} b_1^{(12)} & 0 \\ b_1^{(22)} & 0 \end{pmatrix}\right) = \mathrm{Tr}\left(\begin{pmatrix} (1-\nu_1) b_1^{(12)} & 0 \\ \nu_1 b_1^{(22)} & 0 \end{pmatrix}\right) = (1-\nu_1) b_1^{(12)}$. Hence

$$\omega_A((a_1)^* b) = \nu_1 b_1^{(12)} \prod_{i=2}^d \psi_i(b_i)\,, \quad \omega_A(b (a_1)^*) = (1-\nu_1) b_1^{(12)} \prod_{i=2}^d \psi_i(b_i)\,,$$

which imply (3.23). The case when $i \ne 1$ can be proved in a similar way, using the fact that for all $b = b_1 \otimes b_2 \otimes \ldots \otimes b_d \in \mathcal{A}$, we have $u b_j = b_j u$, $\forall 1 \le j \le i-1$.

Then, for $1 \le i \le d$ we deduce by (3.23) that for all $b \in \mathcal{A}$, we have

$$\nu_i \omega_A((a_i)^* b + b(a_i)^*) = \nu_i \omega_A((a_i)^* b) + (1-\nu_i)\omega_A((a_i)^* b) = \omega_A((a_i)^* b)\,,$$

and, respectively,

$$(1-\nu_i)\omega_A((a_i)^* b + b(a_i)^*) = \nu_i \omega_A(b(a_i)^*) + (1-\nu_i)\omega_A(b(a_i)^*) = \omega_A(b(a_i)^*)\,.$$

The proof is complete. □

**Lemma 3.4.** *Let* $X \in M_n(\mathcal{A})$ *. By letting*

$$x_i \quad := \quad (Id_n \otimes \phi_i^A)(X)\,, \quad \forall 1 \leq i \leq d\,, \tag{3.24}$$

*we have*

$$(Id_n \otimes E_A)(X) \quad = \quad \sum_{i=1}^{d} x_i \otimes e_i\,. \tag{3.25}$$

*Then, with the above notation it follows that*

$$\|X\|_{M_n(\mathcal{A})} \quad \geq \quad \max\left\{ \left\| \sum_{i=1}^{d} \nu_i x_i^* x_i \right\|^{\frac{1}{2}}, \left\| \sum_{i=1}^{d} (1-\nu_i) x_i x_i^* \right\|^{\frac{1}{2}} \right\}. \tag{3.26}$$

*Proof.* Let $X \in M_n(\mathcal{A})$ . Then $X$ is of the form $X = \sum\limits_{j=1}^{r} y_j \otimes b_j$ , where $r \in \mathbb{N}$ , $y_j \in M_n(\mathbb{C})$ and $b_j \in \mathcal{A}$ , $1 \leq j \leq r$ . For all $1 \leq i \leq d$ , let $x_i$ be defined by (3.24) . Then

$$x_i = \sum_{j=1}^{r} \phi_i^A(b_j) y_j\,, \quad 1 \leq i \leq d\,. \tag{3.27}$$

Further set

$$Z := \sum_{i=1}^{d} x_i \otimes a_i \in M_n(\mathcal{A})\,. \tag{3.28}$$

To each state $\omega$ on $M_n(\mathbb{C})$ we can associate a positive sesquilinear form on $M_n(\mathcal{A})$ given by

$$s_\omega(c,d) := (\omega \otimes \omega_A)(d^* c)\,, \quad \forall c, d \in M_n(\mathcal{A})\,.$$

By (3.27), (3.28) and (3.20), we obtain

$$\begin{aligned} s_\omega(X,Z) = \sum_{i=1}^{d}\sum_{j=1}^{r} \omega(x_i^* y_j)\omega_A(a_i^* b_j) \quad &= \quad \sum_{i=1}^{d}\sum_{j=1}^{r} \nu_i \omega(x_i^* y_j)\phi_i^A(b_j) \\ &= \quad \sum_{i=1}^{d} \nu_i \omega(x_i^* x_i) = s_\omega(Z,Z)\,, \end{aligned}$$

where the last equality follows from (3.15) . Hence $s_\omega(X-Z,Z) = 0$ , and therefore

$$s_\omega(X,X) = s_\omega(Z,Z) + s_\omega(X-Z,X-Z) \geq s_\omega(Z,Z)\,.$$

It follows that

$$\omega\left(\sum_{i=1}^{d} \nu_i x_i^* x_i\right) = s_\omega(Z,Z) \leq s_\omega(X,X) \leq \|X\|^2\,,$$

for every state $\omega$ on $M_n(\mathbb{C})$ , and hence

$$\left\| \sum_{i=1}^{d} \nu_i x_i^* x_i \right\| \leq \|X\|^2\,.$$

The same argument applied to the positive sesquilinear form

$$s'_\omega(c,d) := (\omega \otimes \omega_A)(cd^*)\,, \quad \forall c, d \in M_n(\mathcal{A})$$

gives by (3.16) that

$$\omega\left(\sum_{i=1}^{d}(1-\nu_i)x_ix_i^*\right) = s'_\omega(Z,Z) \le s'_\omega(X,X) \le \|X\|^2\,,$$

for every state $\omega$ on $M_n(\mathbb{C})$ , and hence

$$\left\|\sum_{i=1}^{d}(1-\nu_i)x_ix_i^*\right\| \le \|X\|^2\,.$$

This completes the proof. ☐

**Remark 3.5.** For all $X \in M_n(\mathcal{A})$ we have

$$\|(\mathrm{Id}_n \otimes E_A)(X)\|_{M_n(H)} \quad \le \quad \|(\mathrm{Id}_n \otimes q_A)(X)\|_{M_n(\mathcal{A}/\mathrm{Ker}(E_A))}\,. \tag{3.29}$$

This follows by a similar argument as the one used to prove (2.8). In particular, given $x_1,\ldots,x_d \in M_n(\mathbb{C})$ , by letting $X = \sum_{i=1}^{d} x_i \otimes a_i \in M_n(\mathcal{A})$ , an application of (3.29) yields the left hand side inequality in (3.18) .

We now prove the right hand side inequality in (3.18) . Let $y_1\,,\ldots,y_d \in M_n(\mathbb{C})$ . Set

$$Y \quad := \quad \sum_{i=1}^{d} y_i \otimes a_i \in M_n(\mathcal{A})\,.$$

We will compute $(\mathrm{Id}_n \otimes \omega_A)(Y^*Y)$ , $(\mathrm{Id}_n \otimes \omega_A)(YY^*)$ , $(\mathrm{Id}_n \otimes \omega_A)((Y^*Y)^2)$ and $(\mathrm{Id}_n \otimes \omega_A)((YY^*)^2)$ . We have

$$Y^*Y = \sum_{i,j=1}^{d} y_i^*y_j \otimes a_i^*a_j\,, \quad YY^* = \sum_{i,j=1}^{d} y_iy_j^* \otimes a_ia_j^*\,.$$

By (3.15) and (3.16) it follows immediately that

$$(\mathrm{Id}_n \otimes \omega_A)(Y^*Y) \quad = \quad \sum_{i=1}^{d} \nu_i y_i^* y_i \tag{3.30}$$

$$(\mathrm{Id}_n \otimes \omega_A)(YY^*) \quad = \quad \sum_{i=1}^{d} (1-\nu_i) y_i y_i^*\,. \tag{3.31}$$

Furthermore, in order to compute $(\mathrm{Id}_n \otimes \omega_A)((Y^*Y)^2)$ , note that

$$Y^*Y \quad = \quad \sum_{i,j=1}^{d} y_i^*y_j \otimes (a_i^*a_j - \delta_{ij}\nu_i I) + \sum_{i=1}^{d} \nu_i y_i^* y_i \otimes I\,. \tag{3.32}$$

Consider the vectors

$$f_{ij} \quad := \quad a_i^*a_j - \delta_{ij}\nu_i I\,, \quad \forall 1 \le i,j \le d\,.$$

We claim that $\{I, f_{ij}\,, 1 \le i,j \le d\}$ is an orthogonal set in $L^2(\mathcal{A})$ with respect to the positive sesquilinear form on $\mathcal{A}$ given by $\mathcal{A} \times \mathcal{A} \ni (c,d) \mapsto \omega_A(d^*c) \in \mathbb{C}$ , satisfying $\omega_A(I) = 1$ and

$$\omega_A(f_{ij}^*f_{ij}) \quad = \quad \nu_j(1-\nu_i)\,, \quad \forall 1 \le i,j \le d\,. \tag{3.33}$$

Indeed, for $1 \le i,j \le d$ ,

$$\omega_A(f_{ij}^*f_{ij}) \quad = \quad \omega_A(a_j^*a_ia_i^*a_j) - \nu_i\omega_A(a_j^*a_i + a_i^*a_j)\delta_{ij} + \nu_i^2\delta_{ij}\,.$$

By the anticommutation relations (3.6), together with (3.8) we get

$$\omega_A(a_j^* a_i a_i^* a_j) = \omega_A(a_j^*(I - a_i^* a_i) a_j) = \omega_A(a_j^* a_j) - \omega_A(a_j^* a_i^* a_i a_j) = \nu_j - \nu_i\nu_j(1-\delta_{ij}) = \nu_j(1-\nu_i) + \nu_i\nu_j\delta_{ij}\,,$$

wherein we have also used the fact that $a_i^2 = 0\,,\ 1 \le i \le d$. Furthermore, $\omega_A(a_j^* a_i + a_i^* a_j) = 2\nu_i\delta_{ij}$. Hence $\omega_A(f_{ij}^* f_{ij}) = \nu_j(1-\nu_i) + (\nu_i\nu_j - \nu_i^2)\delta_{ij} = \nu_j(1-\nu_i)$, so (3.33) is proved.

We now prove the orthogonality property of the set of vectors $\{I, f_{ij}\,, 1 \le i,j \le d\}$. First, note that for $1 \le i,j \le d$,

$$\omega_A(f_{ij}) \;=\; \omega_A(a_i^* a_j) - \nu_i\delta_{ij} = \nu_i\delta_{ij} - \nu_i\delta_{ij} = 0\,. \tag{3.34}$$

It remains to show that for $1 \le i,j,k,l \le d$,

$$\omega_A(f_{ij}^* f_{kl}) = 0\,, \quad \text{whenever } (i,j) \ne (k,l)\,. \tag{3.35}$$

We have $f_{ij}^* f_{kl} = a_j^* a_i a_k^* a_l - \nu_k a_j^* a_i \delta_{kl} - \nu_i a_k^* a_l \delta_{ij} + \nu_i\nu_k\delta_{ij}\delta_{kl}$. We distinguish the following cases:

1) $i = j \ne k = l$, 2) $i \ne j\,, k = l$, 3) $i = j\,, k \ne l$, 4) $i \ne j\,, k \ne l\,, (i,j) \ne (k,l)$.

Assume 1) $i = j \ne k = l$. Then

$$\begin{aligned}
\omega_A(f_{ii}^* f_{kk}) &= \omega_A(a_i^* a_i a_k^* a_k) - \nu_k\omega_A(a_i^* a_i) - \nu_i\omega_A(a_k^* a_k) + \nu_i\nu_k \\
&= \omega_A(a_i^*(-a_k^* a_i)a_k) - \nu_k\nu_i - \nu_i\nu_k + \nu_i\nu_k \\
&= -\omega_A(a_i^* a_k^*(-a_k a_i)) - \nu_i\nu_k \\
&= \nu_i\nu_k - \nu_i\nu_k = 0
\end{aligned}$$

Cases 2) and 3) are similar, so we only prove one of them. Assume 2) $i \ne j\,, k = l$. Then

$$\omega_A(f_{ij}^* f_{kk}) = \omega_A(a_j^* a_i a_k^* a_k) - \nu_k\omega_A(a_j^* a_i) = \omega_A(a_j^*(I\delta_{ik} - a_k^* a_i)a_k) = \omega_A(a_j^* a_k\delta_{ik}) - \omega_A(a_j^* a_k^* a_i a_k\delta_{ik}) = 0\,.$$

Respectively, assume 4) $i \ne j\,, k \ne l\,, (i,j) \ne (k,l)$. In this case, $\omega_A(f_{ij}^* f_{kl}) = \omega_A(a_j^* a_i a_k^* a_l)$. By considering further the two possible subcases 4$a$) $i \ne k$ and 4$b$) $i = k, j \ne l$, we deduce by (3.6) and (3.8) that $\omega_A(a_j^* a_i a_k^* a_l) = 0$.

Then, based on the expansion (3.32) of $Y^*Y$ in terms of the vectors $\{I, f_{ij}\,, 1 \le i,j \le d\}$, we now get

$$\begin{aligned}
(\mathrm{Id}_n \otimes \omega_A)((Y^*Y)^2) &= \sum_{i,j=1}^d \nu_j(1-\nu_i)(y_i^* y_j)^* y_i^* y_j + \left(\sum_{i=1}^d \nu_i y_i^* y_i\right)^2 \\
&= \sum_{j=1}^d \nu_j y_j^* \left(\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right) y_j + \left(\sum_{i=1}^d \nu_i y_i^* y_i\right)^2 \\
&\le \left(\sum_{i=1}^d \nu_i y_i^* y_i\right)\left(\left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\| + \left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\|\right) \\
&= \left(\left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\| + \left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\|\right)(\mathrm{Id}_n \otimes \omega_A)(Y^*Y)\,,
\end{aligned} \tag{3.36}$$

where the last equality is given by (3.30).

In order to estimate the term $(\mathrm{Id}_n \otimes \omega_A)((YY^*)^2)$, note that

$$YY^* \;=\; \sum_{i,j=1}^d y_i y_j^* \otimes (a_i a_j^* - \delta_{ij}(1-\nu_i)I) + \sum_{i=1}^d (1-\nu_i) y_i y_i^* \otimes I\,. \tag{3.37}$$

We now consider the vectors

$$g_{ij} \quad := \quad a_i a_j^* - \delta_{ij}(1-\nu_i)I\,, \quad \forall 1 \le i,j \le d\,.$$

With a similar proof it can be shown that $\{I, g_{ij}\,, 1 \le i,j \le d\}$ is an orthogonal set in $L^2(\mathcal{A})$ with respect to the positive sesquilinear form on $\mathcal{A}$ given by $\mathcal{A} \times \mathcal{A} \ni (c,d) \mapsto \omega_A(cd^*) \in \mathbb{C}$, satisfying

$$\omega_A(g_{ij}g_{ij}^*) \quad = \quad \nu_j(1-\nu_i)\,, \quad \forall 1 \le i,j \le d\,. \tag{3.38}$$

Thus, based on the expansion (3.37) of $YY^*$ in terms of the vectors $\{I, g_{ij}\,, 1 \le i,j \le d\}$, we obtain

$$\begin{aligned}(\mathrm{Id}_n \otimes \omega_A)((YY^*)^2) \quad &= \quad \sum_{i,j=1}^d \nu_j(1-\nu_i) y_i y_j^* (y_i y_j^*)^* + \left(\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right)^2 \\ &= \quad \sum_{i=1}^d (1-\nu_i) y_i \left(\sum_{j=1}^d \nu_j y_j^* y_j\right) y_i^* + \left(\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right)^2 \\ &\le \quad \left(\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right)\left(\left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\| + \left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\|\right) \\ &= \quad \left(\left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\| + \left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\|\right)(\mathrm{Id}_n \otimes \omega_A)(YY^*)\,,\end{aligned} \tag{3.39}$$

where the last equality is given by (3.31).

As before, the crucial point is to show the following

**Lemma 3.6.** *Let $x_1\,, \dots, x_d \in M_n(\mathbb{C})$. There exists $X \in M_n(\mathcal{A})$ so that $(Id_n \otimes E_A)(X) = \sum_{i=1}^d x_i \otimes e_i$, satisfying, moreover,*

$$\|X\|_{M_n(\mathcal{A})} \quad \le \quad \sqrt{2} \max\left\{\left\|\sum_{i=1}^d \nu_i x_i^* x_i\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^d (1-\nu_i) x_i x_i^*\right\|^{\frac{1}{2}}\right\}. \tag{3.40}$$

For this, we first prove the following

**Lemma 3.7.** *If $y_1\,, \dots, y_d \in M_n(\mathbb{C})$ satisfy*

$$\max\left\{\left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\|^{\frac{1}{2}}\right\} \quad = \quad 1\,, \tag{3.41}$$

*then there exists $Z \in M_n(\mathcal{A})$ such that $\|Z\|_{M_n(\mathcal{A})} \le \frac{1}{\sqrt{2}}$, and, moreover, when $z_1\,, \dots, z_d$ are defined by $(Id_n \otimes E_A)(Z) = \sum_{i=1}^d z_i \otimes e_i$, then*

$$\max\left\{\left\|\sum_{i=1}^d \nu_i (y_i - z_i)^*(y_i - z_i)\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^d (1-\nu_i)(y_i - z_i)(y_i - z_i)^*\right\|^{\frac{1}{2}}\right\} \quad \le \quad \frac{1}{2}\,.$$

*Proof.* Set

$$Y := \sum_{i=1}^d y_i \otimes a_i \in M_n(\mathcal{A})\,.$$

Now let $C > 0$ and define $F_C : \mathbb{R} \to \mathbb{R}$ by formula (2.15). Use functional calculus to define $Z \in M_n(\mathcal{A})$ by (2.16). Then $\|Z\|_{M_n(\mathcal{A})} \leq C$ and, as shown in the proof of Lemma 2.6, it follows that

$$(Y-Z)^*(Y-Z) \leq \frac{1}{16C^2}(Y^*Y)^2\,, \quad (Y-Z)(Y-Z)^* \leq \frac{1}{16C^2}(YY^*)^2\,.$$

By letting $z_i = (\mathrm{Id}_n \otimes \phi_i^A)(Z)$, $1 \leq i \leq d$, we then have

$$(\mathrm{Id}_n \otimes E_A)(Z) = \sum_{i=1}^d z_i \otimes e_i\,, \quad \text{respectively}\,, \quad (\mathrm{Id}_n \otimes E_A)(Y) = \sum_{i=1}^d y_i \otimes E_A(a_i) = \sum_{i=1}^d y_i \otimes e_i\,,$$

and we obtain the estimates

$$\begin{aligned}
\sum_{i=1}^d \nu_i (y_i - z_i)^*(y_i - z_i) &\leq (\mathrm{Id}_n \otimes \omega_A)((Y-Z)^*(Y-Z)) \\
&\leq \frac{1}{16C^2}(\mathrm{Id}_n \otimes \omega_A)((Y^*Y)^2) \\
&\leq \frac{1}{16C^2}\left(\left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\| + \left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\|\right)(\mathrm{Id}_n \otimes \omega_A)(Y^*Y) \\
&\leq \frac{2}{16C^2}(\mathrm{Id}_n \otimes \omega_A)(Y^*Y) \\
&= \frac{1}{8C^2}\sum_{i=1}^d \nu_i y_i^* y_i\,,
\end{aligned}$$

respectively, $\sum_{i=1}^d (1-\nu_i)(y_i - z_i)(y_i - z_i)^* \leq \frac{1}{8C^2}\sum_{i=1}^d (1-\nu_i) y_i y_i^*$. We deduce that

$$\left\|\sum_{i=1}^d \nu_i (y_i - z_i)^*(y_i - z_i)\right\| \leq \frac{1}{8C^2}\left\|\sum_{i=1}^d \nu_i y_i^* y_i\right\| \leq \frac{1}{8C^2}\,,$$

respectively,

$$\left\|\sum_{i=1}^d (1-\nu_i)(y_i - z_i)(y_i - z_i)^*\right\| \leq \frac{1}{8C^2}\left\|\sum_{i=1}^d (1-\nu_i) y_i y_i^*\right\| \leq \frac{1}{8C^2}\,.$$

Hence

$$\max\left\{\left\|\sum_{i=1}^d \nu_i (y_i - z_i)^*(y_i - z_i)\right\|^{\frac{1}{2}}, \left\|\sum_{i=1}^d (1-\nu_i)(y_i - z_i)(y_i - z_i)^*\right\|^{\frac{1}{2}}\right\} \leq \frac{1}{\sqrt{8}C}\,.$$

Now take $C = \frac{1}{\sqrt{2}}$ to obtain the conclusion. □

We are now ready to prove Lemma 3.6. Indeed, Lemma 3.7 shows that if $y := \sum_{i=1}^d y_i \otimes e_i \in M_n(H)$ has norm $\|y\|_{M_n(H)} = 1$, then there exists $Z \in M_n(\mathcal{A})$ such that $\|Z\|_{M_n(\mathcal{A})} \leq \frac{1}{\sqrt{2}}$ and $\|(\mathrm{Id}_n \otimes E_A)(Z) - y\|_{M_n(H)} \leq \frac{1}{2}$. By homogeneity we infer that for all $y \in M_n(H)$, there exists $Z \in M_n(\mathcal{A})$ satisfying the conditions $\|Z\|_{M_n(\mathcal{A})} \leq \frac{1}{\sqrt{2}}\|y\|_{M_n(H)}$ and $\|(\mathrm{Id}_n \otimes E_A)(Z) - y\|_{M_n(H)} \leq \frac{1}{2}\|y\|_{M_n(H)}$. Applying now

Lemma 2.7 with $C = \frac{1}{\sqrt{2}}$ and $\delta = \frac{1}{2}$ we deduce that for all $x \in M_n(H)$ there exists $X \in M_n(\mathcal{A})$ so that $(\mathrm{Id}_n \otimes E_A)(X) = x$, satisfying, moreover,

$$\|X\|_{M_n(\mathcal{A})} \quad \leq \quad \frac{C}{1-\delta}\|x\|_{M_n(H)} = \sqrt{2}\|x\|_{M_n(H)} \, .$$

The proof of Lemma 3.6 is complete.

By Lemmas 3.4 and 3.6 and Remark 3.5 , there exists a linear bijection $\widetilde{E}_A : \mathcal{A}/\mathrm{Ker}(E_A) \to H$ such that

$$\widetilde{E}_A(q_A(a_i)) \quad = \quad e_i \, , \quad 1 \leq i \leq d \, ,$$

making the following diagram commutative:

$$\begin{array}{ccc} M_n(\mathcal{A}) & \xrightarrow{\mathrm{Id}_n \otimes E_A} & M_n(H) \\ & {\scriptstyle \mathrm{Id}_n \otimes q_A} \searrow \quad \nearrow {\scriptstyle \mathrm{Id}_n \otimes \widetilde{E}_A} & \\ & M_n(\mathcal{A}/\mathrm{Ker}(E_A)) & \end{array}$$

Moreover, with respect to the natural operator space structure of the quotient $\mathcal{A}/\mathrm{Ker}(E_A)$ one has for all $x_1 , \dots , x_d \in M_n(\mathbb{C})$ ,

$$\begin{aligned} \max\left\{ \left\| \sum_{i=1}^{d} \nu_i x_i^* x_i \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^{d} (1-\nu_i) x_i x_i^* \right\|^{\frac{1}{2}} \right\} \quad &\leq \quad \left\| \sum_{i=1}^{d} x_i \otimes q_A(a_i) \right\|_{M_n(\mathcal{A}/\mathrm{Ker}(E_A))} \\ &\leq \quad \sqrt{2} \max\left\{ \left\| \sum_{i=1}^{d} \nu_i x_i^* x_i \right\|^{\frac{1}{2}} , \left\| \sum_{i=1}^{d} (1-\nu_i) x_i x_i^* \right\|^{\frac{1}{2}} \right\} \, , \end{aligned}$$

i.e., the inequalities (3.18) hold. This completes the proof of Theorem 3.1 in the finite dimensional case.

We now consider the infinite dimensional case ($\dim(H) = \infty$). Let $V \subset H$ be a finite dimensional subspace, and let $d = \dim(V)$ . Set

$$A_V := P_V A_{|_{P_V H}} \in \mathcal{B}(P_V H) \, ,$$

where $P_V$ is the projection of $H$ onto $V$ . Then $0 \leq A_V \leq I$ .

Let $\mathcal{A}_V$ be the CAR algebra on $V$ , and denote by $\omega_A$ (respectively, $\omega_{A_V}$) the gauge-invariant quasi-free state on $\mathcal{A}$ (respectively, $\mathcal{A}_V$) corresponding to the operator $A$ (respectively, $A_V$). Recall that $\mathcal{A}_V$ is the norm closure of $\mathrm{Span}\{a(e_{i_1})^* \dots a(e_{i_n})^* a(e_{j_1}) \dots a(e_{j_m}) \, ; \, 1 \leq i_1, \dots , i_n, j_1, \dots , j_m, n, m \leq d\}$ . By equation (3.8) it follows that $\omega_{A|_{\mathcal{A}_V}}$ and $\omega_{A_V}$ coincide on all polynomials that generate $\mathcal{A}_V$ . Since states are norm continuous, we conclude that

$$\omega_{A|_{\mathcal{A}_V}} \quad = \quad \omega_{A_V} \, . \tag{3.42}$$

The key point that will allow us to reduce the infinite dimensional case to the finite dimensional one is the fact, which we will justify in the following, that $E_A(b) \in V$ , whenever $b \in \mathcal{A}_V$ . Indeed, given $b \in \mathcal{A}_V$ , we will show that

$$\langle E_A(b), f \rangle_H = 0 \, , \quad \forall f \in V^{\perp} \, .$$

By (3.10), this is equivalent to showing that

$$\omega_A(ba(f)^* + a(f)^* b) \quad = \quad 0 \, , \quad \forall f \in V^{\perp} \, . \tag{3.43}$$

By continuity, it suffices to consider elements $b \in \mathcal{A}_V$ of the form

$$b = a(e_{i_1})^* \dots a(e_{i_n})^* a(e_{j_1}) \dots a(e_{j_m})$$

where $1 \le i_1, \dots, i_n, j_1, \dots, j_m, n, m \le d$. Let $f \in V^\perp$. Since $f \perp e_i$, $1 \le i \le d$, we get by the CAR relations (3.6) that

$$ba(f)^* \quad = \quad (-1)^{n+m} a(f)^* b\,.$$

So if $n+m$ is odd, then $ba(f)^* + a(f)^* b = 0$. If $n+m$ is even, i.e., $n+m+1$ is odd, then by (3.6) and (3.8) (together with (3.42)) it follows that $\omega_A(ba(f)^*) = 0 = \omega_A(a(f)^* b)$. Hence, in both cases (3.43) follows, and our claim is proved. By uniqueness in the construction of the maps $E_A$ and $E_{A_V}$, we conclude that

$$E_{A|_{\mathcal{A}_V}} \quad = \quad E_{A_V}\,. \tag{3.44}$$

Since $\mathcal{A}$ is the $C^*$-algebra generated by the operators $a(\xi)$, $\xi \in H$, it is clear that

$$\mathcal{A} = \overline{\bigcup_V \mathcal{A}_V}\,, \quad \text{(norm closure)} \tag{3.45}$$

where the union is taken over all finite dimensional subspaces $V$ of $H$. Moreover, note that $\mathcal{A}_{V_1} \subseteq \mathcal{A}_{V_2}$ when $V_1 \subseteq V_2$. We also claim that

$$\mathrm{Ker}(E_A) = \overline{\bigcup_V \mathrm{Ker}(E_{A_V})}\,, \quad \text{(norm closure)} \tag{3.46}$$

where the right-hand side is also an increasing union because $\mathrm{Ker}(E_{A_V}) = \mathrm{Ker}(E_A) \cap \mathcal{A}_V$, for all $V \subset H$, finite dimensional subspace. To prove (3.46), let $b \in \mathrm{Ker}(E_A)$ and choose $b_n \in \bigcup_V \mathcal{A}_V$, $n \ge 1$ such that $\|b_n - b\| \to 0$ as $n \to \infty$. Further, set

$$b_n' := b_n - a(E_A(b_n))\,, \quad n \ge 1\,.$$

For $n \ge 1$, since $b_n \in \mathcal{A}_{V_n}$ for some finite dimensional subspace $V_n$ of $H$, we have by (3.44) that $E_A(b_n) = E_{A_{V_n}}(b_n) \in V_n$, and hence $b_n' \in \mathcal{A}_{V_n}$. Moreover, by (3.11), we get

$$E_A(b_n') = E_A(b_n) - E_A(b_n) = 0\,.$$

Therefore, $b_n' \in \mathrm{Ker}(E_A) \cap \mathcal{A}_{V_n} = \mathrm{Ker}(E_{A_{V_n}})$, which proves (3.46). Now, since the union in formula (3.46) is increasing, we also have for all $n \in \mathbb{N}$,

$$M_n(\mathrm{Ker}(E_A)) = \overline{\bigcup_V M_n(\mathrm{Ker}(E_{A_V}))}\,. \quad \text{(norm closure)} \tag{3.47}$$

We are now ready to proceed with the proof of Theorem 3.1 in the case $\dim(H) = \infty$. We shall prove that for all positive integers $n, r$ and all $x_i \in M_n(\mathbb{C})$ and $b_i \in \mathcal{A} = \mathcal{A}(H)$, $1 \le i \le r$, the inequalities (3.12) hold.

Indeed, by (3.45) and the fact that $\bigcup_V \mathcal{A}_V$ is an increasing union, it suffices to prove (3.12) for elements $b_i \in \mathcal{A}_{V_0}$, where $V_0$ is an arbitrary finite dimensional subspace of $H$. Let now such $V_0$ be fixed. Since Theorem 3.1 has been proved in the final dimensional case, we have for each finite dimensional subspace $V$ with $V_0 \subseteq V \subset H$ that

$$\left\| \sum_{i=1}^r x_i \otimes E_{A_V}(b_i) \right\|_{M_n(V)} \le \left\| \sum_{i=1}^d x_i \otimes q_A(b_i) \right\|_{M_n(\mathcal{A}_V/\mathrm{Ker}(E_{A_V}))} \le \sqrt{2} \left\| \sum_{i=1}^r x_i \otimes E_{A_V}(b_i) \right\|_{M_n(V)}. \tag{3.48}$$

By (3.44), $E_{A_V}(b_i) = E_A(b_i)$, $1 \leq i \leq r$, for all such $V$, since $b_i \in V_0 \subseteq V$. Moreover, since the norm in $M_n(\mathcal{A}/\mathrm{Ker}(E_A))$ is the quotient norm of the quotient space $M_n(\mathcal{A})/M_n(\mathrm{Ker}(E_A))$, and likewise for $M_n(\mathcal{A}_V/\mathrm{Ker}(E_{A_V}))$, we get by (3.47) that

$$\lim_V \left\| \sum_{i=1}^r x_i \otimes q_A(b_i) \right\|_{M_n(\mathcal{A}_V/\mathrm{Ker}(E_{A_V}))} = \left\| \sum_{i=1}^r x_i \otimes b_i \right\|_{M_n(\mathcal{A}/\mathrm{Ker}(E_A))},$$

where the limit is taken over the directed set of finite dimensional subspaces $V$ with $V_0 \subseteq V \subset H$, ordered by inclusion. Hence, the inequalities (3.12) follow from (3.48) and the proof of Theorem 3.1 is complete.

**Corollary 3.8.** *Let $P$ be the hyperfinite type $III_1$ factor. For any subspace $H$ of $R \oplus C$, its dual $H^*$ embeds completely isomorphically into the predual $P_*$ of $P$, with cb-isomorphism constant $\leq \sqrt{2}$. In particular, the operator Hilbert space $OH$ cb-embeds into $P_*$ with cb-isomorphism constant $\leq \sqrt{2}$.*

*Proof.* Given a subspace $H$ of $R \oplus C$, let $A$ be the associated operator ($0 \leq A \leq I$) defined by (3.3), $\mathcal{A}$ the CAR algebra over $H$, and $\omega_A$ the corresponding gauge-invariant quasi-free state on $\mathcal{A}$. Denote $\overline{\pi_A(\mathcal{A})}^{\mathrm{sot}}$ by $M$, where $\pi_A$ is the unital $*$-homomorphism from the GNS representation associated to $(\mathcal{A}, \omega_A)$. By Theorem 5.1 in [19], $M$ is a hyperfinite factor. Then the von Neumann algebra tensor product $M \bar{\otimes} P$ is (isomorphic to) the hyperfinite type $\mathrm{III}_1$ factor $P$ (cf. [3] and [7]). Moreover, $M_*$ cb-embeds into $(M \bar{\otimes} P)_*$, the embedding being given by the dual map of a normal conditional expectation from $M \bar{\otimes} P$ onto $M$. Therefore, by Theorem 3.1 it follows that the dual $H^*$ of $H$ embeds completely isomorphically into $P_*$, with cb-isomorphism constant $\leq \sqrt{2}$. Furthermore, note that $H^*$ is completely isometric to a quotient of the dual space $(R \oplus_\infty C)^*$. We infer that any quotient (and further, any sub-quotient, that is, subspace of a quotient) of $(R \oplus_\infty C)^*$ cb-embeds into $P_*$, with cb-isomorphism constant $\leq \sqrt{2}$. As shown by Pisier (cf. [18]), the operator space $OH$ is a subspace of a quotient of $R \oplus_\infty C$. Since $OH$ is self-dual as an operator space (cf. [16]), $OH$ is also a sub-quotient of $(R \oplus_\infty C)^*$. We conclude that $OH$ embeds completely isomorphically into $P_*$, with cb-isomorphism constant $\leq \sqrt{2}$. (See also Junge's results in Section 8 of [10] on the embedding of $OH$ into $P_*$.) □

**Remark 3.9.** Let $H \subset R \oplus C$ be a subspace of dimension $d < \infty$, and let $A$ be the associated operator defined by (3.3), respectively, let $\{e_i\}_{i=1}^d$, $\{\nu_i\}_{i=1}^d$ be defined by (3.14). Assume further that $0 < \nu_i < 1$, for all $1 \leq i \leq d$. Now define for any $x_1, \ldots, x_d \in M_n(\mathbb{C})$,

$$|||\{x_i\}_{i=1}^d|||^* := \inf \left\{ \mathrm{Tr} \left[ \left( \sum_{i=1}^d \frac{1}{\nu_i} v_i v_i^* \right)^{\frac{1}{2}} + \left( \sum_{i=1}^d \frac{1}{(1-\nu_i)} z_i^* z_i \right)^{\frac{1}{2}} \right] ;\ x_i = v_i + z_i \in M_n(\mathbb{C}) \right\}, \tag{3.49}$$

where Tr denotes, as before, the non-normalized trace on $M_n(\mathbb{C})$.

Note that $|||\cdot|||^*$ is the dual norm of $\|\cdot\|_{M_n(H)}$. From the proof of Theorem 3.1 it follows by duality that the transpose $F_A := E_A^*$ of the map $E_A : \mathcal{A} \to H$ becomes a complete injection of $H^*$ into $\mathrm{Span}\{\phi_1^A, \ldots, \phi_d^A\} = \mathcal{A}^*$. More precisely, we obtain that

$$\frac{1}{\sqrt{2}} |||\{x_i\}_{i=1}^d|||^* \leq \left\| \sum_{i=1}^d x_i \otimes \phi_i^A \right\|_{M_n(\mathcal{A})^*} \leq |||\{x_i\}_{i=1}^d|||^*. \tag{3.50}$$

We now discuss estimates for best constants in the inequalities (3.50) above.

**Theorem 3.10.** *Let $c_1, c_2$ denote the best constants in the inequalities*

$$c_1|||\{x_i\}_{i=1}^d|||^* \quad \leq \quad \left\| \sum_{i=1}^d x_i \otimes \phi_i^A \right\|_{M_n(\mathcal{A})^*} \leq c_2|||\{x_i\}_{i=1}^d|||^* . \tag{3.51}$$

*where $d, n$ are arbitrary positive integers, $H \subseteq R \oplus C$ is a Hilbert space of dimension $dim(H) = d$ with associated operator $A$ given by (3.3), $\mathcal{A}$ is the CAR-algebra over $H$, $\phi_1^A, \dots, \phi_d^A$ are defined by (3.19), and $x_1, \dots, x_d \in M_n(\mathbb{C})$. Then*

$$c_1 = \frac{1}{\sqrt{2}}, \qquad c_2 = 1 . \tag{3.52}$$

*Proof.* By (3.50) we obtain immediately the following estimates

$$\frac{1}{\sqrt{2}} \leq c_1 \leq c_2 \leq 1 . \tag{3.53}$$

Next we prove that $c_1 = \frac{1}{\sqrt{2}}$. Take $n = 1$, $d = 1$, in which case $H = \mathbb{C}$, $A = \frac{1}{2}I_H$ and $\mathcal{A} = M_2(\mathbb{C})$, and let $x_1 = I_{M_1(\mathbb{C})} = I_{\mathbb{C}}$. Then $\phi_1^A(b) = \text{Tr}(a_1^* b)$, $\forall b \in \mathcal{A}$, where $a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Since $|a_1|$ is a projection with $\text{Tr}(|a_1|) = 1$, we get $\|\phi_1^A\|_{\mathcal{A}^*} = \|a_1^*\|_{L^1(\mathcal{A}, \text{Tr})} = \|a_1\|_{L^1(\mathcal{A}, \text{Tr})} = \| \, |a_1| \, \|_{L^1(\mathcal{A}, \text{Tr})} = 1$. It is easily checked by the definition (3.49) that $|||x_1|||^* = \sqrt{2}$, hence, $\frac{1}{\sqrt{2}}|||x_1|||^* = 1 = \|\phi_1^A\|_{\mathcal{A}^*} = \|x_1 \otimes \phi_1^A\|_{\mathcal{A}^*}$. It follows that $c_1 \leq \frac{1}{\sqrt{2}}$, which together with (3.53) imply that $c_1 = \frac{1}{\sqrt{2}}$.

We now prove that $c_2 = 1$. For this, given $d \in \mathbb{N}$, let $H = \text{Span}\{e_{1i} \oplus e_{i1}; 1 \leq i \leq d\} \subseteq R \oplus C$. It follows easily by (3.3) that the associated operator is $A = \frac{1}{2}I_H$. Let $\{e_i\}_{i \geq 1}$ be an orthonormal basis of $H$ with respect to which the matrix $A$ is diagonal. As before, let

$$a_i := a(e_i), \quad 1 \leq i \leq d$$

be the generators of the CAR algebra $\mathcal{A} = \mathcal{A}(H)$ built on $H$. We consider the special representation of $\mathcal{A}$ constructed in the proof of Lemma 3.3 and use the identification

$$\mathcal{A} \cong \psi(\mathcal{A}) = \otimes_{i=1}^d M_2(\mathbb{C}) , \tag{3.54}$$

where $\psi$ is the $*$-isomorphism obtained therein. Via this identification, we may assume that the generators $a_i$, $1 \leq i \leq d$, of $\mathcal{A}$ are given by (3.21). Note also that the eigenvalues of $A$ are $\nu_i = \frac{1}{2}$, $\forall 1 \leq i \leq d$, so the corresponding quasi-free state $\omega_A$ on $\mathcal{A}$ is tracial. For simplicity of notation, let $\omega_A$ be denoted by $\tau$.

For all $1 \leq i \leq d$, set $x_i := e_{i1} \in M_d(\mathbb{C})$. In what follows, Tr denotes the non-normalized trace on $M_d(\mathbb{C})$. For $1 \leq i \leq d$, we have $\phi_i^A(b) = \tau(a_i^* b + b a_i^*) = 2\tau(a_i^* b)$, $\forall b \in \mathcal{A}$.

Let $h_i := 2a_i^*$, $1 \leq i \leq d$. Then

$$\begin{aligned} \left\| \sum_{i=1}^d x_i \otimes \phi_i^A \right\|_{(M_d(\mathcal{A}))^*} &= \left\| \sum_{i=1}^d x_i \otimes h_i \right\|_{L^1(M_d(\mathbb{C}) \otimes \mathcal{A}, \text{Tr} \otimes \tau)} \\ &= (\text{Tr} \otimes \tau) \left( \left[ \left( \sum_{i=1}^d x_i^* h_i \right)^* \left( \sum_{i=1}^d x_i^* h_i \right) \right]^{\frac{1}{2}} \right) \\ &= \tau \left( \left( \sum_{i=1}^d h_i^* h_i \right)^{\frac{1}{2}} \right) = 2\,\tau \left( \left( \sum_{i=1}^d a_i a_i^* \right)^{\frac{1}{2}} \right) . \end{aligned}$$

Note that by (3.49) it follows immediately that $|||\{x_i\}_{i=1}^d|||^* \leq \mathrm{Tr}\left(\left(2\sum_{i=1}^d x_i^* x_i\right)^{\frac{1}{2}}\right) \leq \sqrt{2d}$. Therefore, if we show that

$$\lim_{d\to\infty} \sqrt{\frac{2}{d}}\,\tau\left(\left(\sum_{i=1}^d a_i a_i^*\right)^{\frac{1}{2}}\right) = 1\,, \tag{3.55}$$

it then follows by (3.51) that $c_2 \geq 1$, which implies that $c_2 = 1$.

We now prove (3.55). For this, we first show that $a_1a_1^*, \ldots, a_da_d^*$ are independent, self-adjoint random variables with distribution

$$\mu_{a_ia_i^*} = \frac{1}{2}(\delta_{\{0\}} + \delta_{\{1\}})\,, \quad 1 \leq i \leq d\,. \tag{3.56}$$

Using the notation set forth in the proof of Lemma 3.3 and (3.21), a simple computation shows that

$$a_1a_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes (\otimes_{j=2}^d I_2)\,, \quad a_ia_i^* := \left(\otimes_{j=1}^{i-1} I_2\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \left(\otimes_{j=i+1}^d I_2\right)\,, \quad 2 \leq i \leq d\,. \tag{3.57}$$

In particular, $a_ia_i^*$ is a projection, for all $1 \leq i \leq d$. So $a_ia_i^*$ has spectrum $\sigma(a_ia_i^*) = \{0,1\}$, and since $\tau(a_ia_i^*) = \frac{1}{2}$, formula (3.56) follows.

By (3.57), $a_ia_i^*$ and $a_ja_j^*$ do commute, for all $1 \leq i, j \leq d$. Thus, in order to prove the independence of $a_1a_1^*, \ldots, a_da_d^*$ (both in the classical sense and in the sense of Voiculescu (cf. [21])), it remains to show that

$$\tau\left((a_1a_1^*)^{m_1} \ldots (a_da_d^*)^{m_d}\right) = \prod_{i=1}^d \tau((a_ia_i^*)^{m_i})\,, \quad m_1, \ldots, m_d \in \mathbb{N}\,.$$

This follows immediately from the special form (3.57) of the elements $a_ia_i^*$, $1 \leq i \leq d$, and the fact that by the identification (3.54), $\tau$ can be viewed as the tensor product trace on $\otimes_{i=1}^d M_2(\mathbb{C})$.

Now recall that $d$ was arbitrarily chosen. By applying the law of large numbers we deduce that the sequence $\left\{\frac{1}{d}\sum_{i=1}^d a_ia_i^*\right\}_{d\geq 1}$ converges in probability to $\frac{1}{2}I_{M_n(\mathbb{C})}$, as $d \to \infty$. This implies that

$$\lim_{d\to\infty} \sqrt{\frac{1}{d}\sum_{i=1}^d a_ia_i^*} = \frac{1}{\sqrt{2}} I_{M_n(\mathbb{C})} \quad \text{in probability}\,. \tag{3.58}$$

Since, moreover, $0 \leq \frac{1}{d}\sum_{i=1}^d a_ia_i^* \leq 1$, for all $d \geq 1$, it follows that the convergence (3.58) holds also in the 2-norm. Hence $\lim_{d\to\infty} \tau\left(\left(\frac{1}{d}\sum_{i=1}^d a_ia_i^*\right)^{\frac{1}{2}}\right) = \frac{1}{\sqrt{2}}$, which gives (3.55), and the proof is complete. □


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