MOMENT FORMULAS FOR THE QUASI-NILPOTENT DT-OPERATOR

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ABSTRACT. Let T be the quasi-nilpotent DT-operator. By use of Voiculescu's amalgamated R-transform we compute the momets of $(T-\lambda 1)^*(T-\lambda 1)$ where $\lambda \in \mathbb{C}$, and the Brown-measure of $T+\sqrt{\epsilon}Y$, where Y is a circular element $*$-free from T for $\epsilon > 0$. Moreover we give a new proof of Śniady's formula for the moments $\tau(((T^*)^k T^k)^n)$ for $k, n \in \mathbb{N}$.

1. Introduction

The quasi-nilpotent DT-operator T was introduced by Dykema and the second author in [4]. It can be described as the limit in $*$-moments for $n \to \infty$, of random matrices of the form

$$T^{(n)} = \begin{pmatrix} 0 & t_{1,2} & \cdots & t_{1,n} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{n-1,n} \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

where $\{\Re(t_{ij}), \Im(t_{ij})\}_{1\le i<j\le n}$ is a set of $n(n-1)$ independent identically distributed Gaussian random variables with mean 0 and variance $\frac{1}{2n}$. More precisely, T is an element in a finite von Neumann algebra, M, with a faithful normal tracial state, τ, such that for all $s_1, s_2, \ldots, s_k \in \{1, *\}$,

$$\tau(T^{s_1}T^{s_2}\cdots T^{s_k}) = \lim_{n\to\infty} \mathbb{E}[\mathrm{tr}_n((T^{(n)})^{s_1}(T^{(n)})^{s_2}\cdots(T^{(n)})^{s_k})], \tag{1.1}$$

where tr_n is the normalized trace on $M_n(\mathbb{C})$. Moreover the pair $(T, W^*(T))$ is uniquely determined up to $*$-isomorphism by (1.1). The quasi-nilpotent DT-operator can be realized as an element in the free group factor, $L(\mathbb{F}_2)$, in the following way (cf. [4, Sect. 4]): Let (D_0, X) be a pair of free selfadjoint elements in a tracial W^*-probability space

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(M, τ), such that $\mathrm{d}\mu_{\mathrm{D_0}}(\mathrm{t}) = 1_{[0,1]}(\mathrm{t})\mathrm{dt}$ and X is semi-circular distributed, i.e. $\mathrm{d}\mu_{\mathrm{X}}(\mathrm{t}) = \frac{1}{2\pi}\sqrt{4-\mathrm{t}^2}1_{[-2,2]}(\mathrm{t})\mathrm{dt}$. Then $W^*(D_0, X) \simeq W^*(D_0) \star W^*(X) \simeq L(\mathbb{F}_2)$. Put

$$T_N = \sum_{j=1}^{2^N} p_{N,j} X q_{N,j}$$

for $N = 1, 2, \ldots$, where

$$p_{N,j} = 1_{[\frac{j-1}{2^N}, \frac{j}{2^N}]}(D_0), \quad q_{N,j} = 1_{[\frac{j}{2^N}, 1]}(D_0),$$

for $j = 1, 2, \ldots, 2^N$. Then $(T_N)_{N=1}^{\infty}$ converges in norm to an operator $T \in W^*(D_0, X)$, and the $*$-moments of T are given by (1.1), i.e. T is a realization of the quasi-nilpotent DT-operator. In the notation of [4, Sect. 4], $T = \mathcal{UT}(X, \lambda)$, where $\lambda : L^\infty[0,1] \to W^*(D_0)$ is the $*$-isomorphism given by $\lambda(f) = f(D_0)$ for $f \in L^\infty([0,1])$. In the following we put $\mathcal{D} = W^*(D_0) \simeq L^\infty([0,1])$ and let $E_{\mathcal{D}}$ denote the trace-preserving conditional expectation of $W^*(D_0, X)$ onto $\mathcal{D}$.

In this paper we apply Voiculescu's $\mathcal{R}$-transform with amalgamation to compute various $*$-moments of T and of operators closely related to T. First we compute in section 3 moments and the scalar valued $\mathcal{R}$-transform of $(T - \lambda 1)^*(T - \lambda 1)$ for $\lambda \in \mathbb{C}$. The specialized case of $\lambda = 0$ was treated in [4] by more complicated methods. In section 4 we consider the operator

$$T + \sqrt{\epsilon} Y,$$

where Y is a circular operator $*$-free from T and $\epsilon > 0$. By random matrix considerations it is easily seen, that if T_1 and T_2 are two quasi-nilpotent DT-operators, which are $*$-free with respect to amalgamation over the same diagonal, $\mathcal{D}$, then $T + \sqrt{\epsilon} Y$ has the same $*$-distribution as $S = \sqrt{a} T_1 + \sqrt{b} T_2$, when $a = 1 + \epsilon$ and $b = \epsilon$ (cf. [1]). We use this fact to prove, that the Brown measure of $T + \sqrt{\epsilon} Y$ is equal to the uniform distribution on the closed disc $\overline{B}(0, \log(1 + \frac{1}{\epsilon})^{-\frac{1}{2}})$ in the complex plane. Moreover we show, that the spectrum of $T + \sqrt{\epsilon} Y$ is equal to this disc, and that $T + \sqrt{\epsilon} Y$ is not a DT-operator for any $\epsilon > 0$.

In [4] it was conjectured, that

$$\tau(((T^*)^k T^k)^n) = \frac{n^{nk}}{(nk+1)!} \tag{1.2}$$

for $n, k \in \mathbb{N}$. This formula was proved by Śniady in [9]. Śniady's proof of (1.2) is based on Speicher's combinatorial approach to free probability with amalgamation from [11]. The key step in the proof of

(1.2) was to establish a recursion formula for the $\mathcal{D}$-valued moments,

$$E_{\mathcal{D}}\left(((T^*)^k T^k)^n\right) \tag{1.3}$$

for each fixed $k \in \mathbb{N}$. Śniady's recursion formula for the $\mathcal{D}$-valued moments (1.3), was later used by Dykema and the second author to prove, that

$$W^*(T) = W^*(D_0, X) \simeq L(\mathbb{F}_2)$$

and that T admits a one parameter family of non-trivial hyperinvariant subspaces (cf. [5]). In section 5 and section 6 of this paper we give a new proof of Śniady's recursion formula for the $\mathcal{D}$-valued moments (1.3), which at the same time gives a new proof of (1.2). The new proof is based on Voiculescu's $\mathcal{R}$-transform with respect to amalgamation over $M_{2k}(\mathcal{D})$, the algebra of $2k \times 2k$ matrices over $\mathcal{D}$.

2. Preliminaries

In this section we give a few preliminaries on amalgamated probability theory. Let $\mathcal{A}$ be a unital Banach algebra, and let $\mathcal{B}$ be a Banach-sub-algebra containing the unit of $\mathcal{A}$. Then a map, $E_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$, is a conditional expectation if

(a) $E_{\mathcal{B}}$ is linear,
(b) $E_{\mathcal{B}}$ preserves the unit i.e. $E_{\mathcal{B}}(1) = 1$
(c) and $E_{\mathcal{B}}$ has the $\mathcal{B}$, $\mathcal{B}$ bi-module property i.e. $E_{\mathcal{B}}(b_1 a b_2) = b_1 a b_2$ for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$.

If $\mathcal{B}$, $\mathcal{A}$ and $E_{\mathcal{B}}$ are as above we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is a $\mathcal{B}$-probability space. If $\phi : \mathcal{A} \to \mathbb{C}$ is a state on $\mathcal{A}$ which respects $E_{\mathcal{B}}$, i.e. $\tau = \tau \circ E_{\mathcal{B}}$, we say that $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ is compatible to the (non-amalgamated) free probability space $(\mathcal{A}, \phi)$.

If $(\mathcal{B} \subset A, E_{\mathcal{B}})$ is a $\mathcal{B}$-probability space and $a \in \mathcal{A}$ is a fixed variable, we define the amalgamated Cauchy transform of a by

$$G_a(b) = E_{\mathcal{B}}((b-a)^{-1}).$$

for $b \in \mathcal{B}$ and $b - a \in \mathcal{B}_{\text{inv}}$. The Cauchy transform is 1-1 in $\{b \in \mathcal{B}_{\text{inv}} | \, \|b^{-1}\| < \epsilon\}$ for ϵ sufficiently small and Voiculescu's amalgamated $\mathcal{R}$-transform [13] is now defined for $a \in \mathcal{A}$ by

$$\mathcal{R}_a(b) = G_a^{\langle -1 \rangle}(b) - b^{-1}, \tag{2.1}$$

for b being an invertible element of $\mathcal{B}$ suitably close to zero. It turns out that this definition coincides on invertible element with Speicher's definition of the amalgamated $\mathcal{R}$-transform (cf. [11, Th. 4.1.2] and [2]);

$$\mathcal{R}_a(b) = \sum_{n=1}^{\infty} \kappa_n^{\mathcal{B}}(a \otimes_{\mathcal{B}} ba \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} ba). \tag{2.2}$$

We will need the following useful lemma for solving equations involving the amalgamated R-transform and Cauchy-transform.

Lemma 2.1. *Let* $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ *be a* $\mathcal{B}$*-probability space, and let* $a \in \mathcal{A}$*. Then there exists* $\delta > 0$ *such that if* $b \in \mathcal{B}$ *is invertible,* $\|b\| < \delta$, $|\mu| > \frac{1}{\delta}$ *and*

$$\mathcal{R}_a^{\mathcal{B}}(b) + b^{-1} = \mu 1_{\mathcal{A}}$$

then $b = G_a^{\mathcal{B}}(\mu 1_{\mathcal{A}})$.

Proof. Let $\delta = \frac{1}{11\|a\|}$ and define $g_b(b) = G_a^{\mathcal{B}}(b^{-1})$. By [2, Prop. 2.3] we know that g_a maps $\mathcal{B}(0, \frac{1}{4\|a\|})$ bijectively onto a neighboorhood of zero containing $\mathcal{B}(0, \frac{1}{11\|a\|})$ and furthermore that

$$g_a^{\langle -1 \rangle}\left(\mathcal{B}(0, \tfrac{1}{11\|a\|})_{\text{inv}}\right) \subseteq \mathcal{B}(0, \tfrac{2}{11\|a\|})_{\text{inv}}.$$

By definition we know that

$$\mathcal{R}_a^{\mathcal{B}}(b) = G_a^{\mathcal{B}\langle -1 \rangle}(b) + b^{-1} = \left(g_a^{\langle -1 \rangle}(b)\right)^{-1} + b^{-1}$$

so if $\mathcal{R}_a(b) + b^{-1} = \mu 1_{\mathcal{A}}$ then

$$\mu 1_{\mathcal{A}} = g^{\langle -1 \rangle}(b) - b^{-1} + b^{-1} = \left(g_a^{\langle -1 \rangle}(b)\right)^{-1}$$

and thus

$$g_a^{\langle -1 \rangle}(b) = \tfrac{1}{\mu} 1_{\mathcal{A}}. \tag{2.3}$$

If $|\mu| > \frac{1}{\delta}$ then especially $\frac{1}{|\mu|} < \frac{1}{4\|a\|}$ so $\frac{1}{\mu} 1_{\mathcal{A}}$ is in the bijective domain of g_a, so applying g_a on both sides of (2.3) we get exactly

$$G_a^{\mathcal{B}}(\mu 1_{\mathcal{A}}) = g_a(\tfrac{1}{\mu} 1_{\mathcal{A}}) = b$$

since also $\|b\| < \frac{1}{11\|a\|}$. □

If $a \in \mathcal{A}$ is a random variable in the $\mathcal{B}$-probability space $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$, then following Speicher we define a to be $\mathcal{B}$-Gaussian [11, Def 4.2.3] if only $\mathcal{B}$-cumulants of length 2 survive. From (2.2) it follows that in this case the $\mathcal{R}$-transform has a particularly simple form, namely,

$$\mathcal{R}_a(b) = \kappa_2^{\mathcal{B}}(a \otimes_{\mathcal{B}} ba) = E_{\mathcal{B}}(aba). \tag{2.4}$$

In the following theorem (which is probably not a new one we just could not find a proper reference) concerning cumulants we have adopted the notation of Speicher from [11].

Lemma 2.2. *Let* $N \in \mathbb{N}$ *and let* $(\mathcal{B} \subset \mathcal{A}, E_{\mathcal{B}})$ *be a* $\mathcal{B}$*-probability space. Then* $(M_N(\mathcal{B}) \subset M_N(\mathcal{A}), E_{M_n(\mathcal{B})})$ *is a* $M_N(\mathcal{B})$*-probability space with* *cumulants determined by the following formula:*

$$\kappa_n^{M_N(\mathcal{B})}((m_1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (m_n \otimes a_n)) \\ = (m_1 \cdots m_n) \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n)$$

when $m_1, \ldots, m_n \in M_N(\mathbb{C})$ *and* $a_1, \ldots, a_n \in \mathcal{A}$.

We have of course made the identification $M_N(\mathcal{A}) \cong M_N(\mathbb{C}) \otimes \mathcal{A}$.

Proof. Since $M_N(\mathbb{C}) \subset M_N(\mathcal{B})$ we observe that

$$\kappa_n^{M_N(\mathcal{B})}((m_1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (m_n \otimes a_n)) \\ = ((m_1 \cdots m_n) \otimes 1) \cdot \\ \kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)).$$

To finish the proof we claim that

$$\kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)) = \\ 1 \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n). \tag{2.5}$$

The case $n = 1$ is obvious since

$$1_N \otimes \kappa_1^{\mathcal{B}}(a_1) = 1_N \otimes E_{\mathcal{B}}(a_1) = E_{M_N(\mathcal{B})}(1 \otimes a_1) = \kappa_1^{M_N(\mathcal{B})}(1 \otimes a_1).$$

Now assume that the claim is true for $1, 2, \ldots, n-1$. Then (2.5) has an obvious extension to noncrossing partions of length less than or equal to $n-1$. Hence

$$1_N \otimes \kappa_n^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n) \\ = 1_N \otimes E_{\mathcal{B}}(a_1 \cdots a_n) - \sum_{\pi \in NC(n), \pi \neq 1_n} 1 \otimes \kappa_\pi^{\mathcal{B}}(a_1 \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} a_n) \\ = E_{M_N(\mathcal{B})}(1 \otimes_{M_N(\mathcal{B})} a_1 \cdots a_n) \\ - \sum_{\pi \in NC(n), \pi \neq 1_n} \kappa_\pi^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)) \\ = \kappa_n^{M_N(\mathcal{B})}((1 \otimes a_1) \otimes_{M_N(\mathcal{B})} \cdots \otimes_{M_N(\mathcal{B})} (1 \otimes a_n)).$$

By induction this proves the lemma. □

Assume that $\mathcal{M}$ contains a pair (D_0, X) of τ-free selfadjoint elements such that $\mathrm{d}\mu_{\mathrm{D}_0}(\mathrm{t}) = 1_{[0,1]}(\mathrm{t})\mathrm{dt}$ and X is a semicircular distributed. Put $\mathcal{D} = W^*(D_0)$. Then $\lambda : L^\infty([0,1]) \to \mathcal{D}$ given by

$$\lambda(f) = f(D_0),$$

for $f \in L^\infty([0,1])$ is a $*$-isomorphism of $L^\infty([0,1])$ onto $\mathcal{D}$ and

$$\tau \circ \lambda(f) = \int_0^1 f(t)\mathrm{dt}, \quad \mathrm{f} \in \mathrm{L}^\infty([0,1]).$$

We will identify $\mathcal{D}$ with $L^\infty([0,1])$ and thus consider elements of $\mathcal{D}$ as functions. As explained in the introduction, we can realize the quasi-nilpotent DT-operator as the operator $T = \mathcal{UT}(X,\lambda)$ in $W^*(D_0, X) \simeq L(\mathbb{F}_2)$.

Define for $f \in \mathcal{D} \simeq L^\infty([0,1])$

$$(L^*(f))(x) := \int_0^x f(t)\mathrm{d}t \quad \text{and} \quad (L(f))(x) := \int_x^1 f(t)\mathrm{d}t. \tag{2.6}$$

From the appendix of [5] it follows that (T, T^*) is a $\mathcal{D}$-Gaussian pair and that the covariances of (T, T^*) are given by the following lemma

Lemma 2.3. [5, Appendix] *Let $f \in \mathcal{D}$. Then*

$$E_{\mathcal{D}}(TfT^*) = L(f) \quad \textit{and} \quad E_{\mathcal{D}}(T^*fT) = L^*(f)$$

and $E_{\mathcal{D}}(TfT) = E_{\mathcal{D}}(T^*fT^*) = 0.$

3. Moments and $\mathcal{R}$-transform of $(T - \lambda 1)^*(T - \lambda 1)$

Let T be the quasi-diagonal DT-operator and define

$$\tilde{T} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

Since (T, T^*) is a $\mathcal{D}$-Gaussian pair, it follows from lemma 2.2, that cumulants of the form

$$\kappa_n^{M_2(\mathcal{D})}((m_1 \otimes a_1) \otimes_{M_2(\mathcal{D})} \cdots \otimes_{M_2(\mathcal{D})} (m_n \otimes a_n))$$

vanishes when $n \neq 2$, $m_1, m_2, \dots, m_n \in M_2(\mathbb{C})$ and $a_1, a_2, \dots, a_n \in \{T, T^*\}$. Hence by the linearity of $\kappa_n^{M_2(\mathcal{D})}$,

$$\kappa_n^{M_2(\mathcal{D})}(\tilde{T} \otimes_{M_2(\mathcal{D})} \tilde{T} \otimes_{M_2(\mathcal{D})} \cdots \otimes_{M_2(\mathcal{D})} \tilde{T}) = 0$$

when $n \neq 2$, i.e. $\tilde{T}$ is a $M_2(\mathcal{D})$-Gaussian element in $M_2(\mathcal{M})$ under the conditional expectation $E_{M_2(\mathcal{D})} : M_2(\mathcal{M}) \to M_2(\mathcal{D})$ given by

$$E_{M_2(\mathcal{D})} : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{pmatrix} E_{\mathcal{D}}(a_{11}) & E_{\mathcal{D}}(a_{12}) \\ E_{\mathcal{D}}(a_{21}) & E_{\mathcal{D}}(a_{22}) \end{pmatrix}.$$

Since $\tilde{T}$ is $M_2(\mathcal{D})$-Gaussian the $\mathcal{R}$-transform of $\tilde{T}$ is by (2.4) the linear mapping $M_2(\mathcal{D}) \to M_2(\mathcal{D})$ given by

$$\begin{aligned}\mathcal{R}_{\tilde{T}}^{M_2(\mathcal{D})}(z) &= E_{M_2(\mathcal{D})}(\tilde{T} z \tilde{T}) \\ &= E_{M_2(\mathcal{D})}\left(\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}\right) \\ &= E_{M_2(\mathcal{D})}\left(\begin{pmatrix} T^* z_{22} T & 0 \\ 0 & T z_{11} T^* \end{pmatrix}\right) \\ &= \begin{pmatrix} E_{\mathcal{D}}(T^* z_{22} T) & 0 \\ 0 & E_{\mathcal{D}}(T z_{11} T^*) \end{pmatrix} \\ &= \begin{pmatrix} L^*(z_{22}) & 0 \\ 0 & L(z_{11}) \end{pmatrix}.\end{aligned}$$

For $\lambda \in \mathbb{C}$, we put $T_\lambda - T_\lambda 1$ and define

$$\tilde{T}_\lambda = \begin{pmatrix} 0 & T_\lambda^* \\ T_\lambda & 0 \end{pmatrix} = \tilde{T} - \begin{pmatrix} 0 & \overline{\lambda} 1 \\ \lambda 1 & 0 \end{pmatrix}$$

Since $\begin{pmatrix} 0 & \overline{\lambda} 1 \\ \lambda 1 & 0 \end{pmatrix} \in M_2(\mathcal{D})$ we have by $M_2(\mathcal{D})$-freeness that the $\mathcal{R}$-transform is additive [11, Th. 4.1.22] i.e.

$$\mathcal{R}_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(z) = \mathcal{R}_{\tilde{T}}^{M_2(\mathcal{D})} - \begin{pmatrix} 0 & \overline{\lambda} 1 \\ \lambda 1 & 0 \end{pmatrix} = \begin{pmatrix} L^*(z_{22}) & -\overline{\lambda} 1 \\ -\lambda 1 & L(z_{11}) \end{pmatrix}.$$

One easily checks, that if $\delta \in \mathbb{C}$, $\delta \neq 0$, $\delta \neq -\frac{1}{|\lambda|^2}$ and $\mu \in \mathbb{C}$ is one of the two solutions to

$$\mu^2 = \frac{e^\sigma}{\sigma}(1 + |\lambda|^2 \sigma),$$

then

$$\begin{cases} z_{11} = \mu\sigma e^{\sigma(x-1)} \\ z_{12} = -\overline{\lambda}\sigma \\ z_{21} = -\lambda\sigma \\ z_{22} = \mu\sigma e^{-\sigma x} \end{cases} \tag{3.1}$$

is a solution to

$$\mathcal{R}_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(z) + z^{-1} = \mu 1_2.$$

Here x is the variable for the function in $\mathcal{D} = L^\infty([0,1])$. In particular z_{12} and z_{21} are constant operators. If $\sigma \to 0$ then $|\mu| \to \infty$ and $\|z\| \to 0$, so by lemma 2.1 there exists $\rho > 0$ such that $|\sigma| < \rho$ implies

$$G_{\tilde{T}_\lambda}^{M_2(\mathcal{D})}(\mu 1_2) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix},$$

where $(z_{ij})_{i,j\in\{1,2\}}$ is given by (3.1) and

$$\mu = \pm\sqrt{\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma)}.$$

On the other hand the Cauchy-transform of $\tilde{T}$ in $\mu 1_2$ is

$$\begin{aligned}\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix} &= G^{M_2(\mathfrak{D})}_{\tilde{T}_\lambda}(\mu 1_2) \\ &= E_{M_2(\mathfrak{D})}\left(\left(\begin{pmatrix}\mu 1 & 0 \\ 0 & \mu 1\end{pmatrix} - \begin{pmatrix} 0 & T_\lambda^* \\ T_\lambda & 0\end{pmatrix}\right)^{-1}\right) \\ &= E_{M_2(\mathfrak{D})}\left(\begin{pmatrix}\mu 1 & -T_\lambda^* \\ -T_\lambda & \mu 1\end{pmatrix}^{-1}\right) \\ &= E_{M_2(\mathfrak{D})}\left(\begin{pmatrix}\mu(\mu^2 1 - T_\lambda^* T_\lambda)^{-1} & T_\lambda^*(\mu^2 1 - T_\lambda T_\lambda^*)^{-1} \\ T_\lambda(\mu^2 1 - T_\lambda^* T_\lambda)^{-1} & \mu(\mu^2 1 - T_\lambda T_\lambda^*)^{-1}\end{pmatrix}\right).\end{aligned}$$

Thus

$$\begin{cases} z_{11} = \mu E_{\mathfrak{D}}((\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) \\ z_{12} = E_{\mathfrak{D}}(T_\lambda^*(\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) \\ z_{21} = E_{\mathfrak{D}}(T_\lambda(\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) \\ z_{22} = \mu E_{\mathfrak{D}}((\mu^2 1 - T_\lambda T_\lambda^*)^{-1})\end{cases}. \tag{3.2}$$

Combining (3.1) and (3.2) we have

$$\begin{cases} E_{\mathfrak{D}}((\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) = \sigma\mathrm{e}^{\sigma(x-1)} \\ E_{\mathfrak{D}}(T_\lambda^*(\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) = -\overline{\lambda}\sigma \\ E_{\mathfrak{D}}(T_\lambda(\mu^2 1 - T_\lambda^* T_\lambda)^{-1}) = -\lambda\sigma \\ E_{\mathfrak{D}}((\mu^2 1 - T_\lambda T_\lambda^*)^{-1}) = \sigma\mathrm{e}^{-\sigma x}\end{cases}. \tag{3.3}$$

We can now compute the $\mathcal{R}$-transform of $T_\lambda^* T_\lambda$ (wrt. $\mathbb{C}$) from (3.3) and the defining equality for μ^2.

$$\begin{aligned}\mathrm{tr}\left(\left(\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma)1 - T_\lambda^* T_\lambda\right)^{-1}\right) &= \int_0^1 \sigma\mathrm{e}^{\sigma(x-1)}\mathrm{d}x \\ &= \left[\mathrm{e}^{\sigma(x-1)}\right]_0^1 = 1 - \mathrm{e}^{-\sigma}.\end{aligned}$$

Thus

$$G^{\mathbb{C}}_{T_\lambda^* T_\lambda}\left(\frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma)\right) = 1 - \mathrm{e}^{-\sigma}$$

i.e.

$$\mathcal{R}^{\mathbb{C}}_{T_\lambda^* T_\lambda}(1-\mathrm{e}^{-\sigma}) = \frac{\mathrm{e}^{\sigma}}{\sigma}(1+|\lambda|^2\sigma) - \frac{1}{1-\mathrm{e}^{-\sigma}}$$

for σ in a neighboorhood of zero. Substituting $z = 1 - \mathrm{e}^{-\sigma}$ we get $\sigma = -\log(1-z)$, so

$$\mathcal{R}^{\mathbb{C}}_{T^*_\lambda T_\lambda}(z) = -\frac{1}{(1-z)\log(1-z)}(1 - |\lambda|^2 \log(1-z)) - \frac{1}{z}.$$

Hence we have proved the following extension of [4, Theorem 8.7(b)]:

Theorem 3.1. *Let T be the quasinilpotent* DT-operator. Let $\lambda \in \mathbb{C}$ and put $T_\lambda = T - \lambda 1$. Then

$$\mathcal{R}^{\mathbb{C}}_{T^*_\lambda T_\lambda}(z) = -\frac{1}{(1-z)\log(1-z)} - \frac{1}{z} + \frac{|\lambda|^2}{1-z}$$

for z in some neighborhood of 0.

We next determine the $\mathcal{D}$-valued (resp. $\mathbb{C}$-valued) moments of $T^*_\lambda T_\lambda$ for all $\lambda \in \mathbb{C}$. The special case $\lambda = 0$ was treated in [9, Theorem 5] (resp. [4, Theorem 8.7(a)]) by different methods.

Theorem 3.2. *Let $\lambda \in \mathbb{C}$ and let T, T_λ be as in theorem 3.1*

(a) Let Q_n be the sequence of polynomials on $\mathbb{R}$ uniquely determined by the following recursion formula

$$\begin{cases} Q_0(x) = 1, \\ Q_{n+1}(x) = |\lambda|^2 Q_n(x+1) + \int_0^x Q_n(y+1)\,dy \quad \text{for } n \geq 1. \end{cases} \tag{3.4}$$

Then

$$E_{\mathcal{D}}((T^*_\lambda T_\lambda)^n)(x) = Q_n(x), \quad x \in [0,1], \quad n \in \mathbb{N}.$$

(b)

$$\tau((T^*_\lambda T_\lambda)^n) = \sum_{k=0}^{n} \frac{n^k}{(k+1)!}\binom{n}{k}|\lambda|^{2n-2k}, \quad n \in \mathbb{N}.$$

Proof. By (3.3), we have

$$E_{\mathcal{D}}\big((\tfrac{\mathrm{e}^\sigma}{\sigma}(1 + |\lambda|^2\sigma)1 - T^*_\lambda T_\lambda)^{-1}\big) = \sigma \mathrm{e}^{\sigma(x-1)} \tag{3.5}$$

for $\sigma \in B(0,\rho) \setminus \{0\}$ for some $\rho > 0$. Put

$$\psi(\sigma) = \frac{\sigma}{\mathrm{e}^\sigma(1 + |\lambda|^2\sigma)}, \quad \sigma \in \mathbb{C} \setminus \{-\tfrac{1}{|\lambda|^2}\}.$$

Since $\psi(0) = 0$ and $\psi'(0) = 1$, ψ has an analytic invers $\psi^{\langle -1\rangle}$ defined in a neighborhood $B(0,\delta)$ of 0, and we can choose $\delta > 0$, such that $\psi^{\langle -1\rangle}(B(0,\delta)) \subset B(0,\rho)$. By (3.5)

$$E_{\mathcal{D}}((\tfrac{1}{t}1 - T^*_\lambda T_\lambda)^{-1}) = \psi^{\langle -1\rangle}(t)\mathrm{e}^{\psi^{\langle -1\rangle}(t)(x-1)}$$

for $t \in B(0,\delta) \setminus \{0\}$. By power series expansion of the left hand side, we get

$$\sum_{n=0}^{\infty} t^{n+1} E_{\mathfrak{D}}((T_\lambda^* T_\lambda)^n) = \psi^{\langle -1\rangle}(t)\mathrm{e}^{\psi^{\langle -1\rangle}(t)(x-1)} \tag{3.6}$$

for $t \in B(0,\delta')$, where $0 < \delta' \le \delta$ and where the LHS of (3.6) is absolutely convergent in the Banach space $L^\infty([0,1])$. Hence by Cauchy's integral formulas

$$E_{\mathfrak{D}}((T_\lambda^* T_\lambda)^n) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\psi^{\langle -1\rangle}(t)\mathrm{e}^{\psi^{\langle -1\rangle}(t)(x-1)}}{t^{n+2}} \mathrm{d}t \tag{3.7}$$

as a Banach space integral in $L^\infty([0,1])$, where $C = \partial B(0,r)$ with positive orientation and $0 < r < \delta'$. For each fixed $x \in \mathbb{R}$

$$t \mapsto \psi^{\langle -1\rangle}(t)\mathrm{e}^{\psi^{\langle -1\rangle}(t)(x-1)}$$

is an analytic function in $B(0,\delta')$ which is 0 for $t = 0$. Hence the function has a power series expansion of the form

$$\psi^{\langle -1\rangle}(t)\mathrm{e}^{\psi^{\langle -1\rangle}(t)(x-1)} = \sum_{n=0}^{\infty} Q_n(x) t^{n+1} \tag{3.8}$$

for $t \in B(0,\delta')$, where the numbers $(Q_n(x))_{n=0}^\infty$ are given by

$$Q_n(x) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\psi^{\langle -1\rangle}(t)\mathrm{e}^{\psi^{\langle -1\rangle}(t)(x-1)}}{t^{n+2}} \mathrm{d}t. \tag{3.9}$$

In particular the Q_n's are continuous functions of $x \in \mathbb{R}$. Substituting $\sigma = \psi(t)$ in (3.8) we get

$$\sum_{n=0}^{\infty} Q_n(x)\psi(\sigma)^{n+1} = \sigma \mathrm{e}^{\sigma(x-1)}$$

for $\sigma \in B(0,\rho')$, where $\rho' \in (0,\rho)$. Put

$$\begin{cases} R_0(x) = 0 \\ R_{n+1}(x) = |\lambda|^2 Q_n(x+1) + \int_0^x Q_n(x)\mathrm{d}y, \quad n \ge 0. \end{cases}$$

Then

$$\begin{aligned}
\sum_{n=0}^{\infty} R_n(x)\psi(\sigma)^{n+1} &= \psi(\sigma)\left(1+\sum_{n=0}^{\infty} R_{n+1}(x)\psi(\sigma)^{n+1}\right) \\
&= \psi(\sigma)\left(1+|\lambda|^2\left(\sum_{n+0}^{\infty} Q_n(x+1)\right)+\int_0^x\left(\sum_{n+0}^{\infty} Q_n(y+1)\right)\mathrm{dy}\right) \\
&= \psi(\sigma)\left(1+|\lambda|^2\sigma \mathrm{e}^{\sigma x}+\int_0^x \sigma \mathrm{e}^{\sigma y}\mathrm{dy}\right) \\
&= \psi(\sigma)(|\lambda|^2\sigma+1)\mathrm{e}^{\sigma x} = \sigma \mathrm{e}^{\sigma(x-1)} = \sum_{n=0}^{\infty} Q_n(x)\psi(\sigma)^{n+1}
\end{aligned}$$

for all $\sigma \in B(0,\rho')$. Since $\psi(B(0,\rho'))$ is an open neighborhood of 0 in $\mathbb{C}$, it follows that $R_n(x) = Q_n(x)$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$.

Hence $(Q_n(x))_{n=0}^{\infty}$ is the sequence of polynomials given by the recursive formula (3.4). Moreover by (3.7) and (3.9), $E_{\mathfrak{D}}((T_\lambda^* T_\lambda)^n) = Q_n$ as functions in $L^\infty([0,1])$. This proves (a).

(b) By (3.7), we have

$$\tau((T_\lambda^* T_\lambda)^n) = \int_0^1 E_{\mathfrak{D}}((T_\lambda^* T_\lambda)^n)\mathrm{dx} = \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{C}} \frac{1-\mathrm{e}^{-\psi^{\langle -1\rangle}(\mathrm{t})}}{\mathrm{t}^{\mathrm{n}+2}}\mathrm{dt}.$$

Note that $C' = \psi(C)$ is a positively oriented simple path around 0. Hence by the substitution $t = \psi(\sigma)$, we get

$$\begin{aligned}
\tau((T_\lambda^* T_\lambda)^n) &= \frac{1}{2\pi\mathrm{i}}\int_{C'} \frac{\psi'(\sigma)}{\psi(\sigma)^{n+2}}(1-\mathrm{e}^{-\sigma})\mathrm{d}\sigma \\
&= \frac{1}{2\pi\mathrm{i}}\int_{C'} \frac{1}{n+1}\frac{1}{\psi(\sigma)^{n+1}}\frac{\mathrm{d}}{\mathrm{d}\sigma}(1-\mathrm{e}^{-\sigma})\mathrm{d}\sigma \\
&= \frac{1}{2\pi\mathrm{i}(n+1)}\int_{C'} \frac{1}{\psi(\sigma)^{n+1}}\mathrm{e}^{-\sigma}\mathrm{d}\sigma \\
&= \frac{1}{n+1}\left(\frac{1}{2\pi\mathrm{i}}\int_{C'} \frac{\mathrm{e}^{n\sigma}(1+|\lambda|^2\sigma)^{n+1}}{\sigma^{n+1}}\mathrm{d}\sigma\right) \\
&= \frac{1}{n+1}\mathrm{Res}\left(\frac{\mathrm{e}^{n\sigma}(1+|\lambda|^2\sigma)^{n+1}}{\sigma^{n+1}},0\right)
\end{aligned}$$

where the second equation is obtained by partial integration and the last equality is obtained by the Residue theorem.

The above Residue is equal to the coefficient of σ^n in the Power series expansion of

$$e^{n\sigma}(1+|\lambda|^2\sigma)^{-1} = \left(\sum_{k=0}^{\infty}\frac{(n\sigma)^k}{k!}\right)\left(\sum_{i=1}^{n+1}\binom{n+i}{i}(|\lambda|^2\sigma)^i\right).$$

Hence

$$\begin{aligned}\tau((T_\lambda^*T_\lambda)^n) &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{n^k}{k!}\binom{n+1}{n-k}|\lambda|^{2(n-k)} \\ &= \frac{1}{n+1}\sum_{k=0}^{n}\frac{n^k}{(k+1)!}\binom{n}{k}|\lambda|^{2n-2k}.\end{aligned}$$

□

4. Spectrum and Brown-measure of $T+\sqrt{\epsilon}Y$

Let T be the quasinilpotent DT-operator and let Y be a circular operator $*$-free from T. In this section we will show, that

$$\sigma(T+\sqrt{\epsilon}Y) = \overline{B}\Big(0, \frac{1}{\sqrt{\log(1+\epsilon^{-1})}}\Big)$$

and that the Brown-measure $\mu_{T+\sqrt{\epsilon}Y}$ is equal to the uniform distribution on $\overline{B}\left(0, \frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}\right)$, i.e. it has constant density w.r.t. the Lebesque measure on this disk.

Theorem 4.1. *For every* $\epsilon > 0$

$$\sigma(T+\sqrt{\epsilon}Y) = \overline{B}\Big(0, \frac{1}{\sqrt{\log(1+\epsilon^{-1})}}\Big). \tag{4.1}$$

Proof. The result can be obtained by the method of Biane and Lehner [3, Section 5]. Let $a \in \mathbb{C}\setminus\{0\}$. Since $\sigma(T) = \{0\}$ we can write

$$a1-(T+\sqrt{\epsilon})Y = \sqrt{\epsilon}\big(\tfrac{1}{\sqrt{\epsilon}}1 - Y(a1-T)^{-1}\big)(a1-T).$$

Hence

$$a \notin \sigma(T+\sqrt{\epsilon}Y) \text{ iff } \frac{1}{\sqrt{\epsilon}} \notin \sigma\big(Y(a1-T)^{-1}\big). \tag{4.2}$$

Let $Y = UH$ be the polar decomposition of Y. Then $Y(a1-T)^{-1} = UH(a1-T)^{-1}$, where U is $*$-free from $H(a1-T)^{-1}$. Hence $Y(a1-T)^{-1}$ is R-diagonal. Moreover, since $0 \notin \sigma(Y)$, $Y(a1-T)^{-1}$ is not invertible, so by [7, Prop. 4.6.(ii)]

$$\sigma\big(Y(a1-T)^{-1}\big) = B\big(0, \|Y(a1-T)^{-1}\|_2\big). \tag{4.3}$$

By $*$-freeness of Y and $(a1-T)^{-1}$ we have

$$\|Y(a1-T)^{-1}\|_2^2 = \|Y\|_2^2 \|(a1-T)^{-1}\|_2^2 = \|(a1-T)^{-1}\|_2^2 = \left\|\sum_{n=0}^{\infty} \frac{T^n}{a^{n+1}}\right\|_2^2. \tag{4.4}$$

Applying now [4, lemma 7.2] to $D=1$ and $\lambda = \frac{1}{a}$ and $\mu = \delta_0$, we get

$$\left\|\sum_{n=0}^{\infty} \frac{T^n}{a^n}\right\|_2^2 = |a|^2 \left(\exp\left(\frac{1}{|a|^2}\right) - 1\right)$$

Hence by (4.4)

$$\|Y(a1-T)^{-1}\|_2^2 = \exp\left(\frac{1}{|a|^2}\right) - 1.$$

Thus for $a \in \mathbb{C} \setminus \{0\}$ we get by (4.2) and (4.3)

$$a \notin \sigma(T + \sqrt{\epsilon}Y) \Leftrightarrow \frac{1}{\sqrt{\epsilon}} \notin \sigma\big(Y(a1-T)^{-1}\big) \Leftrightarrow \frac{1}{\sqrt{\epsilon}} > \exp\left(\frac{1}{|a|^2}\right) - 1 \Leftrightarrow |a| > \frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}.$$

Hence $\sigma(T+\sqrt{\epsilon}Y) \cup \{0\} = \overline{B}\left(0, \frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}\right)$. Since $\sigma(T+\sqrt{\epsilon}Y)$ is closed it follows that $\sigma(T+\sqrt{\epsilon}Y) = \overline{B}\left(0, \frac{1}{\sqrt{\log(1+\frac{1}{\epsilon})}}\right)$. □

In order to compute the Brown measure of $T+\sqrt{\epsilon}Y$, we first observe that $T+\sqrt{\epsilon}Y$ has the same $*$-distribution as

$$S = \sqrt{a}T_1 + \sqrt{b}T_2^*$$

when T_1 and T_2 are two $\mathcal{D}$-free quasidiagonal operators and $a = 1+\epsilon$ and $b = \epsilon$ [1]. We next compute the Brown measure of S for all values of $a, b \in (0, \infty)$.

Lemma 4.2. *Let μ_Q be the Brown measure of an operator Q in a tracial W^*-probability space (M, tr). Let $r > 0$ and assume that $\mu_Q(\partial B(0,r)) = 0$. Then*

$$\mu_Q(B(0,r)) = -\frac{1}{2\pi} \lim_{\alpha \to 0^+} \Im \left(\int_{\partial B(0,r)} \mathrm{tr}((\mathrm{Q}_\lambda^* \mathrm{Q}_\lambda + \alpha 1)^{-1} \mathrm{Q}_\lambda^*) \mathrm{d}\lambda \right)$$

where $Q_\lambda = Q - \lambda 1$ for $\lambda \in \mathbb{C}$.

Proof. Let $\Delta : M \to [0,\infty)$ be the Fuglede-Kadison determinant on M, and put $L(\lambda) = \log \Delta(Q_\lambda)$ and

$$L_\alpha(\lambda) = \log \Delta((Q_\lambda^* Q_\lambda + \alpha 1)^{1/2}) = \frac{1}{2}\mathrm{tr}(\log(Q_\lambda^* Q_\lambda + \alpha 1))$$

for $\lambda \in \mathbb{C}$.

Put $\lambda_1 = \Re\lambda$, $\lambda_2 = \Im\lambda$ and let $\nabla^2 = \frac{\partial^2}{\partial\lambda_1^2} + \frac{\partial^2}{\partial\lambda_2^2}$ denote the Laplace operator on $\mathbb{C}$. Then by [6, Section 2] $\nabla^2 L_\alpha \geq 0$ and for each $\alpha > 0$, the measure

$$\mu_\alpha = \frac{1}{2\pi}\nabla^2 L_\alpha(\lambda)\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \tag{4.5}$$

is a probability measure on $\mathbb{C}$. Moreover

$$\lim_{\alpha\to 0}\mu_\alpha = \mu \tag{4.6}$$

in the weak* topology on $\mathrm{Prob}(\mathbb{C})$. Also from [6, Section 2] the gradient $(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2})$ of L_α is given by

$$\frac{\partial}{\partial\lambda_1}L_\alpha(\lambda) = -\Re\big(\mathrm{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1}\big) \tag{4.7}$$

$$\frac{\partial}{\partial\lambda_2}L_\alpha(\lambda) = -\Im\big(\mathrm{tr}(Q_\lambda(Q_\lambda^* Q_\lambda + \alpha 1)^{-1}\big) \tag{4.8}$$

By (4.6)

$$\lim_{\alpha\to 0}\int_{\mathbb{C}}\phi\mathrm{d}\mu_\alpha = \int_{\mathbb{C}}\phi\mathrm{d}\mu$$

for all $\phi \in C_0(\mathbb{C})$. Since $1_{B(0,r)}$ is the limit of an increasing sequence $(\phi_n)_{n=1}^\infty$ of $C_0(\mathbb{C})$-functions with $0 \leq \phi_n \leq 1$ for all $n \in \mathbb{N}$ it follows that

$$\begin{aligned}\mu_Q(B(0,r)) &= \lim_{n\to\infty}\int_{\mathbb{C}}\phi_n\mathrm{d}\mu_Q \\ &= \lim_{n\to\infty}\left(\lim_{\alpha\to 0}\int_{\mathbb{C}}\phi_n\mathrm{d}\mu_\alpha\right) \leq \lim_{n\to\infty}\left(\liminf_{\alpha\to 0}\int_{\mathbb{C}}1_{B(0,r)}\mathrm{d}\mu_\alpha\right) \\ &= \liminf_{\alpha\to 0}\mu_\alpha(B(0,r))\end{aligned}$$

Writing $1_{\overline{B(0,r)}}$ as the limit of a decreasing sequence $(\psi_n)_{n=1}^\infty$ of $C_0(\mathbb{C})$-functions, with $0 \leq \psi_n \leq 1$, one gets in the same way

$$\mu_Q(\overline{B}(0,r)) \geq \limsup_{\alpha\to 0}\mu_\alpha(\overline{B}(0,r))$$

Hence if $\mu_Q(\partial B(0,r)) = 0$ we have

$$\limsup_{\alpha\to 0}\mu_\alpha(B(0,r)) \leq \mu_Q(B(0,r)) \leq \liminf_{\alpha\to 0}\mu_\alpha(B(0,r)),$$

and therefore

$$\mu_Q(B(0,r)) = \lim_{\alpha\to 0} \mu_\alpha(B(0,r)).$$

Using (4.5) together with Green's theorem applied to the vector-field $(P_\alpha, Q_\alpha) = (-\frac{\partial L_\alpha}{\partial\lambda_2}, \frac{\partial L_\alpha}{\partial\lambda_1})$ we get

$$\begin{aligned}
\mu_\alpha(B(0,r)) &= \frac{1}{2\pi}\int_{B(0,r)} \nabla^2 L_\alpha(\lambda)\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \\
&= \frac{1}{2\pi}\int_{B(0,r)} \left(\frac{\partial Q_\alpha}{\partial\lambda_1} - \frac{\partial P_\alpha}{\partial\lambda_2}\right)\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \\
&= \frac{1}{2\pi}\int_{\partial B(0,r)} P_\alpha\mathrm{d}\lambda_1 + Q_\alpha\mathrm{d}\lambda_2 \\
&= \frac{1}{2\pi}\int_{\partial B(0,r)} -\frac{\partial L_\alpha}{\partial\lambda_2}\mathrm{d}\lambda_1 + \frac{\partial L_\alpha}{\partial\lambda_1}\mathrm{d}\lambda_2 \\
&= \Im\left(\frac{1}{2\pi}\int_{\partial B(0,r)} \left(\frac{\partial L_\alpha}{\partial\lambda_1} - \mathrm{i}\frac{\partial L_\alpha}{\partial\lambda_2}\right)(\mathrm{d}\lambda_1 + \mathrm{i}\mathrm{d}\lambda_2)\right)
\end{aligned}$$

By (4.7) and (4.8)

$$\frac{\partial L_\alpha}{\partial\lambda_1} - \mathrm{i}\frac{\partial L_\alpha}{\partial\lambda_2} = -\overline{\mathrm{tr}(Q_\lambda(Q_\lambda^*Q_\lambda + \alpha 1)^{-1})} = -\mathrm{tr}((Q_\lambda^*Q_\lambda + \alpha 1)^{-1}Q_\lambda^*).$$

Hence

$$\mu_\alpha(B(0,r)) = -\Im\left(\frac{1}{2\pi}\int_{\partial B(0,r)} \mathrm{tr}((Q_\lambda^*Q_\lambda + \alpha 1)^{-1}Q_\lambda^*)\mathrm{d}\lambda\right)$$

which completes the proof of the lemma. □

Let $S = \sqrt{a}T_1 + \sqrt{b}T_2^*$ with $0 < b < a$. Since cS and S have the same $*$-distribution for all $c \in \mathbb{T}$, the Brown measure μ_S of S is rotation invariant (i.e. invariant under the transformation $z \mapsto cz,\ z \in \mathbb{C}$ when $|c| = 1$). Hence by lemma 4.2 we can compute μ_S, if we can determine

$$\mathrm{tr}((S_\lambda^*S_\lambda + \alpha 1)^{-1}S_\lambda^*)$$

for all $\lambda \in \mathbb{C}$, where $S_\lambda = S - \lambda 1$, and for all α in some interval of the form $(0, \alpha_0)$. This can be done by minor modifications of the methods used in section 3:

Put

$$\tilde{S}_\lambda = \begin{pmatrix} 0 & S_\lambda^* \\ S_\lambda & 0 \end{pmatrix}.$$

Then there exists a $\delta > 0$ (depending on a, b and γ) such that when $\|z\| \le \delta$ and $|\mu| > \frac{1}{\delta}$ the equality

$$\mathcal{R}_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(z) + z^{-1} = \mu 1_2 \tag{4.9}$$

implies that

$$(4.10)\quad z = G^{M_2(\mathcal{D})}_{\tilde{S}_\lambda}(\mu 1_2) = (\mathrm{id}\otimes E_{\mathcal{D}})\begin{pmatrix} \mu(\mu^2 1 - S_\lambda^* S_\lambda)^{-1} & S_\lambda^*(\mu^2 1 - S_\lambda S_\lambda^*)^{-1} \\ S_\lambda(\mu^2 1 - S_\lambda^* S_\lambda)^{-1} & \mu(\mu^2 1 - S_\lambda S_\lambda^*)^{-1}\end{pmatrix}.$$

Moreover, $\tilde{S} = \begin{pmatrix} 0 & S^* \\ S & 0\end{pmatrix}$ is $M_2(\mathcal{D})$-Gaussian by lemma 2.2 since (T_1, T_2^*, T_2, T_2^*) is a $\mathcal{D}$-Gaussian set. Hence for $z = (z_{ij})_{i,j=1}^2 \in M_2(\mathcal{D})$,

$$\mathcal{R}^{M_2(\mathcal{D})}_{\tilde{S}}(z) = E_{M_2(\mathcal{D})}(\tilde{S} z \tilde{S}) = \begin{pmatrix} E_{\mathcal{D}}(S^* z_{22} S) & 0 \\ 0 & E_{\mathcal{D}}(S z_{11} S^*)\end{pmatrix}.$$

Using that (T_1, T_1^*) and (T_2, T_2^*) have the same $\mathcal{D}$-distribution as (T, T^*) and that (T_1, T_1^*) and (T_2, T_2^*) are two $\mathcal{D}$-free sets, we get

$$\begin{aligned} E_{\mathcal{D}}(S^* z_{22} S) &= (aL^* + bL)(z_{22}) \\ E_{\mathcal{D}}(S z_{11} S^*) &= (aL + bL^*)(z_{11}), \end{aligned}$$

where $L(f) : x \mapsto \int_x^1 f(y)\mathrm{d}y$ and $L^*(f) : x \mapsto \int_0^x f(y)\mathrm{d}y$ for $f \in \mathcal{D}$.

Since $\tilde{S}_\lambda = \tilde{S} - \begin{pmatrix} 0 & \overline{\lambda} 1 \\ \lambda 1 & 0\end{pmatrix}$ it follows that

$$\mathcal{R}^{M_2(\mathcal{D})}_{\tilde{S}}(z) = \begin{pmatrix} (aL + bL^*)(z_{22}) & -\overline{\lambda} 1 \\ \lambda 1 & (aL^* + bL)(z_{11})\end{pmatrix}.$$

Thus (4.10) becomes

$$(4.11)\quad \begin{pmatrix} \mu 1 & 0 \\ 0 & \mu 1\end{pmatrix} = \begin{pmatrix} (aL + bL^*)z_{22} & -\overline{\lambda} 1 \\ \lambda 1 & (aL^* + bL)(z_{11})\end{pmatrix} + \frac{1}{\det(z)}\begin{pmatrix} z_{22} & -z_{12} \\ -z_{21} & z_{11}\end{pmatrix}.$$

In analogy with section 3, we look for solutions $z_{ij} \in \mathcal{D} = L^\infty[0,1]$ of the form

$$(4.12)\quad \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22}\end{pmatrix} = \begin{pmatrix} c_{11}\exp(\sigma x) & c_{12} \\ c_{21} & c_{22}\exp(-\sigma x)\end{pmatrix},$$

where $\sigma \in \mathbb{C}$ and $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C})$. It is easy to check that (4.12) is a solution to (4.11) if the following 5 conditions are fulfilled:

$$\begin{aligned} \det(c) &= \frac{\sigma}{a-b} \\ c_{11} &= \frac{\sigma\mu}{a\mathrm{e}^{\sigma}-b} \\ c_{12} &= -\frac{\sigma\overline{\lambda}}{a-b} \\ c_{21} &= -\frac{\sigma\lambda}{a-b} \\ c_{22} &= \frac{\sigma\mu}{a-b\mathrm{e}^{-\sigma}} \end{aligned}$$

The first of these conditions is consistent with the remaining 4 if and only if

$$\frac{(\sigma\mu)^2}{(a\mathrm{e}^{\sigma}-b)(a-b\mathrm{e}^{-\sigma})} - \frac{\sigma^2|\lambda|^2}{(a-b)^2} = \frac{\sigma}{a-b}$$

which is equivalent to

$$\mu^2 = \frac{(a\mathrm{e}^{\sigma}-b)(a-b\mathrm{e}^{-\sigma})(a-b+\sigma|\lambda|^2)}{\sigma(a-b)^2}. \tag{4.13}$$

Put

$$\sigma_0 := -\min\left\{\frac{a-b}{|\lambda|^2}, \log\left(\frac{a}{b}\right)\right\}.$$

Then for $\sigma_0 < \sigma < 0$, the right hand side of (4.13) is negative. Let in this case $\mu(\sigma)$ denote the solution to (4.13) with positive imaginary part, i.e.

$$\mu(\sigma) = \mathrm{i}\frac{a\mathrm{e}^{\sigma/2}-b\mathrm{e}^{-\sigma/2}}{|\sigma|^{1/2}(a-b)}\sqrt{a-b+\sigma|\lambda|^2} \tag{4.14}$$

for $\sigma_0 < \sigma < 0$. Then with

$$c_{11} = \frac{\sigma\mu(\sigma)}{a\mathrm{e}^{\sigma}-b} \qquad c_{12} = -\frac{\sigma\overline{\lambda}}{a-b}$$

$$c_{21} = -\frac{\sigma\lambda}{a-b} \qquad c_{22} = \frac{\sigma\mu(\sigma)}{a-b\mathrm{e}^{-\sigma}}$$

the matrix $z(\sigma) = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$ given by (4.12) is a solution to

$$\mathcal{R}_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(z(\sigma)) + z(\sigma)^{-1} = \mu 1_2.$$

By (4.14) $\lim_{\sigma\to 0^-}|\mu(\sigma)| = \infty$ and $\lim_{\sigma\to 0^-}|\sigma\mu(\sigma)| = 0$ and therefore $\lim_{\sigma\to 0^-}\|z(\sigma)\| = 0$.

Hence for some $\sigma_1 \in (\sigma_0, 0)$ we have $|\mu(\sigma)| > \frac{1}{\delta}$ and $\|z(\sigma)\| > \delta$ when $\sigma \in (\sigma_1, 0)$ where $\delta > 0$ is the number described in connection with (4.9). Thus

$$z(\sigma) = G_{\tilde{S}_\lambda}^{M_2(\mathcal{D})}(\mu(\sigma)1_2) \tag{4.15}$$

for $\sigma \in (\sigma_1, 0)$. But since both $\sigma \mapsto z(\sigma)$ and $\sigma \mapsto \mu(\sigma)$ are analytic functions (of the real variable σ) it follows that (4.15) holds for all $\sigma \in (\sigma_0, 0)$. Note that $\sigma \mapsto -\mathrm{i}\mu(\sigma)$ is a continuous strictly positive function on $(\sigma_0, 0)$, and

$$\lim_{\sigma\to 0^-}(-\mathrm{i}\mu(\sigma)) = +\infty \qquad \lim_{\sigma\to\sigma_0^+}(-\mathrm{i}\mu(\sigma)) = 0.$$

Hence for every fixed real number $\alpha > 0$ we can chose $\sigma \in (\sigma_0, 0)$, such that

$$-\mathrm{i}\mu(\sigma) = \sqrt{\alpha}.$$

Thus by (4.10) and (4.15)

$$E_{\mathcal{D}}(S_\lambda^*(-\alpha 1 - S_\lambda S_\lambda^*)^{-1}) = z(\sigma)_{12} = -\frac{\sigma\overline{\lambda}}{a-b}$$

which is a constant function in $L^\infty[0,1]$. Hence

$$\mathrm{tr}(S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1}) = \frac{\sigma\overline{\lambda}}{a-b}$$

from which

$$\int_{\partial B(0,r)} \mathrm{tr}(S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1})\mathrm{d}\lambda = 2\pi\mathrm{i}\frac{\sigma r^2}{a-b}$$

when $\sigma_0 < \sigma < 0$, where as before $\sigma_0 = -\min\left\{\frac{a-b}{|\lambda|^2}, \log\left(\frac{a}{b}\right)\right\}$.

Now $\alpha \to 0^+$ corresponds to $\sigma \to \sigma_0^+$. Hence

$$\lim_{\alpha\to 0^+}\left(-\frac{1}{2\pi}\Im\int_{\partial B(0,r)} \mathrm{tr}(S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1})\mathrm{d}\lambda\right)$$
$$= -\frac{\sigma_0 r^2}{a-b} = +\min\left\{1, r^2\frac{\log\left(\frac{a}{b}\right)}{a-b}\right\}.$$

Obeserve that $S_\lambda^*(S_\lambda S_\lambda^* + \alpha 1)^{-1} = (S_\lambda^* S_\lambda + \alpha 1)^{-1} S_\lambda^*$. Thus by lemma 4.2 we have for all but countably many $r > 0$, that

$$\mu_S(B(0,r)) = \min\left\{1, r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b}\right\} = \begin{cases} r^2 \frac{\log\left(\frac{a}{b}\right)}{a-b}, & r \le \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}} \\ 1, & r > \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}} \end{cases}.$$

Since the right hand side is a continuous function of r, the formula actually holds for all $r > 0$. This together with the rotation invariance of μ_S shows, that μ_S is equal to the uniform distribution on $\overline{B}\left(0, \sqrt{\frac{a-b}{\log\left(\frac{a}{b}\right)}}\right)$, i.e. has constant density $\frac{1}{\pi} \frac{\log\left(\frac{a}{b}\right)}{a-b}$ on this ball, and vanishes outside the ball. Putting $a = 1 + \epsilon$ and $b = \epsilon$ we get in particular

Theorem 4.3. *The Brown measure of* $T + \sqrt{\epsilon} Y$ *is equal to the uniform distribution on* $\overline{B}\left(0, \frac{1}{\sqrt{\log(1+\epsilon^{-1})}}\right)$.

The Brown mesure of $T + \sqrt{\epsilon} Y$ can be used to give an upper bound of the microstate entropy of $T + \sqrt{\epsilon} Y$. By [8] we have for $S \in \mathcal{M}$

$$\chi(S) \le \int_{\mathbb{C}} \int_{\mathbb{C}} \log|z_1 - z_2| \mathrm{d}\mu_S(z_1) \mathrm{d}\mu_S(z_2) + \frac{5}{4} + \log(\pi \sqrt{2 \mathrm{od}_S}) \tag{4.16}$$

where μ_S is the Brown measure of S on $\mathbb{C}$ and od_S is the off-diagonality of S defined by

$$\mathrm{od}_S := \tau(SS^*) - \int_{\mathbb{C}} |z|^2 \mathrm{d}\mu_S(z). \tag{4.17}$$

Lemma 4.4. *For* $R > 0$ *we have*

$$I := \int_{B(0,R)} \int_{B(0,R)} \log|z_1 - z_2| \, dz_1 dz_2 = \pi^2 (R^2 \log R - \tfrac{1}{4})$$

Proof. Polar substitution in I gives

$$I := 4\pi^2 \int_0^R \int_0^R \left(\frac{1}{2\pi} \int_0^{2\pi} \log|r - \mathrm{e}^{\mathrm{i}\theta} s| \mathrm{d}\theta \right) r \mathrm{d}r s \mathrm{d}s.$$

Let $0 < s < r$. $z \mapsto \log|r - zs|$ is the real value of the complex holomorphic function $z \mapsto \mathrm{Log}(r - zs)$, where Log is the principal branch of the complex logarithm, so $z \mapsto \log|r - zs|$ is a harmonic function in $B(0, \frac{r}{s})$. By the mean value property of harmonic functions

$$\frac{1}{2\pi} \int_0^{2\pi} \log|r - \mathrm{e}^{\mathrm{i}\theta} s| \mathrm{d}\theta = \log(r),$$

so symmetry in r and s reduces I to

$$I := 4\pi^2 \int_0^R \int_0^R \max\{\log(r), \log(s)\} r\mathrm{d}r s\mathrm{d}s$$
$$= 8\pi^2 \int_0^R \left(\int_0^r \log(r) s\mathrm{d}s \right) r\mathrm{d}r$$
$$= 4\pi^2 \int_0^R r^3 \log(r)\mathrm{d}r = \pi^2 R^4 (\log(R) - \tfrac{1}{4}).$$

□

Theorem 4.5.

$$\chi(T + \sqrt{\epsilon}Y) \leq -\frac{1}{2}\log(\log(1+\epsilon^{-1})) - \frac{1}{4} + \log \pi + \frac{1}{2}\log\left(1 + 2\epsilon - \frac{1}{\log(1+\epsilon^{-1})}\right). \tag{4.18}$$

Proof. Let ν_R be the uniform distribution on $\overline{B}(0,R)$. Since ν_R has constant density $(\pi R^2)^{-1}$ on $\overline{B}(0,R)$, we have by lemma 4.4

$$\int_{\mathbb{C}} \int_{\mathbb{C}} \log|z_1 - z_2| \mathrm{d}\nu_{\mathrm{R}}(\mathrm{z}_1)\mathrm{d}\nu_{\mathrm{R}}(\mathrm{z}_2) = \log \mathrm{R} - \frac{1}{4}.$$

The Brown measure of $S = T + \sqrt{\epsilon}Y$ is $\mu_S = \nu_R$ with $R = \log(1 + \epsilon^{-1})^{-\frac{1}{2}}$, and

$$\mathrm{od_S} = \frac{1}{2} + \epsilon - \int_{\mathbb{C}} |\mathrm{z}|^2 \mathrm{d}\nu_{\mathrm{R}} = \frac{1}{2} + \epsilon - \frac{\mathrm{R}^2}{2}.$$

Hence by (4.16)

$$\chi(T + \sqrt{\epsilon}Y) \leq \log R - \frac{1}{4} + \log \pi + \frac{1}{2}\log(1 + 2\epsilon - R^2).$$

This proves (4.18). □

In [1] the first author proved that the microstate-free analog, $\delta_0^*(T)$, of the free entropy dimension is equal to 2. From Theorem 4.5 one gets only the trivial estimate of the free entropy dimension $\delta_0(T)$, namely

$$\delta_0(T) \leq 2 + \lim_{\delta \to 0^+} \frac{\chi(T + \sqrt{2}\delta Y)}{|\log \delta|} = 2. \tag{4.19}$$

If $T + \sqrt{\epsilon}Y$ was a DT-operator for all $\epsilon > 0$ then by [8] equality would hold in (4.18), and hence also in (4.19). In the rest of this section, we prove that unfortunately $T + \sqrt{\epsilon}Y$ is not a DT-operator for any $\epsilon > 0$.

If $R = D + T$ is a $\mathrm{DT}(\mu, 1)$ operator it follows from [4, lemma 7.2] that for $|\lambda| < \|R\|^{-1}$,

$$\left\| \sum_{n=0}^{\infty} \lambda^n R^n \right\|_2^2 = \frac{1}{|\lambda|^2} \left(\exp\left(\sum_{k,l=1}^{\infty} \lambda^{k+1} \overline{\lambda}^{l+1} M_\mu(k,l) - 1 \right) \right),$$

where $M_\mu(k,l) = \int_{\sigma(R)} z^k \overline{z}^l \mathrm{d}\mu_R(z)$.

If thus μ_D is the uniform distribution on a disk with radius d then

$$M_{\mu_D}(k,l) = 0$$

when $k \neq l$ and

$$\begin{aligned} M_{\mu_D}(k,k) &= \frac{1}{\pi d^2} \int_{B(0,d)} |z|^{2k} \mathrm{d}z \\ &= \frac{2\pi}{\pi d^2} \int_0^d r^{2k+1} \mathrm{d}r = \frac{2}{d^2} \left[\frac{r^{2k+2}}{2k+2} \right]_0^r = \frac{d^{2k}}{k+1} \end{aligned}$$

for $k \in \mathbb{N}$. Thus

$$\begin{aligned} (4.20) \quad \left\| \sum_{n=0}^{\infty} \lambda^n (D+T)^n \right\|_2^2 &= \frac{1}{|\lambda|^2} \left[\exp\left(\sum_{k=0}^{\infty} |\lambda|^{2(k+1)} \frac{d^{2k}}{k+1} \right) - 1 \right] \\ &= \frac{1}{|\lambda|^2} \exp\left(\frac{1}{d^2} \left(-\log(1 - d^2|\lambda|^2) \right) \right) \\ &= \frac{1}{|\lambda|^2} \left[(1 - d^2|\lambda|^2)^{-\frac{1}{d^2}} - 1 \right]. \end{aligned}$$

If instead $D + cT$ is a $\mathrm{DT}(\mu_D, c)$ operator with μ_D being the uniform distribution on a disc of radius d then

$$D + cT = c(D' + T)$$

where D' now has the uniform distribution on $B(0, \frac{d}{c})$, so from (4.20) we obtain

$$\begin{aligned} (4.21) \quad & \left\| \sum_{n=0}^{\infty} \lambda^n (D+cT)^n \right\|_2^2 \\ &= \left\| \sum_{n=0}^{\infty} (c\lambda)^n (D'+T)^n \right\|_2^2 = \frac{1}{c^2|\lambda|^2} \left[\left(1 - d^2|\lambda|^2\right)^{-\frac{c^2}{d^2}} - 1 \right]. \end{aligned}$$

Lemma 4.6. *Let $a > b > 0$ and let $S = \sqrt{a}T_1 + \sqrt{b}T_2^*$ where T_1 and T_2 are two $\mathcal{D}$-free quasidiagonal DT-operators. Then*

$$\left\|\sum_{n=0}^{\infty}\lambda^n S^n\right\|_2^2 = \frac{1}{|\lambda|^2}\frac{e^{(a-b)|\lambda|^2}-1}{a-be^{(a-b)|\lambda|^2}}, \quad |\lambda| < \frac{1}{\|S\|^2}.$$

Proof. Let $F_n(x) = E_{\mathcal{D}}((S^*)^n S^n)$ for $n \in \mathbb{N}$ and $x \in [0,1]$. For $t < \frac{1}{\|S\|^2}$ define the $\mathcal{D}$-valued function

$$F(t,x) = \sum_{n=0}^{\infty} F_n(x)t^n. \tag{4.22}$$

By Speicher's cumulant formula we have by $\mathcal{D}$-Gaussianity of S that

$$\begin{aligned} F_n = E_{\mathcal{D}}((S^*)^n S^n) &= \sum_{\pi\in \mathrm{NC}(2n)} \kappa_\pi^{\mathcal{D}}\left((S^*)^{\otimes_{\mathcal{B}} n}\otimes_{\mathcal{B}} S^{\otimes_{\mathcal{B}} n}\right)\\ &= \kappa_2^{\mathcal{D}}\left(S^*\otimes_{\mathcal{B}} E_{\mathcal{D}}((S^*)^{n-1}S^{n-1})S\right)\\ &= (aL^*+bL)(E_{\mathcal{D}}((S^*)^{n-1}S^{n-1})) = (aL^*+bL)(F_{n-1}), \end{aligned}$$

so we get the following recursive algorithm for determining the F_n's.

$$\begin{cases} F_0(x) = 1\\ F_n(x) = aL^*(F_{n-1})(x) + bL(F_{n-1})(x), \quad x\in[0,1] \end{cases},$$

where $L^*(f) : x \mapsto \int_0^x f(y)\mathrm{d}y$ and $L(f) : x \mapsto \int_x^1 f(y)\mathrm{d}y$. Observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}L(f)(x) = -f(x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}L^*(f)(x) = f(x),$$

and that

$$F_n(0) = aL^*(F_{n-1})(0) + bL(F_{n-1})(0) = b\int_0^1 F_{n-1}(x)\mathrm{d}x = b\tau(F_{n-1})$$

for $n \geq 1$. Using (4.22) we have the following differential equation and initial condition in x

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}F(t,x) = (a-b)tF(t,x), \quad x\in[0,1]\\ F(t,0) = f(t), \end{cases}$$

where the function f is given by

$$f(t) = F(t,0) = \sum_{n=0}^{\infty} F_n(0)t^n$$
$$= 1 + \sum_{n=1}^{\infty} \left(aL^*(F_{n-1})(0) + bL(F_{n-1})(0)\right) t^n$$
$$= 1 + b\sum_{n=1}^{\infty} \left(\int_0^1 F_{n-1}(x)\mathrm{d}x\right) t^n$$
$$= 1 + bt\int_0^1 \left(\sum_{n=1}^{\infty} F_{n-1}(x)t^{n-1}\right) \mathrm{d}x$$
$$= 1 + bt\tau(F(t,\cdot))$$

We thus have the unique solution

$$F(t,x) = f(t)\mathrm{e}^{(a-b)tx}, \tag{4.23}$$

where we can now use (4.23) and the initial condition to find the function f.

$$f(t) = 1 + bt\int_0^1 F(t,x)\mathrm{d}x$$
$$= 1 + bt\left[\frac{f(t)}{(a-b)t}\mathrm{e}^{(a-b)tx}\right]_0^1 = 1 + bf(t)\frac{\left(\mathrm{e}^{(a-b)t} - 1\right)}{a-b}.$$

Hence

$$f(t) = \frac{a-b}{a - b\mathrm{e}^{(a-b)t}}$$

so that

$$F(t,x) = \frac{(a-b)\mathrm{e}^{(a-b)tx}}{a - b\mathrm{e}^{(a-b)t}}.$$

Now observe that

$$\left\|\sum_{n=0}^{\infty} \lambda^n S^n\right\|_2^2 = \tau\left(F(|\lambda|^2, x)\right)$$
$$= \int_0^1 F(|\lambda|^2, x)\mathrm{d}x = \frac{1}{|\lambda|^2}\frac{\mathrm{e}^{(a-b)|\lambda|^2} - 1}{a - b\mathrm{e}^{(a-b)|\lambda|^2}}$$

□

Theorem 4.7. *The operator $T + \sqrt{\epsilon}Y$ is not a* DT-operator.

Proof. By substituting $a = 1+\epsilon$ and $b = \epsilon$ in lemma 4.6 we have

$$\left\| \sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon} Y)^n \right\|_2^2 = \frac{1}{|\lambda|^2} \frac{e^{|\lambda|^2} - 1}{1 + \epsilon - \epsilon e^{|\lambda|^2}} \tag{4.24}$$

for all λ in a neighborhood of 0. If $T + \sqrt{\epsilon}Y$ is a DT-operator, then by Theorem 4.3 and (4.21), there exists a $c > 0$, such that when $d = \log(1+\frac{1}{\epsilon})^{-\frac{1}{2}}$

$$\left\| \sum_{n=0}^{\infty} \lambda^n (T + \sqrt{\epsilon} Y)^n \right\|_2^2 = \frac{1}{c^2|\lambda|^2} \left((1 - d^2|\lambda|^2)^{-\frac{c^2}{d^2}} - 1 \right) \tag{4.25}$$

for all λ in a neighborhood of 0. Consider the two analytic functions,

$$\begin{aligned} f(s) &= \frac{e^s - 1}{1 + \epsilon - \epsilon e^s}, \\ g(s) &= \frac{1}{c^2} \left((1 - d^2 s)^{-\frac{c^2}{d^2}} - 1 \right) \end{aligned}$$

which are both defined in the complex disc $U = B(0, \log(1+\frac{1}{\epsilon})^{-\frac{1}{2}})$. By (4.24) and (4.25) $f(s) = g(s)$ for s in some real interval of the form $(0, \delta)$ and hence $f(s) = g(s)$ for all $s \in U$. Moreover f has a meromorphic extension to the full complex plane with a simple pole at $s_0 = \log(1+\frac{1}{\epsilon})$. Hence g also has a meromorphic extension to the full complex plane with a simple pole at $\log(1+\frac{1}{\epsilon}) = d^{-2}$. This implies $c = d$. In this case

$$g(s) = \frac{1}{d^2} \left((1 - d^2 s)^{-1} - 1 \right)$$

which is analytic in $\mathbb{C} \setminus \{s_0\}$. However f has infinitely many poles, namely

$$s_p = \log\left(1 + \frac{1}{\epsilon}\right) + p2\pi, \quad p \in \mathbb{Z}.$$

Since the meromorphic extensions of f and g must coincide, we have reached a contradiction. Therefore $T + \sqrt{\epsilon}Y$ is not a DT-operator. □

5. Śniady's moment formulas. The case $k = 2$.

Let $k \in \mathbb{N}$ be fixed, and let $(P_{k,n})_{n=0}^{\infty}$ be the sequence of polynomials defined recursively by

$$\begin{cases} P_{k,n}(x) = 1, & \\ P_{k,n}^{(k)}(x) = P_{k,n-1}(x+1), & n = 1, 2, \ldots, \\ P_{k,n}(0) = P_{k,n}^{(1)}(0) = \cdots P_{k,n}^{(k-1)}(0) = 0, & n = 1, 2, \ldots \end{cases} \tag{5.1}$$

where $P_{k,n}^{(l)}$ denotes the l'th derivative of $P_{k,n}$. As in the previous sections, T denotes the quasinilpotent DT operator. Śniady's main results from [9] are:

Theorem 5.1. [9, Theorem 5 and Theorem 7]
(a) For all $k, n \in \mathbb{N}$:

$$E_{\mathcal{D}}\left(((T^*)^k T^k)^n\right)(x) = P_{k,n}(x), \quad x \in [0,1]. \tag{5.2}$$

(b) For all $k, n \in \mathbb{N}$:

$$\tau\left(((T^*)^k T^k)^n\right) = \frac{n^{nk}}{(nk+1)!} \tag{5.3}$$

Actually Śniady considers $E_{\mathcal{D}}((T^k(T^*)^k)^n)$ instead of $E_{\mathcal{D}}(((T^*)^k T^k)^n)$, but it is easily seen, that Theorem 5.1 (a) is equivalent to [9, Theorem 5], by the simple change of variable $x \mapsto 1-x$.

Śniady's proof of Theorem 5.1 is a very technical combinatorial proof. In this and the following section we will give an analytical proof of Theorem 5.1 based on Voiculescu's $\mathcal{R}$-transform with amalgamation.

As in [5, (2.11)] we put

$$\rho(z) = -W_0(-z), \qquad z \in \mathbb{C} \setminus [\tfrac{1}{\mathrm{e}}, \infty),$$

where W_0 is the principal branch of Lambert's W-function. Then ρ is the principal branch of the inverse function of $z \mapsto z\mathrm{e}^{-z}$. We shall need the following result from [5, Prop. 4.2].

Lemma 5.2. [5, Prop. 4.2] *Let $(P_{k,n})_{n=0}^{\infty}$ be a sequence of polynomials given by (5.1). Put for $s \in \mathbb{C}$, $|s| < \frac{1}{\mathrm{e}}$ and $j = 1, \dots, k$*

$$\alpha_j(s) = \rho\left(s\mathrm{e}^{\mathrm{i}\frac{2\pi j}{k}}\right), \tag{5.4}$$

$$\gamma_j(s) = \begin{cases} \prod_{l \neq j} \frac{\alpha_l(s)}{\alpha_l(s) - \alpha_j(s)}, & 0 < |s| < \frac{1}{\mathrm{e}} \\ \frac{1}{k}, & s = 0. \end{cases} \tag{5.5}$$

Then

$$\sum_{n=0}^{\infty} (ks)^{nk} P_{k,n}(x) = \sum_{j=1}^{k} \gamma_j(s) \mathrm{e}^{k\alpha_j(s)x} \tag{5.6}$$

for all $x \in \mathbb{R}$ and all $s \in B(0, \frac{1}{\mathrm{e}})$.

The case $k = 1$ of theorem 5.1 is the special case $\lambda = 0$ of theorem 3.2. To illustrate our method of proof of theorem 5.1 for $k \geq 2$, we first consider the case $k = 2$.

Define $\tilde{T} \in M_4(\mathcal{A})$ by

$$\tilde{T} = \begin{pmatrix} 0 & 0 & 0 & T^* \\ T & 0 & 0 & 0 \\ 0 & T & 0 & 0 \\ 0 & 0 & T^* & 0 \end{pmatrix}.$$

Then $\|\tilde{T}\| = \|T\| = \sqrt{\mathrm{e}}$. (cf. [4, Corollary 8.11]) For $\mu \in \mathbb{C}$, $|\mu| < \frac{1}{\mathrm{e}}$ we let $z = z(\mu)$, denote the Cauchy transform of $\tilde{T}$ at $\tilde{\mu} = \mu 1_{M_4(\mathcal{A})}$ wrt. amalgamation over $M_4(\mathcal{D})$ i.e.

$$z = E_{\mathcal{D}}\left((\tilde{\mu} - \tilde{T})^{-1}\right).$$

Clearly

$$(5.7) \quad (\tilde{\mu} - \tilde{T})^{-1} = \sum_{n=0}^{\infty} \mu^{-n-1} \tilde{T}^n = \left(\sum_{n=0}^{3} \mu^{-n-1} \tilde{T}^n\right) \left(\sum_{n=0}^{\infty} \mu^{-4n} \tilde{T}^{4n}\right).$$

By direct computation

$$\tilde{T}^2 = \begin{pmatrix} 0 & 0 & (T^*)^2 & 0 \\ 0 & 0 & 0 & TT^* \\ T^2 & 0 & 0 & 0 \\ 0 & T^*T & 0 & 0 \end{pmatrix},$$

$$\tilde{T}^3 = \begin{pmatrix} 0 & (T^*)^2 T & 0 & 0 \\ 0 & 0 & T(T^*)^2 & 0 \\ 0 & 0 & 0 & T^2 T^* \\ T^* T^2 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\tilde{T}^4 = \begin{pmatrix} (T^*)^2 T^2 & 0 & 0 & 0 \\ 0 & T(T^*)^2 T & 0 & 0 \\ 0 & 0 & T^2 (T^*)^2 & 0 \\ 0 & 0 & 0 & T^* T^2 T^* \end{pmatrix}.$$

Hence using the fact that the expectation $E_{\mathcal{D}}$ of a monomial in T and T^* vanishes unless T and T^* occur the same number of times, we get from (5.7) that z is of the form

$$(5.8) \qquad z = \begin{pmatrix} z_{11} & 0 & 0 & 0 \\ 0 & z_{22} & 0 & z_{24} \\ 0 & 0 & z_{33} & 0 \\ 0 & z_{42} & 0 & z_{44} \end{pmatrix}$$

where $z_{11}, z_{22}, z_{24}, z_{33}, z_{42}, z_{44} \in \mathcal{D}$ are given by

$$\begin{aligned}
z_{11} &= \mu^{-1} E_{\mathcal{D}}((1-\mu^{-4}(T^*)^2T^2)^{-1}),\\
z_{22} &= \mu^{-1} E_{\mathcal{D}}((1-\mu^{-4}T(T^*)^2T)^{-1}),\\
z_{33} &= \mu^{-1} E_{\mathcal{D}}((1-\mu^{-4}T^2(T^*)^2)^{-1}),\\
z_{44} &= \mu^{-1} E_{\mathcal{D}}((1-\mu^{-4}T^*T^2T^*)^{-1}),\\
z_{24} &= \mu^{-3} E_{\mathcal{D}}(T(1-\mu^{-4}(T^*)^2T^2)^{-1}T^*),\\
z_{42} &= \mu^{-3} E_{\mathcal{D}}(T^*(1-\mu^{-4}T^2(T^*)^2)^{-1}T).
\end{aligned}$$

For the last 2 identities, we have used, that

$$A(1-\eta BA)^{-1} = (1-\eta AB)^{-1}A$$

for $A, B \in \mathcal{A}$ and $\eta \in \mathbb{C}$ whenever both sides of this equality are welldefined.

By lemma 2.1, we know, that there exists a $\delta > 0$ such that when $w \in M_4(\mathcal{D})_{\text{inv}}$ and $\mu \in \mathbb{C}$ satisfies $\|w\| < \delta$, $|\mu| > \frac{1}{\delta}$ and

$$\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_4(\mathcal{A})} \tag{5.9}$$

then $w = E_{M_4(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}) = z$. In particular

$$w_{11} = z_{11} = \mu^{-1}((1-\mu^{-4}(T^*)^2T^2)^{-1}),$$

Hence, if we can find a suitable solution to (5.8) for all $\mu \in \mathbb{C}$ in a neighborhood of ∞, we can find $E_{\mathcal{D}}(((T^*)^2T^2)^n)$ for $n = 1, 2, \ldots$ by determining the power series expansion of w_{11} as a function of μ^{-1}.

Since (T, T^*) is a $\mathcal{D}$-Gaussian pair by [5, Appendix] it follows from lemma 2.2 that

$$\kappa_n^{M_4(\mathcal{D})}((m_1 \otimes a_1) \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} (m_n \otimes a_n)) = 0$$

when $n \neq 2$, $m_1, m_2, \ldots, m_n \in M_4(\mathbb{C})$ and $a_1, a_2, \ldots, a_n \in \{T, T^*\}$. By definition

$$\tilde{T} = (e_{21} + e_{32}) \otimes T + (e_{43} + e_{14}) \otimes T^*$$

so by linearity of $\kappa_n^{M_4(\mathcal{D})}$, it follows that

$$\kappa_n^{M_4(\mathcal{D})}(\tilde{T} \otimes_{M_4(\mathcal{D})} \cdots \otimes_{M_4(\mathcal{D})} \tilde{T}) = 0$$

when $n \neq 2$ i.e. $\tilde{T}$ is $M_4(\mathcal{D})$-Gaussian.

Hence using (2.4) we get

$$\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) = \kappa_2^{M_4(\mathcal{D})}(\tilde{T} \otimes_{M_4(\mathcal{D})} w\tilde{T}) = E_{M_4(\mathcal{D})}\left(\tilde{T}w\tilde{T}\right)$$

$$= E_{M_4(\mathcal{D})}\left(\begin{pmatrix} T^*w_{42}T & 0 & T^*w_{44}T^* & 0 \\ 0 & 0 & 0 & Tw_{11}T^* \\ Tw_{22}T & 0 & Tw_{24}T^* & 0 \\ 0 & T^*w_{33}T & 0 & 0 \end{pmatrix}\right)$$

for $w = (w_{ij})_{i,j=1,\dots,4} \in M_4(\mathcal{D})$.

Since $E_{\mathcal{D}}(TfT) = E_{\mathcal{D}}(T^*fT^*) = 0$, and $E_{\mathcal{D}}(T^*fT) = L^*(f)$, $E_{\mathcal{D}}(TfT^*) = L(f)$ for $f \in L^\infty([0,1])$, we have:

$$\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) = \begin{pmatrix} L^*(w_{42}) & 0 & 0 & 0 \\ 0 & 0 & 0 & L(w_{11}) \\ 0 & 0 & L(w_{24}) & 0 \\ 0 & L^*(w_{33}) & 0 & 0 \end{pmatrix}$$

for $w \in M_4(\mathcal{D})$. By (5.8) we only have to consider w of the form

$$w = \begin{pmatrix} w_{11} & 0 & 0 & 0 \\ 0 & w_{22} & 0 & w_{24} \\ 0 & 0 & w_{33} & 0 \\ 0 & w_{42} & 0 & w_{44} \end{pmatrix}. \tag{5.10}$$

For $w \in M_4(\mathcal{D})_{\text{inv}}$ of the form (5.10), (5.9) reduces to the three equations

$$\begin{cases} L^*(w_{42}) + \frac{1}{w_{11}} = \mu 1_{\mathcal{D}} \\ \begin{pmatrix} 0 & L(w_{11}) \\ L^*(w_{33}) & 0 \end{pmatrix} + \begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathcal{D})} \\ L(w_{24}) + \frac{1}{w_{33}} = \mu 1_{\mathcal{D}} \end{cases} \tag{5.11}$$

Definition 5.3. Let $f \in C([0,1])$. We call $(f^{(-n)})_{n=1}^{l}$ for the succesive antiderivatives of f if

$$\frac{\mathrm{d}}{\mathrm{d}x}(f^{(-n)}) = f^{(1-n)} \text{ for } n = 2, 3, \dots, l$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}(f^{(-1)}) = f.$$

Lemma 5.4. *Let $f \in C^2([0,1])$ and let $f^{(-1)}$ and $f^{(-2)}$ be the succesive antiderivatives of f for which*

(i) $f^{(-1)}(1) = 0, \qquad f^{(-2)}(1) = \mu^3.$

Assume further, that

(ii) $f(0) = \mu^{-1}$ *and* $f^{(1)}(0) = 0$.

(iii) For all $x \in [0,1]$,

$$\begin{aligned} f(x) &\neq 0 \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} &\neq 0 \end{aligned}$$

while

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{vmatrix} = 0$$

.

Then $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42} \in C([0,1])$ *given by*

$$\begin{cases} w_{11} = f \\ w_{22} = w_{44} = -\frac{1}{\mu} \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \\ w_{24} = \frac{1}{\mu^2} \frac{f^{(-1)} \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \\ w_{42} = \frac{f^{(1)}}{f^2} \\ w_{33} = \mu^2 \frac{f \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} \end{cases} \tag{5.12}$$

is a solution to (5.11). Moreover

$$\begin{vmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{vmatrix} = -\frac{1}{\mu^2} \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} \tag{5.13}$$

and

$$(5.14)\qquad \begin{cases} L(w_{11}) = -f^{(-1)} \\[2ex] L(w_{24}) = \mu - \frac{1}{\mu^2}\dfrac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f} \\[2ex] L^*(w_{42}) = \mu - \frac{1}{f} \\[2ex] L^*(w_{33}) = -\mu^2 \dfrac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}} \end{cases}.$$

Proof. Assume $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}$ is given by (5.12). Then (5.13) follows immediately. Note that for $f \in C([0,1])$, the functions $g = L(f)$ and $h = L^*(f)$ are characterized by

$$g^{(1)} = -f \quad \text{and} \quad g(1) = 0$$
$$h^{(1)} = f \quad \text{and} \quad h(0) = 0.$$

Hence (5.14) is equivalent to (5.15) and (5.16) below.

$$(5.15)\qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} f^{(-1)} = w_{11} \\[2ex] \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\mu^2}\dfrac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f}\right) = w_{24} \\[2ex] \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{f}\right) = w_{42} \\[2ex] \frac{\mathrm{d}}{\mathrm{d}x}\left(-\mu^2 \dfrac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}\right) = w_{33} \end{cases}$$

$$(5.16)\qquad \begin{cases} f^{(-1)}(1) = 0, \qquad \dfrac{\begin{vmatrix} f^{(-2)}(1) & f^{(-1)}(1) \\ f^{(-1)}(1) & f(1) \end{vmatrix}}{f(1)} = \mu^3 \\[2ex] \frac{1}{f(0)} = \mu, \qquad f^{(1)}(0) = 0 \end{cases}.$$

Now, (5.16) is trivial from (i) and (ii). Next we prove (5.15): Clearly

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{(-1)} = f = w_{11} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\Big(-\frac{1}{f}\Big) = \frac{f^{(1)}}{f^2} = w_{42}.$$

Moreover

$$\text{(5.17)} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f}\right)$$

$$= \frac{f\begin{vmatrix} f^{(-2)} & f \\ f^{(-1)} & f^{(1)} \end{vmatrix} - f^{(1)}\begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{f^2} = \frac{f^{(-1)}\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}{f^2} = \mu^2 w_{24}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}}\right) = \frac{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix} f^{(2)} - \begin{vmatrix} f^{(-1)} & f \\ f^{(1)} & f^{(2)} \end{vmatrix} f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2}$$

$$= -\frac{f\begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} = -\frac{1}{\mu^2} w_{33}.$$

Hence (5.15) holds. It remains to be proved that $w_{11}, w_{22}, w_{33}, w_{44}, w_{24}, w_{42}$ is a solution to (5.11). By (5.12) and (5.14), we have

$$L^*(w_{42}) + \frac{1}{w_{11}} = \left(\mu - \frac{1}{f}\right) + \frac{1}{f} = \mu.$$

Moreover by (5.12) and (5.13)

$$\begin{pmatrix} w_{22} & w_{24} \\ w_{42} & w_{44} \end{pmatrix}^{-1} = \frac{1}{w_{22}w_{44} - w_{24}w_{42}} \begin{pmatrix} w_{44} & -w_{24} \\ -w_{42} & w_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \mu & f^{(-1)} \\ \mu^2 \dfrac{f^{(1)}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}} & \mu \end{pmatrix}$$

which proves that the first and the second inequality in (5.11).

By (5.12) and (5.14),

$$w_{33}(\mu - L(w_{24})) = \frac{\begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix}}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2} = 1 + \frac{\sigma}{\begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2}$$

where

$$\sigma = \begin{vmatrix} f & f^{(1)} \\ f^{(1)} & f^{(2)} \end{vmatrix} \begin{vmatrix} f^{(-2)} & f^{(-1)} \\ f^{(-1)} & f \end{vmatrix} - \begin{vmatrix} f^{(-1)} & f \\ f & f^{(1)} \end{vmatrix}^2 = f \begin{vmatrix} f^{(-2)} & f^{(-1)} & f \\ f^{(-1)} & f & f^{(1)} \\ f & f^{(1)} & f^{(2)} \end{vmatrix}.$$

Hence by (iii), $\sigma = 0$. Therefore $w_{33}(x) \neq 0$ for all $x \in [0,1]$ and $w_{33}^{-1} = \mu - L(w_{24})$, proving the last equality in (5.11). □

Lemma 5.5. *Let $\alpha_j(s), \gamma_j(s)$ for $j = 1, 2$ be as in lemma 5.2 for $k = 2$, i.e. $\alpha_1(0) = \alpha_2(0) = 0$, $\gamma_1(0) = \gamma_2(0) = \frac{1}{2}$ and for $0 < |s| < \mathrm{e}^{-1}$:*

$$\alpha_1(s) = \rho(s), \qquad \alpha_2(s) = \rho(-s),$$
$$\gamma_1(s) = \frac{\alpha_1(s)}{\alpha_1(s) - \alpha_2(s)}, \qquad \gamma_2(s) = \frac{\alpha_2(s)}{\alpha_2(s) - \alpha_1(s)}.$$

Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{e}$, put $s = \frac{1}{2}\mu^{-2}$ and

$$f(x) = \frac{1}{\mu}\left(\sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)x}\right), \quad x \in \mathbb{R} \tag{5.18}$$

$$f^{(-1)}(x) = \frac{1}{2\mu}\left(\sum_{j=1}^{2} \frac{\gamma_j(s)}{\alpha_j(s)} e^{2\alpha_j(s)x}\right), \quad x \in \mathbb{R} \tag{5.19}$$

$$f^{(-2)}(x) = \frac{1}{4\mu}\left(\sum_{j=1}^{2} \frac{\gamma_j(s)}{\alpha_j(s)^2} e^{2\alpha_j(s)x}\right), \quad x \in \mathbb{R} \tag{5.20}$$

Then

(i) $f^{(-1)}, f^{(-2)}$ are succesively antiderivatives of f,

$$f^{(-1)}(1) = 0, \ f^{(-2)}(1) = \mu^3 \tag{5.21}$$

and

$$f(0) = \mu^{-1}, \ f^{(1)}(0) = 0. \tag{5.22}$$

(ii) *The following asymptotic formulas holds for* $|\mu| \to \infty$*:*

$$\begin{aligned} f^{(-2)}(x) &= \mu^3 + \mathcal{O}(\mu^{-1}) \\ f^{(-1)}(x) &= (x-1)\mu^{-1} + \mathcal{O}(\mu^{-5}) \\ f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5}) \\ f^{(1)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}) \\ f^{(2)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}) \end{aligned}$$

where the error estimates holds uniformly in x *on a compact subset in* $\mathbb{R}$.

(iii) *There exists* $\mu_0 \geq \sqrt{\mathrm{e}}$ *such that the restriction of* f *to* $[0,1]$ *satisfies all the conditions in lemma 5.4, when* $|\mu| > \mu_0$.

Proof. Clearly $f^{(-1)}$ and $f^{(-2)}$ are succesively antiderivatives of f and

$$\begin{aligned} f(0) &= \frac{1}{\mu}\sum_{j=1}^{2}\gamma_j(s) = \frac{1}{\mu} \\ f^{(1)}(0) &= \frac{2}{\mu}\sum_{j=1}^{2}\alpha_j(s)\gamma_j(s) = 0. \end{aligned}$$

To prove (5.21), note first, that since $\rho : \mathbb{C} \setminus [\frac{1}{\mathrm{e}}, \infty) \to \mathbb{C}$ is a branch of the inverse function of $z \mapsto z\mathrm{e}^{-z}$, we have

$$\rho(w)\mathrm{e}^{-\rho(w)} = w, \ |w| < \frac{1}{\mathrm{e}}$$

and therefore

$$\mathrm{e}^{2\alpha_j(s)} = \frac{\alpha_j(s)^2}{s^2}, \ j = 1, 2.$$

Since $s^2 = \frac{1}{4}\mu^{-4}$, it follows that

$$f^{(-2)}(x+1) = \mu^4 f(x), \ x \in \mathbb{R} \tag{5.23}$$

$$f^{(-1)}(x+1) = \mu^4 f^{(1)}(x), \ x \in \mathbb{R} \tag{5.24}$$

$$f(x+1) = \mu^4 f^{(2)}(x), \ x \in \mathbb{R}. \tag{5.25}$$

In particular

$$\begin{aligned} f^{(-2)}(1) &= \mu^4 f(0) = \mu^3 \\ f^{(-1)}(1) &= \mu^4 f^{(1)}(0) = 0. \end{aligned}$$

By the proof of [5, Prop. 4.2], $\alpha_j(s)$ and $\rho_j(s)$ are continuous functions of $s \in B(0, \frac{1}{\mathrm{e}})$. Hence, regarding f as a function of μ,

$$\lim_{|\mu|\to\infty}(\mu f(x)) = \sum_{j=1}^{2}\gamma_j(0)\mathrm{e}^{2\alpha_j(0)x} = 1$$

where the limit holds uniformly in x on compact subsets of $\mathbb{R}$. Hence by (5.25) $f^{(2)}(x) = \mathcal{O}(\mu^{-5})$ as $|\mu| \to \infty$ uniformly in x on compact subsets of $\mathbb{R}$. By (5.22),

$$f^{(1)}(x) = \int_0^x f^{(2)}(t)\mathrm{d}t \tag{5.26}$$

$$f(x) = \mu^{-1} + \int_0^x f^{(1)}(t)\mathrm{d}t \tag{5.27}$$

which implies, that $f^{(1)}(x) = \mathcal{O}(\mu^{-5})$ and

$$f(x) = \mu^{-1} + \mathcal{O}(\mu^{-5}) \tag{5.28}$$

uniformly in x on compact subsets of $\mathbb{R}$.

Using again (5.25), (5.26) and (5.27), we get

$$\begin{aligned} f^{(2)}(x) &= \mu^{-5} + \mathcal{O}(\mu^{-9}) \\ f^{(1)}(x) &= x\mu^{-5} + \mathcal{O}(\mu^{-9}). \end{aligned}$$

By (5.21)

$$\begin{aligned} f^{(-1)}(x) &= \int_1^x f(t)\mathrm{d}t \\ f^{(-2)}(x) &= \mu^3 + \int_1^x f^{(-1)}(t)\mathrm{d}t. \end{aligned}$$

Hence by (5.28),

$$\begin{aligned} f^{(-1)}(x) &= (x-1)\mu^{-1} + \mathcal{O}(\mu^{-5}) \\ f^{(-2)}(x) &= \mu^3 + \mathcal{O}(\mu^{-1}) \end{aligned}$$

where all estimates holds uniformly on compact subsets of $\mathbb{R}$. This proves (ii).

By (i), $f^{(-1)}, f^{(-2)}$ coinside with the succesive antiderivatives of f considered in lemma 5.4 and $f(0) = \mu^{-1}, f^{(1)}(0) = 0$.

Moreover, by (ii),

$$\begin{aligned} f(x) &= \mu^{-1} + \mathcal{O}(\mu^{-5}) \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} &= \mu^{-2} + \mathcal{O}(\mu^{-6}) \end{aligned}$$

where the error terms holds uniformly in $x \in [0,1]$. Hence there exists $\mu_0 \geq \sqrt{\mathrm{e}}$, such that

$$f(x) \neq 0 \text{ and } \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f^{(1)}(x) \end{vmatrix} \neq 0$$

for all $x \in [0,1]$. Moreover by the matrix factorization

$$\begin{pmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{pmatrix} \tag{5.29}$$
$$= \begin{pmatrix} 1 & 1 \\ 2\alpha_1(s) & 2\alpha_2(s) \\ 4\alpha_1(s)^2 & 4\alpha_2(s)^2 \end{pmatrix} \begin{pmatrix} \frac{\gamma_1(s)}{4\alpha_1(s)^2}e^{2\alpha_1(s)x} & 0 \\ 0 & \frac{\gamma_2(s)}{4\alpha_2(s)^2}e^{2\alpha_2(s)x} \end{pmatrix} \begin{pmatrix} 1 & 2\alpha_1(s) & 4\alpha_1(s)^2 \\ 1 & 2\alpha_2(s) & 4\alpha_2(s)^2 \end{pmatrix}$$

it follows, that the matrix on the left hand side has rank less than or equal to 2, i.e.

$$\begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) & f(x) \\ f^{(-1)}(x) & f(x) & f^{(1)}(x) \\ f(x) & f^{(1)}(x) & f^{(2)}(x) \end{vmatrix} = 0$$

for $x \in [0,1]$. Hence f satisfies all the conditions in lemma 5.4, when $|\mu| > \mu_0$. □

Proof of Theorem 5.1 in the case $k = 2$: By lemma 2.1 there exists a $\delta > 0$, such that when $w \in M_4(\mathcal{D})_{\text{inv}}$ and $\mu \in \mathbb{C}$ satisfies $\|w\| < \delta$, $|\mu| > \frac{1}{\delta}$ and

$$\mathcal{R}_{\tilde{T}}^{M_4(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_4(\mathcal{D})} \tag{5.30}$$

then $w = E_{\mathcal{D}}((\tilde{\mu} - \tilde{T})^{-1})$. In particular

$$w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-4}(T^*)^2 T^2)^{-1}). \tag{5.31}$$

Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{e}$, put $s = \frac{1}{2}\mu^{-2}$ and

$$f(x) = \frac{1}{\mu}\left(\sum_{j=1}^{2} \gamma_j(s) e^{2\alpha_j(s)x}\right)$$

for $x \in [0,1]$ as in lemma 5.5. By lemma 5.5 (iii) there exists a $\mu_0 > \sqrt{e}$, such that when $|\mu| > \mu_0$, then f satisfies all the requirements af lemma 5.4. Hence by lemma 5.4, the matrix $w \in M_4(\mathcal{D})$ given by (5.10) and (5.12) is a solution to (5.30). Moreover by the asymptotic formulas in lemma 5.5 (ii),

$$\begin{aligned} \begin{vmatrix} f^{(-2)}(x) & f^{(-1)}(x) \\ f^{(-1)}(x) & f(x) \end{vmatrix} &= \mu^2 + \mathcal{O}(\mu^{-2}), \\ \begin{vmatrix} f^{(-1)}(x) & f(x) \\ f(x) & f'(x) \end{vmatrix} &= -\mu^{-2} + \mathcal{O}(\mu^{-6}), \\ \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} &= \mu^{-6} + \mathcal{O}(\mu^{-10}). \end{aligned}$$

Hence by (5.12) and the asymptotic formulas for $f^{(-1)}, f$ and f', we have

$$\begin{aligned} w_{11} &= \mu^{-1} + \mathcal{O}(\mu^{-5}), \\ w_{22} &= w_{44} = \mu^{-1} + \mathcal{O}(\mu^{-5}), \\ w_{24} &= (1-x)\mu^{-3} + \mathcal{O}(\mu^{-3}), \\ w_{42} &= x\mu^{-3} + \mathcal{O}(\mu^{-3}), \\ w_{33} &= \mu^{-1} + \mathcal{O}(\mu^{-5}), \end{aligned}$$

where all the error estimates holds uniformly in $x \in [0,1]$. Hence, there exists $\mu_1 \geq \max\{\mu_0, \frac{1}{\delta}\}$, such that when $|\mu| > \mu_1$ then $\|w\| < \delta$, and hence

$$w = E_{M_4(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

By (5.12), $w_{11} = f$. Hence by (5.31) and (5.18)

$$E_{\mathcal{D}}((1 - \mu^{-4}(T^*)^2 T^2)^{-1})(x) = \mu f(x) = \sum_{j=1}^{2} \gamma_j(s) \mathrm{e}^{2\alpha_j(s)x}$$

where $s = \frac{1}{2}\mu^{-2}$, i.e. for $|s| < \frac{1}{2}\mu_1^{-2}$,

$$E_{\mathcal{D}}((1 - (2s)^2 (T^*)^2 T^2)^{-1})(x) = \sum_{j=1}^{2} \gamma_j(s) \mathrm{e}^{2\alpha_j(s)x}$$

and therefore

$$\sum_{j=0}^{\infty} (2s)^{2n} E_{\mathcal{D}}(((T^*)^2 T^2)^n)(x) = \sum_{j=1}^{2} \gamma_j(s) \mathrm{e}^{2\alpha_j(s)x}. \tag{5.32}$$

Hence by lemma 5.2 and by the uniqueness of the power series expansions of analytic functions, we have

$$E_{\mathcal{D}}(((T^*)^2 T^2)^n)(x) = P_{2,n}(x)$$

for $n \in \mathbb{N}$ and $x \in [0,1]$. This proves theorem 5.1(a) in the case $k = 2$. Theorem 5.1 (b) also follows from (5.32) by integrating the right hand side of (5.32) from 0 to 1 with respect to x (cf. [5, remark 4.3]). □

6. Śniady's moment formulas. The general case.

The above proof of Theorem 5.1 in the case $k = 2$ can fairly easily be generalized to all $k \geq 2$ (Recall that the case $k = 1$ is contained in theorem 3.2).

Let $k \geq 2$ and define $\tilde{T} \in M_{2k}(\mathcal{A})$ by

$$\tilde{T} = \sum_{j=1}^{k} (T \otimes e_{j+1,j} + T^* \otimes e_{k+j+1,k+j})$$

where the indices are computed modulo $2k$, such that $e_{2k+1,2k} = e_{1,2k}$. For $\mu \in \mathbb{C}, |\mu| < \frac{1}{\sqrt{e}}$, we put $\tilde{\mu} = \mu 1_{2k}$ and

$$z = z(\mu) = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

Then only the diagonal entries $z_{11}, \dots, z_{2k,2k}$ and the off-diagonal entries $z_{2,2k}, z_{3,2k-1}, \dots, z_{2k,2}$ can be non-zero. Moreover,

$$z_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k}(T^*)^k T^k)^{-1}).$$

The operator $\tilde{T}$ is $M_{2k}(\mathcal{D})$-Gaussian, and repeating the arguments for $k = 2$, we get that for $w \in M_{2k}(\mathcal{D})$, the matrix

$$u = \mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w) \tag{6.1}$$

can have at most $2k$ non-zero entries, namely the entries

$$\begin{aligned} u_{11} &= L^*(w_{2k,2}) \\ u_{2k,2} &= L^*(w_{2k-1,3}) \\ &\vdots \\ u_{k+2,k} &= L^*(w_{k+1,k+1}) \\ u_{k+1,k+1} &= L(w_{k,k+2}) \\ u_{k,k+2} &= L(w_{k-1,k+3}) \\ &\vdots \\ u_{2,2k} &= L(w_{1,1}). \end{aligned} \tag{6.2}$$

By lemma 2.1 there exists a $\delta > 0$ (depending on k), such that if $w \in M_{2k}(\mathcal{D})_{\text{inv}}, \|w\| < \delta, \mu \in \mathbb{C}, |\mu| > \frac{1}{\delta}$ and

$$\mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_{2k}(\mathcal{D})}, \tag{6.3}$$

then

$$w = z = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1}).$$

In particular

$$w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k}(T^*)^k T^k)^{-1}).$$

Next we construct an explicit solution to (6.3). By the above remarks on z, it is sufficient to consider those $w \in M_{2k}(\mathcal{D})_{\text{inv}}$ for which only the

entries $z_{11}, \dots, z_{2k,2k}$ and $z_{2,2k}, z_{3,2k-1}, \dots, z_{2k,2}$ can be non-zero. For such w, (6.3) can by (6.1) and (6.2) be reduced to the $k+1$ identities:
(6.4)
$$\begin{cases} L^*(w_{2k,2}) + \frac{1}{w_{11}} = \mu 1_{\mathfrak{D}} \\ \begin{pmatrix} 0 & L(w_{j+1,2k+1-j}) \\ L^*(w_{2k-1-j,j+3}) & 0 \end{pmatrix} + \begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathfrak{D})}, \\ \hfill j = 0, 1, \dots, k-2, \\ L(w_{k,k+2}) + \frac{1}{w_{k+1,k+1}} = \mu 1_{\mathfrak{D}}. \end{cases}$$

Definition 6.1. For $j \in \mathbb{N} \cup \{0\}$ and $g \in C^{2j+2}$, we let $\Delta_j(g)$ denote the determinant

$$\Delta_j(g) = \begin{vmatrix} g & g^{(1)} & \cdot^{\cdot^{\cdot}} & g^{(j)} \\ g^{(1)} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & g^{(2j-1)} \\ g^{(j)} & \cdot^{\cdot^{\cdot}} & g^{(2j-1)} & g^{(2j)} \end{vmatrix}. \tag{6.5}$$

In particular $\Delta_0(g) = g$.

Lemma 6.2. *Let $g \in C^{2j+2}(\mathbb{R})$ and $j \in \mathbb{N}$. Then*

$$\Delta_j(g^{(2)})\Delta_j(g) - \Delta_j(g^{(1)})^2 = \Delta_{j-1}(g^{(2)})\Delta_{j+1}(g) \tag{6.6}$$

and
(6.7)
$$\Delta_{j-1}(g^{(2)})\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_j(g)\right) - \Delta_j(g)\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_{j-1}(g^{(2)})\right) = \Delta_{j-1}(g^{(1)})\Delta_j(g^{(1)}).$$

The proof of lemma 6.2 relies on elementary matrix manipulations and is contained in lemma A.1 of appendix A. More specifically (6.6) is a direct consequence of (a) from lemma A.1, and (6.7) follows from (b) of lemma A.1 by using the elementary fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\Delta_j(g)\right) = \begin{vmatrix} g & g^{(1)} & \cdot^{\cdot^{\cdot}} & g^{(j)} \\ g^{(1)} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \\ \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & g^{(2j-1)} \\ g^{(j-1)} & \cdot^{\cdot^{\cdot}} & g^{(2j-2)} & g^{(2j-1)} \\ g^{(j+1)} & \cdot^{\cdot^{\cdot}} & g^{(2j)} & g^{(2j+1)} \end{vmatrix},$$

that is, differentiating (6.5) is the same as differentiating the last row of (6.5).

The next two lemmas are the generalizations of lemma 5.4 and lemma 5.5 to arbitrary $k \geq 2$.

Lemma 6.3. *Let $f \in C^k([0,1])$ and let $(f^{(-j)})_{j=1}^k$ be the antiderivatives of f for which,*

(i)

$$f^{(-j)}(1) = \begin{cases} 0, & 1 \le j \le k-1, \\ \mu^{2k-1}, & j = k. \end{cases}$$

(ii) Assume further that

$$f(0) = \mu^{-1} \text{ and } f^{(-j)}(0) = 0 \text{ for } 1 \le j \le k-1.$$

(iii) For all $x \in [0,1]$,

$$\Delta_j(f^{(-j)})(x) \neq 0, \text{ for } j = 0\dots, k-1$$

and

$$\Delta_k(f^{(-k)})(x) = 0$$

Then the set of $4k-2$ functions listed in (6.8), (6.9) and (6.10) below is a solution to (6.4).

$$\begin{cases} w_{11} = f \\ w_{22} = w_{2k,2k} = -\frac{1}{\mu}\frac{\Delta_1(f^{(-1)})}{f^2} \\ w_{2,2k} = \frac{1}{\mu^2}\frac{f^{(-1)}\Delta_1(f^{(-1)})}{f^2} \\ w_{2k,2} = \frac{f^{(1)}}{f^2} \end{cases} \tag{6.8}$$

For $j = 1, \dots, k-2$

$$\begin{cases} w_{j+2,j+2} = w_{2k-j,2k-j} = -\frac{1}{\mu}\frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\ w_{j+2,2k-j} = \frac{1}{\mu^{2j+2}}\frac{\Delta_j(f^{(-1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2} \\ w_{2k-j,j+2} = \mu^{2j}\frac{\Delta_{j-1}(f^{(1-j)})\Delta_j(f^{(1-j)})}{\Delta_j(f^{(-j)})^2} \end{cases} \tag{6.9}$$

$$w_{k+1,k+1} = \mu^{2k+2}\frac{\Delta_{k-2}\big(f^{(2-k)}\big)\Delta_{k-1}\big(f^{(2-k)}\big)}{\Delta_{k-1}\big(f^{(1-k)}\big)^2} \tag{6.10}$$

Moreover for $j = 0, \dots, k-2$

$$\begin{vmatrix} w_{j+2,j+2} & w_{j+2,2k-2} \\ w_{2k-j,j+2} & w_{2k-j,2k-j} \end{vmatrix} = \frac{1}{\mu}w_{j+2,j+2} \tag{6.11}$$

and

$$(6.12)\qquad \begin{cases} L(w_{11}) = -f^{(-1)} \\ L(w_{j+2,2k-j}) = -\frac{1}{\mu^{2j+2}}\frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}, \quad 0 \le j \le k-3 \\ L(w_{k,k+2}) = \mu - \frac{1}{\mu^{2k-2}}\frac{\Delta_{k-1}(f^{(-k)})}{\Delta_{k-2}(f^{(2-k)})} \end{cases}$$

$$(6.13)\qquad \begin{cases} L^*(w_{2k,2}) = \mu - \frac{1}{f} \\ L^*(w_{2k-j,2+j}) = -\mu^{2j}\frac{\Delta_{j-1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}, \quad 1 \le j \le k-2\,. \\ L^*(w_{k+1,k+1}) = -\mu^{2k-2}\frac{\Delta_{k-2}(f^{(3-k)})}{\Delta_{k-1}(f^{(1-k)})} \end{cases}$$

Proof. Let $w_{11}, w_{22}, \dots, w_{kk}, w_{2,2k}, w_{3,2k-1}, \dots, w_{2k,2}$ be given by (6.8), (6.9) and (6.10). Then for $1 \le j \le k-2$ the left hand side of (6.11) is equal to

$$-\frac{1}{\mu^2}\frac{\Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)})A}{\Delta_j(f^{(-j)})^4},$$

where $A = \Delta_{j-1}(f^{(1-j)})\Delta_{j+1}(f^{(-1-j)}) - \Delta_j(f^{(1-j)})\Delta_j(f^{(-1-j)})$.

By applying (6.6) to $g = f^{(-1-j)}$ it follows that $A = -\Delta_j(f^{(-j)})^2$, which proves (6.11) for $1 \le j \le k-2$. The case $j = 0$ of (6.11) follows immediately from (6.8).

The proofs of (6.12) and 6.13) can be obtained exactly as in the case $k = 2$ provided the following two identities holds: For $j = 0, \dots, k-2$:

$$(6.14)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\Delta_{j+1}(f^{(-2-j)})}{\Delta_j(f^{(-j)})}\right) = \frac{\Delta_j(f^{(-1-j)})\Delta_{j+1}(f^{(-1-j)})}{\Delta_j(f^{(-j)})^2}$$

For $j = 1, \dots, k-1$:

$$(6.15)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\Delta_{j-1}(f^{(2-j)})}{\Delta_j(f^{(-j)})}\right) = \frac{\Delta_{j-1}(f^{(1-j)})\Delta_j(f^{(1-j)})}{\Delta_j(f^{(-j)})^2}$$

However (6.14) follows from (6.7) with $g = f^{(-2-j)}$ after changing j in (6.7) to $j+1$. In the same way (6.15) follows from (6.7) with $g = f^{(-j)}$ and j unchanged. It remaims to be proved, that $w_{11}, \dots, w_{kk}, w_{2,2k}, \dots, w_{2k,2}$ form a solution to (6.4). The proof of the first 2 identities in (6.4) is exactly the same as in the case $k = 2$. Let us check the next $k-2$

identities in (6.4) i.e.

$$(6.16)\quad \begin{pmatrix} 0 & L(w_{j+1,2k+1-j}) \\ L^*(w_{2k-1-j,j+3}) & 0 \end{pmatrix} + \begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \mu 1_{M_2(\mathcal{D})}$$

for $j = 1, \dots, k-2$. By (6.11) and the fact that $w_{2+j,2+j} = w_{2k-j,2k-j}$ (cf. (6.8)) we have

$$\begin{pmatrix} w_{2+j,2+j} & w_{2+j,2k-j} \\ w_{2k-j,2+j} & w_{2k-j,2k-j} \end{pmatrix}^{-1} = \begin{pmatrix} \mu 1_{\mathcal{D}} & \beta \\ \gamma & \mu 1_{\mathcal{D}} \end{pmatrix},$$

where

$$\beta = -\mu \frac{w_{2+j,2k-j}}{w_{2+j,2+j}} = \frac{1}{\mu^{2j}} \frac{\Delta_j(f^{(-1-j)})}{\Delta_{j-1}(f^{(1-j)})}$$

and

$$\gamma = -\mu \frac{w_{2k-j,2+j}}{w_{2+j,2+j}} = \mu^{2j+2} \frac{\Delta_j(f^{(1-j)})}{\Delta_{j+1}(f^{(-1-j)})}.$$

Hence by (6.12) and (6.13)

$$\beta = -L(w_{j+1,2k-j+1}) \text{ and } \gamma = -L^*(w_{2k-1-j,j+3})$$

for $j = 1, \dots, k-2$. This proves (6.16). Observe next that by (6.10) and (6.12)

$$w_{k+1,k+1}(\mu - L(w_{k,k+2})) = \frac{\Delta_{k-1}(f^{(2-k)})\Delta_{k-1}(f^{(-k)})}{\Delta_{k-1}(f^{(1-k)})^2} = 1 + \frac{\sigma}{\Delta_{k-1}(f^{(1-k)})^2},$$

where

$$\sigma = \Delta_{k-1}(f^{(2-k)})\Delta_{k-1}(f^{(-k)}) - \Delta_{k-1}(f^{(1-k)})^2.$$

By (6.6) and the assumptions (iii) in lemma 6.3

$$\sigma = \Delta_{k-2}(f^{(2-k)})\Delta_k(f^{(-k)}) = 0.$$

Hence $w_{k+1,k+1}(\mu - L(w_{k,k+2})) = 1$, which proves the last equality in (6.4). This completes the proof of lemma 6.3. □

Lemma 6.4. *Let $k \in \mathbb{N}, k \geq 2$ and let $\alpha_j(s), \gamma_j(s)$ for $j = 1, \dots, k$ and $0 < |s| < \frac{1}{\mathrm{e}}$ be as in lemma 5.2. Let $\mu \in \mathbb{C}$, $|\mu| > \sqrt{\mathrm{e}}$, put $s = \frac{1}{k}\mu^{-2}$ and*

$$(6.17)\quad \begin{cases} f(x) = \frac{1}{\mu}\left(\sum_{\nu=1}^k \gamma_\nu(s) \mathrm{e}^{k\alpha_\nu(s)x}\right), & x \in \mathbb{R} \\ f^{(-j)}(x) = \frac{1}{\mu k^j}\left(\sum_{\nu=1}^k \frac{\gamma_\nu(s)}{\alpha_\nu(s)^j} \mathrm{e}^{k\alpha_\nu(s)x}\right), & x \in \mathbb{R}, j = 1, \dots, k \end{cases}.$$

Then

(i) $(f^{(-j)})_{j=1}^{k}$ *are succesive antiderivatives of* f*. Moreover*

$$\begin{cases} f^{(-j)}(1) = 0, & 1 \le j \le k-1 \\ f^{(-k)}(1) = \mu^{2k-1} \end{cases} \tag{6.18}$$

and

$$\begin{cases} f(0) = \mu^{-1} \\ f^{(j)}(0) = 0, \quad 1 \le j \le k-1 \end{cases}. \tag{6.19}$$

(ii) The following asymptotic formulas holds for $|\mu| \to \infty$

$$\begin{cases} f^{(-k)}(x) = \mu^{2k-1} + \mathcal{O}(\mu^{-1}) \\ f^{(-j)}(x) = \frac{1}{j!}(x-1)^j \mu^{-1} + \mathcal{O}(\mu^{-2k-1}), & 1 \le j \le k-1 \\ f(x) = \mu^{-1} + \mathcal{O}(\mu^{-2k-1}) \\ f^{(j)}(x) = \frac{1}{j!} x^j \mu^{-2k-1} + \mathcal{O}(\mu^{-4k-1}), & 1 \le j \le k-1 \\ f^{(k)}(x) = \mu^{-2k-1} + \mathcal{O}(\mu^{-4k-1}) \end{cases}, \tag{6.20}$$

where the error estimates holds uniformly in x *on compact subsets of* $\mathbb{R}$*.*

(iii) There exists a $\mu_0 \ge \sqrt{\mathrm{e}}$*, such that the restriction of* f *to* $[0,1]$ *satisfies all the conditions in lemma 6.3, when* $|\mu| > \mu_0$*.*

Proof. From the proof of [5, Prop. 4.2], we know that $\alpha_j(s)$ and $\gamma_j(s)$ are analytic functions of $s \in B(0, \frac{1}{\mathrm{e}})$. Moreover by [4, Prop. 4.1]

$$\begin{cases} \sum_{\nu=1}^{k} \gamma_\nu(s) = 1 \\ \sum_{\nu=1}^{k} \gamma_\nu(s) \alpha_\mu(s)^j = 1, \quad j = 1, \ldots, k-1 \end{cases}. \tag{6.21}$$

Moreover, since $\alpha_j(s) = \rho\big(\mathrm{e}^{\mathrm{i}\frac{2\pi j}{k}} s\big)$, where ρ satisfies

$$\rho(w)\mathrm{e}^{-\rho(w)} = w \text{ for } |w| < \frac{1}{\mathrm{e}}$$

we have $\big(\alpha_\nu(s)\mathrm{e}^{-\alpha_\nu(s)}\big)^k = s^k$ and therefore

$$\mathrm{e}^{k\alpha_\nu(s)} = \frac{s^k}{(\alpha_\nu(s))^k} \tag{6.22}$$

for $\nu = 1, \ldots, k$. Having (6.21) and (6.22) in mind, the proof of (i) and (ii) in lemma 6.4 is now a routine generalization of the proof of lemma

5.5. Concerning (iii) in lemma 6.4, we have

$$(6.23)\qquad \begin{cases} \Delta_j(f^{(-j)}) = \sigma(j)\mu^{-j-1} + \mathcal{O}(\mu^{-2k-j-1}), \quad 0,\dots,k-1, \\ \text{where } \sigma(j) = 1 \text{ for } j = 0,3 \pmod 4 \\ \text{and } \sigma(j) = -1 \text{ for } j = 1,2 \pmod 4 \end{cases}$$

because the leading term in the determinant $\Delta_j(f^{(-j)})$ comes from the antidiagonal, i.e.

$$\Delta_j(f^{(-j)}) = \begin{vmatrix} 0 & \dots & 0 & f \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ f & 0 & \dots & 0 \end{vmatrix} + \mathcal{O}(\mu^{-2k-j-1}) = \sigma(j) f^{j+1} + \mathcal{O}(\mu^{-2k-j-1})$$

since the matrix in question has size $j+1$. Hence $\Delta_j(f^{(-j)})(x) \neq 0$ for $x \in [0,1]$ and $0 \leq j \leq k-1$, when $|\mu|$ is sufficiently large. Moreover $\Delta_k(f^{(-k)}) = 0$ for $x \in [0,1]$, because in analogy with (5.29), $\Delta_k(f^{(-k)}(x))$ is the determinant of the $(k+1)\times(k+1)$ matrix

$$F = (f^{(i+j-k)})_{i,j=0,\dots,k}$$

which has the factorization $F = ADA^t$, where A is the $(k+1)\times k$ matrix with entries

$$a_{il} = (k\alpha_l(s))^i, \quad i = 0,\dots,k, \quad l = 1,\dots,k$$

and D is the $k \times k$ diagonal matrix, with diagonal entries

$$d_{ll} = \frac{\gamma_l(s)}{(k\alpha_l(s))^k} \mathrm{e}^{k\alpha_l(s)}, \quad l = 1,\dots,k.$$

□

Proof of Theorem 5.1 in the general case. Let μ_0 be as in lemma 6.4, let $\mu \in \mathbb{C}, |\mu| > \mu_0$ and put $s = \frac{1}{k}\mu^{-2}$. Put as before

$$f(x) = \frac{1}{\mu}\left(\sum_{\nu=1}^{k} \gamma_j(s)\mathrm{e}^{k\alpha_j(s)x}\right)$$

for $x \in [0,1]$, and define $w_{11}, w_{22}, \dots, w_{k,k}, w_{2,2k}, w_{3,2k-1}, \dots, w_{2k,2}$ by (6.8), (6.9) and (6.10), and put all other entries of $w \in M_{2k}(\mathcal{D})$ equal to 0. Then by lemma 6.4, (6.4) holds, and therefore

$$\mathcal{R}_{\tilde{T}}^{M_{2k}(\mathcal{D})}(w) + w^{-1} = \mu 1_{M_{2k}(\mathcal{D})}.$$

Let $\delta > 0$ be chosen according to lemma 2.1. If we can find a $\mu_1 \geq \max\{\mu_0, \frac{1}{\delta}\}$, such that

$$(6.24)\qquad |\mu| \geq \mu_1 \Rightarrow \|w\| < \delta$$

then $w = E_{M_{2k}(\mathcal{D})}((\tilde{\mu} - \tilde{T})^{-1})$. In particular

$$f = w_{11} = \mu^{-1} E_{\mathcal{D}}((1 - \mu^{-2k}(T^*)^k T^k)^{-1}), \tag{6.25}$$

and the proof of theorem 5.1 for $k \geq 2$ can be completed exactly as in the case $k = 2$. By (6.23)

$$\begin{cases} \Delta_j(f^{(-j)}) = \mathcal{O}(\mu^{-j-1}), & 0 \leq j \leq k-1 \\ \frac{1}{\Delta_j(f^{(-j)})} = \mathcal{O}(\mu^{j+1}), & 0 \leq j \leq k-1 \end{cases} \tag{6.26}$$

uniformly in $x \in [0,1]$ for $|\mu| \to \infty$. We claim that

$$\begin{cases} \Delta_j(f^{(-j-1)}) = \mathcal{O}(\mu^{-j-1}), & 0 \leq j \leq k-2 \\ \Delta_{k-1}(f^{(-k)}) = \mathcal{O}(\mu^k) \\ \Delta_j(f^{(1-j)}) = \mathcal{O}(\mu^{-j-2k-1}), & 0 \leq j \leq k-2 \\ \Delta_{k-1}(f^{(2-k)}) = \mathcal{O}(\mu^{-3k}) \end{cases}. \tag{6.27}$$

Recall by definition 6.1 that

$$\Delta_j(g) = \det\left((g^{(k+l)})_{k,l=0,\dots,j}\right).$$

Hence for $0 \leq j \leq k-2$, $\Delta_j(f^{(-j-1)})$ is the determinant of a $(j+1) \times (j+1)$ matrix, where each entry is equal to one of the functions $f^{(-j-1)}, f^{(-j)}, \dots, f^{(j-1)}$. By (6.20) all these functions are of order $\mathcal{O}(\mu^{-1})$ as $|\mu| \to \infty$. Hence $\Delta_j(f^{(-j-1)}) = \mathcal{O}(\mu^{-j-1})$ proving the first estimate in (6.27). By the same argument, $\Delta_{k-1}(f^{(-k)})$ is the determinant of a $k \times k$ matrix for which the upper left entry is of the order $\mathcal{O}(\mu^{2k-1})$ and all the other entries are of order $\mathcal{O}(\mu^{-1})$. Hence $\Delta_{k-1}(f^{(-k)}) = \mathcal{O}(\mu^{2k-1}(\mu^{-1})^{k-1}) = \mathcal{O}(\mu^k)$. Let $0 \leq j \leq k-1$. Then $\Delta_j(f^{(1-j)})$ is by (6.20) a determinant of a $(j+1) \times (j+1)$ matrix $M = (m_{k,l})_{k,l=0,\dots,j}$ for which

$$\begin{cases} m_{k,l} = \mathcal{O}(\mu^{-1}) & \text{when } k+l < 0 \\ m_{k,l} = \mathcal{O}(\mu^{-2k-1}) & \text{when } k+l \geq 0 \end{cases}.$$

Hence for any permutation π of $\{0, 1, \dots, k\}$ the product

$$m_{0\pi(0)} m_{1\pi(1)} \cdots m_{j\pi(j)}$$

contains at least one factor of order $\mathcal{O}(\mu^{-2k-1})$. Therefore

$$\Delta_j(f^{(1-j)}) = \det(M) = \sum_{\pi \in S_{j+1}} (-1)^{\mathrm{sign}(\pi)} m_{0\pi(0)} m_{1\pi(1)} \cdots m_{k\pi(k)}$$

is of order $\mathcal{O}(\mu^{-2k-1}(\mu^{-1})^j) = \mathcal{O}(\mu^{-2k-j-1})$. This proves the last two estimates in (6.27). Clearly all estimates holds uniformly in $x \in [0,1]$.

Combining (6.8), (6.9), (6.10) and (6.27), we get

$$\begin{cases} w_{l,l} = \mathcal{O}(\mu^{-1}), & 1 \le l \le 2k \\ w_{j+2,2k-j} = \mathcal{O}(\mu^{-2j-3}), & 0 \le j \le k-2 \;. \\ w_{2k-j,j+2} = \mathcal{O}(\mu^{2j+1-2k}), & 0 \le j \le k-2 \end{cases}$$

In particular all the entries of w are of size $\mathcal{O}(\mu^{-1})$ as $|\mu| \to \infty$ uniformly in $x \in [0,1]$. Hence there exists $\mu_1 \ge \max\{\mu_0, \frac{1}{\delta}\}$ such that (6.24) holds. Hence by (6.25) we have for $|s| < \frac{1}{k}\mu_1^{-2}$,

$$\sum_{k=0}^{\infty} (ks)^{nk} E_{\mathcal{D}}(((T^*)^k T^k)^n)(x) = \sum_{\nu=1}^{\infty} \gamma_j(s) \mathrm{e}^{k\alpha_j(s)x}, \quad x \in [0,1].$$

Now Theorem 5.1 follows from lemma 5.2 and [5, remark 4.3] as in the case $k = 2$. □

Appendix A. Determinant-identities on Hankel-matrices

We need the following lemma on Hankel-determinants.

Lemma A.1. *Let* $a_{-(n-1)}, a_{-(n-2)}, \ldots, a_{n-1}, a_n \in \mathbb{C}$ *for some* $n \in \mathbb{N}$. *Then*

(a)

$$\begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \iddots & a_0 \\ a_{-(n-4)} & \iddots & \iddots & \iddots \\ \iddots & \iddots & \iddots & a_{n-4} \\ a_0 & \iddots & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & a_{-(n-3)} & a_{-(n-4)} & \iddots & a_0 \\ a_{-(n-2)} & a_{-(n-3)} & a_{-(n-4)} & \iddots & \iddots & \iddots \\ a_{-(n-3)} & a_{-(n-4)} & \iddots & \iddots & \iddots & a_{n-4} \\ a_{-(n-4)} & \iddots & \iddots & \iddots & a_{n-4} & a_{n-3} \\ \iddots & \iddots & \iddots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_0 & \iddots & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} a_{-(n-1)} & a_{-(n-2)} & \iddots & a_{-1} \\ a_{-(n-2)} & \iddots & \iddots & \iddots \\ \iddots & \iddots & \iddots & a_{n-4} \\ a_{-1} & \iddots & a_{n-4} & a_{n-3} \end{vmatrix} \begin{vmatrix} a_{-(n-3)} & a_{-(n-4)} & \iddots & a_1 \\ a_{-(n-4)} & \iddots & \iddots & \iddots \\ \iddots & \iddots & \iddots & a_{n-2} \\ a_1 & \iddots & a_{n-2} & a_{n-1} \end{vmatrix} - \begin{vmatrix} a_{-(n-2)} & a_{-(n-3)} & \iddots & a_0 \\ a_{-(n-3)} & \iddots & \iddots & \iddots \\ \iddots & \iddots & \iddots & a_{n-3} \\ a_0 & \iddots & a_{n-3} & a_{n-2} \end{vmatrix}^2 .$$

(b)

$$
\begin{vmatrix}
a_{-(n-2)} & a_{-(n-3)} & \iddots & a_1 \\
a_{-(n-3)} & \iddots & \iddots & \iddots \\
\iddots & \iddots & \iddots & a_{n-1} \\
a_1 & \iddots & a_{n-1} & a_n
\end{vmatrix}
\begin{vmatrix}
a_{-(n-2)} & a_{-(n-3)} & \iddots & a_0 \\
a_{-(n-3)} & \iddots & \iddots & \iddots \\
\iddots & \iddots & \iddots & a_{n-3} \\
a_0 & \iddots & a_{n-3} & a_{n-2}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_{-(n-1)} & a_{-(n-2)} & \iddots & a_0 \\
a_{-(n-2)} & \iddots & \iddots & \iddots \\
\iddots & \iddots & \iddots & a_{n-3} \\
a_{-1} & \cdots & a_{n-3} & a_{n-2} \\
a_1 & a_2 & \cdots & a_n
\end{vmatrix}
\begin{vmatrix}
a_{-(n-3)} & a_{-(n-4)} & \iddots & a_1 \\
a_{-(n-4)} & \iddots & \iddots & \iddots \\
\iddots & \iddots & \iddots & a_{n-2} \\
a_1 & \iddots & a_{n-2} & a_{n-1}
\end{vmatrix}
$$

$$
- \begin{vmatrix}
a_{-(n-1)} & a_{-(n-2)} & \iddots & a_0 \\
a_{-(n-2)} & \iddots & \iddots & \iddots \\
\iddots & \iddots & \iddots & a_{n-2} \\
a_0 & \iddots & a_{n-2} & a_{n-1}
\end{vmatrix}
\begin{vmatrix}
a_{-(n-3)} & a_{-(n-4)} & \iddots & a_1 \\
a_{-(n-4)} & \iddots & \iddots & \iddots \\
\iddots & \iddots & \iddots & a_{n-3} \\
a_0 & \cdots & a_{n-3} & a_{n-2} \\
a_2 & a_3 & \cdots & a_n
\end{vmatrix}.
$$

Proof. To prove (a) we actually prove the more general equation

$$
\text{(A.1)} \quad
\begin{vmatrix}
a_{22} & a_{23} & \cdots & a_{2,n-1} \\
a_{32} & a_{33} & \cdots & a_{3,n-1} \\
\vdots & \vdots & & \vdots \\
a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1}
\end{vmatrix}
\begin{vmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1,n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2,n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3,n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1,n-1} \\
a_{21} & a_{22} & \cdots & a_{2,n-1} \\
\vdots & \vdots & & \vdots \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1}
\end{vmatrix}
\begin{vmatrix}
a_{22} & a_{23} & \cdots & a_{2,n} \\
a_{32} & a_{33} & \cdots & a_{3,n} \\
\vdots & \vdots & & \vdots \\
a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{vmatrix}
$$

$$
- \begin{vmatrix}
a_{12} & a_{13} & \cdots & a_{1,n} \\
a_{22} & a_{23} & \cdots & a_{2,n} \\
\vdots & \vdots & & \vdots \\
a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n}
\end{vmatrix}
\begin{vmatrix}
a_{21} & a_{22} & \cdots & a_{2,n-1} \\
a_{31} & a_{32} & \cdots & a_{3,n-1} \\
\vdots & \vdots & & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n-1}
\end{vmatrix}
$$

for $a_{ij} \in \mathbb{C}$ and $i, j \in \{1, \ldots, n\}$.

We first add some zero terms to the left-hand side (LHS) of (A.1).

$$
\text{LHS} = \begin{vmatrix}
a_{22} & \cdots & a_{2,n-1} \\
\vdots & & \vdots \\
a_{n-1,2} & \cdots & a_{n-1,n-1}
\end{vmatrix}
\begin{vmatrix}
a_{11} & \cdots & a_{1,n} \\
\vdots & & \vdots \\
a_{n,1} & \cdots & a_{n,n}
\end{vmatrix}
$$

$$
+ \sum_{k=2}^{n-1}
\begin{vmatrix}
a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\
a_{31} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n-1} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1}
\end{vmatrix}
\begin{vmatrix}
a_{12} & \cdots & a_{1,k} & a_{1,k} & a_{1,k+1} & \cdots & a_{2,n-1} \\
a_{22} & \cdots & a_{2,k} & a_{2,k} & a_{2,k+1} & \cdots & a_{3,n-1} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots \\
a_{n,2} & \cdots & a_{n,k} & a_{n,k} & a_{n,k+1} & \cdots & a_{n-1,n-1}
\end{vmatrix}
$$

We note that the last matrix in the sum is zero because coloumn $k-1$ and k are equal. Now we expand LHS after the k'th coloumn of the second matrix in the k'th addent. We get

$$\mathrm{LHS} = \sum_{j=1}^{n} (-1)^{1+j} a_{j,1} \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix}$$

$$+ \sum_{k=2}^{n-1} \sum_{j=1}^{n} (-1)^{k+j} a_{j,k} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix}$$

where $j = 1$ and $j = n$ means leave out row 1 and n respectively. Switching the indices we have

$$\text{(A.2)} \quad \mathrm{LHS} = \sum_{j=1}^{n} \begin{vmatrix} a_{12} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{j-1,2} & \cdots & a_{j-1,n} \\ a_{j+1,2} & \cdots & a_{j+1,n} \\ \vdots & & \vdots \\ a_{n,2} & \cdots & a_{n,n} \end{vmatrix} \Bigg((-1)^{1+j} a_{j,1} \begin{vmatrix} a_{22} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix}$$
$$\sum_{k=2}^{n-1} (-1)^{k+j} a_{j,k} \begin{vmatrix} a_{21} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n-1} \end{vmatrix} \Bigg)$$

But the parenthesis on the right-hand side is exactly expansion along the j'th row of the following determinants

$$\text{(A.3)} \qquad \begin{cases} \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} \end{vmatrix}, & j = 1 \\ \begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{j,1} & \cdots & a_{j,n-1} \\ a_{j,1} & \cdots & a_{j,n-1} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix} = 0, & 2 \le j \le n-1 \\ -\begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n-1} \end{vmatrix}, & j = n. \end{cases}$$

Combining (A.2) and (A.3) we obtain the right-hand side of (A.1) and thus also the proof of (a).

To prove (b) we prove the more general equation

$$
\text{(A.4)}\quad
\begin{vmatrix}
a_{21} & a_{22} & \cdots & a_{2,n} \\
a_{31} & a_{32} & \cdots & a_{3,n} \\
a_{41} & a_{42} & \cdots & a_{4,n} \\
\vdots & \vdots & & \vdots \\
a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{vmatrix}
\begin{vmatrix}
a_{12} & a_{13} & \cdots & a_{1,n} \\
a_{22} & a_{23} & \cdots & a_{2,n} \\
\vdots & \vdots & & \vdots \\
a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1,n} \\
a_{21} & a_{22} & \cdots & a_{2,n} \\
\vdots & \vdots & & \vdots \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\
a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{vmatrix}
\begin{vmatrix}
a_{22} & a_{23} & \cdots & a_{2,n} \\
a_{32} & a_{33} & \cdots & a_{3,n} \\
\vdots & \vdots & & \vdots \\
a_{n,2} & a_{n,3} & \cdots & a_{n,n}
\end{vmatrix}
$$

$$
- \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1,n} \\
a_{21} & a_{22} & \cdots & a_{2,n} \\
\vdots & \vdots & & \vdots \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{vmatrix}
\begin{vmatrix}
a_{22} & a_{23} & \cdots & a_{2,n} \\
a_{32} & a_{33} & \cdots & a_{3,n} \\
\vdots & \vdots & & \vdots \\
a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\
a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n}
\end{vmatrix}
$$

for $a_{ij} \in \mathbb{C}$, $i \in \{1, \dots n+1\}$ and $j \in \{1, \dots, n\}$. We remark that Hankel-matrices are symmetric and for these (A.4) reduces to (b). Observe that for $k \in \{2, \dots, n\}$ we have

$$
0 = (-1)^k \begin{vmatrix}
a_{1,k} & a_{11} & a_{12} & \cdots & a_{1,n} \\
a_{2,k} & a_{12} & a_{22} & \cdots & a_{2,n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{n,k} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} \\
a_{n+1,k} & a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n}
\end{vmatrix}
$$

$$
= (-1)^k \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1,n} \\
\vdots & \vdots & & \vdots \\
a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\
a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\
\vdots & \vdots & & \vdots \\
a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{vmatrix},
$$

where the $j = 1$ and $j = n + 1$ are interpreted as remove the 1^{st} and $(n+1)^{\text{th}}$ coloumn respectively. Thus also

$$
0 = \sum_{k=2}^{n}
\begin{vmatrix}
a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n} \\
a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n} \\
\vdots & & \vdots & \vdots & & \vdots \\
a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n}
\end{vmatrix}
$$

$$
\cdot \left((-1)^k \sum_{j=1}^{n+1} a_{j,k} (-1)^{j+1}
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1,n} \\
\vdots & \vdots & & \vdots \\
a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n} \\
a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n} \\
\vdots & \vdots & & \vdots \\
a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}
\end{vmatrix} \right)
$$

Switching the indices we have

$$
\text{(A.5)}\quad 0=\sum_{j=1}^{n+1}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n}\\ \vdots & \vdots & & \vdots\\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,n}\\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,n}\\ \vdots & \vdots & & \vdots\\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n}\end{vmatrix}
\cdot\left(\sum_{k=2}^{n}(-1)^{k+j-1}a_{j,k}\begin{vmatrix} a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2,n}\\ a_{32} & \cdots & a_{3,k-1} & a_{3,k+1} & \cdots & a_{3,n}\\ \vdots & & \vdots & \vdots & & \vdots\\ a_{n-1,2} & \cdots & a_{n-1,k-1} & a_{n-1,k+1} & \cdots & a_{n-1,n}\end{vmatrix}\right)
$$

The parenthesis of (A.5) is expansion along the j^{th} row of the following expression except for $j=n+1$ where we expand along the n^{th} row.

$$
\text{(A.6)}\quad\begin{cases}\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n}\\ a_{22} & a_{23} & \cdots & a_{2,n}\\ \vdots & \vdots & & \vdots\\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n}\end{vmatrix}, & j=1\\ 0, & j\in\{2,\dots,n-1\}\\ \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n}\\ a_{32} & a_{33} & \cdots & a_{3,n}\\ \vdots & \vdots & & \vdots\\ a_{n,2} & a_{n,3} & \cdots & a_{n,n}\end{vmatrix}, & j=n\\ -\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n}\\ \vdots & \vdots & & \vdots\\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n}\\ a_{n+1,2} & a_{n+1,3} & \cdots & a_{n+1,n}\end{vmatrix} & j=n+1.\end{cases}
$$

Combining (A.5) and (A.6) we obtain (A.4) and this finishes the proof of (b). ☐

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