Lectures in Noncommutative Geometry Seminar 2005

TRACE FUNCTIONALS AND TRACE DEFECT FORMULAS ...

I. Traces on classical ψdo’s.

We consider:

\[ X \] — compact boundaryless \( n \)-dimensional manifold (closed).

\[ E \] — hermitian vector bundle over \( X \).

\[ \mathcal{A} \] — the ‘algebra’ of classical ψdo’s \( \mathcal{A} \) acting in \( E \).

On pseudodifferential operators:

Recall that a differential operator of order \( m \geq 0 \) on \( \mathbb{R}^n \) can be written:

\[
Au = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = \mathcal{F}_{\xi \rightarrow x}^{-1} \left( \sum_\alpha a_\alpha(x) \xi^\alpha \hat{u}(\xi) \right)
\]

\[
= \text{OP}(a(x, \xi))u, \text{ with } a(x, \xi) = \sum_\alpha a_\alpha(x) \xi^\alpha.
\]

A classical pseudodifferential symbol of order \( \nu \in \mathbb{R} \):

\[
a(x, \xi) \sim a_\nu(x, \xi) + a_{\nu-1}(x, \xi) + \cdots + a_{\nu-j}(x, \xi) + \cdots
\]
\[ a_{\nu-j}(x, t\xi) = t^{\nu-j} a(x, \xi) \text{ for } |\xi| \geq 1, \ t \geq 1. \]

Elliptic, when the principal symbol \( a_{\nu}(x, \xi) \neq 0 \) for \( |\xi| \geq 1 \).

Defines a pseudodifferential operator (\( \psi \text{do} \)):

\[ Au = \text{Op}(a)u = \mathcal{F}_{\xi \rightarrow x}^{-1}(a(x, \xi)\hat{u}(\xi)) \]

Continuous from \( H^s(\mathbb{R}^n) \) to \( H^{s-\nu}(\mathbb{R}^n) \). Composition:

\[ \text{Op}(a(x, \xi)) \text{Op}(b(x, \xi)) = \text{Op}(a \# b), \text{ where} \]

\[ a \# b \sim a \cdot b + \sum_{\alpha \neq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_x^\alpha a \partial_x^\alpha b. \]

Elliptic operators have (approximate) inverses.

\( \Psi \text{do}'s \) are defined on manifolds by use of local coordinates.

Consider \( A \) on a closed manifold \( X \). For \( \nu < -n \), \( A \) is trace-class, \( \text{Tr} \ A = \int_X \text{tr} \ K_A(x, x)dx \). Here \( \text{Tr}([A, A']) = 0 \), where \( [A, A'] = AA' - A'A \).

A trace functional \( \ell(A) \) is a linear functional that vanishes on commutators: \( \ell([A, A']) = 0 \). Search for nontrivial trace functionals on higher-order \( \psi \text{do}'s \)!
(I) Wodzicki, Guillemin ca. ‘84: *The noncommutative residue*

\[ \res(A) = \int_X \int_{|\xi|=1} \tr a_{-n}(x, \xi) \varphi S(\xi) dx; \]

it has a coordinate invariant meaning. \( (\varphi = (2\pi)^{-n} d.) \)

- **Local** (depends only on certain homogeneous terms in \( a \)).
- Defined for all \( A \in A \), unique up to a factor.
- **Vanishes** for \( \nu \notin \mathbb{Z} \).
- **Vanishes** for \( \nu < -n \); does **not** extend \( \text{Tr} \ A \)!

(II) Kontsevich and Vishik ca. ‘94: *The canonical trace TR(A)*

- **Global** (depends on the full structure).
- Defined only for some \( A \), namely in the cases:
  
  (1) \( \nu < -n \), then \( \text{TR} \ A = \text{Tr} \ A \);
  
  (2) \( \nu \notin \mathbb{Z} \);
  
  (3) \( \nu \in \mathbb{Z}, \ n \text{ odd}, A \text{ has even-even parity}; \)
  
  (4) \( \nu \in \mathbb{Z}, \ n \text{ even}, A \text{ has even-odd parity}. \) (Added by GG.)

(Will give formula later.)
Parity properties:

**even-even** alternating parity: Even order terms are even in \( \xi \),
\[
a_{\nu-j}(x, -\xi) = (-1)^{\nu-j} a_{\nu-j}(x, \xi) \text{ for } |\xi| \geq 1.
\]
Example: Differential operators and their parametrices.

**even-odd** alternating parity: Even order terms are odd in \( \xi \),
\[
a_{\nu-j}(x, -\xi) = (-1)^{\nu-j-1} a_{\nu-j}(x, \xi) \text{ for } |\xi| \geq 1.
\]
Example: \( D|D|^{-1}, D \) a selfadj. first-order elliptic diff. op.

The trace property holds in the following sense:
\[
\text{TR}([A, A']) = 0 \text{ in the cases}
\]
\[
(1') \quad \nu + \nu' < -n,
\]
\[
(2') \quad \nu + \nu' \in \mathbb{R} \setminus \mathbb{Z}.
\]
\[
(3') \quad \nu \text{ and } \nu' \in \mathbb{Z}, n \text{ is odd, } A \text{ and } A' \text{ are both even-even or both even-odd.}
\]
\[
(4') \quad \nu \text{ and } \nu' \in \mathbb{Z}, n \text{ is even, } A \text{ is even-odd and } A' \text{ is even-even.}
\]
Both res $A$ and TR $A$ were originally defined by use of

**complex powers:**

Let $P$ be elliptic of even order $m > 0$, say $P > 0$.

Define $\zeta(A, P, s) = \text{Tr}(AP^{-s})$, the *generalized zeta function*, holomorphic for Re $s > (n + \nu)/m$, extends meromorphically to $\mathbb{C}$ with simple poles in

\[ \{(n + \nu - j)/m \mid j \in \mathbb{N}\} \cup \{-k \mid k \in \mathbb{N}\}; \]

here $\mathbb{N} = \{0, 1, 2, \ldots \}$.

In particular, $\zeta$ has a Laurent expansion at $s = 0$:

\[ \zeta(A, P, s) \sim \frac{1}{s} C_{-1}(A, P) + C_0(A, P) + \sum_{l \geq 1} C_l(A, P) s^l. \]

Then

(I) res $A = m \cdot C_{-1}(A, P)$, the residue at $s = 0$.

(II) In the cases (1)–(4) (with $P$ even-even for (3)–(4)),

$C_{-1}(A, P) = 0$ and

\[ \text{TR } A = C_0(A, P). \]

NB! Independent of $P$!

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**Three operator families:**

$P$: strongly elliptic ps.d.o. on $X$ of even order $m > 0$.

- *Resolvent* $(P - \lambda)^{-1}$,
- *Heat operator* $e^{-tP}$,
- *Power operator* $P^{-s}$ (defined as 0 on ker $P$).

Can be obtained from one another:

\[
\text{Resolvent } (P - \lambda)^{-1} \quad \xrightarrow{\text{Cauchy int.}} \quad e^{-tP} \quad \text{Heat operator} \\
\text{Laplace transf.}
\]

\[
\text{Cauchy int. } \sim \quad \sim \text{ Mellin transf.}
\]

\[
\Gamma(s)P^{-s}
\]

*Power operator*

Example of Cauchy integral:

\[
P^{-s} = \frac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-s}(P - \lambda)^{-1} d\lambda.
\]
THREE EQUIVALENT ASYMPTOTIC TRACE EXPANSIONS:

The resolvent trace expansion:

\[
\text{Tr}(A(P - \lambda)^{-N}) \sim \sum_{j \geq 0} \tilde{c}_j (-\lambda)^{-\frac{\nu+n-j}{m}} - N
\]
\[+ \sum_{k \geq 0} (\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k) (-\lambda)^{-k-N},
\]
for \(\lambda \to \infty\) in \(\mathbb{C} \setminus \overline{\mathbb{R}}_+.\) \((N > (\nu + n)/m).\)

The heat trace expansion:

\[
\text{Tr}(Ae^{-tP}) \sim \sum_{j \geq 0} c_j t^{\frac{j-\nu-n}{m}} + \sum_{k \geq 0} (-c'_k \log t + c''_k)t^k
\]
for \(t \to 0^+.\)

The complex power trace expansion:

\[
\Gamma(s) \text{Tr}(AP^{-s}) \sim \sum_{j \geq 0} \frac{c_j}{s + \frac{j-\nu-n}{m}} + \sum_{k \geq 0} \left( \frac{c'_k}{(s+k)^2} + \frac{c''_k}{s+k} \right);
\]
where the right-hand side gives the pole structure of the meromorphic extension. Division by \(\Gamma(s)\) gives simple poles, and

\[
C_{-1}(A, P) = \tilde{c}'_0 = c'_0, \quad C_0(A, P) = \tilde{c}_{n+\nu} + \tilde{c}''_0 = c_{n+\nu} + c''_0;
\]
where we set \(\tilde{c}_{n+\nu} = c_{n+\nu} = 0\) if \(n + \nu \notin \mathbb{N}.\) In cases (1)–(4),

\[
C_0(A, P) = c''_0 = \text{TR} A.
\]
Moreover, in cases (1)–(4),

\[ c_0'' = \text{TR}(A) = \int_X \int \text{tr} \, a(x, \xi) \, d\xi \, dx; \]

it has a coordinate invariant meaning. Here \( \int f(x, \xi) \, d\xi \) is a \emph{partie finie} integral, defined as follows: When \( f(x, \xi) \) is a classical symbol of order \( \nu \), then

\[
\int_{|\xi| \leq R} f(x, \xi) \, d\xi \sim \sum_{j \in \mathbb{N}, j \neq n+\nu} a_j(x) R^{n+\nu-j} + a'_0(x) \log R + a''_0(x)
\]

for \( R \to \infty \), and one sets \( \int f(x, \xi) \, d\xi = a''_0(x) \).

Instead of considering powers \( AP^{-s} \), one can deduce these results directly from trace expansions of resolvents \( A(P - \lambda)^{-1} \), using the calculus of G-Seeley ‘95. Details in vol. 366 of AMS Comtemp. Math. Proc., 2005.
II. Trace defect formulas.

Consider $C_0(A, P)$ in general. When (1)–(4) do not hold, $C_0(A, P)$ will depend on $P$ and need not vanish on $[A, A']$. However, there are formulas for the trace defects:

(a) $C_0(A, P) - C_0(A, P') = -\frac{1}{m} \text{res}(A(\log P - \log P'))$,

(b) $C_0([A, A'], P) = -\frac{1}{m} \text{res}(A[A', \log P])$,

showing in particular that they are local. ((a) by Okikiolu ‘95, Konts.-V. ‘95, (a)+(b) by Melrose-Nistor ‘96 unpublished.)

Their proofs go via the holomorphic family $P^{-s}$, with

$$\frac{d}{ds} P^{-s}|_{s=0} = -\log P.$$ 

$\log P$ has symbol $m \log |\xi| + b(x, \xi)$, where $b$ is classical of order 0. Thus

$$A(\log P - \log P')$$

is classical of order $\nu$,

$$A(A' \log P - \log PA')$$

is classical of order $\nu$,

so res is defined.
**Question:** Do similar formulas hold for manifolds with boundary?

A reasonable choice of boundary operator calculus containing elliptic differential boundary problems and their solution operators is the Boutet de Monvel calculus. Can we show similar formulas for such operators?

**Problematic fact:** Even for $P_T = (-\Delta)_{\text{Dirichlet}}$, the complex powers $(P_T)^s$ and the logarithm $\log(P_T)$ are not in the BdM calculus. But the resolvent $(P_T - \lambda)^{-1}$ does belong to a parameter-dependent version of the BdM calculus.

**Subquestion:** Can we prove the formulas (a) and (b) using only resolvent information?
III. Some applications of res, TR and $C_0(A, P)$.

Recall: res $A$ is proportional to the residue of $\zeta(A, P, s)$ at $s = 0$, so

$$\text{res } A = 0 \iff \zeta(A, P, s) \text{ is regular at } 0.$$  

Holds when $A$ is a diff. op., in particular for $\zeta(I, P, s) \equiv \zeta(P, s)$.

The eta function of a selfadjoint, not semibounded elliptic $\psi$do:

$$\eta(P, s) = \sum_{\lambda \text{ ev. } \neq 0} \text{sign } \lambda \lambda^{-s} = \zeta(P|P|^{-1}, |P|, s),$$

is not covered by this. Deep result by Atiyah-Patodi-Singer and Gilkey:

$$\text{(\star) res}(P|P|^{-1}) = 0,$$

i.e., $\eta(P, s)$ is regular at $s = 0$. For a Dirac operator $D$, with

$$D = \sigma(\partial_{x_n} + A) \text{ near } \partial X, \text{ A tangential selfadjoint},$$

the value $\eta(A, 0)$ for $A$ on $\partial X$ is a nonlocal term entering in the index formula for the APS realization of $D$.

From (\star) one can moreover deduce that res $\Pi = 0$ for any classical $\psi$do projection $\Pi$, a fact with various applications.
Concerning TR and $C_0(A, P)$:

Some people call $C_0(A, P)$ a regularized trace of $A$, with notation e.g. $\overline{\text{Tr}}(A)$ (Melrose). Enters in an index formula for $A$:

When $B$ is an approximate inverse (a parametrix),

$$\text{ind } A = \text{Tr}(AB - I) - \text{Tr}(BA - I)$$

$$= C_0(AB - I, P) - C_0(BA - I, P) = C_0([A, B], P)$$

$$= -\frac{1}{m} \text{res}(A[B, \log P]),$$

by the trace defect formula. This is a point of departure for further calculations.
IV. Manifolds with boundary.

Now let $X$ be a compact $n$-dimensional manifold with smooth boundary $X' = \partial X$ (itself a closed manifold).

Typical operators when $X \subset \mathbb{R}^n$:

$$
\begin{pmatrix}
1 - \Delta \\ \gamma_0
\end{pmatrix}
$$
and its inverse $(Q_+ + G \quad K)$;

$\gamma_0$ is the trace operator $u \mapsto u|_{X'}$,

$Q = (1 - \Delta)^{-1} = \text{Op}(\frac{1}{1 + |\xi|^2})$ on $\mathbb{R}^n$,

$Q_+ = r^+ Q e^+ \ (e^+ \text{ extends by 0, } r^+ \text{ restricts to } X)$,

$G$ is a singular Green operator (the “boundary correction”),

$K$ is a Poisson operator.

Here $R = Q_+ + G$ and $K$ solve the respective semi-homogeneous problems:

$$
\begin{cases}
(1 - \Delta)u = f \text{ in } X, \\
\gamma_0 u = 0 \text{ on } X';
\end{cases}
\quad
\begin{cases}
(1 - \Delta)u = 0 \text{ in } X, \\
\gamma_0 u = g \text{ on } X'.
\end{cases}
$$

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Boutet de Monvel ‘71 defined pseudodifferential boundary operators (ψdbo’s) in general as systems (Green operators):

\[
\begin{pmatrix}
P_+ + G & K \\
T & S
\end{pmatrix} : C^\infty(X, E) \times C^\infty(X', F) \to C^\infty(X, E') \times C^\infty(X', F'),
\]

where

- \( P \) is a ψdo on a closed manifold \( \tilde{X} \supset X \), \( P_+ = r^+ Pe^+ \),
- \( G \) is a singular Green operator,
- \( T \) is a trace operator from \( X \) to \( X' \),
- \( K \) is a Poisson operator from \( X' \) to \( X \),
- \( S \) is a ψdo on \( X' \).

\( P \) must satisfy the \textit{transmission condition} at \( X' \), assuring that \( P_+ \) preserves smoothness on \( X \). Consider operators of order \( \nu \) with polyhomogeneous symbols of suitable types.

Traces can be studied when \( E = E' \), \( F = F' \); the new object is \( A = P_+ + G : C^\infty(X, E) \to C^\infty(X, E) \). Transmission requires integer order. For \( G \) alone one can study all real orders.
Technical condition: $G$ should be of class 0 (well-defined on $L_2(X)$), for otherwise, order $<-n$ does not assure trace-class.

The noncommutative residue was defined by Fedosov, Golse, Leichtnam and Schrohe ‘96 for $A = P_+ + G$ by:

$$
\text{res}(A) = \int_X \int_{|\xi|=1} \text{tr} p_{-n}(x, \xi) \phi S(\xi) dx \\
+ \int_{X'} \int_{|\xi'|=1} \text{tr}(\text{tr}_n g)_{1-n}(x', \xi') \phi S(\xi') dx';
$$

here $\text{tr}_n$ takes the trace in the normal direction to $X'$; in fact $\text{tr}_n G$ is a classical $\psi$do on $X'$.

That this is indeed a residue was shown by G-Schrohe ‘01: As auxiliary operator we can take an elliptic differential operator $P_1$ of order $m > n + \nu$ on $\tilde{X}$ having a sector $V$ around $\mathbb{R}_-$ in its resolvent set. Then

$$
\text{Tr}(A(P_1 - \lambda)_+^{-1}) \sim \sum_{0 \leq j \leq n+\nu} c_j (-\lambda)^{\frac{n+\nu-j}{m}-1} \\
+ (c'_0 \log(-\lambda) + c''_0)(-\lambda)^{-1} + O(\lambda^{-1-\varepsilon}), \text{ for } \lambda \to \infty \text{ in } V.
$$
There is a corresponding expansion for $\Gamma(s) \text{Tr}(A(P_1^{-s})_+)$. In particular,

$$\text{Tr}(A(P_1^{-s})_+) = \frac{1}{s} C_{-1}(A, P_1, +) + C_0(A, P_1, +) + O(s)$$

for $s \to 0$, with $C_{-1}(A, P_1, +) = c'_0$, $C_0(A, P_1, +) = c_{n+\nu} + c''_0$ (as usual, we set $c_{n+\nu} = 0$ if $\nu + n \notin \mathbb{N}$). By G-Schrohe ‘01,

$$\text{res}(A) = m \cdot C_{-1}(A, P_1, +).$$

Searching for a canonical trace, G-Schrohe ‘04 showed:

(i) $C_0(A, P_1, +)$ is a quasi-trace, in the sense that $C_0(A, P_1, +) - C_0(A, P_2, +)$ and $C_0([A, A'], P_1, +)$ are local.

(ii) The value of $C_0(A, P_1, +)$ is a finite part integral

$$\int_X \int \text{tr} p(x, \xi) d\xi dx + \int_X \int \text{tr} (\text{tr}_n g)(x', \xi') d\xi' dx',$$

modulo local contributions.

But $C_0(A, P_1, +)$ is rarely a canonical trace. Yes, if $\nu < -n$. Yes, if $\nu \notin \mathbb{Z}$, but then only $G$ enters. When $\nu \in \mathbb{Z}$ and $P \neq 0$, parity does not help much, for both dimensions $n$ and $n - 1$. 

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enter at the same time. Cf. the closed manifold conditions:

(3) \( n \) odd and \( A \) even-even, or (4) \( n \) even and \( A \) even-odd.

So, \( C_0(A, P_{1,+}) \) itself becomes the important object!

Trace defect formulas? (See slide 9 for closed manifolds.)

By a proof that completely avoids the issue of how the operators \((P_1^{-s})_+\) and \((\log P_1)_+\) really act, relying instead on resolvent formulations, we have managed to show (G ‘05):

**Theorem.** Let \( A = P_+ + G, A' = P'_+ + G' \) be given, with two auxiliary elliptic differential operators \( P_1 \) and \( P_2 \).

One can construct \( \psi \)do’s \( S \) and \( S' \) on \( X' \) in a specific way from the given operators such that

(a) \[
C_0(A, P_{1,+}) - C_0(A, P_{2,+})
= -\frac{1}{m} \text{res}_X((P(\log P_1 - \log P_2))_+) - \frac{1}{m} \text{res}_{X'}(S),
\]

(b) \[
C_0([A, A'], P_{1,+})
= -\frac{1}{m} \text{res}_X((P[P', \log P_1])_+) - \frac{1}{m} \text{res}_{X'}(S').
\]
V. Ingredients in the proofs.

$P$ is assumed elliptic with $\mathbb{R}_-$ in the resolvent set.

$Q_\lambda = (P - \lambda)^{-1}$ defined in sector $V$ around $\mathbb{R}_-$, symbol $q(x, \xi, \lambda) \sim \sum_{j \geq 0} q_{-m-j}(x, \xi, \lambda)$. Then

$P^{-s}$ is a classical $\psi$do of order $-ms$ (Seeley '67), symbol $p^{(-s)}(x, \xi) \sim \sum_{j \geq 0} p_{-ms-j}^{(-s)}(x, \xi)$, where

$$p_{-ms-j}^{(-s)}(x, \xi) = \frac{i}{2\pi} \int_C \lambda^{-s} q_{-m-j}(x, \xi, \lambda) \, d\lambda;$$

$C$ a closed curve in $\mathbb{C} \setminus \mathbb{R}_-$ encircling the eigenvalues of $p_m(x, \xi)$.

$$\log P = \text{Op}(m \log[\xi] + b(x, \xi)); \quad [\xi] = |\xi| \text{ for } |\xi| \geq 1;$$

$$b(x, \xi) \sim \sum_{j \geq 0} b_{-j}(x, \xi) \text{ classical of order 0,}$$

$$b_{-j}(x, \xi) = \frac{i}{2\pi} \int_C \log \lambda q_{-m-j}(x, \xi, \lambda) \, d\lambda \text{ for } j > 0.$$
Remarkable fact (Scott ‘04, partially known earlier):

\[ C_0(I, P) = -\frac{1}{m} \text{res}(\log P). \]

We can show it without using complex powers, by observing:

**Lemma 1.** The strictly homogeneous symbol \( q_{-m-n}^h(x, \xi, \lambda) \) is integrable at \( \xi = 0 \) and \( \infty \), and

\[ C_0(I, P) = \int_X c_n(x), \text{ with } c_n(x) = \int_{\mathbb{R}^n} q_{-m-n}^h(x, \xi, -1) \, d\xi. \]

**Lemma 2.** When \( f(x, \xi, \lambda) \) is holomorphic in \( \lambda \) on a nbd. of \( \mathbb{R}_- \), with suitable bounds, then (with a curve \( \mathcal{C} \) in \( \mathbb{C} \setminus \mathbb{R}_- \))

\[ \frac{1}{2\pi i} \int_{\mathcal{C}} \log \lambda f(x, \xi, \lambda) \, d\lambda = \int_{-\infty}^0 f(x, \xi, t) \, dt. \]

For, \( \log \lambda \) gives a jump of \( 2\pi i \) at \( \mathbb{R}_- \); the contributions from \( \log |\lambda| \) cancel out.

Combine this with homogeneity, polar coordinates:

\[ \int_{\mathbb{R}^n} q_{-m-n}^h(x, \xi, -1) \, d\xi = \frac{1}{m} \int_{|\eta|=1} \int_{-\infty}^0 q_{-m-n}^h(x, \eta, t) \, dt \phi S(\eta) \]

\[ = -\frac{1}{m} \int_{|\eta|=1} \frac{i}{2\pi} \int_{\mathcal{C}} \log \lambda q_{-m-n}^h(x, \eta, \lambda) \, d\lambda \phi S(\eta) \]

\[ = -\frac{1}{m} \int_{|\eta|=1} b_n(x, \eta) \phi S(\eta), \]

the inner integral in the residue of \( \log P \)!
The trace defect formulas in the closed manifold case can be proved by calculations where this type of argument is central; we never need to consider $P^{-s}$, and $\log P$ enters only in a very rudimentary way.

Finally, for the case with boundary, this argument is again central, but a lot of extra efforts are needed to master the contributions from the boundary.

Application e.g. to index formulas:

If $A = P_+ + G \colon C^\infty(X, E) \to C^\infty(X, F)$ is elliptic of order and class 0, and $B$ is a parametrix, then with auxiliary elliptic operators $P_1$ in $E$ and $P_2$ in $F$,

\[
\text{ind } A = C_0(AB - I, P_{2,+}) - C_0(BA - I, P_{1,+}) = C_0(AB, P_{2,+}) - C_0(BA, P_{1,+}) + \frac{1}{m} \text{res}((\log P_2)_+) - \frac{1}{m} \text{res}((\log P_1)_+).
\]

Here $C_0(AB, P_{2,+}) - C_0(BA, P_{1,+})$ is a res with $\psi$do part $\text{res}((B \log P_2 A - BA \log P_1)_+)$. 

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References


