Matematisk Afdeling
Københavs Universitet
Gerd Grubb

5. Elliptic problems.

With the characterization of L^2 Sobolev spaces over $\mathbb{R}^n$ given in Theorem 4.9, we easily get a complete description of how $I - \Delta$ (also denoted $1 - \Delta$) acts in these spaces:

Theorem 5.1. *The operator $I - \Delta$ defines a homeomorphism of $H^{k+2}(\mathbb{R}^n)$ onto $H^k(\mathbb{R}^n)$ for any $k \geq 0$. Moreover,*

$$u \in \bigcap_{k \geq 0} H^k(\mathbb{R}^n) \iff (1 - \Delta)u \in \bigcap_{k \geq 0} H^k(\mathbb{R}^n), \tag{5.1}$$

where $\bigcap_{k\geq 0} H^k(\mathbb{R}^n)$ can also be described as the space of funtions in $C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that all derivatives are in $L^2(\mathbb{R}^n)$.

In particular, if $f \in C_c^\infty(\mathbb{R}^n)$, the solution $u \in L^2(\mathbb{R}^n)$ of $(1 - \Delta)u = f$ is in $C^\infty(\mathbb{R}^n)$.

Proof. Let $k \geq 0$. Then by Theorem 4.9 and (3.9),

$$\begin{aligned} u \in H^{k+2}(\mathbb{R}^n) &\iff (1 + |y|^2)^{(k+2)/2}\hat{u} \in L^2(\mathbb{R}^n) \\ &\iff (1 + |y|^2)^{k/2}\hat{f} \in L^2(\mathbb{R}^n) \iff f \in H^k(\mathbb{R}^n), \end{aligned} \tag{5.2}$$

with $\|u\|_{k+2,\wedge} = \|f\|_{k,\wedge}$. This shows the first statement, and (5.1) follows by applying this for all k. The last statement follows from Theorem 4.10. $\square$

This is a simple case where we see the usefulness of the Sobolev spaces: They give precise information on the relation between the regularity of f and the regularity of u.

Let us give some comments to the application of the Lax-Milgram Theorem in [E98, Section 6.2]. The situation is the following: There is given a bilinear form $B[u, v]$ on a Hilbert space H, which is continuous:

$$|B[u, v]| \leq \alpha \|u\|_H \|v\|_H \tag{5.3}$$

(with $\alpha > 0$), and positive:

$$B[u, u] \geq \beta \|u\|_H^2 \tag{5.4}$$

(with $\beta > 0$). The role of H will be played by $H_0^1(U)$. There is another, equally important Hilbert space involved, namely $L^2(U)$; let us denote it X in the abstract explanations. Then a slightly more general situation is where B merely satisfies the coerciveness inequality:

$$B[u, u] \geq \beta \|u\|_H^2 - \gamma \|u\|_X^2 \tag{5.5}$$

(with $\gamma \geq 0$; if $\gamma < 0$ we can omit the term).

Now the point, not stressed very explicitly in [E98], is that whereas we identify X with its dual space by the Riesz representation theorem for Hilbert spaces, we *do not identify* H with its dual space. In contrast, we keep considering H and H^* as two different spaces, although there certainly exists an isometry J of H onto H^*, namely the mapping $J\colon u \mapsto l_u$ sending u into the functional l_u defined by

$$l_u(v) = (u, v)_H \text{ for } u \in H. \tag{5.6}$$

Here

$$\|l_u\|_{H^*} = \sup_{v \in H \setminus \{0\}} \frac{|l_u(v)|}{\|v\|_H} = \sup_{v \in H \setminus \{0\}} \frac{|(u, v)|}{\|v\|_H} = \|u\|_H, \tag{5.7}$$

by the sharpness of the Cauchy-Schwarz inequality.

The relation between H and X is that we have H injected in X "algebraically, topologically and densely": algebraically because it is a linear subspace, topologically because the norm is stronger:

$$\|u\|_H \ge c\|u\|_X, \tag{5.8}$$

with $c > 0$ (here $c = 1$ in the case of $H = H_0^1$ and $X = L^2$), and densely (in the case of $H = H_0^1$ and $X = L^2$ it holds since $C_c^\infty(U)$ is dense in $L^2(U)$).

As for H^*, we shall view it as a space containing X:

Lemma 5.2. *There is a continuous injection of X into H^*, where $f \in X$ is identified with the functional $\tilde{f}$ on H defined by:*

$$\tilde{f}(v) = (f, v)_X. \tag{5.9}$$

Proof. When $f \in X$, the right hand side of (5.9) defines a continuous linear functional $\tilde{f}$ on H thanks to (5.8):

$$|(f, v)_X| \le \|f\|_X \|v\|_X \le c^{-1} \|f\|_X \|v\|_H;$$

note that

$$\|\tilde{f}\|_{H^*} \le c^{-1} \|f\|_X. \tag{5.10}$$

This gives us a continuous mapping from X to H^*, but in order to speak of an injection, we also have to show that it is *injective*, i.e., that $\tilde{f} = \tilde{f}_1$ implies $f = f_1$. So let $\tilde{f} = \tilde{f}_1$; then for all $v \in H$:

$$0 = \tilde{f}(v) - \tilde{f}_1(v) = (f, v)_X - (f_1, v)_X = (f - f_1, v)_X.$$

Since H is dense in X, it follows that $f - f_1 = 0$.

Now since we have the injectiveness of the mapping $f \mapsto \tilde{f}$, we can identify X with the subspace of H^* consisting of the functionals that come from elements of X in this way. □

Summing up, we have inclusions

$$H \subset X \subset H^* \tag{5.11}$$

(with decreasing norms). In the concrete application, $(H_0^1(U))^*$ is denoted $H^{-1}(U)$; a further description is given in [E98, Sect. 5.9.1]. So the inclusions here take the form

$$H_0^1(U) \subset L^2(U) \subset H^{-1}(U). \tag{5.12}$$

Unfortunately, [E98] does not prove the second inclusion although it is tacitly understood in the applications (for example when a differential operator L is considered as acting from H_0^1 to H^{-1} and is next replaced by $L + \mu I$ — to speak of the identity operator here, one needs to know that there is an inclusion of H_0^1 in H^{-1}).

The Lax-Milgram Theorem as it is written in [E98] solely in terms of H and H^* is of course elementary, but part of the point of it is the application to functionals defined from elements of X, which is somewhat more refined. (In earlier texts — e.g. in the books of J. L. Lions — this point is made clearly and explicitly.)

The action of the functional $\tilde{f}$ in Lemma 5.2 can be written as $\langle \tilde{f}, v\rangle$, or even as $\langle f, v\rangle$ when the injection $X \subset H^*$ is understood.

The solution operator determined in Theorem 6.2.3 can be clarified as follows: Adding $\mu \cdot (u,v)_X$ to B gives us a positive continuous form $B_\mu[u,v]$ on H. Then the Lax-Milgram Theorem provides us with a solution operator $M\colon H^* \to H$, in fact a homeomorphism, such that

$$B_\mu[Mf, v] = \langle f, v\rangle \text{ for all } v \in H,$$

when f is given in H^*. In particular, M maps the linear subset X into a linear subset of H, and here

$$\|Mf\|_H \le C\|f\|_{H^*} \le c^{-1}C\|f\|_X, \tag{5.13}$$

cf. (5.10). The restriction of M to X is denoted L_μ^{-1} in the concrete application (cf. [E98, 6.2(14)]). It is the solution operator for the weak Dirichlet problem with a right-hand side f given in $X = L^2(U)$, and it is continuous from $L^2(U)$ to $H_0^1(U)$ in view of (5.13).

Later, in Section 6.2.3, the formal adjoint — let us cautiously denote it L' — of the differential operator is introduced in the definition on page 302. Here also the adjoint bilinear form $B^*[u,v]$ is defined in a precise way; it is again continuous on $H = H_0^1(U)$ and satisfies the same coerciveness inequality. Let us explain the adjoint of K occurring in the proof of Theorem 6.2.4:

From $B^*_\gamma[u,v] = B[u,v] + \gamma\cdot(u,v)_{L^2}$ we get by the above construction a solution operator $(L'_\gamma)^{-1}$,continuous from $L^2(U)$ into $H_0^1(U)$.

Lemma 5.3. *L_γ^{-1} and $(L'_\gamma)^{-1}$ are each other's adjoints as bounded operators in $L^2(U)$.*

Proof. Let $f, g \in L^2(U)$ and let $v = L_\gamma^{-1} f$, $w = (L'_\gamma)^{-1} g$. Then

$$(f, (L'_\gamma)^{-1} g)_{L^2} = (f, w)_{L^2} = B[v,w] = B^*[w,v] = (g,v)_{L^2} = (v,g)_{L^2} = (L_\gamma^{-1} f, g)_{L^2}. \quad \square$$

These two operators serve as solution operators for the weak Dirichlet problems for $L + \gamma$ resp. $L' + \gamma$. Since $K = \gamma L_\gamma^{-1}$, $K^* = \gamma (L'_\gamma)^{-1}$.

Here are some comments on the claim that K is compact. We assume that the reader has an elementary knowledge of compact operators that we can appeal to for the following elementary proof of the Rellich (-Kondratev) theorem, used in the proof of Theorem 6.2.4.

First a little bit about Sobolev spaces of periodic functions. Consider the space $L^2(\mathbb{T}^n)$ of functions $u(x)$ on $\mathbb{R}^n$ that have period 2π in each variable x_j and are square integrable on $Q =]-\pi,\pi[^n$; the space $L^2(\mathbb{T}^n)$ is a Hilbert space when provided with the scalar product and norm

$$(f,g) = (2\pi)^{-n}\int_Q f(x)\overline{g(x)}\,dx, \quad \|f\| = \Big((2\pi)^{-n}\int_Q |f(x)|^2\,dx\Big)^{\frac{1}{2}}, \tag{5.14}$$

and with the usual convention that functions differing on a set of Lebesgue measure zero are identified with one another. Clearly, $L^2(\mathbb{T}^n) \simeq L^2(Q,(2\pi)^{-n}dx)$. The functions $u \in L^2(\mathbb{T}^n)$ can be expanded in Fourier series with respect to the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}^n}$, where $e_k = e^{ik\cdot x} = e^{i(k_1x_1+\cdots+k_nx_n)}$. In fact, there is a 1–1 correspondence between the functions in $L^2(\mathbb{T}^n)$ and the elements of $\ell^2(\mathbb{Z}^n)$, where f corresponds to $\{c_k\}_{k\in\mathbb{Z}^n}$ when

$$f = \sum_{k\in\mathbb{Z}^n} c_k e_k, \text{ convergence in } L^2(\mathbb{T}^n); \tag{5.15}$$

here $c_k = c_k(f) = (f,e_k)$ for each k, and $\|f\|^2_{L^2} = \sum_{k\in\mathbb{Z}^n}|c_k|^2$. So in fact we have an isometric isomorphism of $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$.

For m integer ≥ 0, we denote by $C^m(\mathbb{T}^n)$ the (Banach) space of functions in $C^m(\mathbb{R}^n)$ that have period 2π in each variable; this space identifies with the subspace of functions in $C^m(\overline{Q})$ such that all derivatives up to order m match at the boundary in such a way that the functions extend to continuous periodic functions. $C^\infty(\mathbb{T}^n)$ is then $\bigcap_{m\in\mathbb{N}_0} C^m(\mathbb{T}^n)$.

Moreover, we define $H^m(\mathbb{T}^n)$ as the functions in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ that have weak derivatives up to order m in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and are periodic in each coordinate x_j with period 2π. It is provided with the norm $\|f\|_{H^m(\mathbb{T}^n)} = \big(\sum_{|\alpha|\leq m}\|D^\alpha f\|^2_{L^2(\mathbb{T}^n)}\big)^{\frac{1}{2}}$.

It is easy to see (by mollification on compact subsets of $\mathbb{R}^n$) that $C^\infty(\mathbb{T}^n)$ is dense in $H^m(\mathbb{T}^n)$. For the functions in $C^\infty(\mathbb{T}^n)$, the Fourier series (5.15) converges uniformly along with all the pointwise differentiated series, so here, since $D^\alpha e^{ik\cdot x} = i^{|\alpha|}k^\alpha e^{ik\cdot x}$, $D^\alpha f$ has the system of Fourier coefficients $\{i^{|\alpha|}k^\alpha c_k\}_{k\in\mathbb{Z}^n}$. Thus the $H^m(\mathbb{T}^n)$-norm satisfies

$$\|f\|^2_{H^m} = \sum_{|\alpha|\leq m}\|D^\alpha f\|^2_{L^2} = \sum_{k\in\mathbb{Z}^k}\sum_{|\alpha|\leq m}|k^\alpha c_k|^2 \simeq \sum_{k\in\mathbb{Z}^k}|\langle k\rangle^m c_k|^2, \tag{5.16}$$

as in the proof of Theorem 4.9 above.

NB! Note that the Hilbert spaces appearing here are all separable (in the sense that they have a countable orthonormal basis).

Proposition 5.4. *Let U be bounded. If a linear operator K is continuous from $L^2(U)$ to $H^1_0(U)$, then it is compact as an operator in $L^2(U)$.*

Proof. We assume that the following result is known: A bounded selfadjoint operator T in a separable Hilbert space is compact, if and only if the eigenvectors of T form a complete orthonormal system $(e_j)_{j\in\mathbb{N}}$ such that the associated eigenvalues μ_j go to 0 for $j\to\infty$. (In particular, nonzero eigenvalues have finite multiplicity.) The composition of a bounded operator and a compact operator is compact. Therefore, we shall look for operators like T, and show that K can be written as a composition of such a T and bounded operators.

If U is replaced by the torus $\mathbb{T}^n$ it is not hard to find such an operator. Indeed, the functions in $H^1(\mathbb{T}^n)$ satisfy

$$\|f\|^2_{H^1(\mathbb{T}^n)} = \sum_{k\in\mathbb{Z}^n}(1+k_1^2+\cdots+k_n^2)|c_k|^2 = \sum_{k\in\mathbb{Z}^n}|\langle k\rangle c_k|^2, \tag{5.17}$$

so if we define T_s as the operator

$$T_s\colon f = \sum_{k\in\mathbb{Z}^n} c_k e_k \mapsto T_s f = \sum_{k\in\mathbb{Z}^n} \langle k\rangle^s c_k e_k \tag{5.18}$$

(corresponding to the operator of multiplication by $\langle k\rangle^s$ in $\ell^2(\mathbb{Z}^n)$), then T_{-1} is an isometry of $L^2(\mathbb{T}^n)$ onto $H^1(\mathbb{T}^n)$ with inverse T_1. Here T_{-1} is selfadjoint in $L^2(\mathbb{T}^n)$, with the orthonormal basis of eigenfunctions e_k, $k\in\mathbb{Z}^n$, and associated eigenvalues $\langle k\rangle^{-1}$ that go to 0 for $|k|\to\infty$. Thus T_{-1} is a compact operator in $L^2(\mathbb{T}^n)$!

For our general operator K in $L^2(U)$, we proceed as follows: We can assume that $U \subset\subset Q^n$ (by a translation and scaling if necessary). The map $E_0\colon C_c^\infty(U)\to C^\infty(\mathbb{T}^n)$ that consists of extending by 0 on $Q^n\setminus U$ and extending by periodicity to $\mathbb{R}^n$, is continuous with respect to H^1-norms, so it extends to a continuous injection E_0 of $H_0^1(U)$ into $H^1(\mathbb{T}^n)$. After composing K with this map, we compose with $T_{-1}T_1$, noting that $T_{-1}T_1$ preserves supports. All together we find that K can be written as a mapping

$$K = r_U T_{-1} T_1 E_0 K, \tag{5.19}$$

where T_1E_0K is bounded from $L^2(U)$ to $L^2(\mathbb{T}^n)$, the restriction map r_U is bounded from $L^2(\mathbb{T}^n)$ to $L^2(U)$, and T_{-1} is compact in $L^2(\mathbb{T}^n)$. It follows that K is compact in $L^2(U)$. $\square$

The method of proof also works for $H^1(U)$ if U has C^1-boundary; then we replace E_0 by the extension operator $E\colon H^1(U)\to H^1(\mathbb{R}^n)$ followed by multiplication by a C^∞ function ζ which is 1 on U and is supported in a slightly larger open set U'.

E5. Exercises

Exercise E5.1.

Consider the operator Δ^2 (usually called the biharmonic operator).

(1) Show that the problem
$$(\Delta^2+1)u = f$$
can be solved in Sobolev spaces over $\mathbb{R}^n$.

(2) Let $f\in L^2(\mathbb{R}^n)$. In which dimensions n will the solution u be continuous?

(3) Let $f\in H^k$ for some integer $k\ge 0$. How large should one take k to be sure that $u\in C^4(\mathbb{R}^n)$?

(The answers should of course be presented with detailed arguments.)

Exercise E5.2.

Consider the second order operator L on $\mathbb{R}^2$ defined by

$$Lu = -\tfrac{\partial^2}{\partial x_1^2}u - \tfrac{\partial^2}{\partial x_1\partial x_2}u - \tfrac{\partial^2}{\partial x_2^2}u$$

(also written as $Lu = -\partial_1^2 u - \partial_1\partial_2 u - \partial_2^2 u$ or $Lu = -u_{x_1x_1} - u_{x_1x_2} - u_{x_2x_2}$).

(1) Show that L is elliptic on $\mathbb{R}^2$.

(2) Find the associated bilinear form $B[u,v]$.

(3) Let $U = B^0(0,2)$. Find constants α and β such that $B[u,v]$ satifies the inequalities in the Lax-Milgram Theorem with $H = H_0^1(U)$.

(4) Formulate some important conclusions for the Dirichlet problem for L on U.