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4. Auxiliary tools in the study of Sobolev spaces.

As noted in [E98], page 629, the function

$$\eta(x) = \begin{cases} C\exp(\frac{1}{|x|^2-1}) & \text{if } |x|<1, \\ 0 & \text{if } |x|\ge 1, \end{cases} \tag{4.1}$$

where C is chosen so that $\int \eta(x)\,dx = 1$, is in $C^\infty(\mathbb{R}^n)$ (with support in $B(0,1)$).

To see this, we first observe the basic fact that

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t>0, \\ 0 & \text{if } t\le 0, \end{cases} \tag{4.2}$$

is in $C^\infty(\mathbb{R})$. For, it is continuous at 0 since $e^{-y}\to 0$ for $y\to\infty$, and all its derivatives are of the form $p(\frac{1}{t})e^{-\frac{1}{t}}$ for $t>0$, with polynomials p; they likewise go to 0 for $t\searrow 0$ (since $p(y)e^{-y}\to 0$ for $y\to\infty$). It follows that f is C^∞ at 0 with all derivatives equal to 0 there.

Now $g(t)$, defined as

$$g(t) = e^{\frac{1}{t^2-1}} = e^{\frac12(\frac{1}{t-1}-\frac{1}{t+1})} = e^{-\frac{1}{2(1-t)}}e^{-\frac{1}{2(1+t)}}, \text{ for } |t|<1,$$

and as 0 for $|t|\ge 1$, is a product of two functions like f, one with t replaced by $t+1$ (changing from zero to positive values at $t=-1$) and the other with t replaced by $1-t$ (changing from positive values to zero at $t=1$). So g is C^∞ on $\mathbb{R}$ (and positive for $|t|<1$). It is then clear that $\eta(x) = Cg(|x|^2) = Cg(x_1^2+\cdots+x_n^2)$ is likewise C^∞.

The function $\eta(x)$ and its variant $\eta_\varepsilon(x)$ are used at described in [E98, Sect. C.4] and in other places in the book; we have met it on page 28 and on page 250ff. Note that it is an unusual function if one is mainly used to analytic functions (functions that are the sum of their Taylor series locally — they extend to holomorphic functions of complex variables). For example, $f(t)$ is NOT the sum of its Taylor series at $t=0$, since all the Taylor coefficients at 0 are zero, whereas f is not identically zero.

But such non-analytic but C^∞ functions are plentiful. Here is another useful example:

Lemma 4.1. *Let U be an open subset of $\mathbb{R}^n$ and let K be a compact subset of U. There exists a function $\psi\in C_c^\infty(U)$ taking values in $[0,1]$ such that ψ is 1 on K.*

Proof. For an $\varepsilon>0$, let

$$K' = \overline{K+B(0,\varepsilon)} = \text{ the closure of } \{x=y+z \mid y\in K,\ z\in B(0,\varepsilon)\}, \tag{4.3}$$

and set

$$\psi(x) = (\eta_\varepsilon * 1_{K'})(x) = \int_{K'} \eta_\varepsilon(x-y)\,dy. \tag{4.4}$$

Since $\int_{\mathbb{R}^n} \eta_\varepsilon \, dx = 1$, $\psi(x) \in [0,1]$ for all x. Since $1_{K'} \in L^1(\mathbb{R}^n)$, ψ is in $C^\infty(\mathbb{R}^n)$ by [E98, Th. C.4.6]. Moreover, since $\operatorname{spt} \eta_\varepsilon(x-y) \subset K'$ when $x \in K$, and $\operatorname{spt} \eta_\varepsilon(x-y) \cap K' = \emptyset$ when $\operatorname{dist}(x, K) > 2\varepsilon$,

$$\psi(x) = \begin{cases} 1 & \text{when } x \in K, \\ 0 & \text{when } x \notin K + B(0, 2\varepsilon). \end{cases} \tag{4.5}$$

Since K is compact, $\operatorname{dist}(K, \partial U) > 0$. For $2\varepsilon < \operatorname{dist}(K, \partial U)$, $\psi \in C_c^\infty(U)$. $\square$

Note that by this construction, we can get the support of ψ to be very close to K; we could also obtain that ψ is 1 on a *neighborhood of* K by modifying the construction a little.

We often use the convention that when $\psi \in C_c^\infty(U)$, it identifies with the function in $C_c^\infty(\mathbb{R}^n)$ (or in $C_c^\infty(U')$, $U' \supset U$) obtained by extending by zero outside U.

We can now explain how the partition of unity used in the proof of [E98, Th. 5.3.2] is constructed. It is important here that the sets V_i, which together fill out U, have been chosen so that any compact subset K of U meets only finitely many of them (we then say that the cover $\{V_i\}_{i \in \mathbb{N}_0}$ is *locally finite*). This is so, because $\operatorname{dist}(K, \partial U) > 0$, so $K \cap V_i = \emptyset$ when $\frac{1}{i} < \operatorname{dist}(K, \partial U)$; we have taken

$$\begin{aligned} V_i &= \{x \in U \mid \operatorname{dist}(x, \partial U) \in \,]\tfrac{1}{i+3}, \tfrac{1}{i+1}[\,\} \text{ for } i \geq 1, \\ V_0 &= \{x \in U \mid \operatorname{dist}(x, \partial U) > \tfrac{1}{3}\}. \end{aligned} \tag{4.6}$$

We can moreover choose V_i' with $\overline{V_i'} \subset V_i$ such that $\bigcup_{i \in \mathbb{N}_0} V_i' = U$. For example, take

$$\begin{aligned} V_i' &= \{x \in U \mid \operatorname{dist}(x, \partial U) \in \,]\tfrac{1}{i+2.9}, \tfrac{1}{i+1.1}[\,\} \text{ for } i \geq 1, \\ V_0' &= \{x \in U \mid \operatorname{dist}(x, \partial U) > \tfrac{1}{2.9}\}. \end{aligned} \tag{4.7}$$

Lemma 4.2. *Let the open set U be a union of bounded open sets V_i, $i \in \mathbb{N}_0$, for which there are open subsets V_i' such that $\overline{V_i'} \subset V_i$ and we still have $\bigcup_{i \in \mathbb{N}_0} V_i' = U$. Assume moreover, that the cover $\{V_i\}_{i \in \mathbb{N}_0}$ is locally finite (each compact subset of U has nonempty intersection with only a finite number of the V_i). Then there is a family of nonnegative functions $\zeta_i \in C_c^\infty(V_i)$ such that*

$$\textstyle\sum_{i \in \mathbb{N}_0} \zeta_i(x) = 1 \text{ for all } x \in U. \tag{4.8}$$

Proof. Since $\overline{V_i'}$ is a compact subset of V_i, we can for each i choose a nonnegative function $\psi_i \in C_c^\infty(V_i)$ that is 1 on V_i', by Lemma 4.1. Now

$$\Psi(x) = \textstyle\sum_{i \in \mathbb{N}_0} \psi_i(x)$$

is a well-defined C^∞-function on U, since any point $x \in U$ has a compact neighborhood in U where only finitely many of the functions ψ_i are nonzero. Moreover, $\Psi(x) \geq 1$ at all $x \in U$, since each x is in V_i' for some i. Then let

$$\zeta_i(x) = \frac{\psi_i(x)}{\Psi(x)} \text{ for } x \in U,\ i \in \mathbb{N}_0. \tag{4.9}$$

The system $\{\zeta_i\}_{i \in \mathbb{N}_0}$ has the desired properties. $\square$

We say that $\{\zeta_i\}_{i \in \mathbb{N}_0}$ is a *partition of unity* (cf. (4.8)) for U subordinate to the cover $\{V_i\}_{i \in \mathbb{N}_0}$.

Remark 4.3. Theorem 5.3.2 holds also for unbounded open sets U. In fact, one can then let

$$U_i = \{x \in U \mid \operatorname{dist}(x, \partial U) > \tfrac{1}{i},\ |x| < i\}$$

for all $i \in \mathbb{N}$ and again define V_i for $i > 0$ as $U_{i+3} \setminus \bar{U}_{i+1}$, $V_0 = U_3$. The proof goes in the same way as before, with obvious modifications.

There is a slightly different type of partition of unity that we shall also need (for example in the proof of [E98, Th. 5.3.3]).

Lemma 4.4. *Let K be a compact subset of $\mathbb{R}^n$, and let $\{V_i\}_{i=0}^N$ be a bounded open cover of K (i.e., the V_i are bounded and open, and $K \subset \bigcup_{i=0}^N V_i$). There exists a family of functions $\zeta_i \in C_c^\infty(V_i)$ taking values in $[0,1]$ such that*

$$\textstyle\sum_{i=0}^N \zeta_i(x) = 1 \text{ for } x \in K. \tag{4.10}$$

Proof. Let $V_{ij} = \{x \in V_i \mid \operatorname{dist}(x, \partial V_i) > \frac{1}{j}\}$, then the family of sets $\{V_{ij}\}_{i=0,\dots,N;j\in\mathbb{N}}$ forms an open cover of K. Since K is compact, there is a finite subfamily that still covers K; here since $V_{ij} \subset V_{ij'}$ for $j < j'$, we can reduce to a system where there is at most one j for each i, and we can adjoin one set, e.g. V_{i1}, for each of those values of i that are not represented in the system. In this way we get an open cover $\{V_i'\}_{i=0}^N$ of K such that $V_i' \subset\subset V_i$ (i.e., $\overline{V_i'}$ is a compact subset of V_i) for each i.

Now use Lemma 4.1 for each i to choose $\psi_i \in C_c^\infty(V_i)$, equal to 1 on V_i' and taking values in $[0,1]$. Then

$$\Psi(x) = \textstyle\sum_{i=0}^N \psi_i(x) \ge 1 \text{ for } x \in \bigcup_{i=0}^N V_i' \supset K. \tag{4.11}$$

Since $\bigcup_{i=0}^N V_i'$ is an open set containing K, we can use Lemma 4.1 once more to find a function $\varphi \in C_c^\infty(\bigcup_{i=0}^N V_i')$ that is 1 on K and takes values in $[0,1]$. Now set

$$\zeta_i(x) = \begin{cases} \psi_i(x)\frac{\varphi(x)}{\Psi(x)} & \text{on } \bigcup_{i=0}^N V_i', \\ 0 & \text{elsewhere;} \end{cases} \tag{4.12}$$

it is a well-defined C^∞ function supported in a compact subset of V_i and taking values in $[0,1]$, and the family of functions ζ_i clearly satisfies (4.10). □

In this case we say that $\{\zeta_i\}_{i=0}^N$ is a partition of unity for K subordinate to the cover $\{V_i\}_{i=0}^N$. If $K = \bar{U}$ for some open set, one could also say (and [E98] does so) that $\{\zeta_i\}_{i=0}^N$ is a partition of unity for U subordinate to the cover $\{V_i\}_{i=0}^N$.

When Lemma 4.4 is applied in connection with studies of $W^{k,p}(U)$, we need the fact that the Leibniz formula holds for $D^\alpha(\zeta u)$ not just when $\zeta \in C_c^\infty(U)$, but also when $\zeta \in C^\infty(\mathbb{R}^n)$ with ζ and all its derivatives bounded. This is a straightforward extension of the case treated in Theorem 5.2.1 (iv) using that in the integrals with a testfunction φ, the integrand is supported in $\operatorname{spt}\varphi$ at any rate. We then also have:

$$\|D^\alpha(\zeta u)\|_{L^p(U)} = \|\sum_{\beta\le\alpha} \tbinom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u\|_{L^p(U)} \le C\|u\|_{W^{|\alpha|,p}(U)}, \tag{4.12a}$$

where the constant depends on the bounds on ζ and its derivatives. Thus the mapping $u \mapsto \zeta u$ is continuous in $W^{k,p}(U)$.

There is an error in the proof of Theorem 5.3.3. $\{V_i\}_{i=0}^N$ is not a cover of the compact set $\bar{U}$ and the ζ_i cannot be constructed to lie in $C_c^\infty(V_i)$ and add up to 1 on U. Instead one can apply Lemma 4.4 to the cover defined by use of the full balls $B^0(x_i^0, \frac{r_i}{2})$:

Corrections to the proof of [E98, Th. 5.3.3].

Write B^0 instead of B in line 8 from below on page 253. Replace lines 2–1 from below on page 253 by:

“5. Now let $W_i = B^0(x_i^0, \frac{r_i}{2})$, $W_0 = V_0$, and let $\{\zeta_i\}_{i=0}^N$ be a partition of unity for U subordinate to the cover $\{W_0, \dots, W_N\}$. For each i, $\zeta_i v_i$ is defined on $\bar{V}_i$; it extends by 0 on $\bar{U} \setminus \bar{V}_i$ to a function in $C^\infty(\bar{U})$, since $\operatorname{dist}(\operatorname{spt} \zeta_i, \bar{U} \setminus \bar{V}_i) > 0$. Then also the sum $v = \sum_{i=0}^N \zeta_i v_i$ is in $C^\infty(\bar{U})$. In”

In lines 3–4 of page 254, use (4.12a) and replace “$= CN\delta$” by “$\leq C(N+1)\delta$”.

The textbook is mostly concerned with Sobolev spaces over bounded sets U. However, it will be useful to have some specific information at least about the case where U is the unbounded set $\mathbb{R}^n$. In this case one has the following improvement of the approximation results in Theorems 5.3.1–3:

Theorem 4.5. *For any $p \in [1, \infty[$ and any integer $k \geq 0$, $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{k,p}(\mathbb{R}^n)$; we shall show how it can be approximated in $W^{k,p}$-norm by functions in $C_c^\infty(\mathbb{R}^n)$.

First we take a function $\chi(x) \in C_c^\infty(\mathbb{R}^n)$ taking values in $[0,1]$ such that

$$\chi(x) = 1 \text{ for } |x| \leq 1, \quad \chi(x) = 0 \text{ for } |x| \geq 2; \tag{4.13}$$

such a function exists by Lemma 4.1. For $N \in \mathbb{N}$, let

$$u_N(x) = \chi(x/N)u(x); \tag{4.14}$$

it is supported in $B(0, 2N)$ and equals $u(x)$ on $B(0, N)$. By Theorem 5.2.1 in [E98], $u_N \in W^{k,p}(\mathbb{R}^n)$ and we have for $|\alpha| \leq k$:

$$\begin{aligned} D^\alpha u_N(x) &= \sum_{\beta \leq \alpha} \tbinom{\alpha}{\beta} D^\beta[\chi(x/N)] D^{\alpha-\beta} u(x) \\ &= \chi(x/N) D^\alpha u(x) + \sum_{\beta \leq \alpha, |\beta| \geq 1} \tbinom{\alpha}{\beta} N^{-|\beta|} [D^\beta \chi](x/N) D^{\alpha-\beta} u(x). \end{aligned} \tag{4.15}$$

Here $\chi(x/N) D^\alpha u \to D^\alpha u$ in $L^p(\mathbb{R}^n)$ for $N \to \infty$ by the theorem of Lebesgue. Noting that for $|\beta| \geq 1$,

$$N^{-|\beta|} |[D^\beta \chi](x/N) D^{\alpha-\beta} u(x)| \leq N^{-1} C |D^{\alpha-\beta} u(x)|, \tag{4.16}$$

we can also use Lebesgue's theorem to conclude that each term in the sum over $\beta \leq \alpha, |\beta| \geq 1$, goes to 0 in $L^p(\mathbb{R}^n)$ for $N \to \infty$. It follows that

$$D^\alpha u_N \to D^\alpha u \text{ in } L^p(\mathbb{R}^n) \text{ for } N \to \infty, \text{ each } |\alpha| \leq k, \tag{4.17}$$

so $\|u_N - u\|_{W^{k,p}(\mathbb{R}^n)} \to 0$ for $N \to \infty$.

Since u_N and its derivatives are supported in $B(0, 2N)$,

$$\|\eta_\varepsilon * u_N - u_N\|_{W^{k,p}(\mathbb{R}^n)} = \|\eta_\varepsilon * u_N - u_N\|_{W^{k,p}(B(0,3N))} \to 0 \text{ for } \varepsilon \to 0; \tag{4.18}$$

cf. Theorem 5.3.1. □

Recall that $W_0^{k,p}(U)$ is defined as the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. Then Theorem 4.5 implies

Corollary 4.6. *For any* $p \in [1, \infty[$ *and any integer* $k \geq 0$,

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n). \tag{4.19}$$

Note that in the situation of Theorem 5.3.3, $C^\infty(\bar{U}) \subset C^k(\bar{U}) \subset W^{k,p}(U)$, so also $C^k(\bar{U})$ is dense in $W^{k,p}(U)$.

The proofs of the extension theorem (Theorem 5.4.1) and the trace theorem (Theorem 5.5.1) both use the procedure of reducing the situation to the case of a "flat" boundary. Let us describe the steps in detail:

(1) Cover ∂U with a finite number of balls $W_i = B^0(x_i^0, \frac{r_i}{2})$, $i = 1, \dots, N$, around boundary points x_i^0 such that each $B^0(x_i^0, r_i)$ is as in Section C.1, with a C^1 mapping Φ carrying $B^0(x_i^0, r_i)$ over into an open neighborhood of $y_i^0 = \Phi(x_i^0)$, where $\partial U \cap B^0(x_i^0, r_i)$ is mapped into a bounded open subset of $\mathbb{R}^{n-1}$ (the latter identified with $\{y \in \mathbb{R}^n \mid y_n = 0\}$). Supply the cover with one set $W_0 \subset\subset U$ to get an open cover of $\bar{U}$, and choose a partition of unity $\{\zeta_i\}_{i=0}^N$ for $\bar{U}$ subordinate to $\{W_i\}$.

(2) For a given function $u \in C^1(\bar{U})$, write $u = \sum_{i=0}^N \zeta_i u$, and consider each $u_i = \zeta_i u$ separately. For $i = 0$, the function vanishes on a neighborhood of ∂U, so it can simply be extended by 0 beyond U, and its boundary value is 0. Now consider an $i > 0$. Here u_i is by the coordinate change defined by Φ carried into a function $u_i' \in C^1(\overline{\mathbb{R}}^n_+)$ with compact support in $\overline{\mathbb{R}}^n_+$; we can assume that the support is contained in the interior of a ball $B = B(y_i^0, s_i)$. Let $B^\pm = \{y \in B \mid y_n \gtrless 0\}$. Simple rules for the behavior of derivatives and L^p norms under coordinate changes give that there are constants C_1 and C_2, independent of u_i, such that

$$\|u_i\|_{W^{1,p}(U)} \leq C_1 \|u_i'\|_{W^{1,p}((B^+)^\circ)} \leq C_2 \|u_i\|_{W^{1,p}(U)}. \tag{4.20}$$

For the extension theorem we proceed in the "flat" situation as described on page 255-256.

The remark (ii) on page 257 leaves the impression that it is much harder to find a continuous extension operator from $W^{k,p}(U)$ to $W^{k,p}(\mathbb{R}^n)$ (in the case where U is C^k) when $k > 2$. This is not so. In fact, the crucial construction again takes place in the situation of a flat piece of the boundary. Choose $k+1$ positive numbers $\lambda_1, \dots, \lambda_{k+1}$, *for example* $\lambda_j = \frac{1}{j}$ (this is the choice used on page 255 for $k = 1$). With coefficients c_j yet to be determined, we define the extension operator from B^+ to B by

$$E_0 u(x) = \bar{u}(x) = \begin{cases} u(x) & \text{if } x_n \geq 0, \\ c_1 u(x', -\lambda_1 x_n) + \cdots + c_{k+1} u(x', -\lambda_{k+1} x_n) & \text{if } x_n < 0, \end{cases} \tag{4.21}$$

where we as usual denote $(x_1, \dots, x_{n-1})$ by x'. For functions $u \in C^k(B^+)$, the extension will be in $C^k(B)$ when the x_n-derivatives up to order k are continuous at $x_n = 0$ (the other derivatives follow along), and this requires that

$$\begin{aligned} c_1 + \dots + c_{k+1} &= 1, \\ \lambda_1 c_1 + \dots + \lambda_{k+1} c_{k+1} &= -1, \\ \lambda_1^2 c_1 + \dots + \lambda_{k+1}^2 c_{k+1} &= 1, \\ &\vdots \\ \lambda_1^k c_1 + \dots + \lambda_{k+1}^k c_{k+1} &= (-1)^k. \end{aligned} \tag{4.22}$$

This system of equations can be solved for $c_1, \dots, c_{k+1}$. In fact, the coefficient matrix V is known as the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_{k+1} \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \dots & \lambda_{k+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^k & \lambda_2^k & \lambda_3^k & \dots & \lambda_{k+1}^k \end{pmatrix}, \text{ with } \det V = \prod_{1 \le i < j \le k+1} (\lambda_j - \lambda_i); \tag{4.23}$$

it is nonzero when the λ_j's are distinct, as we assumed.

With the extension defined in this way for smooth functions, one finds again that E_0 is a bounded linear operator with respect to $W^{k,p}$-norm. The rest of the proof is as in (1)–(2) above, now with C^1 replaced by C^k.

This shows:

Theorem 4.7. *When U is bounded and C^k, there exists a continuous linear mapping $E\colon W^{p,k}(U) \to W^{p,k}(\mathbb{R}^n)$ such that $(Eu)|_U = u$ for all $u \in W^{p,k}(U)$.*

For any open set $V \supset\supset U$ one can achieve that E maps into functions supported in V by composing it with the multiplication by a function ψ constructed as in Lemma 4.1 (for $\bar{U} \subset V$).

It may be observed that the extension defined in (4.21) works also for $W^{k+1,p}$-norms. In fact, when $v \in C^k(B)$ and has $k+1$-order derivatives that extend to continuous functions on B^+ and on B^- individually, but possibly with a jump along $B^+ \cap B^-$, then v is in $W^{k+1,p}(B^0)$, since, in integrations by part against a test function, the continuity of the derivatives of order k suffices to assure that "boundary contributions" along $B^+ \cap B^-$ cancel each other (as in Example 1 on page 243). For $k = 1$, this explains (i) of the remark on page 257.

Remark 4.8. There exist more refined extension results. One can for example construct one extension operator that works for all k when U is merely C^1 (or even just Lipschitz), see e.g. D. E. Edmunds and W. D. Evans: Spectral Theory and Differential Operators, Oxford University Press 1987, Theorem V.4.11.

Concerning the proof of the trace theorem in the "flat" situation, we observe:

When $p > 1$, the function $|s|^p$ of the variable $s \in \mathbb{R}$ is in $C^1(\mathbb{R})$ with derivative

$$\tfrac{d}{ds} |s|^p = p |s|^{p-1} \operatorname{sgn} s. \tag{4.24}$$

For,

$$|s|^p = \begin{cases} s^p & \text{for } s \geq 0, \\ (-s)^p & \text{for } s < 0. \end{cases} \quad \tfrac{d}{ds}|s|^p = \begin{cases} ps^{p-1} & \text{for } s > 0, \\ -p(-s)^{p-1} & \text{for } s < 0, \end{cases} \tag{4.25}$$

and the derivative at $s = 0$ is 0. Then if $u(x)$ is C^1, the chain rule gives, when $p > 1$:

$$\partial_{x_n}|u(x)|^p = p|u(x)|^{p-1}\operatorname{sgn}|u(x)|\partial_{x_n}u(x), \tag{4.26}$$

as a continuous derivative. This formula is used in (1) on page 258.

But in the case $p = 1$, $|s|$ is not C^1, and the formula (4.26) does not hold in the classical sense at the isolated zeroes of u (of which there can be many). Instead of speculating over a justification of (1) on page 258 in this case, one can pass via the inequalities

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |\zeta(x',0)u(x',0)|\,dx' &= \int_{\mathbb{R}^{n-1}} |\int_0^r \partial_{x_n}(\zeta(x',x_n)u(x',x_n))\,dx_n|\,dx' \\ &\leq \int_{B^+} [|(\partial_{x_n}\zeta)u| + |\zeta\partial_{x_n}u|]\,dx,\end{aligned}$$

from which the proof is completed as in the book.

Our applications of Sobolev spaces will primarily be concerned with the case $p = 2$. Here one uses the notation

$$W^{k,2}(U) = H^k(U), \quad W_0^{k,2}(U) = H_0^k(U). \tag{4.27}$$

Theorem 5.8.7 in [E98] shows how to characterize the elements of L^2 Sobolev spaces over $\mathbb{R}^n$ by weighted L^2 norms on their Fourier transforms. For completeness, we go through the result here, showing also how it connects with "having derivatives in the Fourier sense" as introduced in Section 3.

Theorem 4.9. *Let $k \in \mathbb{N}$. A function $u \in L^2(\mathbb{R}^n)$ is in $H^k(\mathbb{R}^n)$ if and only if, for all $|\alpha| \leq k$, $y^\alpha \hat{u}(y) \in L^2(\mathbb{R}^n)$ (i.e., $D^\alpha u$ exists in the Fourier sense according to Definition 3.1).*

The following norms are equivalent on $H^k(\mathbb{R}^n)$:

$$\begin{aligned}\|u\|_{H^k(\mathbb{R}^n)} &= (\sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n)})^{\frac{1}{2}}, \\ \|u\|_{\wedge,k} &= \|(1+|y|^2)^{\frac{k}{2}}\hat{u}(y)\|^2_{L^2(\mathbb{R}^n)}, \\ \|u\|_{\wedge',k} &= \|(1+|y|^k)\hat{u}(y)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{4.28}$$

Proof. It follows from Lemma 3.5 that when $y^\alpha\hat{u}(y) \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq k$ then u has weak derivatives in $L^2(\mathbb{R}^n)$ for $|\alpha| \leq k$, hence lies in $H^k(\mathbb{R}^n)$.

Conversely, let $u \in H^k(\mathbb{R}^n)$. By Theorem 4.5, there is a sequence of functions $u^j \in C_c^\infty(\mathbb{R}^n)$, $j \in \mathbb{N}$, such that $u^j \to u$ in $H^k(\mathbb{R}^n)$. We have for $v \in C_c^\infty(\mathbb{R}^n)$ (cf. Theorem 4.3.2):

$$\|v\|^2_{H^k(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \|D^\alpha v\|^2_{L^2(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \|y^\alpha \hat{v}(y)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sum_{|\alpha|\leq k} y^{2\alpha}|\hat{v}(y)|^2\,dy. \tag{4.29}$$

Now observe that

$$(1+|y|^2)^k = (1+y_1^2+\cdots+y_n^2)^k = \sum_{|\alpha|\le k} C_\alpha y^{2\alpha},$$

with positive integer coefficients C_α, so for $C_1 = \max_{|\alpha|\le k} C_\alpha$,

$$\sum_{|\alpha|\le k} y^{2\alpha} \le \sum_{|\alpha|\le k} C_\alpha y^{2\alpha} = (1+|y|^2)^k \le C_1 \sum_{|\alpha|\le k} y^{2\alpha}, \text{ when } y\in\mathbb{R}^n. \tag{4.30}$$

Note also that:

$$\frac{(1+|y|^2)^k}{(1+|y|^k)^2} = \frac{(|y|^{-2}+1)^k}{(|y|^{-k}+1)^2} \to 1 \text{ for } |y|\to\infty,$$

so there are positive constants C_2 and C_3 such that

$$(1+|y|^2)^k \le C_2(1+|y|^k)^2 \le C_3(1+|y|^2)^k, \text{ when } y\in\mathbb{R}^n. \tag{4.31}$$

From (4.29), (4.30) and (4.31) it follows that

$$\|v\|^2_{H^k(\mathbb{R}^n)} \le \|(1+|y|^2)^{\frac{k}{2}}\hat v\|^2_{L^2(\mathbb{R}^n)} \le C_2\|(1+|y|^k)\hat v\|^2_{L^2(\mathbb{R}^n)} \le C_3C_1\|v\|^2_{H^k(\mathbb{R}^n)}, \tag{4.32}$$

for $v\in C_c^\infty(\mathbb{R}^n)$; note that the expressions $\|(1+|y|^2)^{\frac{k}{2}}\hat v\|_{L^2(\mathbb{R}^n)}$ and $\|(1+|y|^k)\hat v\|^2_{L^2(\mathbb{R}^n)}$ are weighted L^2 norms (where the Lebsegue measure dy is replaced by $(1+|y|^2)^k\,dy$ resp. $(1+|y|^k)^2\,dy$).

Now since u^j is a Cauchy sequence with respect to the norm in H^k, $\hat u^j$ is a Cauchy sequence with respect to the norm $\|(1+|y|^2)^{\frac{k}{2}}\hat v\|_{L^2(\mathbb{R}^n)}$. Thus it has a limit $\hat u_0$ in the weighted L^2 space, with $\|(1+|y|^2)^{\frac{k}{2}}\hat u_0\|_{L^2(\mathbb{R}^n)}<\infty$. In particular, $\hat u^j\to\hat u$ and $\hat u^j\to\hat u_0$ in $L^2(\mathbb{R}^n)$, so we see that $\hat u=\hat u_0$. The estimates for $\hat u_0$ then imply (cf. (4.30)) that $y^\alpha\hat u\in L^2$ for $|\alpha|\le k$.

This shows the first statement in the theorem, and the second follows since (4.32) now holds for all $v\in H^k(\mathbb{R}^n)$. $\square$

The third norm in (4.28) was only included since this is the one used in [E98, Th. 5.8.7]. The second norm is more customarily used in the literature on PDE based on the Fourier transform. It allows a generalization of the spaces $H^s(\mathbb{R}^n)$ not only to noninteger values of $s\ge 0$, but also to negative values (by duality, somewhat like in the definition of H^{-1}-spaces in [E98, Sect. 5.9] — these definitions are in fact primitive versions of distribution theory).

Many refined estimates relating Sobolev spaces to other Banach spaces are shown in [E98, Ch. 5]. When working solely with L^2 Sobolev spaces, one can often make do with the following special case:

Theorem 4.10. *Let k and m be integers ≥ 0 such that*

$$k > m+\frac{n}{2}. \tag{4.33}$$

1° *The space $H^k(\mathbb{R}^n)$ identifies with a subspace of $C^m(\mathbb{R}^n)$, and there is a constant C such the following estimate holds for the elements:*

$$\sup_{|\beta|\le m}\sup_{x\in\mathbb{R}^n}|D^\beta u(x)|\le C\|u\|_{H^k(\mathbb{R}^n)}. \tag{4.34}$$

2° *When U is a bounded open set with C^k-boundary, then the space $H^k(U)$ identifies with a subspace of $C^m(\bar U)$, and there is a constant C' such the following estimate holds for the elements:*

$$\sup_{|\beta|\le m}\sup_{x\in U}|D^\beta u(x)|\le C'\|u\|_{H^k(U)}. \tag{4.35}$$

Proof. 1°. In view of Theorem 4.10, the elements u of $H^k(\mathbb{R}^n)$ have

$$\int_{\mathbb{R}^n}(1+|y|^2)^k|\hat u(y)|^2\,dy<\infty. \tag{4.36}$$

Then for $|\beta|\le m$, we can use the Cauchy-Schwarz inequality to see that

$$\begin{aligned}\int_{\mathbb{R}^n}|y^\beta\hat u(y)|\,dy&=\int_{\mathbb{R}^n}\frac{(1+|y|^2)^{\frac{k-m}{2}}}{(1+|y|^2)^{\frac{k-m}{2}}}|y^\beta\hat u(y)|\,dy\\&\le\Big(\int_{\mathbb{R}^n}\frac{1}{(1+|y|^2)^{k-m}}\,dy\Big)^{\frac12}\Big(\int_{\mathbb{R}^n}(1+|y|^2)^{k-m}y^{2\beta}|\hat u(y)|^2\,dy\Big)^{\frac12}<\infty,\end{aligned} \tag{4.37}$$

where the first factor is finite since $k-m>\frac{n}{2}$, and the second factor is finite since $2(k-m)+2|\beta|\le 2k$, cf. (4.36).

When $y^\beta\hat u\in L^1$, the inverse Fourier transform of $(iy)^\beta\hat u$ has a continuous, bounded representative u_β, with

$$\sup_{x\in\mathbb{R}^n}|u_\beta(x)|\le(2\pi)^{-\frac{n}{2}}\|y^\beta\hat u\|_{L^1(\mathbb{R}^n)}. \tag{4.38}$$

So we now have that u as well as its derivatives in the Fourier sense $D^\beta u$ have continuous representatives u resp. u_β for $|\beta|\le k$. In view of Theorem 4.5, there is a sequence $u^j\in C_c^\infty(\mathbb{R}^n)$ converging to u in $H^k(\mathbb{R}^n)$ for $j\to\infty$. Then the above estimates imply that u^j and its derivatives $D^\beta u^j$ for $|\beta|\le m$ converge to u and the u_β in sup-norm. It follows that $u\in C^m(\mathbb{R}^n)$ with $D^\beta u=u_\beta$ for $|\beta|\le m$, and the estimate (4.35) is valid. This shows 1°.

2°. We here use the extension theorem (Theorem 4.7). For $u\in H^k(U)$, apply the preceding result to Eu; it shows that Eu can be identified with a C^m function on $\mathbb{R}^n$ and the estimate (4.34) holds for Eu. Since $\|Eu\|_{H^k(\mathbb{R}^n)}\le C_1\|u\|_{H^k(U)}$, the estimate (4.35) follows. □

The hypothesis that U should be C^k can be weakened to C^1 (or Lipschitz) by appealing to an extension operator as described in Remark 4.8.

We shall need the following simple version of the Poincaré inequality:

Theorem 4.11. *When U is contained in $B(0,R)$, one has for $u \in H_0^1(U)$:*

$$\|u\|_{L^2(U)} \leq \sqrt{2}\, R\|Du\|_{L^2(U)}. \tag{4.39}$$

Proof. When $u \in C_c^\infty(U)$ (extended as usual as 0 outside U), we have for the points $x = (x', x_n) \in B(0,R)$, using the Cauchy-Schwarz inequality:

$$\begin{aligned} |u(x', x_n)|^2 &= \Big| \int_{x_n}^R \partial_{y_n} u(x', y_n)\, dy_n \Big|^2 \leq \int_{x_n}^R 1\, dy_n \int_{x_n}^R |\partial_{y_n} u(x', y_n)|^2\, dy_n \\ &= (R - x_n) \int_{x_n}^R |\partial_{y_n} u(x', y_n)|^2\, dy_n \leq (R - x_n) \int_{-R}^R |\partial_{y_n} u(x', y_n)|^2\, dy_n. \end{aligned}$$

Integration with respect to $x_n \in [-R, R]$ gives:

$$\begin{aligned} \int_{-R}^R |u(x', x_n)|^2\, dx_n &\leq \int_{-R}^R (R - x_n)\, dx_n \int_{-R}^R |\partial_{y_n} u(x', y_n)|^2\, dy_n \\ &= 2R^2 \int_{-R}^R |\partial_{x_n} u(x', x_n)|^2\, dx_n. \end{aligned}$$

This already shows the inequality if $n = 1$. If $n > 1$, we integrate both sides with respect to x' and find, since u is supported in $B(0,R)$:

$$\begin{aligned} \|u\|_{L^2(U)}^2 &= \int_{|x'| \leq R, |x_n| \leq R} |u|^2\, dx' dx_n \\ &\leq 2R^2 \int_{|x'| \leq R, |x_n| \leq R} |\partial_{x_n} u|^2\, dx' dx_n \leq 2R^2 \|Du\|_{L^2(U)}^2, \end{aligned}$$

which shows (4.39) for $u \in C_c^\infty(U)$. The result follows for general $u \in H_0^1(U)$ by approximation by C_c^∞ functions. $\square$

In comparison with [E98, Th. 5.6.3], our p equals 2 and n can take any value.