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3. On the Fourier transformation

The Fourier transformation $\mathcal{F}$, defined for functions in $L^1(\mathbb{R}^n)$ by the formula

$$(\mathcal{F}u)(y) = \hat{u}(y) = \tfrac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} u(x)\, dx, \tag{3.1}$$

has three very important properties:

(I) $\mathcal{F}$ extends from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to an isometric isomorphism (i.e., a unitary operator), also denoted $\mathcal{F}$, from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$; here the inverse $\mathcal{F}^{-1}$ is the analogous extension of the operator where $-ix\cdot y$ is replaced by $ix \cdot y$ in (3.1). In particular,

$$\int_{\mathbb{R}^n} u\overline{v}\, dx = \int_{\mathbb{R}^n} \hat{u}\overline{\hat{v}}\, dy, \text{ for } u, v \in L^2(\mathbb{R}^n). \tag{3.2}$$

(II) For $u \in C_c^\infty(\mathbb{R}^n)$, one has:

$$\mathcal{F}(D^\alpha u) = (iy)^\alpha \hat{u}(y), \ \ (\text{where } D^\alpha u = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u,\ (iy)^\alpha = i^{|\alpha|} y_1^{\alpha_1} \dots y_n^{\alpha_n}). \tag{3.3}$$

This also holds if $u \in C^k(\mathbb{R}^n)$, $k = |\alpha|$, with $D^\beta u \in L^2(\mathbb{R}^n)$ for all $|\beta| \le k$. (It can be shown by approximation: Let χ be a function $\in C^\infty$ such that

$$\chi(x) = 1 \text{ for } |x| \le 1,\ \chi(x) = 0 \text{ for } |x| \ge 2, \tag{3.4}$$

then one can check, using the Leibniz formula, that $D^\beta(\chi(x/N)u) \to D^\beta u$ in L^2 when $N \to \infty$, for all $|\beta| \le k$, and formula (3.3) holds for the functions $\chi(x/N)u$.)

(III) When u and v are in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\mathcal{F}(u * v) = (2\pi)^{n/2}\, \hat{u} \cdot \hat{v}. \tag{3.5}$$

The properties in (II) and (III) extend to more general situations. One can extend the entire calculus of Fourier transformation to the space $\mathcal{S}'$ of *temperate distributions* on $\mathbb{R}^n$. The rule in (3.3) extends to all temperate distributions. The rule in (3.5) extends to cases where one of the factors is a general temperate distribution, but the other factor is subject to some restrictions (e.g. compact support or certain bounds). Both factors can be taken as temperate distributions with support in the quadrangle $\{x \in \mathbb{R}^n \mid x_i \ge 0 \text{ for } i = 1, \dots, n\}$.

The textbook [E98] has the point of view of not appealing to distribution theory, which could otherwise be a nice framework making many operations "legal". Instead, it manages with the concept of *weak derivatives* ([E98, Section 5.2.1]). For the present purposes, let us begin with another generalization of derivatives:

Definition 3.1. *When $u \in L^2(\mathbb{R}^n)$, we say that $D^\alpha u$ exists in the Fourier sense, when $y^\alpha \hat{u}(y) \in L^2(\mathbb{R}^n)$, and we then define this derivative by*

$$D^\alpha u = \mathcal{F}^{-1}[(iy)^\alpha \hat{u}(y)]. \tag{3.6}$$

This generalization is made in order to give a sense to the statement "when u and $D^\alpha u$ are in L^2, then (3.3) holds". We simply define what it means to say that "$D^\alpha u \in L^2$" in this way. One can show that this generalization is a special case of the definition of weak derivatives, see further below.

In the study of Examples 1–5 on pages 186–190 in [E98], the definition "in the Fourier sense" is very useful. We extend the definition to $-\Delta$ by saying that Δu is defined in the Fourier sense as $\mathcal{F}^{-1}(|y|^2\hat{u})$, when $|y|^2\hat{u}$ is in $L_2(\mathbb{R}^n)$. What is found in Example 1 is then that for any $f \in L^2(\mathbb{R}^n)$ there is a unique solution u of equation (6) in $L^2(\mathbb{R}^n)$ with Δu defined in the Fourier sense.

One can then *moreover* see that this u has *all derivatives up to order* 2 defined in the Fourier sense, for

$$|y_i| \le \tfrac{1}{2}(1 + y_i^2) \le 1 + |y|^2, \quad |y_i y_j| \le \tfrac{1}{2}(y_i^2 + y_j^2) \le |y|^2, \tag{3.7}$$

so $(iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \le 2$.

In each of the examples on pages 185–190 we find the solution of the problem in the form

$$u(x) = \mathcal{F}^{-1}[p(y)\hat{f}(y)], \tag{3.8}$$

possibly including a t-dependence, for example:

$$\begin{aligned}
u(x) &= \mathcal{F}^{-1}\Big[\frac{1}{1+|y|^2}\hat{f}(y)\Big] \text{ solves (6)},\\
u(x,t) &= \mathcal{F}^{-1}_{y\to x}[e^{-t|y|^2}\hat{g}(y)] \text{ solves (16)},\\
u(x,t) &= \mathcal{F}^{-1}_{y\to x}\Big[\hat{g}(y)\cos(t|y|)\hat{g}(y) + \hat{h}(y)\frac{\sin(t|y|)}{|y|}\Big] \text{ solves (24)}
\end{aligned} \tag{3.9}$$

(in the last example, we have replaced the initial condition on $\partial_t u$ by the general form $\partial_t u(x,0) = h(x)$). The differential operator $-\Delta$ itself is also of the form (3.8), with $p(y) = |y|^2$. There is a general name for such operators: When $p(y)$ is a function on $\mathbb{R}^n$, we call the operator defined by (3.8) the *pseudodifferential operator* with *symbol* $p(y)$ and denote it $\mathrm{OP}(p)$; in other words,

$$\mathrm{OP}(p)u = \mathcal{F}^{-1}(p\mathcal{F}u) = \tfrac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{ix\cdot y}p(y)\hat{u}(y)\,dy. \tag{3.10}$$

In the last expression we have written out the formula for the inverse Fourier transform. The variable called y there is more often denoted ξ (the Greek letter xi) in the literature.

Pseudodifferential operators are used in much of the advanced literature on PDE. The functions $p(y)$ that are considered there are subject to further restrictions, but the point is that they can be more general functions that just *polynomials* in y, which are the ones that define *differential operators.*

Actually, the literature is concerned with a generalization where the function $p(y)$ in formula (3.10) is allowed to depend on x also (the "variable coefficient" case). This gives certain difficulties that we shall not go into here. Let us just mention some other interesting examples of ("constant coefficient") pseudodifferential operators:

Example 3.2. We can define a "square root of the Laplacian" by taking the square root of $|y|$ as symbol:

$$\sqrt{-\Delta} = \mathrm{OP}(|y|). \tag{3.11}$$

Its square is $-\Delta$, for

$$(\sqrt{-\Delta}\,)^2 u = \sqrt{-\Delta}\,\sqrt{-\Delta}\,u = \mathcal{F}^{-1}(|y|\mathcal{F}(\mathcal{F}^{-1}(|y|\mathcal{F}u))) = \mathcal{F}^{-1}(|y|^2\mathcal{F}u) = -\Delta u.$$

The operator is well-defined on the functions $u \in L^2(\mathbb{R}^n)$ for which $|y|\hat{u}(y) \in L^2(\mathbb{R}^n)$. We can let it enter in a generalization of the heat equation problem:

$$\begin{aligned} \partial_t u(x,t) + \sqrt{-\Delta}\,u(x,t) &= 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ u(x,0) &= g(x) \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{3.12}$$

By Fourier transformation in the x-variable, this problem is turned into the problem

$$\begin{aligned} \partial_t \hat{u}(y,t) + |y|\hat{u}(y,t) &= 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ \hat{u}(y,0) &= \hat{g}(y) \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{3.13}$$

For each fixed y, (3.13) has the unique solution

$$\hat{u}(y,t) = e^{-|y|t}\hat{g}(y). \tag{3.14}$$

So the problem (3.12) has the solution

$$u(x,t) = \mathcal{F}^{-1}_{y\to x}(e^{-|y|t}\hat{g}(y)) = \mathrm{OP}(e^{-|y|t})g. \tag{3.15}$$

In the case $n = 1$, it is not hard to show (see Exercise 3.1) that

$$\mathcal{F}^{-1}(e^{-|y|t}) = (\tfrac{2}{\pi})^{\frac12}\frac{t}{x^2+t^2}, \tag{3.16}$$

so then

$$u(x,t) = (\tfrac{2}{\pi})^{\frac12}\frac{t}{x^2+t^2} * g(x) \tag{3.17}$$

is an explicit formula for the solution of (3.12).

Example 3.3. The case studied in Example 3.2 is not without interest in the theory of Finance. In fact, the Black-Sholes equation has for some purposes been found inadequate, and other equations have been proposed. In the framework where the Black-Sholes equation is turned into the heat equation, one of the proposed variants is

$$\begin{aligned} \partial_t u(x,t) + (\sqrt{\alpha^2 - \partial_x^2} - \alpha)u(x,t) &= 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) &= g(x) \quad \text{on } \mathbb{R}, \end{aligned} \tag{3.18}$$

where

$$(\sqrt{\alpha^2 - \partial_x^2} - \alpha) = \mathrm{OP}((\alpha^2 + y^2)^{\frac12} - \alpha), \tag{3.19}$$

for some positive constant α. There are also variants with symbols

$$p(y) = (\alpha^2 + y^2)^{\nu/2} - \alpha^\nu, \text{ with } 0 < \nu < 1. \tag{3.20}$$

We can of course write the solution of these problems as

$$u(x,t) = \mathcal{F}^{-1}_{y\to x}(e^{-p(y)t}\hat{g}(y)) = \mathrm{OP}(e^{-p(y)t})g, \tag{3.21}$$

but there remains an anlysis of how the solution really looks. Moreover, in the applications to Finance, one needs to consider the problem on a half-line $x \in [a,\infty[$, with a boundary condition at the point a, and here the solvability discussion is considerably more difficult than in the case of the heat equation.

Let us show how Definition 3.1 relates to the definition of weak derivatives. Recall that $C_c^\infty(U)$ denotes the space of C^∞ functions on U that have compact (closed and bounded) support inside U — so the support has positive distance from ∂U. $C_c^\infty(U)$ is often called the space of *test functions* on U.

Definition 3.4. *When u and $v \in L^1_{\mathrm{loc}}(U)$, we say that v is the weak α'th derivative $D^\alpha u$ (on U), if*

$$\int_U v\varphi\,dx = (-1)^{|\alpha|}\int_U uD^\alpha\varphi\,dx, \quad \textit{all } \varphi \in C_c^\infty(U). \tag{3.22}$$

A motivation for this definition is that when $u \in C^k(U)$ for $k = |\alpha|$, then (3.22) holds with v equal to the classically defined derivative $D^\alpha u$; this is seen by integration by parts, noting that there are no boundary contributions since φ is 0 in a neighborhood of ∂U.

Lemma 3.5. *When $u \in L^2(\mathbb{R}^n)$ has the derivative $D^\alpha u \in L^2(\mathbb{R}^n)$ in the Fourier sense, then $D^\alpha u$ is also the α'th weak derivative of u on $\mathbb{R}^n$.*

Proof. Let $v = D^\alpha u$ in the Fourier sense; by definition, $v = \mathcal{F}^{-1}((iy)^\alpha\hat{u}(y))$ and is in $L^2(\mathbb{R}^n)$. When $\varphi \in C_c^\infty(\mathbb{R}^n)$, we find by use of (I):

$$\begin{aligned}\int_{\mathbb{R}^n} v\overline{\varphi}\,dx &= \int_{\mathbb{R}^n}\hat{v}\overline{\hat{\varphi}}\,dy = \int_{\mathbb{R}^n}(iy)^\alpha\hat{u}\overline{\hat{\varphi}}\,dy\\ &= (-1)^\alpha\int_{\mathbb{R}^n}\hat{u}\overline{(iy)^\alpha\hat{\varphi}}\,dy = (-1)^\alpha\int_{\mathbb{R}^n}u\overline{D^\alpha\varphi}\,dx.\end{aligned} \tag{3.23}$$

This shows that (3.22) holds with this v. $\square$

One can also show the converse, namely that when u and $v \in L^2(\mathbb{R}^n)$ with $v = D^\alpha u$ in the weak sense on $\mathbb{R}^n$, then v is the α'th derivative of u in the Fourier sense. We leave out the proof since it demands a few more tools.

E3. Exercises

Exercise E3.1. We denote by $H(x)$ the Heaviside function $1_{\mathbb{R}_+}$ (with $\mathbb{R}_+ =]0,\infty[$). For $t > 0$, show that

$$\mathcal{F}^{-1}(H(y)e^{-ty}) = \tfrac{1}{(2\pi)^{\frac12}}\frac{1}{t - ix}. \tag{3.24}$$

Find $\mathcal{F}^{-1}(H(-y)e^{ty})$, and show that

$$\mathcal{F}^{-1}(e^{-t|y|}) = (\tfrac{2}{\pi})^{\frac12}\frac{t}{x^2 + t^2}. \tag{3.25}$$

Exercise E3.2.

(1) Show that if $f(y) \in L^1(\mathbb{R})$ with $yf(y) \in L^1(\mathbb{R})$, then $\check{f} = \mathcal{F}^{-1}f$ is in $C^1(\mathbb{R})$ with $\check{f}$ and $D\check{f}$ bounded.

(2) Show that if $y^k f(y) \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}_0$, then $\check{f} = \mathcal{F}^{-1}f$ is in $C^\infty(\mathbb{R})$ with $D^k\check{f}$ bounded for all k.

(3) Let $p(y)$ be a continuous real function on $\mathbb{R}$ for which there exists positive constants c and C such that

$$p(y) \geq c|y| \text{ for } |y| \geq C. \tag{3.26}$$

Show that when $f(y) \in L^2(\mathbb{R})$ and $t > 0$, then $y^k e^{-tp(y)} f(y) \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}_0$.

(4) Use this to conclude, with

$$p(y) = (\alpha^2 + y^2)^{\frac{1}{2}} - \alpha, \tag{3.27}$$

that the solution (3.21) of the problem (3.18) is C^∞ in x.[1]

(*Hints.* For (1), it is useful to know that

$$|e^{ia} - 1| = |\int_0^a e^{is}\, ds| \leq |a|, \text{ when } a \in \mathbb{R}. \tag{3.28}$$

For (3), one can apply the Cauchy-Schwarz inequality to $y^k e^{-tp(y)} \cdot f$.)

[1]For this p one can also show that $u(x,t)$ is C^∞ in t for $t > 0$, but we leave that out of the exercise. The results can be shown also for the more general functions in (3.20).