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2. Comments to the treatment of the Dirichlet problem

It is useful for the calculations in Section 2.2 to observe, once and for all:

Lemma 2.1. *When the sphere $\partial B(x,r)$ is provided with the interior normal, then*

$$\frac{\partial}{\partial \nu_y}\Phi(y-x) = \frac{1}{n\alpha(n)r^{n-1}} \quad \text{on } \partial B(x,r), \tag{2.1}$$

and hence the following integral is an average:

$$\int_{\partial B(x,r)} \tfrac{\partial}{\partial \nu_y}\Phi(y-x)f(y)\,dS = ⨍_{\partial B(x,r)} f(y)\,dS. \tag{2.2}$$

This converges to $f(x)$ for $r \to 0$ when f is continuous at x.

Proof. The formula (2.1) follows from the first three lines of [E98, p. 25]. Then (2.2) is seen from (1.12), and the convergence is shown in Lemma 1.2. $\square$

We shall often make use of the following general result from integration theory:

Lemma 2.2. *Let $(M, \mathbb{E}, \mu)$ be a measure space and let I be an interval of $\mathbb{R}$. Let $f(y,a)$ be a family of functions of $y \in M$ depending on the parameter $a \in I$, such that for each $a \in I$, $f(y,a) \in \mathcal{L}^1(M,\mathbb{E},\mu)$. If $\frac{\partial}{\partial a}f(y,a)$ exists for all $(y,a) \in M \times I$ and there is a function $g(y) \in \mathcal{L}^1(M,\mathbb{E},\mu)$ such that $|\frac{\partial}{\partial a}f(y,a)| \le g(y)$ for all $(y,a) \in M \times I$, then $F(a) = \int_M f(y,a)\,d\mu$ is a differentiable function of $a \in I$, and*

$$\frac{d}{da}F(a) = \int_M \frac{\partial}{\partial a} f(y,a)\,d\mu. \tag{2.4}$$

The differentiations leading to (10) on page 24 in [E98] can be justified using this lemma: To study the derivatives of $u(x)$ at a point x_0, we only need to let x run in a neighborhood of the point, say the ball $B(x_0,1)$. The support of $f(y)$ is contained in some ball $B(0,R)$; then $f(x-y)$ vanishes for y outside the ball $B(x_0,R+1)$ when $|x-x_0| < 1$. So the integrals can be restricted to the set $B(x_0,R+1)$, where $\Phi(y)$ is in $\mathcal{L}^1$. Now use that the functions $\Phi(y)\partial_{x_i}f(x-y)$ and $\Phi(y)\partial_{x_i}\partial_{x_j}f(x-y)$ are all bounded by $C|\Phi(y)|$.

More generally, we often consider a function defined by an integral of a given function against a kernel (a function of two variables). An example is the Poisson formula for $\mathbb{R}^n_+$,

$$u(x) = \int_{\mathbb{R}^{n-1}} K(x,y')g(y')\,dy'; \tag{2.3}$$

also here we need to show that differentiations with respect to x can be performed as differentiations inside the integral sign.

Let us see how the lemma can be applied to (2.3) when $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. Here x is an n-tuple of parameters $x_1, \dots x_n$, and $M = \mathbb{R}^{n-1}$. Consider a point $x^0 \in \mathbb{R}^n_+$; note that $x^0_n \geq \delta$ for some $\delta > 0$. Let $x \in B(x^0, \frac{1}{2}\delta)$. Then since $|x - (y', 0)| \geq x_n \geq \delta$, we have:

$$\begin{aligned} |x - (y', 0)| &= |x - x^0 + x^0 - (y', 0)| \geq |x^0 - (y', 0)| - |x - x^0| \\ &\geq |x^0 - (y', 0)| - \tfrac{1}{2}\delta \geq \tfrac{1}{2}|x^0 - (y', 0)|, \text{ so} \\ |x - (y', 0)|^{-n} &\leq 2^n |x^0 - (y', 0)|^{-n}. \end{aligned}$$

Moreover, when $|x - x^0| \leq \frac{1}{2}\delta$, $x_n \leq x^0_n + \frac{1}{2}\delta$. For the integrand in (2.3), there is then an estimate

$$|K(x, y')g(y')| \leq 2^{n+1}(x^0 + \tfrac{1}{2}\delta)\|g\|_{L^\infty} K(x^0, y'), \text{ when } |x - x^0| \leq \tfrac{1}{2}\delta; \tag{2.5}$$

here the right hand side is a fixed integrable function of $y' \in \mathbb{R}^{n-1}$. There are similar estimates for all the partial derivatives with respect to x (try a couple of them!). Then we can use Lemma 2.2 to conclude that $u(x)$ is in $C^\infty(\mathbb{R}^n_+)$.

Theorem 14 in [E98, Section 2.2] shows how we can solve the Dirichlet problem for $\mathbb{R}^n_+$ with $f = 0$, for a reasonable class of given boundary values g. It is not hard to include a nice class of nonzero functions f, when $n \geq 3$:

Theorem 2.3. *Let $n \geq 3$. Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, and let $f = r_{\mathbb{R}^n_+} F$ (the restriction of F to $\mathbb{R}^n_+$) for some $F \in C^\infty_c(\mathbb{R}^n)$. Then the Dirichlet problem*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \mathbb{R}^n_+, \\ u &= g \quad \text{on } \partial\mathbb{R}^n_+, \end{aligned} \tag{2.6}$$

has the solution

$$u(x) = v(x) + w(x) \quad \text{on } \overline{\mathbb{R}}^n_+, \tag{2.7}$$

where

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^n} \Phi(x - y)F(y)\,dy, \\ w(x) &= \int_{\mathbb{R}^{n-1}} K(x, y')(g(y') - v(y', 0))\,dy'. \end{aligned} \tag{2.8}$$

The function u is in $C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}}^n_+)$.

Proof. First we use Theorem 1 in [E98, Section 2.2] to construct v such that it satisfies $-\Delta v(x) = F(x)$ for all $x \in \mathbb{R}^n$, by the first formula in (2.8) above. We have already seen in the proof of Theorem 1 that $v \in C^2$; the same considerations can be used for the higher derivatives, showing that $D^\alpha v(x) = \int \Phi(y) D^\alpha_x F(x - y)\,dy$ exists for all α. As for boundedness of v, we can even show that $v(x) \to 0$ for $|x| \to \infty$. In fact, let $M = \operatorname{spt} F$, it is a compact subset of $\mathbb{R}^n$. When $x \notin M$, then $\Phi(x - y)$ is $O(\operatorname{dist}(x, M)^{2-n})$ for $y \in M$ (see [E98, p. 620] for the big-oh notation), so

$$|v(x)| = |\int_M \Phi(x - y)F(y)\,dy| \leq C \operatorname{dist}(x, M)^{2-n} \to 0 \text{ for } |x| \to \infty. \tag{2.9}$$

In particular, $g_1(x') = v(x', 0)$ is in $C^\infty(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$. Let $w(x) = u(x) - v(x)$ on $\overline{\mathbb{R}}^n_+$. Then we see that u solves (2.6) if and only if w solves

$$\begin{aligned} -\Delta w &= 0 && \text{in } \mathbb{R}^n_+, \\ w &= g - g_1 && \text{on } \partial\mathbb{R}^n_+. \end{aligned} \tag{2.10}$$

Now problem (2.10) is solved by Theorem 14, by the second formula in (2.8). Then $u = r_{\overline{\mathbb{R}}^n_+} v + w$ solves (2.6). Theorem 14 shows that $w \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}}^n_+)$, and the same clearly holds for $r_{\overline{\mathbb{R}}^n_+} v$. $\square$

E2. Exercises

Exercise E2.1. Give a detailed deduction of the Poisson kernel $K(x, y') = K(x_1, x_2, y_1)$ in the case $U = \mathbb{R}^2_+ = \{ (x_1, x_2) \mid x_2 > 0 \}$ (with explicit constants), and verify the properties:

(1) $K \in C^\infty$ (as a function of $x \in \mathbb{R}^2_+$, $y_1 \in \mathbb{R}$).
(2) $K > 0$.
(3) $\int_{\mathbb{R}} K(x, y_1)\, dy_1 = 1$ for all $x \in \mathbb{R}^2_+$.
(4) $\Delta_x K(x, y_1) = 0$.

Exercise E2.2. In the case $n = 2$, you are asked to give a detailed deduction of the Poisson kernel $K(x, y)$ for the disc $U = B^0(0, 1)$ ($x \in B^0(0, 1)$, $y \in \partial B(0, 1)$), with explicit constants. Show that it has the properties:

(1) $K > 0$.
(2) $\int_{\partial B(0,1)} K(x, y)\, dS(y) = 1$ for all $x \in B^0(0, 1)$.
(3) $\Delta_x K(x, y) = 0$.