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SUPPLEMENT TO THE COURSE IN PARTIAL DIFFERENTIAL EQUATIONS, FALL 2002

We here supply the explanations in [E98] (short for L. C. Evans: "Partial Differential Equations", 1998) with some further details.

Besides the notation $\frac{\partial u}{\partial x_i} = u_{x_i}$ we shall also write $\partial_{x_i} u$, or $\partial_i u$ when this does not lead to confusion.

1. Integration in several variables

Consider an open set $U \subset \mathbb{R}^n$. Appendix C.1 gives the definition:

Definition 1.1. *U is said to have a C^k-boundary, when every boundary point x^0 has a neighborhood $B(x^0, r)$ for which there exists a C^k-function f from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ such that — after a relabelling and reorientation of the coordinate axes if necessary — one has:*

$$U \cap B(x^0, r) = \{ x \in B(x^0, r) \mid x_n > f(x_1, \dots, x_{n-1}) \}. \tag{1.1}$$

Note that then

$$\partial U \cap B(x^0, r) = \{ x \in B(x^0, r) \mid x_n = f(x_1, \dots, x_{n-1}) \}; \tag{1.2}$$

this shows how the boundary surface in the neighborhood of x^0 is described as the graph of a C^k-function f.

Next, [E98] states that when ∂U is C^1 (we shall then also say that U is C^1), there is a well-defined outward pointing unit *normal vector* ν at each boundary point x^0. In fact, we can describe it in terms of the function in Definition 1.1; we claim that

$$\nu = (\nu^1, \dots, \nu^n) = \frac{1}{(|\partial_1 f|^2 + \dots + |\partial_{n-1} f|^2 + 1)^{\frac{1}{2}}} (\partial_1 f, \dots, \partial_{n-1} f, -1) \tag{1.3}$$

holds at the points $x = (x', f(x'))$ in $\partial U \cap B(x^0, r)$. With the short notation $x' = (x_1, \dots, x_{n-1})$, $D' = (\partial_1, \dots, \partial_{n-1})$, we can write the formula as

$$\nu = \frac{1}{(|D'f|^2 + 1)^{\frac{1}{2}}} (D'f, -1). \tag{1.4}$$

To justify this formula, note that the surface in $\mathbb{R}^n$ described by the graph of f has the tangent vectors $(1, 0, \dots, 0, \partial_1 f(x'))$, $(0, 1, 0, \dots, 0, \partial_2 f(x'))$, … ,$(0, \dots, 0, 1, \partial_{n-1} f(x'))$ at $x = (x', f(x'))$ (they span the tangent hyperplane at x). The vector in (1.3) is clearly orthogonal to all of these, moreover it is directed downwards (away from U) and has norm 1.

On page 627, the mapping $\Phi\colon (x', x_n) \mapsto (x', x_n - f(x'))$ and its inverse $\Psi\colon (y', y_n) \mapsto (y', y_n + f(y'))$ are considered, and it is shown how Φ replaces $\partial U \cap B(x^0, r)$ by a flat piece in $\mathbb{R}^{n-1} \times \{0\}$. Let us denote it $V' \times \{0\}$. The gradient matrix of Φ (as defined in A.4 (ii) and C.5) is

$$D\Phi = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -\partial_1 f & -\partial_2 f & \dots & -\partial_{n-1} f & 1 \end{pmatrix}, \tag{1.5}$$

with determinant 1, so the Jacobian $|\det D\Phi|$ equals 1; also $|\det D\Psi| = 1$. The mapping Φ and its inverse allow us to consider $\partial U \cap B(x^0, r)$ as a topological space — with the topology carried over from the flat piece $V' \subset \mathbb{R}^{n-1}$. This gives just a hint on how one can consider the boundary of ∂U as a topological space.

Since f is C^1, the normal ν at $x = (x', f(x'))$ depends continuously on x'; by use of the mapping Φ we can also consider it as a continuous function of $x \in \partial U \cap B(x^0, r)$.

In C.2 it is used without further explanation that there exists a suitable *surface measure* dS on ∂U such that the *Gauss-Green formula*

$$\int_{x\in U} \partial_{x_i} u(x)\, dx = \int_{x \in \partial U} u(x) \nu^i(x)\, dS \tag{1.6}$$

holds for C^1-functions, when U is bounded. To explain this further, we appeal again to Definition 1.1.

Let us first explain what integration with respect to dS stands for. If f is a linear function of two variables, $f(x') = c_1 x_1 + c_2 x_2 + c_0$, its graph is a plane H in $\mathbb{R}^3$, and a circle $C = \{x' \mid |x' - x'_0| \le a\}$ in $\mathbb{R}^2$ is mapped onto an ellipse $E = \{(x', f(x')) \mid x' \in C\}$ in H. Consider both C and E as cuts in the cylinder $C \times \mathbb{R}$, then a little geometry shows that the ratio between the area of E and the area of C must equal the ratio between 1 and $|\nu^3|$ (the last component of the unit normal to H). In other words, E is a stretching of C by the factor $|\nu^3|^{-1} = (1 + |D'f|^2)^{\frac12}$ (cf. (1.4)).

In the case of n variables and a linear function $f(x')$, one likewise finds that the graph of f is a hyperplane H, and that a ball $C = \{x' \mid |x' - x'_0| \le a\}$ in $\mathbb{R}^{n-1}$ is mapped onto an ellipsoid $E = \{(x', f(x')) \mid x' \in C\}$ in H, such that the ratio between the "areas" of E and C — the volumes, seen from an $(n-1)$-dimensional point of view — is:

$$\frac{\mathrm{vol}_{n-1}(E)}{\mathrm{vol}_{n-1}(C)} = \frac{1}{|\nu^n|} = (1 + |D'f|^2)^{\frac12}.$$

A general C^1-function f is close to being linear in the neighborhood of each point, so a small ball C of $\mathbb{R}^{n-1}$ is mapped into a piece $E = \{\, (x', f(x')) \mid x' \in C \,\}$ of the graph such that the ratio between the areas of E and C is approximated very well by $|\nu^n|^{-1} = (1 + |D'f(x')|^2)^{\frac12}$ there. This leads us to *define*

$$\int_{\partial U \cap B(x^0, r)} v(x)\, dS = \int_{V'} v(x', f(x'))(1 + |D'f(x')|^2)^{\frac12}\, dx'. \tag{1.7}$$

One can verify that this definition is quite satisfactory, in the sense that another choice of coordinate representation of ∂U, as the graph of a function of another coordinate, will give the same value of the left hand side.

Now consider the formula (1.6). Let us relabel i as n. Assume moreover that we are in the situation of Definition 1.1 and that u is supported[1] in a smaller ball $B(x^0,r')$ (with $r'<r$). We extend u by 0 at the points (x',x_n) where $x'\in V'$ and x_n is larger than the values covered by $B(x^0,r)$; the extension will be denoted $\tilde u$ and is C^1 in x_n for $x_n>f(x')$. Then we can calculate the left hand side of (1.6) as follows:

(1.8)
$$\begin{aligned}\int_U \partial_{x_n}u(x)\,dx &= \int_{U\cap B(x^0,r)} \partial_{x_n}u(x)\,dx = \int_{x'\in V'}\Big(\int_{f(x')}^{\infty}\partial_{x_n}\tilde u(x',x_n)\,dx_n\Big)\,dx'\\ &= -\int_{V'}u(x',f(x'))\,dx' = \int_{V'}u(x',f(x'))\tfrac{-1}{(|D'f|^2+1)^{\frac12}}(|D'f|^2+1)^{\frac12}\,dx'\\ &= \int_{\partial U\cap B(x^0,r)}u(x)\nu^n(x)\,dS;\end{aligned}$$

the third equality comes from integrating the derivative in x_n, and the last equality comes from (1.4) and (1.7).

This shows that formula (1.6) holds for $i=n$ in the situation of Definition 1.1, when u is supported in a slightly smaller ball $B(x^0,r')$.

This is the basic piece of information we shall need. In the treatment of more general cases, we can usually decompose the function u into a finite sum of C^1-functions that are supported inside balls where the boundary surface can be defined as in Definition 1.1, plus a C^1-function u_0 supported in the interior of U. The latter gives 0 since the extension $\tilde u_0$ of u_0 that equals 0 on $\mathbb{R}^n\setminus \operatorname{spt} u$ is C^1, and hence

$$\int_U \partial_{x_i}u_0\,dx = \int_{\mathbb{R}^n}\partial_{x_i}\tilde u_0\,dx = \int_{K_R}\partial_{x_i}\tilde u_0\,dx = 0,$$

where we have taken a large box $K_R=[-R,R]^n$ containing $\operatorname{spt} u$ in $K_{R/2}$; here the integration of $\partial_{x_i}\tilde u_0$ in the x_i-direction gives 0. The decomposition of u is achieved by use of a suitable "partition of unity".[2]

In very general cases, we may have to deal with some pieces where the boundary cannot be represented by a function of the i'th coordinate for any i. One can show that such troublesome pieces can be taken to have arbitrarily small volume (and surface area) so that the formula (1.6) is obtained in general by approximation.

Instead of decomposing u into pieces with small support, one could try to divide U into nice box-like domains where one can use integration in the i'th coordinate. Here all the boundaries may contribute, but the contributions from the new boundaries that are interior to U cancel each other, since the normals for two adjacent domains have opposite

[1]The support of a function u — denoted $\operatorname{spt} u$ in A.3 — is the complement of the largest open set where $u=0$.

[2]When $\bar U$ is contained in a union of open sets $V_1,\dots,V_N$, one can find a system of C^∞ functions $\zeta_1,\dots,\zeta_N$ such that ζ_j is supported in V_j for each j and $\sum_{j=1}^N\zeta_j=1$ on $\bar U$; see Problems 5.10.2 and 4 in [E98]. Then $u=\zeta_1u+\dots+\zeta_Nu$ where ζ_ju is in $C^1(\bar U)$ with support in V_j. We take this up in supplementing notes later on.

signs along the common surface. Also in this strategy, there may be some boxes left over that are not suited to the desired integration, but they can be taken with small volume so that the general result is obtained by approximation.

Next, we comment on the formulas for integration in polar coordinates in C.3. When we merely integrate over the ball $B(0,R)$, the formula takes the form

$$\int_{B(0,R)} u(x)\,dx = \int_0^R \Big(\int_{\partial B(0,r)} u(x)\,dS\Big)\,dr. \tag{1.9}$$

Let us see how this looks in the case where $u(x) = |x|^a$ for some $a \in \mathbb{R}$. Then

$$\begin{aligned}\int_{B(0,R)} u(x)\,dx &= \int_0^R \Big(\int_{\partial B(0,r)} r^a\,dS\Big)\,dr = \int_0^R r^a r^{n-1}\Big(\int_{\partial B(0,1)} 1\,dS\Big)\,dr\\ &= \int_0^R r^{a+n-1} n\alpha(n)\,dr = \begin{cases} +\infty & \text{if } a \le -n,\\ \frac{n}{a+n}\alpha(n)R^{a+n} & \text{if } a > -n.\end{cases}\end{aligned} \tag{1.10}$$

Here $n\alpha(n)$ stands for the area of $\partial B(0,1)$, and we used that the area of $\partial B(0,r)$ is r^{n-1} times as large. By taking $a = 0$ and $R = 1$ in (1.10), we see that $\alpha(n)$ equals the volume of $B(0,1) \subset \mathbb{R}^n$. The exact formula formula for $\alpha(n)$ is as follows (here $\Gamma(s)$ denotes the Gamma function which satisfies $\Gamma(s+1) = s\Gamma(s)$, $\Gamma(k+1) = k!$ for k integer ≥ 0, $\Gamma(\frac12) = \pi^{\frac12}$):

$$\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}, \quad \text{in particular, } \alpha(1) = 2,\ \alpha(2) = \pi,\ \alpha(3) = \tfrac43\pi. \tag{1.11}$$

A consequence of (1.10) is that

$$\int_{B(0,R)} |u(x)|\,dx < \infty, \text{ if } |u(x)| \le C|x|^a \text{ for some } a > -n. \tag{1.12}$$

We shall also add some information to the definition of *averages* in A.3 (vi). One often needs to let r go to 0 in the formulas

$$\begin{aligned}&⨍_{B(x,r)} u(y)\,dy = \tfrac{1}{\alpha(n)r^n}\int_{B(x,r)} u(y)\,dy,\\ &⨍_{\partial B(x,r)} u(y)\,dS = \tfrac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} u(y)\,dS.\end{aligned} \tag{1.13}$$

Lemma 1.2. *When u is continuous at x, the averages in* (1.13) *converge to $u(x)$ for $r \to 0$.*

Proof. We have that for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|u(y) - u(x)| \le \varepsilon \text{ for } |y - x| \le \delta. \tag{1.14}$$

Now since $\int_{B(x,r)} 1\,dy = \alpha(n)r^n$,

$$|\fint_{B(x,r)} u(y)\,dy - u(x)| = |\tfrac{1}{\alpha(n)r^n}\int_{B(x,r)} (u(y)-u(x))\,dy|$$
$$\le \tfrac{1}{\alpha(n)r^n}\int_{B(x,r)} \sup_{y\in B(x,r)} |u(y)-u(x)|\,dz = \sup_{y\in B(x,r)} |u(y)-u(x)|.$$

For a given ε, this is $\le \varepsilon$ when $r \le \delta$, if δ is chosen according to (1.14). Thus

$$\fint_{B(x,r)} u(y)\,dy \to u(x) \text{ for } r \to 0. \tag{1.15}$$

The proof for $\fint_{\partial B(x,r)} u(y)\,dS$ is similar, using that $\int_{\partial B(x,r)} 1\,dS = n\alpha(n)r^{n-1}$. □

For general locally integrable functions there is convergence almost everywhere; see E.3.

E1. Exercises

Exercise E1.1. Let U be the open unit ball $B^0(0,1)$ in $\mathbb{R}^n$. One would expect from the geometry that the normal ν at the point $x \in \partial U$ is simply $\frac{x}{|x|}$. Verify this for the points on the upper half sphere

$$S_+^{n-1} = \{\, x \in \mathbb{R}^n \mid |x| = 1,\ x_n > 0\,\}, \tag{E1.1}$$

whose points can be represented as $x = (x', f(x'))$ with $f(x') = (1-|x'|^2)^{\frac12}$, by performing the calculations in (1.3) for this f.

How does formula (1.7) look in this case?

Exercise E1.2. The Black-Sholes equation is as follows, for a function $v(s,t)$ of two variables $s > 0$ and $t \le T$:

$$\frac{\partial v}{\partial t} - rv + rs\frac{\partial v}{\partial s} + cs^2\frac{\partial^2 v}{\partial s^2} = 0; \tag{E1.2}$$

here T, r and c are given positive constants (c is often written as $\frac12\sigma^2$).

Show that it can be reduced to the heat equation in the following steps:

(1) Set $s = e^x$, where $x \in \mathbb{R}$, and let $v_1(x,t) = v(e^x,t)$. Show that when v satisfies (E1.2), then v_1 satisfies an equation of the form

$$\frac{\partial v_1}{\partial t} - rv_1 + c_1\frac{\partial v_1}{\partial x} + c\frac{\partial^2 v_1}{\partial x^2} = 0; \tag{E1.3}$$

find c_1.

(2) Set $v_2(x,t) = e^{\alpha x+\beta t}v_1(x,t)$, where α and β are real constants. Show that there is a (unique) choice of α and β for which $v_2(x,t)$ solves

$$\frac{\partial v_2}{\partial t} + c\frac{\partial^2 v_2}{\partial x^2} = 0; \tag{E1.4}$$

find α and β.

(3) Set $t = T - \tau$, where $\tau \geq 0$, and $x = c^{\frac{1}{2}} y$, and let $v_3(y, \tau) = v_2(c^{\frac{1}{2}} y, T - \tau)$. Show that if v_2 solves (E1.4), then v_3 solves the heat equation

$$\frac{\partial v_3}{\partial \tau} - \frac{\partial^2 v_3}{\partial y^2} = 0. \tag{E1.5}$$

(4) Show that conversely, if $v_3(y, \tau)$ solves (E1.5), then $v(s, t)$ can be defined by performing the above modifications backwards and solves (E1.2).

(5) Find out what an initial condition $v_3(y, 0) = g(y)$ for (E1.5) corresponds to for (E1.2).