ZETA AND ETA FUNCTIONS FOR ATIYAH-PATODI-SINGER OPERATORS

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1. Introduction

[1]The zeta function of a Laplacian Δ is $\zeta(\Delta, s) = \operatorname{trace}(\Delta^{-s})$, where Δ^{-s} is taken to be zero in the nullspace of Δ.

Minakshisundaram and Pleijel [MP] showed that for the Laplacian on a compact manifold X with empty boundary, the zeta function has a meromorphic extension to the complex plane $\mathbb{C}$. Later, Atiyah and Bott suggested the use of zeta functions in proving an index formula

$$\operatorname{ind}(P) = \int_X \alpha(x)\, dx, \tag{1.1}$$

for an elliptic operator P. The density $\alpha(x)\, dx$ is determined locally by the symbol of P. One version of this proof, given in Section 2 below, uses the zeta functions of $\Delta_1 = P^*P$ and $\Delta_2 = PP^*$.

If $\partial X = X'$ is *not* empty, the zeta method gives an analogous formula for the index of a realization P_B of P given by a *differential* boundary condition $Bu = 0$ on ∂X,

$$\operatorname{ind}(P_B) = \int_X \alpha(x)\, dx + \int_{X'} \beta(x')\, dx', \tag{1.2}$$

with densities $\alpha\, dx$ and $\beta\, dx'$ locally determined by the symbols of P and B.

However, good differential boundary conditions do not always exist. In particular, they do not always exist for the geometrically interesting first order operators studied by Atiyah, Patodi and Singer [APS]. Near ∂X, these have the form

$$P = \sigma(\tfrac{\partial}{\partial x_n} + A) \tag{1.3}$$

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where x_n is a coordinate normal to the boundary, σ is a morphism between vector bundles, and A is a selfadjoint elliptic first order differential operator on $X' = \partial X$. The operator A determines an orthogonal projection

$$\Pi_{\geq} = \text{projection on eigenspaces of } A \text{ with eigenvalue } \geq 0.$$

The boundary condition

$$\Pi_{\geq} u = 0 \quad \text{on} \quad \partial X \tag{1.4}$$

determines a realization of P, which we call $P_{\geq}$. In the case that σ and A are independent of x_n (the "cylindrical" case), [APS] found

$$\operatorname{ind}(P_{\geq}) = \int_X \alpha(x)\,dx - \tfrac{1}{2}\eta(A,0) - \tfrac{1}{2}\nu_0(A). \tag{1.5}$$

Here $\alpha(x)\,dx$ is the same density as in (1.1) and (1.2); $\eta(A,0)$ is the analytic continuation to $s=0$ of the eta function

$$\eta(A,s) = \operatorname{trace}(A|A|^{-s-1}),$$

a regularized signature of A; and $\nu_0(A)$ is the nullity of A. The proof in [APS] used heat asymptotics of the Laplacians

$$\Delta_1 = {P_{\geq}}^* P_{\geq} \quad \text{and} \quad \Delta_2 = P_{\geq} {P_{\geq}}^*, \tag{1.6}$$

but did not obtain separate expansions for $\operatorname{Tr}(e^{-t\Delta_1})$ and $\operatorname{Tr}(e^{-t\Delta_2})$. Partial expansions for these (with $n+1$ terms plus a remainder), sufficient to obtain the index formula, have been obtained recently in [G2], even for the case where σ and A depend on x_n.

The main result of the present paper is to obtain, in the cylindrical case, and by relatively elementary means, a complete description of the singularities of $\Gamma(s)\zeta(\Delta_i,s)$, and more generally of $\Gamma(s)\operatorname{Tr}(D\Delta_1^{-s})$, $\Gamma(s)\operatorname{Tr}(D\Delta_2^{-s})$, $\Gamma(s)\operatorname{Tr}(DP\Delta_1^{-s})$, and $\Gamma(s)\operatorname{Tr}(DP^*\Delta_2^{-s})$, where D is a differential operator on X'. We also allow some variation in the boundary condition (1.4). The singularities of $\Gamma(s)\zeta(\Delta_i,s)$ at $s=0$ give the index formula (1.5). The full set of singularities gives the complete expansion of the corresponding heat traces $\operatorname{Tr} De^{-t\Delta_i}$ and so on, as $t \to 0$. However, in the actual calculation of the full set of coefficients it is advantageous to work with the power functions $\zeta(\Delta_i,s)$, etc., because they are directly related to the power functions of A by product formulas.

The factor D plays no role in the index question, but has been useful in the study of other invariants by Branson and Gilkey [BG] in the case of differential operators, and will presumably be useful in this more general case as well.

The simplifying feature of the cylindrical case is that we can write a very accurate approximation of the resolvent using functions of the tangential operator A in (1.3). The noncylindrical case requires a somewhat new approach to these resolvent expansions; this is taken up in another article [GS].

Another recent approach to the index formula (1.5), due to Melrose [M], imposes the boundary condition implicitly by adding an infinite cylinder to X. Then $e^{-t\Delta_i}$ is no longer trace class, but a modified notion of trace can still be used. Our approach is less sophisticated.

As to organization, Section 2 establishes notation, states the main results and some consequences, explains the relation between zeta functions and resolvent traces, and recalls the essential facts about these functions for the case of boundaryless manifolds. Section 3 analyzes the $\mathrm{Tr}(D\Delta_i^{-s})$, giving the zeta function of Δ_i when $D = I$. Section 4 treats $\mathrm{Tr}(DP\Delta_1^{-s})$ and $\mathrm{Tr}(DP^*\Delta_2^{-s})$, giving the eta functions of $P_\geq$ and $P_\geq{}^*$ when $D = I$. Section 5 gives the expansions of the corresponding heat traces.

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2. Notation and statement of results

2.1 General representation formulas.

Suppose that Q is a closed operator in a Hilbert space having a resolvent $(Q-\lambda)^{-1}$ which is meromorphic at $\lambda = 0$ and holomorphic in some sector $|\arg(-\lambda)| < \alpha$, with $\|(Q-\lambda)^{-1}\| = O(|\lambda|^{-1})$. Then the "power function" $Z(Q,s)$ is defined for $\mathrm{Re}\, s > 0$ by

$$Z(Q,s) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-s}(Q-\lambda)^{-1} d\lambda \tag{2.1}$$

where $\mathcal{C}$ is a curve

$$\begin{aligned} \mathcal{C}_{\theta,r_0} = \{\, \lambda = re^{\mathrm{i}\theta} \mid \infty > r \geq r_0 \,\} + \{\, \lambda = r_0 e^{\mathrm{i}\theta'} \mid \theta \geq \theta' \geq -\theta \,\} \\ + \{\, \lambda = re^{\mathrm{i}(2\pi-\theta)} \mid r_0 \leq r < \infty \,\}, \end{aligned} \tag{2.2}$$

with $\pi - \alpha < \theta \leq \pi$, and r_0 chosen so that $(Q-\lambda)^{-1}$ is holomorphic for $0 < |\lambda| \leq r_0$. If Q is invertible then $Z(Q,s) = Q^{-s}$; in any case, $Z(Q,s) = 0$ on the nullspace of Q, since $\int_{\mathcal{C}} \lambda^{-s-1} d\lambda = 0$.

If $Z(Q,s)$ is trace class for some s, then Q has a zeta function

$$\zeta(Q,s) = \mathrm{Tr}\, Z(Q,s) \tag{2.3}$$

and, for appropriate operators D, a "modified" zeta function

$$\zeta(D,Q,s) = \mathrm{Tr}\, DZ(Q,s). \tag{2.4}$$

Similarly, under appropriate conditions, we define

$$Y(Q,s) = QZ(Q^*Q, \tfrac{s+1}{2}) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-(s+1)/2} Q(Q^*Q-\lambda)^{-1} d\lambda \tag{2.5}$$

and the eta functions

$$\eta(Q,s) = \mathrm{Tr}\, Y(Q,s), \quad \eta(D,Q,s) = \mathrm{Tr}\, DY(Q,s). \tag{2.6}$$

When Q is selfadjoint,

$$\sum_{\lambda\in\operatorname{sp}(Q)\setminus\{0\}} |\lambda|^{-s} = \zeta(Q^2, \tfrac{s}{2}), \qquad \sum_{\lambda\in\operatorname{sp}(Q)\setminus\{0\}} \operatorname{sign}\lambda|\lambda|^{-s} = \eta(Q,s), \tag{2.7}$$

with summation over the eigenvalues, repeated according to multiplicities.

The operators Q for our zeta and eta functions arise as follows. On an n-dimensional manifold X with boundary $\partial X = X'$, we consider a first order differential operator

$$P : C^\infty(E_1) \to C^\infty(E_2)$$

between sections of hermitian vector bundles E_1 and E_2 over X. The restriction of E_i to the boundary X' is denoted E_i'. A neighborhood of the boundary has the form $X' \times [0,c]$, and we assume that E_i is isomorphic to the pull-back of E_i' there, and P is represented as

$$P = \sigma(\partial_{x_n} + A) \ \text{ on } \ X' \times [0,c], \tag{2.8}$$

where σ is a unitary morphism from E_1' to E_2', and A is an elliptic first order differential operator in $C^\infty(E_1')$, selfadjoint in $L_2(X', E_1')$ with respect to some smooth measure dx'. σ and A are independent of x_n, and the measure dx on X equals $dx'dx_n$ on $X' \times [0,c]$ (the "cylindrical" case). Also in the rest of X, P is assumed elliptic.

Let V_λ denote the eigenspace of A with eigenvalue λ. We have orthogonal projections

$$\begin{aligned} \Pi_> &= \text{ projection on } \bigoplus_{\lambda>0} V_\lambda, \\ \Pi_0 &= \text{ projection on } V_0, \quad \nu_0 = \dim V_0, \\ \Pi_\geq &= \Pi_> + \Pi_0. \end{aligned} \tag{2.9}$$

The notation $V_0(Q)$, $\Pi_0(Q)$, $\nu_0(Q)$, will be used also for other selfadjoint operators in the following.

We denote by $P_\geq$ the operator P acting on the domain

$$D(P_\geq) = \{\, u \in H^1(X, E_1) \mid \Pi_\geq(u|_{X'}) = 0 \,\}; \tag{2.10}$$

here $H^1(X,E)$ is the Sobolev space of order 1 of sections in E. More generally, for an orthogonal projection B with

$$\text{Range } (\Pi_>) \subset \ \text{Range } (B) \subset \ \text{Range } (\Pi_\geq) \tag{2.11}$$

we shall study P_B, defined as the operator P acting on the domain where $B(u|_{X'}) = 0$.

We extend the operator P to an operator $\widetilde{P}$ on the double $\widetilde{X}$ of X as indicated in [APS, p. 55] (switching the roles of E_1 and E_2 on $\widetilde{X} \setminus X$); the extended bundles are denoted by $\widetilde{E}_1$ and $\widetilde{E}_2$. Denote $\widetilde{P}^*\widetilde{P} = \widetilde{\Delta}_1$ and $\widetilde{P}\widetilde{P}^* = \widetilde{\Delta}_2$. For an operator Q on $C^\infty(\widetilde{X}, \widetilde{E}_i)$, we denote by Q_+ its restriction to X, i.e.,

$$Q_+ = r^+ Q e^+, \tag{2.12}$$

where e^+ denotes extension by zero on $\widetilde{X} \setminus X$ and r^+ denotes restriction to X. When Q is of trace class with a continuous kernel $\mathcal{K}(x, y, Q)$ (so that $\operatorname{Tr} Q = \int_{\widetilde{X}} \operatorname{tr} \mathcal{K}(x, x, Q)\, dx$, where tr denotes the trace of the endomorphism in the vector bundle fibre), we set

$$\operatorname{Tr}_+(Q) = \operatorname{Tr}(Q_+) = \int_X \operatorname{tr} \mathcal{K}(x, x, Q) dx. \tag{2.13}$$

Likewise, we set

$$\zeta_+(D, Q, s) = \operatorname{Tr}_+(DZ(Q, s)). \tag{2.14}$$

The operators D in the modified zeta and eta functions in (2.4) and (2.6) will be differential operators (for suitable choices of i, j)

$$D_{ij} : C^\infty(E_j) \to C^\infty(E_i) \quad (i, j = 1, 2), \tag{2.15}$$

that on $X' \times [0, c]$ act like tangential operators D'_{ij} on X', constant in x_n. (Concerning non-tangential choices, see Remark 4.3 below.)

In stating our main results, we use the notation (explained in more detail in (3.14) ff.)

$$F_t(s) = \frac{\Gamma(s+t)}{\Gamma(t)\Gamma(s+1)}. \tag{2.16}$$

Theorem 2.1. *The generalized zeta functions $\zeta(D_{ii}, \Delta_i, s)$ decompose as follows:*

$$\begin{aligned} &\Gamma(s)\zeta(D_{ii}, \Delta_i, s) = \\ &\quad \Gamma(s)[\zeta_+(D_{ii}, \widetilde{\Delta}_i, s) + \tfrac14 (F_{\frac12}(s) - 1)\zeta(D''_{ii}, A^2, s) + (-1)^i \tfrac14 F_{\frac12}(s)\eta(D''_{ii}, A, 2s)] \\ &\qquad + \tfrac1s [\operatorname{Tr}_+(D_{ii}\Pi_0(\widetilde{\Delta}_i)) - \operatorname{Tr}(D_{ii}\Pi_0(\Delta_i)) + (-1)^i \tfrac14 \operatorname{Tr}(D''_{ii}\Pi_0(A))] + h_i(s); \end{aligned} \tag{2.17}$$

with $D''_{11} = D'_{11}$, $D''_{22} = \sigma^* D'_{22}\sigma$, and $h_i(s)$ entire. In particular, the zeta functions satisfy:

$$\begin{aligned} \Gamma(s)\zeta(\Delta_i, s) = \Gamma(s)[\zeta_+(\widetilde{\Delta}_i, s) + \tfrac14 (F_{\frac12}(s) - 1)\zeta(A^2, s) + (-1)^i \tfrac14 F_{\frac12}(s)\eta(A, 2s)] \\ + \tfrac1s [\operatorname{Tr}_+(\Pi_0(\widetilde{\Delta}_i)) - \nu_0(\Delta_i) + (-1)^i \tfrac14 \nu_0(A)] + h_i(s). \end{aligned} \tag{2.18}$$

Theorem 2.2. *The generalized eta functions decompose as follows:*

$$\begin{aligned} \Gamma(s) \operatorname{Tr}(D_{12} P \Delta_1^{-s}) &= \Gamma(s)\big[\operatorname{Tr}_+(D_{12} P \widetilde{\Delta}_1^{-s}) + \tfrac14 (F_{\frac12}(s-1) - 1)\eta(D'_{12}\sigma, A, 2s-1)\big] \\ &\quad + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(D'_{12}\sigma\Pi_0(A))(s - \tfrac12)^{-1} + h_1(s), \\ \Gamma(s) \operatorname{Tr}(D_{21} P^* \Delta_2^{-s}) &= \Gamma(s)\big[\operatorname{Tr}_+(D_{21} P^* \widetilde{\Delta}_2^{-s}) + \tfrac14 (F_{\frac12}(s-1) - 1)\eta(\sigma^* D'_{21}, A, 2s-1)\big] \\ &\quad + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\sigma^* D'_{21}\Pi_0(A))(s - \tfrac12)^{-1} + h_2(s), \end{aligned} \tag{2.19}$$

where $h_i(s)$ is entire.

In these theorems, the zeta and eta functions on the right hand side have meromorphic extensions to $\mathbb{C}$; see Lemma 2.5 below. Hence, so do those on the left.

The index theorem in [APS] can be deduced from Theorem 2.1, as follows.

Corollary 2.3. *The index of $P_\geq$ is given by*

$$\operatorname{ind}(P_\geq) = \int_X \alpha_0 - \tfrac{1}{2}\eta(A),$$

where α_0 is the standard index form for P as an operator on a manifold without boundary, and $\eta(A)$ is the eta-invariant

$$\eta(A) = \eta(A,0) + \nu_0(A).$$

Proof. The nonzero eigenvalues of $\Delta_1 = P_\geq{}^* P_\geq$ and $\Delta_2 = P_\geq P_\geq{}^*$ coincide, so $\zeta(\Delta_1, s) = \zeta(\Delta_2, s)$. We use (2.18) to express these zeta functions. The other zeta functions appearing there are regular at 0, and the eta function appears with opposite sign on each side of our equality, so the eta function is also regular at $s = 0$. Since $F_{\frac{1}{2}}(0) = 1$, we find

$$\begin{aligned}\zeta_+(\widetilde{\Delta}_1, 0) + \operatorname{Tr}_+(\Pi_0(\widetilde{\Delta}_1)) - \tfrac{1}{4}\eta(A,0) - \tfrac{1}{4}\nu_0(A) - \nu_0(\Delta_1)&\\ = \zeta_+(\widetilde{\Delta}_2, 0) + \operatorname{Tr}_+(\Pi_0(\widetilde{\Delta}_2)) + \tfrac{1}{4}\eta(A,0) + \tfrac{1}{4}\nu_0(A) - \nu_0(\Delta_2).&\end{aligned}$$

So

$$\begin{aligned}\operatorname{ind}(P_\geq) &= \nu_0(\Delta_1) - \nu_0(\Delta_2)\\ &= \Big[\zeta_+(\widetilde{\Delta}_1, 0) + \operatorname{Tr}_+ \Pi_0(\widetilde{\Delta}_1) - \zeta_+(\widetilde{\Delta}_2, 0) - \operatorname{Tr}_+ \Pi_0(\widetilde{\Delta}_2)\Big] - \tfrac{1}{2}\left[\eta(A,0) + \nu_0(A)\right].\end{aligned} \tag{2.20}$$

The first sum in brackets is the standard index form for $\widetilde{P}$ on $\widetilde{X}$ (see below), but integrated just over X. This proves the Corollary.

To identify the bracketed term in (2.20), we use the well-known Lemma 2.5 below. In fact, by (2.23), the term is the integral over X of the fibre trace of

$$\begin{aligned}\mathcal{K}(x,x,Z(\widetilde{\Delta}_1,0)) + \mathcal{K}_0(x,x,\widetilde{\Delta}_1) - \mathcal{K}(x,x,Z(\widetilde{\Delta}_2,0)) - \mathcal{K}_0(x,x,\widetilde{\Delta}_2)&\\ = c_n(x,\widetilde{\Delta}_1) - c_n(x,\widetilde{\Delta}_2).&\end{aligned}$$

The integral over $\widetilde{X}$ of this trace is

$$\begin{aligned}\int_{\widetilde{X}} \operatorname{tr}[c_n(x,\widetilde{\Delta}_1) - c_n(x,\widetilde{\Delta}_2)]\,dx &= \zeta(\widetilde{\Delta}_1) + \operatorname{Tr}\Pi_0(\widetilde{\Delta}_1) - \zeta(\widetilde{\Delta}_2) - \operatorname{Tr}\Pi_0(\widetilde{\Delta}_2)\\ &= \operatorname{Tr}\Pi_0(\widetilde{\Delta}_1) - \operatorname{Tr}\Pi_0(\widetilde{\Delta}_2) = \operatorname{ind}(\widetilde{P}).\end{aligned}$$

Thus $\operatorname{tr}[c_n(x,\widetilde{\Delta}_1) - c_n(x,\widetilde{\Delta}_2)]dx$ is the standard index form for $\widetilde{P}$ on $\widetilde{X}$ (which has been calculated explicitly when $\widetilde{P}$ is a twisted Dirac operator, cf. e.g. [APS]). $\square$

Note that this proof, like that in [APS], shows that $\eta(A,s)$ is regular at $s = 0$, since the existence of $\lim_{s\to 0}\eta(A,s)$ is assured by the convergence of the other terms.

When $E_1 = E_2$, Theorem 2.2 gives the behavior of the eta functions

$$\eta(P_\geq, 2s) = \operatorname{Tr}(P\Delta_1^{-s-\frac{1}{2}}) \ \text{ and } \ \eta(P_\geq{}^*, 2s) = \operatorname{Tr}(P^*\Delta_2^{-s-\frac{1}{2}}).$$

Douglas and Wojciechowski [DW, Th. 4.3] showed, in the case of generalized Dirac operators with $\dim X$ odd, $\nu_0(A) = 0$, that $\eta(P_\geq, 2s)$ is meromorphic with the same singularity structure as $\operatorname{Tr}_+(P\widetilde{\Delta}_1^{-s})$. This also follows from:

Corollary 2.4. 1° *Suppose that* $E_1 = E_2$ *and, in* (2.8), $\sigma^* = -\sigma$, $\sigma A = -A\sigma$. *Then* $\eta(A,s) \equiv 0$, *and*

$$\begin{aligned}\Gamma(s+\tfrac12)\eta(P_{\geq},2s) &= \Gamma(s+\tfrac12)\operatorname{Tr}_+(P\widetilde{\Delta}_1^{-s}) + \tfrac{1}{4\sqrt{\pi}\,s}\operatorname{Tr}(\sigma\Pi_0(A)) + h_1(s),\\ \Gamma(s+\tfrac12)\eta(P_{\geq}{}^*,2s) &= \Gamma(s+\tfrac12)\operatorname{Tr}_+(P^*\widetilde{\Delta}_2^{-s}) - \tfrac{1}{4\sqrt{\pi}\,s}\operatorname{Tr}(\sigma\Pi_0(A)) + h_2(s),\end{aligned} \tag{2.22}$$

with h_i *entire.*

2° *Moreover, when* P *is a generalized Dirac operator, the residue of* $\eta(P_{\geq},2s)$ *at* $s=0$ *equals* $\frac{1}{4\pi}\operatorname{Tr}(\sigma\Pi_0(A))$ *and the residue of* $\eta(P_{\geq}{}^*,2s)$ *at* $s=0$ *equals* $\frac{-1}{4\pi}\operatorname{Tr}(\sigma\Pi_0(A))$. *In the situation of [DW, Theorem A.1], these residues are zero.*

Proof. Since $\sigma A = -A\sigma$,

$$\operatorname{Tr}(\sigma A|A|^{-2s-1}) = -\operatorname{Tr}(A\sigma|A|^{-2s-1}) = -\operatorname{Tr}(\sigma|A|^{-2s-1}A) = -\operatorname{Tr}(\sigma A|A|^{-2s-1}),$$

using a circular permutation, so $\eta(A,s)\equiv 0$. Then (2.22) follows from Theorem 2.2.

For generalized Dirac operators, $\operatorname{Tr}(P\widetilde{\Delta}_i^{-s})$ is regular at $s=0$ by Bismut and Freed [BF] (or see Branson and Gilkey [BG, Theorem 3.4]). Dividing by $\Gamma(s+\frac12)$, we see that the residue of $\eta(P_{\geq},2s)$ at $s=0$ is indeed $\frac{1}{4\pi}\operatorname{Tr}(\sigma\Pi_0(A))$. Similarly, the residue of $\eta(P_{\geq}{}^*,2s)$ at $s=0$ is $\frac{1}{4\pi}\operatorname{Tr}(\sigma^*\Pi_0(A)) = \frac{-1}{4\pi}\operatorname{Tr}(\sigma\Pi_0(A))$, since $\sigma^*=-\sigma$.

In the situation of [DW, Th. A.1], $\ker A = V_0(A)$ can be decomposed into an orthogonal direct sum $V_{0,+}\oplus V_{0,-}$ with $\sigma\colon V_{0,+}\xrightarrow{\sim} V_{0,-}$. Then $\operatorname{Tr}(\sigma\Pi_0(A))=0$. □

Generalizations of these results to the realizations P_B (cf. (2.11)) are given at the end of Sections 3 and 4.

For Dirac operators on manifolds X of odd dimension, there also exist local elliptic boundary conditions (cf. e.g. Singer [Si]), where the singularity structure of the zeta and eta functions is found as in Gilkey and Smith [GiS].

2.2 Description of the singularities.

In the case where D_{ij} is just a morphism φ we will list explicitly all the singularities of $\Gamma(s)\zeta(\varphi,\Delta_i,s)$, leaving other cases to the interested reader. For this, we need the following well-known result. It will be used with $n_1=n$, $X_1=\widetilde{X}$, $Q=\widetilde{\Delta}_i$, and with $n_1=n-1$, $X_1=X'$, $Q=A^2$.

Lemma 2.5. *Let* Q *be a second order elliptic differential operator in a hermitian* C^∞ *vector bundle* E *over a compact* n_1*-dimensional manifold* X_1 *without boundary, such that* Q *is selfadjoint* ≥ 0 *in* $L_2(X_1,E)$ *with respect to some smooth measure on* X_1. *Let* D *be a differential operator in* E *of order* $d\geq 0$.

Denote the orthogonal projection onto the nullspace $V_0(Q)$ *of* Q *by* $\Pi_0(Q)$; *it is the integral operator with* C^∞ *kernel* $\mathcal{K}_0(x,y,Q)=\sum_{1\leq l\leq\nu_0}u_l(x)\otimes\overline{u}_l(y)$, *where the* u_l *are a smooth orthonormal basis of* V_0. *Denote* $\sum_{1\leq l\leq\nu_0}(Du_l(x))\otimes\overline{u}_l(y) = D_{(x)}\mathcal{K}_0(x,y,Q)$.

For $\operatorname{Re} s > \frac{n_1}{2}+d$, $DZ(Q,s)$ *is trace class and has a continuous kernel* $\mathcal{K}(x,y,DZ(Q,s))$. *The kernel at* $x=y$, *and the trace of* $DZ(Q,s)$ *(also denoted* $\zeta(D,Q,s)$*), extend mero-*

morphically to $\mathbb{C}$, as follows:

$$\begin{aligned}\Gamma(s)\mathcal{K}(x,x,DZ(Q,s)) + \frac{D_{(x)}\mathcal{K}_0(x,x,Q)}{s} &\sim \sum_{j=0}^{\infty} \frac{c_j(x,D,Q)}{s+\frac{j-n_1-d}{2}},\\ \Gamma(s)\operatorname{Tr}(DZ(Q,s)) + \frac{\operatorname{Tr}(D\Pi_0(Q))}{s} &\sim \sum_{j=0}^{\infty} \frac{c_j(D,Q)}{s+\frac{j-n_1-d}{2}},\end{aligned} \tag{2.23}$$

with $c_j(D,Q) = \int_{X_1} \operatorname{tr} c_j(x,D,Q)\,dx$, where $\sim$ means that the left hand side minus the sum for $j \le N$ in the right hand side is holomorphic for $\operatorname{Re} s > \frac{n_1+d-N-1}{2}$, any N. The coefficients $c_j(x,D,Q)$ are C^∞ sections of $\operatorname{Hom}(E,E)$, determined by differential operators in the symbols of Q and D in local coordinates, and the hermitian metric in E and the smooth measure on X_1. For $j+d$ odd, $c_j(x,D,Q)=0$.

The functions $\Gamma(s)\mathcal{K}(x,x,DZ(Q,s))$ and $\Gamma(s)\operatorname{Tr}(DZ(Q,s))$ are $O(e^{-\delta|\operatorname{Im} s|})$, any $\delta < \frac{\pi}{2}$, for $|\operatorname{Im} s| \ge 1$, $C_1 \le \operatorname{Re} s \le C_2$ (any real C_1 and C_2).

Remark 2.6. When Q is a *pseudodifferential operator*, there is a related result, where one must however include double poles at the negative integers $-k$ for $\Gamma(s)$ times the zeta function, as pointed out in Duistermaat and Guillemin [DG]. This means that the zeta function itself can have nonvanishing simple poles at the negative integers. Only for *differential operators* can one be sure that these poles vanish. In [S], the restriction to differential operators for this property is left out of the summary, bottom of page 290.

At a certain point we shall need the next coefficient in the Laurent series for $\Gamma(s)\zeta(D,Q,s)$ at a pole (it is generally not locally determined). We denote it by c'_j:

$$c'_j(D,Q) = \lim_{s\to\frac{-j+n_1+d}{2}} \Big[\Gamma(s)\zeta(D,Q,s) - \frac{c_j(D,Q)}{s+\frac{j-n_1-d}{2}}\Big]. \tag{2.24}$$

Now, to work out the singularities of $\Gamma(s)\zeta(\varphi,\Delta_1,s)$, we need the constants

$$\begin{aligned}\beta_m &= [\text{residue of } \tfrac14 F_{\frac12}(s) \text{ at } s=-\tfrac12-m] = \frac{(-1)^m}{4m!\sqrt{\pi}\,\Gamma(\frac12-m)},\\ \gamma_k &= \tfrac14(F_{\frac12}(\tfrac{k}{2})-1) = \tfrac14\Big[\frac{\Gamma(\frac{k+1}{2})}{\sqrt{\pi}\,\Gamma(1+\frac{k}{2})}-1\Big],\\ \varepsilon_m &= [\text{residue of } \tfrac14 F_{\frac12}(s)\Gamma(s) \text{ at } s=-\tfrac12-m] = \frac{(-1)^{m+1}}{4m!\sqrt{\pi}\,(m+\frac12)};\end{aligned} \tag{2.25}$$

here $m=0,1,2,\dots$, and the k are integers avoiding negative odd numbers. Define moreover β'_m as the residue of $\frac14 F_{\frac12}(s)(s+\frac12+m)^{-1}$ at $s=-\frac12-m$.

Let φ be a C^∞ morphism in E_1 that equals $\varphi^0 := \varphi|_{X'}$ on $X'\times[0,c]$.

From Lemma 2.5 we have, omitting vanishing coefficients,

$$\Gamma(s)\zeta_+(\varphi,\widetilde\Delta_1,s) \sim \sum_{k=0}^{\infty} \frac{c_{2k,+}(\varphi,\widetilde\Delta_1)}{s+k-\frac{n}{2}} - \frac{\operatorname{Tr}_+\big(\varphi\Pi_0(\widetilde\Delta_1)\big)}{s},$$

where $c_{j,+}(\varphi,\widetilde{\Delta}_1)=\int_X \operatorname{tr} c_j(x,\varphi,\widetilde{\Delta}_1)\,dx$. Likewise, since A acts on X' of dimension $n-1$,

$$\Gamma(s)\zeta(\varphi^0,A^2,s)\sim\sum_{k=0}^{\infty}\frac{c_{2k}(\varphi^0,A^2)}{s+k-\frac{n-1}{2}}-\frac{\operatorname{Tr}\big(\varphi^0\Pi_0(A)\big)}{s},$$

and

$$\begin{aligned}\Gamma(s)F_{\frac12}(s)\eta(\varphi^0,A,2s)&=\frac{1}{\sqrt{\pi}\,s}\Gamma(s+\tfrac12)\zeta(\varphi^0A,A^2,s+\tfrac12)\\ &\sim\sum_{0\le k}\frac{c_{2k+1}(\varphi^0A,A^2)}{\sqrt{\pi}\,(\frac n2-k-1)(s+k+1-\frac n2)}+\frac{\eta(\varphi^0,A,0)}{s}\quad\text{if } n \text{ is odd},\\ &\sim\sum_{0\le k\ne\frac n2-1}\frac{c_{2k+1}(\varphi^0A,A^2)}{\sqrt{\pi}\,(\frac n2-k-1)(s+k+1-\frac n2)}\\ &\qquad+\frac{c_{n-1}(\varphi^0A,A^2)}{\sqrt{\pi}\,s^2}+\frac{c'_{n-1}(\varphi^0A,A^2)}{\sqrt{\pi}\,s}\quad\text{if } n \text{ is even},\end{aligned}\tag{2.26}$$

where $c'_{n-1}(\varphi^0,A^2)$ is defined as in (2.24). When $\varphi^0=1$ then $\eta(A,2s)$ is regular at $s=0$ (Corollary 2.3), i.e. $c_{n-1}(A,A^2)=0$ and $c'_{n-1}(A,A^2)=\sqrt{\pi}\,\eta(A,0)$.

We use these expansions in Theorem 2.1 with $i=1$, to find:

Corollary 2.7. When n is even, *the singularities of $\Gamma(s)\zeta(\varphi,\Delta_1,s)$ consist of the following four sums:*

(i) *From* $\Gamma(s)\zeta_+(\varphi,\widetilde{\Delta}_1,s)+\frac1s\operatorname{Tr}_+(\varphi\Pi_0(\widetilde{\Delta}_1))$,

$$\sum_{k\ge0}\frac{c_{2k,+}(\varphi,\widetilde{\Delta}_1)}{s+k-\frac n2}.$$

(ii) *From* $\frac14(F_{\frac12}(s)-1)\Gamma(s)\zeta(\varphi^0,A^2,s)$, *noting that* $F_{\frac12}(0)=1$,

$$\sum_{0\le k<\frac n2}\frac{\gamma_{n-1-2k}c_{2k}(\varphi^0,A^2)}{s+k-\frac{n-1}{2}}+\sum_{k\ge\frac n2}\Big[\frac{\beta_{k-\frac n2}c_{2k}(\varphi^0,A^2)}{(s+k-\frac{n-1}{2})^2}+\frac{\beta_{k-\frac n2}c'_{2k}(\varphi^0,A^2)+(\beta'_{k-\frac n2}-\frac14)c_{2k}(\varphi^0,A^2)}{s+k-\frac{n-1}{2}}\Big].$$

(iii) *From* $-\frac14F_{\frac12}(s)\Gamma(s)\eta(\varphi^0,A,2s)$,

$$-\frac14\sum_{0\le k\ne\frac n2-1}\frac{c_{2k+1}(\varphi^0A,A^2)}{\sqrt{\pi}\,(\frac n2-k-1)(s+k+1-\frac n2)}-\frac{c_{n-1}(\varphi^0A,A^2)}{4\sqrt{\pi}\,s^2}-\frac{c'_{n-1}(\varphi^0A,A^2)}{4\sqrt{\pi}\,s}.$$

(iv) *From the remaining terms in* $\frac1s$,

$$-\frac1s\big[\operatorname{Tr}(\varphi\Pi_0(\Delta_1))+\tfrac14\operatorname{Tr}(\varphi^0\Pi_0(A))\big].$$

The poles of $F_{\frac12}(s)$ at the negative half-integers here give rise to double poles since they coincide with poles of $\Gamma(s)\zeta(\varphi^0,A^2,s)$.

Corollary 2.8. When n is odd, *the singularities of* $\Gamma(s)\zeta(\varphi,\Delta_1,s)$ *consist of the following four sums:*

(i) *From* $\Gamma(s)\zeta_+(\varphi,\widetilde{\Delta}_1,s)+\frac{1}{s}\operatorname{Tr}_+(\varphi\Pi_0(\widetilde{\Delta}_1))$,

$$\sum_{k\geq 0}\frac{c_{2k,+}(\varphi,\widetilde{\Delta}_1)}{s+k-\frac{n}{2}}.$$

(ii) *From* $\frac14(F_{\frac12}(s)-1)\Gamma(s)\zeta(\varphi^0,A^2,s)$,

$$\sum_{k\geq 0}\frac{\gamma_{n-1-2k}c_{2k}(\varphi^0,A^2)}{s+k-\frac{n-1}{2}}+\sum_{m\geq 0}\frac{\varepsilon_m\zeta(\varphi^0,A^2,-m-\frac12)}{s+m+\frac12}.$$

(iii) *From* $-\frac14 F_{\frac12}(s)\Gamma(s)\eta(\varphi^0,A,2s)$,

$$-\tfrac14\sum_{k\geq 0}\frac{c_{2k+1}(\varphi^0 A,A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)(s+k+1-\frac{n}{2})}-\frac{\eta(\varphi^0,A,0)}{4s}.$$

(iv) *From the remaining terms in* $\frac1s$,

$$-\tfrac1s\big[\operatorname{Tr}(\varphi\Pi_0(\Delta_1))+\tfrac14\operatorname{Tr}(\varphi^0\Pi_0(A))\big].$$

The poles of $F_{\frac12}(s)$ at the negative half-integers here give rise to simple poles, picking up the values $\zeta(\varphi^0,A^2,-m-\frac12)$ of $\zeta(\varphi^0,A^2,s)$ between its poles.

There are similar formulas for Δ_2, with φ^0 replaced by $\sigma^*\varphi^0\sigma$ and a change of sign in the contributions from the eta function and the nullspace of A.

Note that the coefficients c_j and $c_{j,+}$ are determined by the symbols of the operators, whereas the coefficients c'_j and $\zeta(\varphi^0,A^2,-m-\frac12)$ cannot be expected to be so.

One can likewise describe the pole structure of the eta functions of Δ_i determined in Theorem 2.2; see Section 4.

We conclude this section with a result used in the proofs of Theorems 2.1 and 2.2, relating resolvents and zeta functions. This same proposition can be used to deduce Lemma 2.5 from the resolvent expansion in [S] or [Sh].

Proposition 2.9. *Suppose that f is meromorphic at* 0 *with Laurent expansion*

$$f(\lambda)=\sum_{-k}^{\infty}b_j(-\lambda)^j,\ |\lambda|\leq\rho,$$

that f is holomorphic in the open sector $S_{\delta_0}=\{\,\lambda\in\mathbb{C}\mid|\arg\lambda-\pi|<\delta_0\,\}$ *(for some* $\delta_0\leq\pi$*), and* $f(\lambda)=O(|\lambda|^{-\alpha})$ *for some* $\alpha\in]0,1]$ *as* $\lambda\to\infty$*, uniformly in each sector* S_δ *for* $\delta<\delta_0$*. Let* $\mathcal{C}$ *be a curve* $\mathcal{C}_{\pi,r_0}$ *as in (2.2) (a Laurent loop), with* $0<r_0<\varrho$*. Set* $f_0(\lambda)=f(\lambda)-\sum_{-k}^{-1}b_j(-\lambda)^j$*, and*

$$\zeta(s)=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}f(\lambda)\,d\lambda,\quad \operatorname{Re}s>1-\alpha, \tag{2.27}$$

with $\lambda^{-s} = r^{-s}e^{-is\theta}$, $r > 0$ *and* $|\theta| \leq \pi$. *Then*

$$\zeta(s) = \frac{\sin \pi s}{\pi} \int_0^\infty r^{-s} f_0(-r)\, dr, \quad 1-\alpha < \operatorname{Re} s < 1, \text{ and} \tag{2.28}$$

$$f_0(-\lambda) = \tfrac{1}{2\mathrm{i}} \int_{\operatorname{Re} s = \sigma} \lambda^{s-1} \frac{\zeta(s)}{\sin \pi s}\, ds, \quad 1-\alpha < \sigma < 1. \tag{2.29}$$

The function $\frac{\pi\zeta(s)}{\sin \pi s}$ *is meromorphic for* $\operatorname{Re} s > 1-\alpha$, *having simple poles at* $s = j+1$ *with residues* $(-1)^{j+1}\zeta(j+1) = -b_j$, $j = 0, 1, 2, \dots$.

Moreover, the following properties a) and b) are equivalent:

a) f *has an asymptotic expansion as* λ *goes to infinity*

$$f(-\lambda) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} a_{j,l} \lambda^{-\alpha_j} (\log \lambda)^l, \quad 0 < \alpha_j \nearrow +\infty, \ m_j \in \{0, 1, 2, \dots\}, \tag{2.30}$$

uniformly for $-\lambda$ *in* S_δ, *for each* $\delta < \delta_0$.

b) $\dfrac{\pi\zeta(s)}{\sin \pi s}$ *is meromorphic on* $\mathbb{C}$ *with the singularity structure*

$$\frac{\pi\zeta(s)}{\sin \pi s} \sim -\sum_{j=-k}^{\infty} \frac{b_j}{s-j-1} + \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} \frac{a_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}} \tag{2.31}$$

(in the sense that for large N, *the left hand side minus the sums for* $j \leq N$ *in the right hand side is holomorphic for* $1-\alpha_N < \operatorname{Re} s < N+1$*); and for each real* C_1, C_2 *and each* $\delta < \delta_0$,

$$\left|\frac{\zeta(s)}{\sin \pi s}\right| \leq C(C_1, C_2, \delta) e^{-\delta|\operatorname{Im} s|}, \text{ for } |\operatorname{Im} s| \geq 1, \ C_1 \leq \operatorname{Re} s \leq C_2. \tag{2.32}$$

Thus the singularities of $\varphi(s) = \frac{\pi\zeta(s)}{\sin \pi s}$ *in* $\operatorname{Re} s < 1$ *are determined by the expansion (2.30) and the singular Laurent terms of* $f(\lambda)$ *at* $\lambda = 0$. *In particular, the coefficient of* $\lambda^{-\alpha_j}$ *in (2.30) is the residue of* $\frac{\pi\zeta(s)}{\sin \pi s}$ *at* $s = 1-\alpha_j$ *plus, if* α_j *is integer, the coefficient of* $\lambda^{-\alpha_j}$ *in the Laurent expansion of* $f(-\lambda)$ *at* $\lambda = 0$. *The coefficient of* $\lambda^{-\alpha_j}(\log \lambda)$ *is the coefficient of* $(s-(1-\alpha_j))^{-2}$ *in the Laurent expansion of* $\frac{-\pi\zeta(s)}{\sin \pi s}$ *at* $s = 1-\alpha_j$.

Proof. For $j \leq -1$ and $\operatorname{Re} s > 0$, $\int_{\mathcal{C}} \lambda^{j-s} d\lambda = 0$, since the contour can be closed at ∞ in $\{|\arg \lambda| < \pi\}$. So the singular part of f, $\sum_{-k}^{-1} b_j(-\lambda)^j$, is "killed" by the integral over $\mathcal{C}$ in (2.27). For the remaining part f_0, the circular part of $\mathcal{C}$ can be reduced to the origin if $\operatorname{Re} s < 1$, reducing (2.27) to (2.28) (note that f_0 is $O(|\lambda|^{-\alpha})$ too).

The inversion (2.29) requires growth estimates for $\zeta(s)$. Replacing the integration curve by $\mathcal{C}(\delta) := \mathcal{C}_{\pi-\delta,0}$, $0 < \delta < \delta_0$, we have that

$$|\zeta(s)| = \left|\tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}(\delta)} \lambda^{-s} f_0(\lambda)\, d\lambda\right| = O(e^{(\pi-\delta)|\operatorname{Im} s|}), \ 1-\alpha < C_1 \leq \operatorname{Re} s \leq C_2 < 1. \tag{2.33}$$

For, when $\lambda = re^{\mathrm{i}(\pi-\delta)}$, we can use the estimate

$$\Big|\int_0^\infty r^{-s} e^{\mathrm{i}(\pi-\delta)(1-s)} O((1+r)^{-\alpha})\, dr\Big| \le Ce^{(\pi-\delta)|\operatorname{Im} s|}, \tag{2.34}$$

and there is a similar estimate on the other half of $\mathcal{C}(\delta)$.

Now let

$$\varphi(s) = \int_0^\infty r^{-s} f_0(-r)\, dr = \frac{\pi\zeta(s)}{\sin \pi s}. \tag{2.35}$$

Since $(\sin \pi s)^{-1}$ is $O(e^{-\pi|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \ge 1$, we have by (2.33) that $\varphi(\sigma+i\tau) = O(e^{-\delta|\tau|})$ for $1-\alpha < C_1 \le \sigma \le C_2 < 1$. Also, $\varphi(\sigma+i\tau)$ is the Fourier transform $\hat F(\tau)$ of the function $F(x) = e^{(1-\sigma)x} f_0(-e^x)$. Since $f_0(\lambda) = O(|\lambda|^{-\alpha})$, $F(x)$ decays exponentially as $x \to \pm\infty$, for $1-\alpha < \sigma < 1$. By Fourier inversion, $F(x) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ix\tau}\varphi(\sigma+i\tau)d\tau$, giving (2.29), for $\lambda > 0$. It extends to $|\arg\lambda| < \delta_0$ by analytic continuation.

It is seen from (2.27) that $\zeta(s)$ is holomorphic for $\operatorname{Re} s > 1-\alpha$; and since $\zeta(j+1) = \frac{-i}{2\pi}\int_{|\lambda|=r_0} \lambda^{-j-1} f(\lambda)\, d\lambda = (-1)^j b_j$, $j = 0, 1, 2, \dots$, $\frac{\pi\zeta(s)}{\sin \pi s}$ is meromorphic for $\operatorname{Re} s > 1-\alpha$, having simple poles with residues $-b_j$.

Now suppose that a) holds; then

$$f_0(-\lambda) = \sum_{j=0}^{N-1}\sum_{l=0}^{m_j} a_{j,l}\lambda^{-\alpha_j}(\log\lambda)^l - \sum_{j=-k}^{-1} b_j\lambda^j + O(|\lambda|^{-\alpha_N+\varepsilon}) \text{ for } \lambda \to \infty, \tag{2.36}$$

for $\alpha_N \ge k$, any $\varepsilon > 0$. Note that

$$\int_0^1 r^{j-s}\, dr = \frac{-1}{s-j-1} \quad \text{for } \operatorname{Re} s < j+1,$$
$$\int_1^\infty r^{\beta-s}(\log r)^l\, dr = \frac{l!}{(s-\beta-1)^{l+1}} \quad \text{for } \operatorname{Re} s > \beta+1$$

(the cases $l > 0$ follow from the case $l = 0$ by application of ∂_s^l); the right hand sides extend meromorphically to $\mathbb{C}$. Then we get from (2.35), for arbitrarily large N:

$$\begin{aligned}
\varphi(s) &= \int_0^1 \Big[\sum_{j=0}^{N-1} b_j r^{j-s} + r^{-s}O(r^N)\Big]dr \\
&\quad + \int_1^\infty \Big[\sum_{j=0}^{N-1}\sum_{l=0}^{m_j} a_{j,l} r^{-\alpha_j-s}(\log r)^l - \sum_{j=-k}^{-1} b_j r^{j-s} + r^{-s}O(r^{-\alpha_N+\varepsilon})\Big]dr \\
&= -\sum_{j=-k}^{N-1} \frac{b_j}{s-j-1} + \sum_{j=0}^{N-1}\sum_{l=0}^{m_j} \frac{a_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}} + h_N(s)
\end{aligned}$$

where h_N is holomorphic for $1-\alpha_N+\varepsilon < \operatorname{Re} s < N+1$, and the other terms are meromorphic on $\mathbb{C}$. This gives the singularities (2.31).

To show the decay, we use the integral in (2.33) and expand on each piece of $\mathcal{C}(\delta)$:

$$\begin{aligned}\zeta(s) &= -\tfrac{\mathrm{i}}{2\pi}\Big(\int_0^1 + \int_1^\infty (re^{\mathrm{i}(\pi-\delta)})^{-s} f_0(re^{\mathrm{i}(\pi-\delta)})e^{\mathrm{i}(\pi-\delta)}\,dr\Big)\\ &\quad + \tfrac{\mathrm{i}}{2\pi}\Big(\int_0^1 + \int_1^\infty (re^{\mathrm{i}(-\pi+\delta)})^{-s} f_0(re^{\mathrm{i}(-\pi+\delta)})e^{\mathrm{i}(-\pi+\delta)}\,dr\Big).\end{aligned} \tag{2.37}$$

The first integral from 0 to 1 is written as

$$\begin{aligned}&\tfrac{-\mathrm{i}}{2\pi}\int_0^1 (re^{\mathrm{i}(\pi-\delta)})^{-s} f_0(re^{\mathrm{i}(\pi-\delta)})e^{\mathrm{i}(\pi-\delta)}\,dr\\ &\qquad = \tfrac{-\mathrm{i}}{2\pi}\int_0^1 \sum_{j=0}^{N-1} e^{\mathrm{i}(j+1-s)(\pi-\delta)} b_j r^{j-s}\,dr + \int_0^1 r^{-s} e^{\mathrm{i}(\pi-\delta)(1-s)} O(r^N)\,dr\\ &\qquad = \sum_{j=0}^{N-1} \frac{-e^{\mathrm{i}(j+1-s)(\pi-\delta)} b_j}{j+1-s} + e^{\mathrm{i}(\pi-\delta)(1-s)}\int_0^1 r^{-s} O(r^N)\,dr.\end{aligned}$$

Let $|\operatorname{Im} s| \geq 1$. The sum over j extends meromorphically to $\mathbb{C}$, and its terms are $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ for $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$. The last term exists and is $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ when $\operatorname{Re} s < N+1$. Similar considerations hold for the other integral from 0 to 1. In the integrals from 1 to ∞ we expand as in (2.36), obtaining functions that are $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ for $\operatorname{Re} s > 1-\alpha_N+\varepsilon$. We conclude that the estimate in (2.33) extends to $1-\alpha_N < \operatorname{Re} s < N+1$, $|\operatorname{Im} s| \geq 1$, for arbitrarily large N. Dividing by $\sin \pi s$ we find that $\varphi(s)$ satisfies (2.32). This shows a) $\Longrightarrow$ b).

Conversely, assume b). Then $f_0(-\lambda)$ is given by (2.29), and we obtain the expansion (2.30) by shifting the contour of integration past the poles of $\zeta(s)/\sin \pi s$. The remainder after all terms up to the singularity $s = 1 - \alpha_N$ is given by the integral (2.29) but with $\sigma < 1 - \alpha_N$; it is $O(|\lambda|^{-\alpha_N+\varepsilon})$ on S_δ. □

Corollary 2.10. *When $f(\lambda)$ and $\zeta(s)$ are as in Proposition 2.9, then $\Gamma(s)\zeta(s)$ is meromorphic on $\mathbb{C}$ with the singularity structure*

$$\Gamma(s)\zeta(s) \sim \sum_{j=-k}^{j=-1} \frac{-\tilde{b}_j}{s-j-1} + \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} \frac{\tilde{a}_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}},\quad \tilde{b}_j = \frac{b_j}{\Gamma(-j)},\quad \tilde{a}_{j,l} = \frac{a_{j,l}}{\Gamma(\alpha_j)}. \tag{2.38}$$

When $\delta_0 > \frac{\pi}{2}$, one has moreover, for any $\delta' < \delta_0 - \frac{\pi}{2}$, any real C_1 and C_2:

$$|\Gamma(s)\zeta(s)| \leq C'(C_1, C_2, \delta)e^{-\delta'|\operatorname{Im} s|},\quad \textit{for } |\operatorname{Im} s| \geq 1,\ C_1 \leq \operatorname{Re} s \leq C_2. \tag{2.39}$$

Proof. Since $\pi(\sin \pi s)^{-1} = \Gamma(s)\Gamma(1-s)$, (2.38) results from (2.31) by multiplication by $\Gamma(1-s)^{-1}$, whose zeros cancel the poles $b_j/(s-j-1)$, $j \geq 0$. If $\delta - \pi/2 = \delta' > 0$, the estimate $|\zeta(s)| \leq Ce^{(\pi-\delta)|\operatorname{Im} s|}$ shown in the proof of Proposition 2.9 (and assured by (2.32)) implies (2.39), since $\Gamma(s)$ is $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \geq 1$, $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$, any $\varepsilon > 0$. (Cf. e.g. the assertion in Bourbaki [B, p. 182]:

$$|\Gamma(s)| \sim \sqrt{2\pi}\, |\operatorname{Im} s|^{\operatorname{Re} s - \frac{1}{2}} e^{-\frac{\pi}{2}|\operatorname{Im} s|} \text{ for } |\operatorname{Im} s| \to \infty, \tag{2.40}$$

valid for fixed $\operatorname{Re} s$ or $\operatorname{Re} s$ in compact intervals of $\mathbb{R}$.) □

3. The zeta function expansions

3.1 Approximating the resolvent.

For $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ and A as in (1.1), set

$$A_\lambda = (A^2 - \lambda)^{1/2}, \quad A' = A + \Pi_0(A),$$

cf. (2.9) ff. As in (2.10), $P_\geq$ is the realization of P with boundary condition $\Pi_\geq(u|_{\partial X}) = 0$. Then $(P_\geq)^*$ is the realization of P^* defined by the boundary condition $\Pi_<(\sigma^* u|_{\partial X}) = 0$, $\Pi_< = I - \Pi_\geq$; and the selfadjoint operators

$$\Delta_1 = P_\geq{}^* P_\geq \text{ and } \Delta_2 = P_\geq P_\geq{}^*$$

are the realizations of P^*P resp. PP^* defined by the boundary condition

$$\begin{cases} \Pi_\geq \gamma_0 u = 0, \\ (\Pi_< \gamma_1 + A\Pi_< \gamma_0)u = 0; \end{cases} \quad \text{resp.} \quad \begin{cases} \Pi_< \gamma_0 \sigma^* u = 0, \\ (-\Pi_\geq \gamma_1 + A\Pi_\geq \gamma_0)\sigma^* u = 0; \end{cases}$$

where $\gamma_j u = \partial_n^j u|_{\partial X}$. They are elliptic (cf. e.g. [G2, Th. 2.1]), and the $\Delta_i - \lambda$ are invertible for $\lambda \in \mathbb{C} \setminus \overline{\mathbb{R}}_+$ since the Δ_i are selfadjoint ≥ 0. Denoting $\gamma u = \{\gamma_0 u, \gamma_1 u\}$ and adding the first line composed with the invertible operator A_λ to the second line (as noted in [G2, (3.8) ff.], this gives the most convenient λ-dependence of the solution operator for the inhomogeneous problem), we can also write the boundary conditions as

$$B_{1,\lambda}\gamma u = 0 \text{ resp. } B_{2,\lambda}\gamma\sigma^* u = 0,$$

where

$$\begin{aligned} B_{1,\lambda}\gamma u &= (A_\lambda \Pi_\geq + A\Pi_<)\gamma_0 u + \Pi_< \gamma_1 u, \\ B_{2,\lambda}\gamma u &= (A_\lambda \Pi_< + A\Pi_\geq)\gamma_0 u - \Pi_\geq \gamma_1 u. \end{aligned} \tag{3.1}$$

Along with the realizations, we consider the full systems describing inhomogeneous boundary problems

$$\mathcal{A}_{1,\lambda} = \begin{pmatrix} P^*P - \lambda \\ B_{1,\lambda}\gamma \end{pmatrix}, \quad \mathcal{A}_{2,\lambda} = \begin{pmatrix} PP^* - \lambda \\ B_{2,\lambda}\gamma\sigma^* \end{pmatrix}. \tag{3.2}$$

The resolvents of Δ_1 and Δ_2 enter as the first blocks in their inverses:

$$\mathcal{A}_{i,\lambda}^{-1} = \mathcal{B}_{i,\lambda} = (\, R_{i,\lambda} \quad K_{i,\lambda}\,) = (\,(\Delta_i - \lambda)^{-1} \quad K_{i,\lambda}\,); \tag{3.3}$$

here $K_{i,\lambda}$ is a Poisson operator, cf. [BM], [G1]. For the operators extended to the double $\widetilde{X}$, we denote

$$Q_{i,\lambda} = (\widetilde{\Delta}_i - \lambda)^{-1}; \tag{3.4}$$

then (cf. (2.12))

$$R_{i,\lambda} = Q_{i,\lambda,+} + G_{i,\lambda}, \tag{3.5}$$

where $G_{i,\lambda}$ is the singular Green operator (s.g.o) part adjusting the boundary values. $R_{i,\lambda}$ is holomorphic in λ outside the spectrum of Δ_i, and $Q_{i,\lambda,+}$ is holomorphic outside the spectrum of $\widetilde{\Delta}_i$, so $G_{i,\lambda}$ is holomorphic outside a discrete subset of $\overline{\mathbb{R}}_+$. The expansion of $Q_{i,\lambda,+}$ is known, and we will obtain the expansion of $G_{i,\lambda}$ by comparison to the corresponding s.g.o on the cylinder

$$X^0 = X' \times \overline{\mathbb{R}}_+ \tag{3.6}$$

(where the liftings of E_i' are denoted E_i^0).

On this cylinder, consider the operator $P^0 = \sigma(\partial_n + A)$ and its realization $P^0_\geq$, with Laplacians $\Delta_1^0 = P^{0*}_\geq P^0_\geq$ and $\Delta_2^0 = P^0_\geq P^{0*}_\geq$. We then have formulas similar to (3.2), (3.3), where the operators are provided with an upper index ^{0}; in particular, the resolvents are:

$$R^0_{i,\lambda} = Q^0_{i,\lambda,+} + G^0_{i,\lambda}, \quad Q^0_{i,\lambda} = (\widetilde{\Delta}^0_i - \lambda)^{-1},$$

where the $\widetilde{\Delta}^0_i$ act in bundles over the double $\widetilde{X}^0 = X' \times \mathbb{R}$, and subscript "+" denotes restriction to X^0 as in (2.12); note that

$$\widetilde{\Delta}^0_1 = -\partial_n^2 + A^2, \quad \widetilde{\Delta}^0_2 = -\partial_n^2 + \sigma A^2 \sigma^*, \quad \text{on } X^0.$$

We can write the s.g.o $G^0_{i,\lambda}$ explicitly in terms of the special operator

$$(G_\lambda u)(x', x_n) = \int_0^\infty e^{-(x_n+y_n)A_\lambda} u(x', y_n)\, dy_n. \tag{3.7}$$

When G is an operator defined by $Gu = \int_0^\infty \mathcal{G}(x_n, y_n) u(x', y_n)\, dy_n$, where $\mathcal{G}$ is a function of x_n, y_n valued in operators in x', we call $\mathcal{G}(x_n, y_n)$ the *normal kernel* of G, and define its *normal trace* as

$$\operatorname{tr}_n G = \int_0^\infty \mathcal{G}(x_n, x_n)\, dx_n,$$

when it exists. The normal kernel of G_λ is $e^{-(x_n+y_n)A_\lambda}$, and the normal trace is

$$\operatorname{tr}_n G_\lambda = \int_0^\infty e^{-2x_n A_\lambda}\, dx_n = (2A_\lambda)^{-1}. \tag{3.8}$$

The following formulas are derived in [G2]; they can be verified directly:

$$\begin{aligned} &G^0_{1,\lambda} = G_{\mathrm{e},\lambda} + G_{\mathrm{o},\lambda} - \tfrac{\Pi_0(A)}{2\sqrt{-\lambda}} G_\lambda, \quad G^0_{2,\lambda} = \sigma\big(G_{\mathrm{e},\lambda} - G_{\mathrm{o},\lambda} + \tfrac{\Pi_0(A)}{2\sqrt{-\lambda}} G_\lambda\big)\sigma^*, \\ &G_{\mathrm{e},\lambda} = \tfrac{-|A|}{2A_\lambda(|A|+A_\lambda)} G_\lambda = \big(\tfrac{-A^2}{2\lambda A_\lambda} + \tfrac{|A|}{2\lambda}\big) G_\lambda, \\ &G_{\mathrm{o},\lambda} = \tfrac{-1}{2(|A|+A_\lambda)} \tfrac{A}{|A'|} G_\lambda = \big(-\tfrac{A}{2\lambda} + \tfrac{A_\lambda}{2\lambda} \tfrac{A}{|A'|}\big) G_\lambda. \end{aligned} \tag{3.9}$$

In the last expressions we have used that $1/(|A|+A_\lambda) = (|A|-A_\lambda)/(A^2-(A^2-\lambda)) = |A|/\lambda - A_\lambda/\lambda$. The indices e and o refer to the evenness and oddness of the principal symbols with respect to ξ'. (The parity alternates between even and odd in the sequences of lower order symbols.) We have changed the definition of $G_{\mathrm{o},\lambda}$ slightly from [G2]; the present $G_{\mathrm{o},\lambda}$ is 0 in the nullspace $V_0(A)$, and the contribution from $V_0(A)$ to $G^0_{i,\lambda}$ is taken out as a separate term. Moreover, the conjugation by σ is now included in $G^0_{2,\lambda}$. All the operators are defined and holomorphic for $\lambda \in \mathbb{C}\setminus\overline{\mathbb{R}}_+$. From (3.8), (3.9) follow:

$$\operatorname{tr}_n G_{\mathrm{e},\lambda} = \tfrac{-A^2}{4\lambda A_\lambda^2} + \tfrac{|A|}{4A_\lambda\lambda}, \quad \operatorname{tr}_n G_{\mathrm{o},\lambda} = \tfrac{-A}{4\lambda A_\lambda} + \tfrac{1}{4\lambda}\tfrac{A}{|A'|}. \tag{3.10}$$

For our purposes, near ∂X, the true s.g.o $G_{i,\lambda}$ can be replaced by the cylindrical version $G^0_{i,\lambda}$.

Lemma 3.1. *Let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(x_n) = 1$ for $|x_n| < \frac{c}{3}$, $\chi(x_n) = 0$ for $|x_n| > \frac{2c}{3}$. Then $G_{1,\lambda} - \chi G^0_{1,\lambda}\chi$ is trace class in $L_2(X,E_1)$ with norm $O(|\lambda|^{-N})$ for $|\lambda|\to\infty$ with $\arg\lambda \in [\delta, 2\pi-\delta]$, any $\delta>0$. The same is true of $\partial_\lambda^k[G_{1,\lambda} - \chi G^0_{1,\lambda}\chi]$ for $k=1,2,\dots$, and of expressions $DG_{1,\lambda} - \chi D' G^0_{1,\lambda}\chi$, where D is a differential operator, constant in x_n near X' and equal to D' there.*

Similar estimates hold for $G_{2,\lambda} - \chi G^0_{2,\lambda}\chi$ in $L_2(X,E_2)$, and for the operators $(1-\chi)G^0_{i,\lambda}$ and $G^0_{i,\lambda} - \chi G^0_{i,\lambda}\chi$ in $L_2(X^0, E^0_i)$. Here the $G^0_{i,\lambda}$ can be replaced by $G_{\mathrm{e},\lambda}$ or $G_{\mathrm{o},\lambda}$.

All these functions are holomorphic in $\lambda\in\mathbb{C}\setminus\overline{\mathbb{R}}_+$.

Proof. That $(1-\chi)G_{\mathrm{e},\lambda}$, $(1-\chi)G_{\mathrm{o},\lambda}$ and $(1-\chi)G^0_{i,\lambda}$ are trace class with trace norm $O(|\lambda|^{-N})$, any N, follows from the fall-off of $(1-\chi(x_n))e^{-x_nA_\lambda}$ for $x_n\to\infty$, as accounted for in [G2, Lemma 5.1]. (They are even exponentially decreasing in $|\lambda|$.) This holds on all rays with argument $\theta \in]0,2\pi[$, uniformly for θ in compact subintervals. The same holds for $G_{\mathrm{e},\lambda}(1-\chi)$, etc., and since $G_{\mathrm{e},\lambda} - \chi G_{\mathrm{e},\lambda}\chi = (1-\chi)G_{\mathrm{e},\lambda} + \chi G_{\mathrm{e},\lambda}(1-\chi)$, such expressions are likewise covered.

Now consider $G_{1,\lambda} - \chi G^0_{1,\lambda}\chi$. In the following, write $\chi(x_n/\varepsilon) = \chi_\varepsilon(x_n)$. As in [G2] we shall compare the true inverse of $\mathcal{A}_{1,\lambda}$ with an approximate inverse $\mathcal{B}'_{1,\lambda}$ containing the *true* interior contribution $Q_{1,\lambda,+}$ and the *cylindrical* boundary contribution:

$$\mathcal{B}'_{1,\lambda} = (\, Q_{1,\lambda,+} + \chi G^0_{1,\lambda}\chi \quad \chi K^0_{1,\lambda}\,).$$

$K^0_{1,\lambda}$ is of the form $\Psi(A,\lambda)e^{-x_nA_\lambda}$, cf. [G2, Sect. 3] for details. Now

$$\mathcal{A}_{1,\lambda}\mathcal{B}'_{1,\lambda} = \begin{pmatrix} P^*P-\lambda \\ B_{1,\lambda}\gamma \end{pmatrix} (\, Q_{1,\lambda,+} + \chi G^0_{1,\lambda}\chi \quad \chi K^0_{1,\lambda}\,) = I + \mathcal{S}_\lambda,$$

$$\text{with } \mathcal{S}_\lambda = \begin{pmatrix} \mathcal{S}_{11,\lambda} & \mathcal{S}_{12,\lambda} \\ \mathcal{S}_{21,\lambda} & 0 \end{pmatrix}, \quad \mathcal{S}_{11,\lambda} = [D_n^2 + A^2, \chi]G^0_{1,\lambda}\chi,$$

$$\mathcal{S}_{21,\lambda} = B_{1,\lambda}\gamma(Q_{1,\lambda,+} - Q^0_{1,\lambda,+}\chi), \quad \mathcal{S}_{12,\lambda} = [D_n^2 + A^2,\chi]K^0_{1,\lambda};$$

see in particular [G2, (4.17), (4.21)]. Since $[D_n^2+A^2,\chi]$ vanishes for $x_n \le \frac{c}{3}$, we can write

$$\mathcal{S}_{11,\lambda} = [D_n^2+A^2,\chi]\chi(1-\chi_{\frac13})G^0_{1,\lambda}\chi, \quad \mathcal{S}_{12,\lambda} = [D_n^2+A^2,\chi]\chi(1-\chi_{\frac13})K^0_{1,\lambda},$$

and it follows as above that these terms are trace class (considered as operators $\begin{pmatrix} \mathcal{S}_{11,\lambda} & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & \mathcal{S}_{12,\lambda} \\ 0 & 0 \end{pmatrix}$ in $L_2(X,E_1)\times L_2(X',E_1'))$ with $O(|\lambda|^{-N})$ estimates. For the term $\mathcal{S}_{21,\lambda}$, we appeal instead to the properties of $Q_{1,\lambda,+}-Q^0_{1,\lambda,+}\chi$. This is a pseudodifferential operator whose symbol is rapidly decreasing in (ξ,λ) for x near X', so the composition with trace operators γ_j maps $L_2(X,E_1)$ into $H^s(X',E_1')$, any s, with norm rapidly decreasing in λ for $\arg\lambda$ in compact intervals of $]0,2\pi[$. (This is an easy special case of the norm estimates shown in [G1, Sect. 2.5] — the case of regularity $+\infty$ — and can also be verified directly.)

This shows that $\mathcal{S}_\lambda$ is trace class with $O(|\lambda|^{-N})$ estimates for all N. For sufficiently large $|\lambda|$, the true inverse $\mathcal{B}_{1,\lambda}$ is then defined via a Neumann series

$$\mathcal{B}_{1,\lambda}=\mathcal{B}'_{1,\lambda}(I+\mathcal{S}'_\lambda),\text{ where } \mathcal{S}'_\lambda=\textstyle\sum_{k\ge1}(-\mathcal{S}_\lambda)^k=-\mathcal{S}_\lambda(I+\mathcal{S}'_\lambda);$$

here $\mathcal{S}'_\lambda$ is likewise trace class with $O(|\lambda|^{-N})$ estimates. It follows that

$$\begin{aligned}G_{1,\lambda}-\chi G^0_{1,\lambda}\chi&=R_{1,\lambda}-Q_{1,\lambda,+}-\chi G^0_{1,\lambda}\chi\\&=(Q_{1,\lambda,+}+\chi G^0_{1,\lambda}\chi)(I+\mathcal{S}'_{11,\lambda})+\chi K^0_{1,\lambda}\mathcal{S}'_{21,\lambda}-Q_{1,\lambda,+}-\chi G^0_{1,\lambda}\chi=\mathcal{S}''_\lambda,\end{aligned}$$

where $\mathcal{S}''_\lambda$ is trace class with $O(|\lambda|^{-N})$ estimates. The rules of calculus assure that such estimates likewise hold for the operators composed with D.

There is a similar proof for $G_{2,\lambda}-\chi G^0_{2,\lambda}\chi$.

The functions are holomorphic by virtue of their construction. Moreover, the decrease in λ is for each of the operators improved by an application of $\partial/\partial\lambda$. This completes the proof. □

One can also show that the operators map into H^s, any s, with $O(|\lambda|^{-N})$ estimates.

3.2 Construction of the zeta functions.

We integrate $\lambda^{-s}R_{i,\lambda}$ along an appropriate curve $\mathcal{C}$ as in Proposition 2.9, running along the negative axis and around a small circle of radius

$$r_0<\min\{\lambda_1(\Delta_i),\lambda_1(\widetilde\Delta_i),\lambda_1(A^2)\} \tag{3.11}$$

where λ_1 denotes the smallest nonzero eigenvalue. Then

$$\begin{aligned}Z(\Delta_i,s)&=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}R_{i,\lambda}\,d\lambda=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}Q_{i,\lambda,+}\,d\lambda+\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G_{i,\lambda}\,d\lambda\\&=Z(\widetilde\Delta_i,s)_++G_{Z,i,s},\ \text{ where we have set}\\G_{Z,i,s}&=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G_{i,\lambda}\,d\lambda.\end{aligned} \tag{3.12}$$

Define the transforms

$$G_{Z,\mathrm{e},s}=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G_{\mathrm{e},\lambda}\,d\lambda\quad\text{and}\quad G_{Z,\mathrm{o},s}=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G_{\mathrm{o},\lambda}\,d\lambda. \tag{3.13}$$

Note that the operators $G_{\mathrm{e},\lambda}$ and $G_{\mathrm{o},\lambda}$ are holomorphic in $\mathbb{C}\setminus[\lambda_1(A^2),\infty[$.

To describe the various G_Z, we use the function defined for $\operatorname{Re}(-t)<\operatorname{Re}s<0$ by

$$
\begin{aligned}
F_t(s) &= \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}_{\pi,r_0}} \tau^{-s-1}(1-\tau)^{-t}\,d\tau \\
&= \tfrac{\mathrm{i}}{2\pi}(e^{(-s-1)\mathrm{i}\pi}-e^{(s+1)\mathrm{i}\pi})\int_0^\infty u^{-s-1}(1+u)^{-t}\,du \\
&= \tfrac{1}{\pi}\sin\pi(s+1)\,\tfrac{\Gamma(-s)\Gamma(s+t)}{\Gamma(t)} = \tfrac{\Gamma(s+t)}{\Gamma(t)\Gamma(s+1)};
\end{aligned}
\tag{3.14}
$$

$\mathcal{C}_{\pi,r_0}$ is taken with $r_0\in\,]0,1[$, cf. (2.2). $F_t(s)$ coincides with the binomial coefficient $\binom{s+t-1}{t-1}$, also equal to $(sB(t,s))^{-1}$, where B is the beta function. $F_t(s)$ extends meromorphically to general s and $t\in\mathbb{C}$. In particular,

$$
\begin{aligned}
F_{\frac12}(s) &= \tfrac{\Gamma(s+\frac12)}{\sqrt{\pi}\,\Gamma(s+1)} = \tbinom{s-\frac12}{-\frac12}, \quad F_1(s)=1, \quad F_0(s)=0 \text{ if } s\neq 0,\\
F_{-\frac12}(s) &= \tfrac{-\Gamma(s-\frac12)}{2\sqrt{\pi}\,\Gamma(s+1)} = \tbinom{s-\frac32}{-\frac32}, \quad F_t(0)=1 \text{ if } \Gamma(t)\neq\infty.
\end{aligned}
\tag{3.15}
$$

That $F_1(s)=1$ follows directly from the first integral in (3.14), and the formula for $F_0(s)$ follows from the fact that $\frac{\mathrm{i}}{2\pi}\int_{\mathcal{C}_{\pi,r_0}}\tau^{-s-1}\,d\tau=0$ for $\operatorname{Re}s>0$.

The formulas for the singular Green operator terms are greatly simplified when we take normal traces.

Proposition 3.2. *Define $G_{Z,\mathrm{e},s}$ and $G_{Z,\mathrm{o},s}$ by* (3.13), *cf. also* (3.9), (3.7). *Then*

$$
\begin{aligned}
\operatorname{tr}_n G_{Z,\mathrm{e},s} &= \tfrac14(F_{\frac12}(s)-1)Z(A^2,s),\\
\operatorname{tr}_n G_{Z,\mathrm{o},s} &= -\tfrac14 F_{\frac12}(s)Y(A,2s).
\end{aligned}
\tag{3.16}
$$

Proof. Expand the operators on ∂X with respect to the orthogonal eigenprojections $\{\Pi_\mu\}_{\mu\in\mathrm{sp}(A)}$ for A. Our $G_{Z,\mathrm{e},s}$ and $G_{Z,\mathrm{o},s}$ are both 0 in the zero eigenspace. Using (3.10) we find, by replacing λ by $\mu^2\tau$ for each μ,

$$
\begin{aligned}
\operatorname{tr}_n G_{Z,\mathrm{e},s} &= \operatorname{tr}_n \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G_{\mathrm{e},\lambda}\,d\lambda = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}\big(\tfrac{-A^2}{4\lambda A_\lambda^2}+\tfrac{|A|}{4A_\lambda\lambda}\big)\,d\lambda\\
&= \sum_\mu \tfrac14\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s-1}\big(-\tfrac{\mu^2}{\mu^2-\lambda}+\tfrac{|\mu|}{(\mu^2-\lambda)^{\frac12}}\big)\,d\lambda\cdot\Pi_\mu\\
&= \sum_\mu \tfrac14|\mu|^{-2s}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\tau^{-s-1}(-(1-\tau)^{-1}+(1-\tau)^{-\frac12})\,d\tau\cdot\Pi_\mu\\
&= \tfrac14(-F_1(s)+F_{\frac12}(s))Z(A^2,s) = \tfrac14(-1+F_{\frac12}(s))Z(A^2,s);
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
\operatorname{tr}_n G_{Z,\mathrm{o},s} &= \operatorname{tr}_n \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{-s}G_{\mathrm{o},\lambda}\,d\lambda = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{-s}\big(\tfrac{-A}{4\lambda A_\lambda}+\tfrac{1}{4\lambda}\tfrac{A}{|A'|}\big)\,d\lambda\\
&= \sum_\mu \tfrac14\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{-s-1}\big(-\tfrac{\mu}{(\mu^2-\lambda)^{\frac12}}+\tfrac{\mu}{|\mu|}\big)\,d\lambda\cdot\Pi_\mu\\
&= \sum_\mu \tfrac14\mu|\mu|^{-2s-1}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\tau^{-s-1}\big(-(1-\tau)^{-\frac12}+1\big)\,d\tau\cdot\Pi_\mu\\
&= \tfrac14\big(-F_{\frac12}(s)+F_0(s)\big)Y(A,2s) = -\tfrac14F_{\frac12}(s)Y(A,2s). \qquad \square
\end{aligned} \tag{3.18}
$$

Note that the *even part* produces a function derived from the *zeta* function of A, and the *odd part* produces a function derived from the *eta* function of A. This is the fundamental observation for the following, relating the power functions of the boundary value problem to those of A.

Now we combine this with the interior contribution, taken from the doubled manifold $\widetilde X$.

Theorem 3.3. *The zeta functions decompose as in* (2.18)*:*

$$
\begin{aligned}
\Gamma(s)\zeta(\Delta_i,s) &= \Gamma(s)[\zeta_+(\widetilde\Delta_i,s)+\tfrac14(F_{\frac12}(s)-1)\zeta(A^2,s)+(-1)^i\tfrac14F_{\frac12}(s)\eta(A,2s)]\\
&\quad+\tfrac1s[\operatorname{Tr}_+(\Pi_0(\widetilde\Delta_i))-\nu_0(\Delta_i)+(-1)^i\tfrac14\nu_0(A)]+h_i(s).
\end{aligned} \tag{3.19}
$$

Moreover, $\Gamma(s)\zeta(\Delta_i,s)$ *is* $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im}s|})$ *for* $|\operatorname{Im}s|\ge1$, $-\infty<C_1\le\operatorname{Re}s\le C_2<\infty$, *any* $\varepsilon>0$.

Proof. The basic idea is this. By Lemma 3.1, the resolvent $(\Delta_i-\lambda)^{-1}=(\widetilde\Delta_i-\lambda)^{-1}_++G_{i,\lambda}$ has the same asymptotic behavior for λ going to infinity as $(\widetilde\Delta_i-\lambda)^{-1}_++\chi G^0_{i,\lambda}\chi$, and the last term behaves like $G^0_{i,\lambda}$. Here the contribution from $\widetilde\Delta_i$ is well-known; and the contributions from $G_{\mathrm{e},\lambda}$ and $G_{\mathrm{o},\lambda}$ in $G^0_{i,\lambda}$ have been dealt with in Proposition 3.2; they give the terms involving $F_{\frac12}(s)$. It turns out that what remains is some adjustments due to the Laurent expansions of the resolvents at $\lambda=0$ and the trace of $G^0_{i,\lambda}$ restricted to the null-space of A; these adjustments yield the coefficient of $\frac1s$ in (3.19).

To check the details, we start with (3.12). In order to take traces, we integrate by parts to obtain integrands of trace class:

$$
\begin{aligned}
Z(\Delta_i,s) &= \tfrac{1}{(s-1)\cdots(s-k)}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{k-s}\partial_\lambda^kR_{i,\lambda}\,d\lambda\\
&= \tfrac{1}{(s-1)\cdots(s-k)}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{k-s}\partial_\lambda^k(\widetilde\Delta_i-\lambda)^{-1}_+\,d\lambda+\tfrac{1}{(s-1)\cdots(s-k)}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{k-s}\partial_\lambda^kG_{i,\lambda}\,d\lambda;
\end{aligned} \tag{3.20}
$$

here

$$\partial_\lambda^k(\Delta_i-\lambda)^{-1}=k!(\Delta_i-\lambda)^{-k-1},\quad \partial_\lambda^k(\widetilde\Delta_i-\lambda)^{-1}_+=[k!(\widetilde\Delta_i-\lambda)^{-k-1}]_+.$$

For $k>\frac12\dim X-1$ these operators are trace class, and their traces are holomorphic except at the eigenvalues of Δ_i resp. $\widetilde\Delta_i$, all in $\{\lambda\ge0\}$. They are meromorphic at $\lambda=0$, with singularity

$$\partial_\lambda^k(-\lambda)^{-1}\Pi_0(\Delta_i)\quad\text{resp.}\quad\partial_\lambda^k(-\lambda)^{-1}\Pi_0(\widetilde\Delta_i)_+.$$

So for large $\operatorname{Re} s$ we can take traces in (3.20) and find

$$\zeta(\Delta_i, s) = \tfrac{1}{(s-1)\cdots(s-k)} \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{k-s} \operatorname{Tr}_X \partial_\lambda^k R_{i,\lambda}\, d\lambda. \tag{3.21}$$

We also have

$$\operatorname{Tr}_{X^0} G_{Z,\mathrm{e},s} = \tfrac{1}{(s-1)\cdots(s-k)} \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{k-s} \operatorname{Tr}_{X^0} \partial_\lambda^k G_{\mathrm{e},\lambda}\, d\lambda, \tag{3.22}$$

and the analogous formula for $G_{Z,\mathrm{o},s}$.

By Lemma 3.1, $\operatorname{Tr}_X \partial_\lambda^k G_{i,\lambda} - \operatorname{Tr}_X \chi \partial_\lambda^k G^0_{i,\lambda}\chi$ and $\operatorname{Tr}_{X^0} \partial_\lambda^k G^0_{i,\lambda} - \operatorname{Tr}_{X^0} \chi\partial_\lambda^k G^0_{i,\lambda}\chi$ and their λ-derivatives are $O(\lambda^{-N})$, any N, for λ going to infinity, so in view of (3.9),

$$\begin{aligned}\operatorname{Tr}_X \partial_\lambda^k R_{i,\lambda} = \operatorname{Tr}_X \partial_\lambda^k (\widetilde{\Delta}_i - \lambda)_+^{-1} + \operatorname{Tr}_{X^0}(\partial_\lambda^k [G_{\mathrm{e},\lambda} - (-1)^i G_{\mathrm{o},\lambda}]) \\ + (-1)^i \operatorname{Tr}_{X^0} \partial_\lambda^k \tfrac{\Pi_0(A)}{2\sqrt{-\lambda}} G_\lambda + \tilde{g}_i(\lambda),\end{aligned}$$

where $\tilde{g}_i(\lambda)$ and its λ-derivatives are $O(\lambda^{-N})$ at infinity for any N. (Note that the conjugation by the unitary morphism σ in $G^0_{2,\lambda}$ is eliminated when we take traces.) Here

$$\operatorname{Tr}_{X^0} \partial_\lambda^k \tfrac{\Pi_0(A)}{2\sqrt{-\lambda}} G_\lambda = \operatorname{Tr}_{X'} \partial_\lambda^k \tfrac{1}{2\sqrt{-\lambda}} \operatorname{tr}_n \Pi_0(A) G_\lambda = \partial_\lambda^k \tfrac{1}{-4\lambda} \nu_0(A)$$

is meromorphic at 0 (in contrast to $\frac{\Pi_0(A)}{2\sqrt{-\lambda}} G_\lambda$ itself, when $\nu_0(A) \neq 0$). For some $r_1 > 0$, $\operatorname{Tr} \partial_\lambda^k R_{i,\lambda}$ and $\operatorname{Tr} \partial_\lambda^k (\widetilde{\Delta}_i - \lambda)_+^{-1}$ are meromorphic on $\mathbb{C} \setminus [r_1, \infty[$, with just the singularity

$$\begin{aligned}\operatorname{Tr} \partial_\lambda^k (-\lambda)^{-1} \Pi_0(\Delta_i) = \partial_\lambda^k (-\lambda)^{-1} \nu_0(\Delta_i) \quad \text{resp.} \\ \operatorname{Tr} \partial_\lambda^k (-\lambda)^{-1} \Pi_0(\widetilde{\Delta}_i)_+ = \partial_\lambda^k (-\lambda)^{-1} \operatorname{Tr}_+ \Pi_0(\widetilde{\Delta}_i),\end{aligned}$$

at $\lambda = 0$; and $G_{\mathrm{e},\lambda}$ and $G_{\mathrm{o},\lambda}$ are holomorphic on $\mathbb{C} \setminus [r_1, \infty[$. It follows that $\tilde{g}_i(\lambda)$ is meromorphic on $\mathbb{C} \setminus [r_1, \infty[$ with just a pole $c\partial_\lambda^k (-\lambda)^{-1}$ at 0, and we can define $\widetilde{\widetilde{g}}_i(\lambda)$ (by integrating from infinity) such that $\partial_\lambda^k \widetilde{\widetilde{g}}_i = \tilde{g}_i$ and $\widetilde{\widetilde{g}}_i$ is $O(\lambda^{-N})$ for $|\lambda| \to \infty$ with $\arg\lambda \in [\delta, 2\pi - \delta]$. Let

$$g_i(\lambda) = (-1)^i (-4\lambda)^{-1} \nu_0(A) + \widetilde{\widetilde{g}}_i(\lambda).$$

This is meromorphic on $\mathbb{C} \setminus [r_1, \infty[$, and

$$\operatorname{Tr}_X \partial_\lambda^k R_{i,\lambda} = \operatorname{Tr}_X \partial_\lambda^k (\widetilde{\Delta}_i - \lambda)_+^{-1} + \operatorname{Tr}_{X^0}(\partial_\lambda^k [G_{\mathrm{e},\lambda} - (-1)^i G_{\mathrm{o},\lambda}]) + \partial_\lambda^k g_i(\lambda). \tag{3.23}$$

Then we find altogether that $g_i(\lambda)$ behaves at infinity and at zero as follows:

$$\begin{aligned} g_i(\lambda) &\sim (-1)^i (-4\lambda)^{-1} \nu_0(A) \text{ for } |\lambda| \to \infty, \text{ uniformly for } \arg\lambda \in [\delta, 2\pi - \delta]; \\ g_i(\lambda) &\sim (-\lambda)^{-1} (\nu_0(\Delta_i) - \operatorname{Tr}_+ \Pi_0(\widetilde{\Delta}_i)) + \sum_{j \geq 0} b_j (-\lambda)^j, \text{ for } \lambda \to 0,\end{aligned} \tag{3.24}$$

with arbitrary $\delta > 0$, so it is as in Proposition 2.9 with $\delta_0 = \pi$. Consider the "zeta transforms" of the three terms in (3.23) (the functions obtained by insertion in the right hand side of (3.21)). The contribution from the first term is well-known and equals $\zeta_+(\widetilde{\Delta}_i, s)$. The second is covered by Proposition 3.2, cf. (3.22). For the third term we apply Proposition 2.9 to $g_i(\lambda)$, using (3.24):

$$\tfrac{\pi}{(s-1)\cdots(s-k)\sin\pi s}\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{k-s}\partial_\lambda^k g_i(\lambda)\,d\lambda = \tfrac{\pi}{\sin\pi s}\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}g_i(\lambda)\,d\lambda$$
$$\sim -\frac{\nu_0(\Delta_i) - \operatorname{Tr}_+\Pi_0(\widetilde{\Delta}_i)}{s} - \sum_{j\ge 0}\frac{b_j}{s-j-1} + \frac{(-1)^i\nu_0(A)}{4s},$$

where $\sim$ means indication of the singularity structure. Altogether, this gives

$$\zeta(\Delta_i, s) \sim \zeta_+(\widetilde{\Delta}_i, s) + \tfrac14(F_{\frac12}(s)-1)\zeta(A^2,s) + (-1)^i\tfrac14 F_{\frac12}(s)\eta(A,2s)$$
$$+\frac{\sin\pi s}{\pi}\Big(\frac{-\nu_0(\Delta_i) + \operatorname{Tr}_+\Pi_0(\widetilde{\Delta}_i) + (-1)^i\frac14\nu_0(A)}{s} - \sum_{j\ge 0}\frac{b_j}{s-j-1}\Big).$$

We multiply this by $\Gamma(s)$, and obtain the main statement of Theorem 3.3, since $\frac{1}{\pi}\Gamma(s)\sin\pi s = \Gamma(1-s)^{-1}$ cancels the simple poles in the sum over $j \ge 0$ and equals 1 at $s = 0$.

The exponential decrease on vertical strips follows since it holds for the functions stemming from each of the three terms in the right hand side of (3.23). For the first term we have Lemma 2.5, for the third term we use Corollary 2.10, and for the middle term we have from Lemma 2.5 the exponential decrease of $\zeta(A^2, s)$ and $\eta(A, 2s)$ and combine this with the fact that $F_{\frac12}(s)$ is polynomially bounded for $|\operatorname{Im} s| \ge 1$, $-\infty < C_1 \le \operatorname{Re} s \le C_2 < \infty$, in view of (2.40). □

Now, consider the more general case $\zeta(D_i, \Delta_i, s)$, with $D_i = D_{ii}$ and $D'_i = D'_{ii}$ as in (2.15) ff. Note that D'_i commutes with the normal trace operation tr_n, so Proposition 3.2 generalizes to give analogues of (3.17), (3.18), with $D''_1 = D'_1$, $D''_2 = \sigma D'_2\sigma$:

$$\operatorname{tr}_n D''_i G_{Z,\mathrm{e},s} = \tfrac14(F_{\frac12}(s)-1)D''_i Z(A^2,s);$$
$$\operatorname{tr}_n D''_i G_{Z,\mathrm{o},s} = -\tfrac14 F_{\frac12}(s)D''_i Y(A,2s).$$

Then Theorem 3.3 generalizes to:

Theorem 3.4. *The generalized zeta function $\zeta(D_i, \Delta_i, s)$ decomposes as follows:*

(3.25) $\Gamma(s)\zeta(D_i, \Delta_i, s) =$

$$\Gamma(s)[\zeta_+(D_i, \widetilde{\Delta}_i, s) + \tfrac14(F_{\frac12}(s)-1)\zeta(D''_i, A^2, s) + (-1)^i\tfrac14 F_{\frac12}(s)\eta(D''_i, A, 2s)]$$
$$+\frac{1}{s}[\operatorname{Tr}_+(D_i\Pi_0(\widetilde{\Delta}_i)) - \operatorname{Tr}(D_i\Pi_0(\Delta_i)) + (-1)^i\tfrac14\operatorname{Tr}(D''_i\Pi_0(A))] + h_i(s),$$

with $D''_1 = D'_1$, $D''_2 = \sigma^* D'_2\sigma$, and $h_i(s)$ entire.

Here $\Gamma(s)\zeta(D_i,\Delta_i,s)$ *is* $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im} s|})$ *for* $|\operatorname{Im} s|\ge 1$, $-\infty < C_1 \le \operatorname{Re} s \le C_2 < \infty$, *any* $\varepsilon > 0$.

From the knowledge of the poles of $\Gamma(s)\zeta_+(D_i,\widetilde{\Delta}_1,s)$, $\Gamma(s)\zeta(D_i'',A^2,s)$ and $\Gamma(s)\eta(D_i'',A,2s)$ (cf. Lemma 2.5) we can now find the pole structure of $\Gamma(s)\zeta(D_i,\Delta_i,s)$, taking the new factors $F_{\frac12}(s)$ into account. The general result is that the poles are contained in $\{s=\frac{n+d-j}{2}\mid j\ge 0\}$ ($d=$ the order of D_i); they are simple for $\operatorname{Re} s>\frac{d}{2}$ and at most double for $\operatorname{Re} s\le\frac{d}{2}$. The detailed analysis for the case where D_i is a morphism is given in Corollaries 2.7 and 2.8.

We note that Proposition 2.9 can be applied to deduce a full expansion of the resolvent trace:

Corollary 3.5. *The generalized resolvent traces* $\operatorname{Tr} D_i\partial_\lambda^k(\Delta_i-\lambda)^{-1}$ *(defined for* $k>\frac12(n+d)$*, where* d *is the order of* D_i*) have expansions*

$$\operatorname{Tr} D_i\partial_\lambda^k(\Delta_i-\lambda)^{-1}\sim\sum_{0\le j<n}c_j(-\lambda)^{\frac{n+d-j}{2}-1-k}+\sum_{j=n}^{\infty}(c_j\log\lambda+c_j')(-\lambda)^{\frac{n+d-j}{2}-1-k}. \tag{3.26}$$

Proof. The resolvent trace and the zeta function are related by the formula

$$\zeta(D_i,\Delta_i,s)=\tfrac{1}{(s-1)\cdots(s-k)}\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{k-s}\operatorname{Tr} D_i\partial_\lambda^k(\Delta_i-\lambda)^{-1}\,d\lambda.$$

Since $\Gamma(s)\zeta(D_i,\Delta_i,s)$ is exponentially decreasing on vertical strips in $\mathbb{C}$,

$$\frac{\pi\zeta(D_i,\Delta_i,s)}{\sin\pi(s-k)}=\frac{(-1)^k\pi\zeta(D_i,\Delta_i,s)}{\sin\pi s}=(-1)^k\Gamma(1-s)\Gamma(s)\zeta(D_i,\Delta_i,s)$$

is *a fortiori* so, cf. (2.40). The singularity structure of this function is seen from (3.19) multiplied by $\Gamma(1-s)$, and we observe that the poles are at most double, since $\Gamma(1-s)F_{\frac12}(s)=\frac{\Gamma(1-s)\Gamma(s+\frac12)}{\sqrt{\pi}\,\Gamma(s+1)}$ has simple poles. We know from general considerations that $f(\lambda)=\operatorname{Tr} D_i\partial_\lambda^k(\Delta_i-\lambda)^{-1}$ is holomorphic on $\mathbb{C}\setminus\overline{\mathbb{R}}_+$ and has a Laurent expansion at 0 with the singular part $c(-\lambda)^{-k-1}$, $c=\operatorname{Tr}(D_i'\Pi_0(\Delta_i))k!$. We also need an estimate $f(\lambda)=O(|\lambda|^{-\alpha})$ for λ going to ∞ (with $\alpha>0$), in order to apply Proposition 2.9. A rough estimate can be obtained (for large enough k) by use of the spectral asymptotics and the elliptic regularity of Δ_1. Or one can appeal to the fact that $f(\lambda)$ has an expansion

$$\operatorname{Tr} D_i\partial_\lambda^k(\Delta_i-\lambda)^{-1}=\sum_{0\le j<n}c_j(-\lambda)^{\frac{n+d-j}{2}-1-k}+O(|\lambda|^{\frac d2-\frac38-k}),\ \text{for } |\lambda|\to\infty; \tag{3.27}$$

shown by analyzing (3.23) (with D_i inserted) as in [G1, Sect. 3.3]; it enters here that $G_{e,\lambda}$ and $G_{o,\lambda}$ are of "regularity 0" ([G2, Cor. 3.2]), so that the traces have expansions

$$\operatorname{Tr} D_i'\partial_\lambda^k(G_{e,\lambda}+(-1)^iG_{o,\lambda})=\sum_{1\le j<n}c_{i,j}(-\lambda)^{\frac{n+d-j}{2}-1-k}+O(|\lambda|^{\frac d2-\frac38-k}), \tag{3.28}$$

for $\lambda \to \infty$. In particular, the trace in (3.27) is $O(|\lambda|^{-\alpha})$ with $\alpha = \min\{k - \frac{1}{2}(n+d), 1\}$.

Thus $f(\lambda)$ satisfies the hypotheses of Proposition 2.9, and the Laurent coefficients b_j at 0, $j \ge 0$, give the poles of $\frac{\pi}{\sin \pi\sigma} \frac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-\sigma} f(\lambda)\, d\lambda$ for $\operatorname{Re} \sigma > 1 - \alpha$. Now we can apply the passage from b) to a) in Proposition 2.9, obtaining an expansion (3.26); there are no logarithmic terms for $j < n$ in view of (3.27) or the general information on $\Gamma(s)\zeta(D_i, \Delta_i, s)$ given above. $\square$

By a further analysis as in Corollaries 2.7 and 2.8 one can say more about the coefficients. A full expansion of the resolvent is also obtained in [GS], by a more direct method that allows x_n-dependent operators.

Remark 3.6. Although $\widetilde{\Delta}_i$ is an extension of P^*P or PP^* to the doubled manifold $\widetilde{X}$, the singularities of $\Gamma(s)\zeta_+(D_i, \widetilde{\Delta}_i, s)$, as given by Lemma 2.5, are completely determined by P in the original manifold X. Then since all terms in (3.25) except possibly $\frac{1}{s} \operatorname{Tr}_+(D_i \Pi_0(\widetilde{\Delta}_i))$ and $h_i(s)$ depend only on P in X, this must hold also for the coefficient $\operatorname{Tr}_+(D_i \Pi_0(\widetilde{\Delta}_i))$ (and hence for all terms). Note also that the proof of (3.25) shows that it is valid with $\widetilde{\Delta}_i$ replaced by any other selfadjoint elliptic extension $\widetilde{\Delta}'_i \ge 0$ of P^*P resp. PP^* to a boundaryless compact manifold extending X. We can use this to give a formula for $\operatorname{Tr}_+(D_i \Pi_0(\widetilde{\Delta}_i))$ without reference to extensions: Extend the bundles E_1 and E_2, and the operators P^*P and PP^*, to $\widetilde{X}$ by simply reflecting in the boundary X', using that in the cylinder, $P^*P = -\partial_n^2 + A^2$ and $PP^* = -\partial_n^2 + \sigma A^2 \sigma^*$. Denote these extended operators by $\widetilde{\Delta}'_i$. They commute with reflection, so the nullspace of each is the direct sum of "even" and "odd" eigensections. The even part of the nullspace gives the null space of $\Delta_{i,N}$, the Neumann realization of P^*P (resp. PP^*), while the odd part gives the nullspace of the Dirichlet realization $\Delta_{i,D}$. Thus

$$\operatorname{Tr}_+(D_i \Pi_0(\widetilde{\Delta}_i)) = \operatorname{Tr}_+(D_i \Pi_0(\widetilde{\Delta}'_i)) = \operatorname{Tr}(D_i \Pi_0(\Delta_{i,N})) + \operatorname{Tr}(D_i \Pi_0(\Delta_{i,D})). \tag{3.29}$$

3.3 Other boundary conditions.

The operator $P_{\ge}$ with boundary condition $\Pi_{\ge} \gamma_0 u = 0$ can easily be replaced by P_B with boundary condition $B\gamma_0 u = 0$, where B is any orthogonal projection in $L_2(E'_1)$ with

$$\operatorname{Range}(\Pi_>) \subset \operatorname{Range}(B) \subset \operatorname{Range}(\Pi_{\ge}).$$

This changes the boundary conditions only in V_0, the nullspace of A. The adjoint $(P_B)^*$ is the realization of P^* with boundary condition $\sigma B^\perp \sigma^* \gamma_0 u = 0$, $B^\perp = I - B$.

In $V_0 \otimes L_2(\mathbb{R}_+)$, the singular Green operators for $P^{0*}_{\ge} P^0_{\ge}$ and $P^0_{\ge} P^{0*}_{\ge}$ are, from (3.9),

$$G^0_{1,\lambda} = \tfrac{-1}{2\sqrt{-\lambda}} G_\lambda \quad \text{and} \quad G^0_{2,\lambda} = \tfrac{1}{2\sqrt{-\lambda}} \sigma G_\lambda \sigma^* \quad \text{in} \quad V_0 \otimes L_2(\mathbb{R}_+),$$
$$\text{with } \ G_\lambda u(x_n) = \int_0^\infty e^{-(x_n + y_n)\sqrt{-\lambda}} u(y_n)\, dy_n \text{ there.}$$

The corresponding operators for $P^{0*}_B P^0_B$ and $P^0_B P^{0*}_B$ are:

$$G^0_{B,1,\lambda} = \tfrac{-1}{2\sqrt{-\lambda}} G_\lambda (B - B^\perp) \text{ and } G^0_{B,2,\lambda} = \tfrac{1}{2\sqrt{-\lambda}} \sigma G_\lambda (B - B^\perp) \sigma^* \text{ in } V_0 \otimes L_2(\mathbb{R}_+).$$

The singular part of $\operatorname{tr}_n \partial_\lambda^k G^0_{B,i,\lambda}$ at $\lambda = 0$ is $(-1)^i \frac{1}{4}\partial_\lambda^k(-\lambda)^{-1}\operatorname{Tr}((B - B^\perp)\Pi_0(A))$.

Going through the proof of Theorem 3.4 with these modifications *in the null-space,* one finds:

Corollary 3.7. *For the generalized zeta functions* $\zeta(D_1, P_B{}^* P_B, s)$ *and* $\zeta(D_2, P_B P_B{}^*, s)$ *there are decompositions of* $\Gamma(s)\zeta(D_1, P_B{}^* P_B, s)$ *and* $\Gamma(s)\zeta(D_2, P_B P_B{}^*, s)$ *as in* (3.25)*, except that* $\operatorname{Tr}(D_i''\Pi_0(A))$ *must be replaced by* $\operatorname{Tr}(D_i''(B - B^\perp)\Pi_0(A))$*. Then in Corollary 2.3,* $\eta(A) = \eta(A, 0) + \nu_0(A)$ *is replaced by*

$$\eta_B(A) := \eta(A, 0) + \operatorname{Tr}((B - B^\perp)\Pi_0(A)). \tag{3.30}$$

Here (3.30) is a regularized signature of A wherein the nullspace V_0 has been split into positive and negative parts by the involution $B - B^\perp$.

By use of the exact formulas in (3.17)–(3.18), one can also calculate the singularity structure for boundary conditions as in [GS], where $\Pi_\geq$ is modified on other eigenspaces.

Remark 3.8. Consider the case where $E_1 = E_2$ and P is selfadjoint, so in particular $\sigma = -\sigma^*$, $\sigma A = -A\sigma$. Assume that $V_0(A)$ admits a decomposition into an orthogonal direct sum $V_0 = V_{0,+} \oplus V_{0,-}$ such that $\sigma\colon V_{0,+} \xrightarrow{\sim} V_{0,-}$, as in the case of [DW, Th. A.1]. For any such decomposition one gets a *selfadjoint realization* P_B by taking $B = \Pi_> + \Pi_{0,+}$, where $\Pi_{0,\pm}$ denotes the orthogonal projection onto $V_{0,\pm}$. For in this case, $\sigma\Pi_> = \Pi_<\sigma$ and $\sigma\Pi_{0,+} = \Pi_{0,-}\sigma$, so

$$\sigma B^\perp \sigma^* = \sigma(I - \Pi_> - \Pi_{0,+})(-\sigma) = -\sigma(\Pi_< + \Pi_{0,-})\sigma = \Pi_> + \Pi_{0,+} = B.$$

Here

$$\operatorname{Tr}((B - B^\perp)\Pi_0(A)) = \operatorname{Tr}(\Pi_{0,+} - \Pi_{0,-}) = \dim V_{0,+} - \dim V_{0,-} = 0, \tag{3.31}$$

and hence $\eta_B(A) = \eta(A, 0)$. (See Corollary 2.4 2°.)

4. The eta functions

4.1 Construction of the eta functions.

The eta functions associated with $P_\geq$ are

$$\eta(D_1, P_\geq, s) = \operatorname{Tr}(D_1 P \Delta_1^{-\frac{s+1}{2}}) \quad \text{and} \quad \eta(D_2, P_\geq{}^*, s) = \operatorname{Tr}(D_2 P^* \Delta_2^{-\frac{s+1}{2}}), \tag{4.1}$$

where $\Delta_1 = P_\geq{}^* P_\geq$, $\Delta_2 = P_\geq P_\geq{}^*$, and $D_i = D_{i,3-i}$ is a differential operator as in (2.15). This is not covered by Theorem 3.4 since P involves the normal derivative ∂_n.

As in Section 3, our analysis depends on the resolvents $(\Delta_i - \lambda)^{-1}$ and Proposition 2.9. There is some simplification here because $P(\Delta_1 - \lambda)^{-1}$ and $P^*(\Delta_2 - \lambda)^{-1}$ are holomorphic at $\lambda = 0$; P resp. P^* annihilates the nullspace of Δ_1 resp. Δ_2. But there is a contribution from the nullspace of A that needs special treatment; it takes the form $\Pi_0(A)/4\sqrt{-\lambda}$, which is not meromorphic at 0. So we analyze the contributions from the interior, and from the s.g.o. *outside* the nullspace of A, by means of Proposition 2.9; the remaining part is easy to handle.

The heart of the calculation is for the s.g.o. outside $V_0(A) \otimes L_2(\mathbb{R}_+)$. Denote the projection for that part by

$$\Pi_0^\perp(A) = I - \Pi_0(A).$$

Since $P^0 = \sigma(\partial_n + A)$ and $P^{0*} = (-\partial_n + A)\sigma^*$, we get from (3.7)–(3.10)

$$\begin{aligned} \operatorname{tr}_n D_1' P^0 G^0_{1,\lambda} &= D_1'\sigma(A - A_\lambda)\tfrac{1}{4\lambda}\left(\tfrac{-A^2}{A_\lambda^2} + \tfrac{|A|-A}{A_\lambda} + \tfrac{A}{|A'|}\right) + \tfrac{1}{4\sqrt{-\lambda}} D_1'\sigma\Pi_0(A), \\ \operatorname{tr}_n D_2' P^{0*} G^0_{2,\lambda} &= D_2'\sigma(A + A_\lambda)\tfrac{1}{4\lambda}\left(\tfrac{-A^2}{A_\lambda^2} + \tfrac{|A|+A}{A_\lambda} - \tfrac{A}{|A'|}\right)\sigma^* + \tfrac{1}{4\sqrt{-\lambda}} D_2'\Pi_0(A)\sigma^*. \end{aligned} \tag{4.2}$$

To evaluate the zeta-transform of these, we need more formulas as in Proposition 3.2:

$$\Pi_0^\perp(A)\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s-1} A_\lambda^j \, d\lambda = Z(A^2, s - \tfrac{j}{2})F_{-\frac{j}{2}}(s), \tag{4.3}$$

where $F_t(s)$ is defined in (3.14), and $F_0(s) = 0$. Noting that

$$A^2 Z(A^2, s) = Z(A^2, s-1), \quad AZ(A^2, s) = Y(A, 2s-1), \quad \tfrac{A}{|A'|} Z(A^2, s) = Y(A, 2s),$$

while $F_1(s) = 1$ and

$$F_{\frac12}(s) - F_{-\frac12}(s) = \pi^{-\frac12}\frac{(s-\frac12)\Gamma(s-\frac12) + \frac12\Gamma(s-\frac12)}{s\Gamma(s)} = \frac{\Gamma(s-\frac12)}{\sqrt{\pi}\,\Gamma(s)} = F_{\frac12}(s-1) \tag{4.4}$$

(see (3.15)), we find:

Lemma 4.1. *The operators* $G^0_{i,\lambda}\Pi_0^\perp(A)$ *satisfy:*

$$\begin{aligned} \operatorname{tr}_n P^0 \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G^0_{1,\lambda}\Pi_0^\perp(A)\, d\lambda &= \tfrac14[F_{\frac12}(s-1) - 1]\sigma Y(A, 2s-1) \\ \operatorname{tr}_n P^{0*} \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G^0_{2,\lambda}\sigma\Pi_0^\perp(A)\sigma^*\, d\lambda &= \tfrac14[F_{\frac12}(s-1) - 1] Y(A, 2s-1)\sigma^*. \end{aligned} \tag{4.5}$$

The integrals are well-defined since $G^0_{i,\lambda}\Pi_0^\perp(A)$ is holomorphic near 0. Note that the zeta terms have dropped out, and $\Pi_0^\perp(A)$ is not needed on the right hand side, since $Y(A, 2s-1)$ annihilates the nullspace of A. From (4.5) we conclude:

$$\begin{aligned} \operatorname{Tr} D_1' P^0 \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G^0_{1,\lambda}\Pi_0^\perp(A)\, d\lambda &= \tfrac14[F_{\frac12}(s-1) - 1]\eta(D_1'\sigma, A, 2s-1) \\ \operatorname{Tr} D_2' P^{0*} \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G^0_{2,\lambda}\sigma\Pi_0^\perp(A)\sigma^*\, d\lambda &= \tfrac14[F_{\frac12}(s-1) - 1]\eta(\sigma^* D_2', A, 2s-1). \end{aligned} \tag{4.6}$$

So when we take into account the contributions from the nullspace of A, we will get:

Theorem 4.2. *The generalized eta functions decompose as follows:*

$$
\begin{aligned}
\Gamma(s)\eta(D_1,P_\geq,2s-1) &= \Gamma(s)\operatorname{Tr}(D_1P\Delta_1^{-s})\\
&= \Gamma(s)\big[\operatorname{Tr}_+(D_1P\widetilde{\Delta}_1^{-s}) + \tfrac14(F_{\frac12}(s-1)-1)\eta(D_1'\sigma,A,2s-1)\big]\\
&\quad + \tfrac{1}{4\sqrt{\pi}}\operatorname{Tr}(D_1'\sigma\Pi_0(A))(s-\tfrac12)^{-1} + h_1(s),
\end{aligned}\tag{4.7}
$$

$$
\begin{aligned}
\Gamma(s)\eta(D_2,P_\geq{}^*,2s-1) &= \Gamma(s)\operatorname{Tr}(D_2P^*\Delta_2^{-s})\\
&= \Gamma(s)\big[\operatorname{Tr}_+(D_2P^*\widetilde{\Delta}_2^{-s}) + \tfrac14(F_{\frac12}(s-1)-1)\eta(\sigma^*D_2',A,2s-1)\big]\\
&\quad + \tfrac{1}{4\sqrt{\pi}}\operatorname{Tr}(\sigma^*D_2'\Pi_0(A))(s-\tfrac12)^{-1} + h_2(s),
\end{aligned}\tag{4.8}
$$

where $h_i(s)$ is entire.

The functions $\Gamma(s)\eta(D_1,P_\geq,2s-1)$ and $\Gamma(s)\eta(D_2,P_\geq{}^*,2s-1)$ are $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im}s|})$ for $|\operatorname{Im}s|\geq 1$, $-\infty<C_1\leq\operatorname{Re}s\leq C_2<\infty$, any $\varepsilon>0$.

Proof. It suffices to discuss the case of $D_1P\Delta_1^{-s}$. As in the proof of Theorem 3.3, we write

$$
\operatorname{Tr}_X D_1P\Delta_1^{-s} = \tfrac{1}{(s-1)\cdots(s-k)}\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{k-s}\operatorname{Tr}_X\partial_\lambda^k D_1PR_{1,\lambda}\,d\lambda,\tag{4.9}
$$

where k and s are taken so large that the operators are trace class. We also have

$$
\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}\operatorname{Tr}_{X^0}D_1'P^0G^0_{1,\lambda}\Pi_0^\perp(A)\,d\lambda = \tfrac{1}{(s-1)\cdots(s-k)}\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{k-s}\operatorname{Tr}_{X^0}\partial_\lambda^kD_1'P^0G^0_{1,\lambda}\Pi_0^\perp(A)\,d\lambda.\tag{4.10}
$$

Now, by (3.5), (4.2) and Lemma 3.1,

$$
\begin{aligned}
\operatorname{Tr}_X\partial_\lambda^kD_1PR_{1,\lambda} &= \operatorname{Tr}_X\partial_\lambda^kD_1P(\widetilde{\Delta}_1-\lambda)_+^{-1}+\\
&\operatorname{Tr}_{X^0}\partial_\lambda^kD_1'P^0G^0_{1,\lambda}\Pi_0^\perp(A) + \operatorname{Tr}_{X'}\partial_\lambda^k\tfrac14D_1'\sigma\Pi_0(A)(-\lambda)^{-\frac12} + \tilde{g}_1(\lambda),
\end{aligned}
$$

where $\tilde{g}_1(\lambda)$ is $O(\lambda^{-N})$ for λ going to infinity with $\arg\lambda\in[\delta,2\pi-\delta]$, any N. Let $\tilde{\tilde{g}}_1$ be a k'th primitive of $\tilde{g}_1$, such that $\tilde{\tilde{g}}_1$ is $O(\lambda^{-N})$ for λ going to infinity, any N, and set

$$
g_1(\lambda) = c(-\lambda)^{-\frac12} + \tilde{\tilde{g}}_1(\lambda),\quad c = \tfrac14\operatorname{Tr}_{X'}(D_1'\sigma\Pi_0(A)).
$$

Then

$$
\operatorname{Tr}_X\partial_\lambda^kD_1PR_{1,\lambda} = \operatorname{Tr}_X\partial_\lambda^kD_1P(\widetilde{\Delta}_1-\lambda)_+^{-1} + \operatorname{Tr}_{X^0}\partial_\lambda^kD_1'P^0G^0_{1,\lambda}\Pi_0^\perp(A) + \partial_\lambda^kg_1(\lambda).\tag{4.11}
$$

Since $\operatorname{Tr}_X\partial_\lambda^kD_1PR_{1,\lambda}$, $\operatorname{Tr}_X\partial_\lambda^kD_1P(\widetilde{\Delta}_1-\lambda)_+^{-1}$ and $\operatorname{Tr}_{X^0}\partial_\lambda^kD_1'P^0G^0_{1,\lambda}\Pi_0^\perp(A)$ are regular at 0, $\partial_\lambda^kg_1$ is holomorphic on a set $\mathbb{C}\setminus[r_1,\infty[$ with $r_1>0$; then so is g_1 (but not $\tilde{g}_1$). It satisfies:

$$
\begin{aligned}
&g_1(\lambda)\sim c(-\lambda)^{-\frac12}\text{ for }|\lambda|\to\infty,\text{ uniformly for }\arg\lambda\in[\delta,2\pi-\delta];\\
&g_1(\lambda)\sim\sum_{j\geq0}b_j(-\lambda)^j,\text{ for }\lambda\to0.
\end{aligned}\tag{4.12}
$$

When the terms in the right hand side of (4.11) are inserted in (4.9), the first term gives $\mathrm{Tr}_+(D_1P\widetilde{\Delta}_1^{-s})$, the second is dealt with in Lemma 4.1 ff. and (4.10), and the third gives, by Proposition 2.9 and (4.12),

$$\tfrac{1}{(s-1)\cdots(s-k)}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{k-s}\partial_\lambda^k g_1(\lambda)\,d\lambda = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}g_1(\lambda)\,d\lambda$$
$$\sim \frac{\sin\pi s}{\pi}\Big(-\sum_{j\ge 0}\frac{b_j}{s-j-1}+\frac{c}{s-\frac12}\Big).$$

Altogether, we find

$$\mathrm{Tr}_X(D_1P\Delta_1^{-s}) \sim \mathrm{Tr}_+(D_1P\widetilde{\Delta}_1^{-s}) + \tfrac14[F_{\frac12}(s-1)-1]\eta(D_1'\sigma, A, 2s-1))$$
$$+\frac{\sin\pi s}{\pi}\Big(-\sum_{j\ge 0}\frac{b_j}{s-j-1}+\frac{\mathrm{Tr}_{X'}(D_1'\sigma\Pi_0(A))}{4(s-\frac12)}\Big).$$

Multiplication of this by $\Gamma(s)$ gives Theorem 4.2, since $\frac1\pi\Gamma(s)\sin\pi s=\Gamma(1-s)^{-1}$ cancels the simple poles in the sum over $j\ge 0$ and equals $(\pi)^{-\frac12}$ at $s=\frac12$. The exponential decrease is accounted for as in the proof of Theorem 3.3. □

Remark 4.3. Since any differential operator D on X, that is constant in x_n on $X'\times[0,c]$, can be written as a polynomial in ∂_{x_n} with tangential x_n-independent differential operator coefficients there, and ∂_{x_n} just gives a factor $-A_\lambda$ when applied to $G^0_{i,\lambda}$, the above methods also allow the analysis of $\mathrm{Tr}(D\Delta_i^{-s})$ for such general D. The general structure is

$$\mathrm{Tr}(D\Delta_i^{-s}) \sim \sum_{0\le l<n}\frac{c_l(D,\Delta_i)}{s+\frac{l-n-d}{2}}+\sum_{l\ge n}\Big[\frac{c_l(D,\Delta_i)}{(s+\frac{l-n-d}{2})^2}+\frac{c_l'(D,\Delta_i)}{s+\frac{l-n-d}{2}}\Big], \tag{4.13}$$

where d is the order of D; the coefficients can be described more precisely by methods as in Corollaries 2.7–2.8 and 4.4.

4.2 Description of the singularities.

One finds (as after Theorem 3.4) that the poles of (4.7)–(4.8) are contained in the set $\{s=\frac{n+d+1-j}{2}\mid j\ge 0\}$, and are simple for $j<n$ and at most double in general.

We shall now list explicitly the singularities of the eta functions of $P_\ge$ in the case where D_i is just a morphism φ. We set

$$\delta_m = [\text{residue of } \tfrac14F_{\frac12}(s-1)\Gamma(s)=\tfrac{1}{4\sqrt\pi}\Gamma(s-\tfrac12) \text{ at } s=\tfrac12-m]=\frac{(-1)^m}{4\sqrt\pi\, m!}. \tag{4.14}$$

Let φ be a C^∞ morphism from E_2 to E_1 that equals $\varphi^0:=\varphi|_{X'}$ on $X'\times[0,c]$.

From Lemma 2.5 we have

$$\Gamma(s)\,\mathrm{Tr}_+(\varphi P\widetilde{\Delta}_1^{-s}) \sim \sum_{k=0}^\infty\frac{c_{2k+1,+}(\varphi P,\widetilde{\Delta}_1)}{s+k-\frac n2},$$
$$\Gamma(s)\eta(\varphi^0\sigma, A, 2s-1)=\Gamma(s)\zeta(\varphi^0\sigma A, A^2, s)\sim\sum_{k=0}^\infty\frac{c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k-\frac{n-1}{2}}. \tag{4.15}$$

As in the proof of Corollaries 2.7 and 2.8 we use this for the expansions in Theorem 4.2, cf. also (2.24) and (2.25), to find:

Corollary 4.4.

1° **When n is even**, *the singularities of* $\Gamma(s)\eta(D_1, P_{\geq}, 2s-1) = \Gamma(s)\operatorname{Tr}(\varphi P \Delta_1^{-s})$ *consist of the following sums:*

From $\Gamma(s)\operatorname{Tr}_+(\varphi P \widetilde{\Delta}_1^{-s})$,

$$\sum_{k\geq 0} \frac{c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)}{s+k-\frac{n}{2}}.$$

From $\frac{1}{4}(F_{\frac{1}{2}}(s-1)-1)\Gamma(s)\eta(\varphi^0\sigma, A, 2s-1) = \frac{1}{4}(F_{\frac{1}{2}}(s-1)-1)\Gamma(s)\zeta(\varphi^0\sigma A, A^2, s)$,

$$\sum_{0\leq k<\frac{n}{2}-1} \frac{\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k-\frac{n-1}{2}}$$

$$+\sum_{k\geq\frac{n}{2}-1}\Big[\frac{\beta_{k+1-\frac{n}{2}}c_{2k+1}(\varphi^0\sigma A, A^2)}{(s+k-\frac{n-1}{2})^2} + \frac{\beta_{k+1-\frac{n}{2}}c'_{2k+1}(\varphi^0\sigma A, A^2) + (\beta'_{k+1-\frac{n}{2}}-\frac{1}{4})c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k-\frac{n-1}{2}}\Big];$$

here the singularity at $s=\frac{1}{2}$ *(i.e. for* $k=\frac{n}{2}-1$*) is*

$$\frac{\frac{1}{4\pi}c_{n-1}(\varphi^0\sigma A, A^2)}{(s-\frac{1}{2})^2} + \frac{\frac{1}{4\pi}c'_{n-1}(\varphi^0\sigma A, A^2) + (\beta'_0-\frac{1}{4})c_{n-1}(\varphi^0\sigma A, A^2)}{s-\frac{1}{2}}.$$

From the remaining term,

$$\frac{\operatorname{Tr}(\varphi^0\sigma\Pi_0(A))}{4\sqrt{\pi}\,(s-\frac{1}{2})}.$$

2° **When n is odd**, *the singularities of* $\Gamma(s)\eta(D_1, P_{\geq}, 2s-1) = \Gamma(s)\operatorname{Tr}(\varphi P \Delta_1^{-s})$ *consist of the following sums:*

From $\Gamma(s)\operatorname{Tr}_+(\varphi P \widetilde{\Delta}_1^{-s})$,

$$\sum_{k\geq 0} \frac{c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)}{s+k-\frac{n}{2}}.$$

From $\frac{1}{4}(F_{\frac{1}{2}}(s-1)-1)\Gamma(s)\eta(\varphi^0\sigma, A, 2s-1)$,

$$\sum_{k\geq 0} \frac{\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k-\frac{n-1}{2}} + \sum_{m\geq 0}\frac{\delta_m\eta(\varphi^0\sigma, A, -2m)}{s+m-\frac{1}{2}};$$

here the singularity at $s=\frac{1}{2}$ *equals*

$$\frac{\eta(\varphi^0\sigma, A, 0)}{4\sqrt{\pi}\,(s-\frac{1}{2})}.$$

From the remaining term,

$$\frac{\operatorname{Tr}(\varphi^0\sigma\Pi_0(A))}{4\sqrt{\pi}\,(s-\frac{1}{2})}.$$

3° *There are similar formulas for* $\Gamma(s)\eta(D_2, P_{\geq}{}^*, 2s-1) = \Gamma(s)\operatorname{Tr}(\varphi P^*\Delta_2^{-s})$*, with* $P\widetilde{\Delta}_1$ *replaced by* $P^*\widetilde{\Delta}_2$ *and* $\varphi^0\sigma$ *replaced by* $\sigma^*\varphi^0$*.*

In 1°, the poles of $F_{\frac12}(s-1)$ at half-integer s give rise to double poles, since they coincide with poles of $\Gamma(s)\eta(\varphi^0\sigma, A, 2s-1)$. In 2°, the poles of $F_{\frac12}(s-1)$ at half-integer s give rise to simple poles, picking up the values $\eta(\varphi^0\sigma, A, -2m)$ of $\eta(\varphi^0\sigma, A, 2s-1)$ between its poles.

The corollary is formulated for the eta functions as functions of $2s-1$; for convenience, we also show how the formulas look when $2s-1$ is replaced by $2s$:

Corollary 4.5.
When n is even,

$$\Gamma(s+\tfrac12)\eta(\varphi, P_{\geq}, 2s) \sim \sum_{k\geq 0}\frac{c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)}{s+k-\frac{n-1}{2}} + \sum_{0\leq k<\frac n2 -1}\frac{\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k+1-\frac n2}$$
$$+\sum_{k\geq\frac n2-1}\Big[\frac{\beta_{k+1-\frac n2}c_{2k+1}(\varphi^0\sigma A, A^2)}{(s+k+1-\frac n2)^2} + \frac{\beta_{k+1-\frac n2}c'_{2k+1}(\varphi^0\sigma A, A^2) + (\beta'_{k+1-\frac n2}-\frac14)c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k+1-\frac n2}\Big]$$
$$+\frac{1}{4\sqrt{\pi}\, s}\operatorname{Tr}(\varphi^0\sigma\Pi_0(A)). \tag{4.16}$$

When n is odd,

$$\Gamma(s+\tfrac12)\eta(\varphi, P_{\geq}, 2s) \sim \sum_{k\geq 0}\frac{c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)}{s+k-\frac{n-1}{2}} + \sum_{k\geq 0}\frac{\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k+1-\frac n2}$$
$$+\sum_{m\geq 0}\frac{\delta_m\eta(\varphi^0\sigma A, -2m)}{s+m} + \frac{1}{4\sqrt{\pi}\, s}\operatorname{Tr}(\varphi^0\sigma\Pi_0(A)). \tag{4.17}$$

As in Corollary 3.5, one can deduce a full expansion of the corresponding resolvent expressions by use of Proposition 2.9; we leave this to the interested reader.

One can also modify the boundary condition as in Section 3.3, and then one finds in a similar way:

Corollary 4.6. *For the generalized eta function* $\eta(D_1, P_B, s) = \operatorname{Tr}(D_1 P_B(P_B{}^*P_B)^{-\frac{s+1}{2}})$ *one has a decomposition of* $\Gamma(s)\eta(D_1, P_B, 2s-1)$ *as in* (4.7)*, except that* $\operatorname{Tr}(D_1'\sigma\Pi_0(A))$ *must be replaced by* $\operatorname{Tr}(D_1'\sigma(B-B^\perp)\Pi_0(A))$*.*

When P *is as in Corollary 2.4 2°, then the residue of* $\eta(P_B, 2s)$ *at* $s=0$ *satisfies*

$$\operatorname{Res}\eta(P_B, 0) = \frac{1}{4\pi}\operatorname{Tr}(\sigma(B-B^\perp)\Pi_0(A)). \tag{4.18}$$

Remark 4.7. In the situation considered in Remark 3.8, where P_B is *selfadjoint,*

$$\operatorname{Tr}(\sigma(B-B^\perp)\Pi_0(A)) = \operatorname{Tr}(\sigma(\Pi_{0,+}-\Pi_{0,-})) = 0 \tag{4.19}$$

(since $\sigma\Pi_{0,\pm}$ have zero diagonal blocks when written as block matrices with respect to the decomposition $V_{0,+}\oplus V_{0,-}$). Thus $\eta(P_B, 2s)$ is regular at $s=0$, as noted in [DW, App. 1].

4.3 The variation of eta at 0.

Assume that a boundary condition $B\gamma_0 u = 0$ has been chosen, as in Remarks 3.8 and 4.6, such that P_B is selfadjoint. Then $\sigma = -\sigma^*$, $\sigma A = -A\sigma$, and $\Delta_1 = \Delta_2 =: \Delta$, $\widetilde{\Delta}_1 = \widetilde{\Delta}_2 =: \widetilde{\Delta}$. Assume that P varies smoothly with a parameter w. Denote the derivatives with respect to w by $\dot{P}$, $\dot{\sigma}$ and $\dot{A}$. Suppose that $\Pi_>(A)$ and $\Pi_<(A)$ do not vary, while $\Pi_0(\widetilde{\Delta})$ and $\Pi_0(\Delta)$ vary smoothly, and $\eta(\dot{\sigma}, A, s)$ is regular at $s = 0$. (This last condition is automatically satisfied if $n = \dim X$ is odd, or if $\dot{\sigma} = 0$.) Then for s sufficiently large,

$$\dot{\eta}(P_B, s) = -s \operatorname{Tr}(\dot{P} P_B^{-s-1}), \tag{4.20}$$

as one finds by differentiating $\frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-\frac{s+1}{2}} P_B(P_B^2 - \lambda)^{-1}\, d\lambda$ with respect to w, integrating by parts, and taking traces. For

$$\dot{Y}(P_B, s) = \tag{4.21}$$
$$\frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-\frac{s+1}{2}} [\dot{P}_B(P_B^2 - \lambda)^{-1} - P_B(P_B^2 - \lambda)^{-1}(\dot{P}_B P_B + P_B\dot{P}_B)(P_B^2 - \lambda)^{-1}]\, d\lambda.$$

Now $\frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s} P_B(P_B^2 - \lambda)^{-1}\dot{P}_B P_B(P_B^2 - \lambda)^{-1}\, d\lambda$ is trace class for s sufficiently large, and its trace is the same as for $\frac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}\dot{P}_B P_B^2(P_B^2 - \lambda)^{-2}\, d\lambda$;

$$\begin{aligned}
&\operatorname{Tr} \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s} P_B(P_B^2 - \lambda)^{-1}\dot{P}_B P_B(P_B^2 - \lambda)^{-1}\, d\lambda \\
&\quad = \tfrac{1}{(s-1)\dots(s-k)}\tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{k-s}\operatorname{Tr}[P_B \textstyle\sum C_j^k \partial_\lambda^j (P_B^2 - \lambda)^{-1}\dot{P}_B P_B^2 \partial_\lambda^{k-j}(P_B^2 - \lambda)^{-2}]\, d\lambda \\
&\quad = \tfrac{1}{(s-1)\dots(s-k)}\tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{k-s}\operatorname{Tr}[\dot{P}_B P_B^2 \partial_\lambda^k (P_B^2 - \lambda)^{-2}]\, d\lambda \\
&\quad = \operatorname{Tr}\tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}\dot{P}_B P_B^2 (P_B^2 - \lambda)^{-2}\, d\lambda.
\end{aligned}$$

A similar argument applies to the term in (4.21) with $P_B\dot{P}_B$. Moreover, since B is constant, $\dot{P}_B$ can be replaced by $\dot{P}$. Putting all this in (4.21), and replacing P_B^2 by $(P_B^2 - \lambda) + \lambda$, yields (4.20).

Now in our case $\dot{P} = \dot{\sigma}\sigma^* P + \sigma\dot{A}$ near X'. Take a cut-off function $\psi \equiv 0$ near X', $\psi \equiv 1$ away from X'. Then

$$\dot{P} = [\psi\dot{P} + (1-\psi)\sigma\dot{A}] + (1-\psi)\dot{\sigma}\sigma^* P.$$

The term in brackets can serve as D_1 in Theorem 3.4, and $(1-\psi)\dot{\sigma}\sigma^*$ can be D_2 in Theorem 4.2. We use the versions amended as in Remark 4.7, and compute (4.20) at $s = 0$. The interior contributions from Theorems 3.4 and 4.2 combine to give

$$-2\operatorname{Res}\zeta_+(\dot{P}, \widetilde{\Delta}, \tfrac12) - \tfrac{1}{\sqrt{\pi}}\operatorname{Tr}_+(\dot{P}\Pi_0(\widetilde{\Delta})) + \tfrac{1}{\sqrt{\pi}}\operatorname{Tr}(\dot{P}\Pi_0(\Delta)). \tag{4.22}$$

(Notice that P annihilates $\Pi_0(\Delta)$ and $\Pi_0(\widetilde{\Delta})$.) The boundary contributions from Theorem 3.4 are

$$(\tfrac12 - \tfrac1\pi)\operatorname{Res}\zeta(\sigma\dot{A}, A^2, \tfrac12) + \tfrac{1}{2\pi}\operatorname{Res}\eta(\sigma\dot{A}, A, 1) + \tfrac{1}{4\sqrt{\pi}}\operatorname{Tr}(\sigma\dot{A}(B - B^\perp)\Pi_0(A)). \tag{4.23}$$

(Note from (3.15) that $F_{\frac12}(\frac12) = \frac{2}{\pi}$.) The boundary contribution from Theorem 4.2 comes from $\operatorname{Tr}(\dot\sigma\sigma^* P\Delta^{-\frac{s+1}{2}})$, which involves the pole of $F_{\frac12}(\frac{s-1}{2})$ at $s=0$, with residue $\frac{2}{\pi}$. Since we assume $\eta(\dot\sigma, A, s)$ is regular at $s=0$, this contribution is

$$\frac{-1}{2\pi}\eta(\dot\sigma, A, 0) - \frac{1}{2\pi}\operatorname{Tr}(\dot\sigma(B - B^\perp)\Pi_0(A)). \tag{4.24}$$

Altogether, we find:

Theorem 4.8. *In the considered selfadjoint case, the variation of* $\eta(P_B, 0)$ *is the sum of* (4.22), (4.23) *and* (4.24).

When n is odd, then $\operatorname{Res}\zeta(\sigma\dot A, A^2, \frac12) = 0 = \operatorname{Res}\eta(\sigma\dot A, A, 1)$, by Lemma 2.5. When n is even, then $\operatorname{Res}\zeta_+(\dot P, \widetilde\Delta, \frac12) = 0$.

5. Heat kernel expansions

5.1 The relation between exponential functions and and power functions.

We shall now study the exponential functions $e^{-t\Delta_i}$, the heat operators associated with $P_\geq$. When Q is lower bounded selfadjoint, the exponential function is described by

$$e^{-tQ} = \frac{\mathrm{i}}{2\pi}\int_{\mathcal{C}'} e^{-t\lambda}(Q-\lambda)^{-1}\,d\lambda, \quad t>0; \tag{5.1}$$

where $\mathcal{C}'$ is a curve encircling the full spectrum in the positive direction and such that $e^{-t\lambda}$ falls off for $|\lambda|\to\infty$ on the curve (e.g. one can let $\mathcal{C}'$ begin with a ray with argument $\in]0, \frac{\pi}{2}[$ and end with a ray with argument $\in]-\frac{\pi}{2}, 0[$).

The exponential function and the power function of an operator $Q\ge 0$ with compact resolvent are related to one another by the formulas (where $\Pi_0^\perp(Q) = I - \Pi_0(Q)$):

$$\begin{aligned} Z(Q,s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tQ}\Pi_0^\perp(Q)\,dt, \quad \operatorname{Re} s>0,\\ e^{-tQ}\Pi_0^\perp(Q) &= \frac{1}{2\pi\mathrm{i}}\int_{\operatorname{Re} s=c} t^{-s}Z(Q,s)\Gamma(s)\,ds, \quad c>0, \end{aligned} \tag{5.2}$$

that follow from the scalar formulas valid in each eigenspace. (The transition between the formulas is analyzed in Proposition 5.1 below.)

Taking $Q = S^*S$ for suitable operators S, we have accordingly (cf. (2.5), note that $\Pi_0(S^*S) = \Pi_0(S)$):

$$\begin{aligned} Z(S^*S,s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-tS^*S}\Pi_0^\perp(S)\,dt,\\ e^{-tS^*S}\Pi_0^\perp(S) &= \frac{1}{2\pi\mathrm{i}}\int_{\operatorname{Re} s=c} t^{-s}Z(S^*S,s)\Gamma(s)\,ds,\\ Y(S,2s) = SZ(S^*S, s+\tfrac12) &= \frac{1}{\Gamma(s+\frac12)}\int_0^\infty t^{s-\frac12}Se^{-tS^*S}\,dt,\\ Se^{-tS^*S} &= \frac{1}{2\pi\mathrm{i}}\int_{\operatorname{Re} s=c} t^{-s}Y(S,2s-1)\Gamma(s)\,ds. \end{aligned} \tag{5.3}$$

In the last two formulas, $\Pi_0^\perp(S)$ is left out, since S vanishes on $V_0(S)$.

We often use these formulas composed with a differential operator D. When the expressions are trace class (usually for $\operatorname{Re} s$ resp. c sufficiently large) one can take the trace on both sides in (5.3) (composed with D), obtaining the formulas relating zeta and eta functions to exponential function traces:

$$
\begin{aligned}
\zeta(D, S^*S, s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} De^{-tS^*S}\Pi_0^\perp(S)\, dt,\\
\operatorname{Tr} De^{-tS^*S}\Pi_0^\perp(S) &= \tfrac{1}{2\pi i} \int_{\operatorname{Re} s=c} t^{-s}\zeta(D, S^*S, s)\Gamma(s)\, ds,\\
\eta(D, S, 2s) = \zeta(DS, S^*S, s+\tfrac12) &= \frac{1}{\Gamma(s+\frac12)} \int_0^\infty t^{s-\frac12} \operatorname{Tr} DSe^{-tS^*S} dt,\\
\operatorname{Tr} DSe^{-tS^*S} &= \tfrac{1}{2\pi i} \int_{\operatorname{Re} s=c} t^{-s}\eta(D, S, 2s-1)\Gamma(s)\, ds.
\end{aligned} \tag{5.4}
$$

(There are similar transition formulas for the symbols and kernels of the operators.)

These formulas will allow a translation between properties of the power function traces and of the exponential function traces, as used in [DG] for the passage from the power functions to the exponential functions, and e.g. in [BG] for the passage from the exponential functions to the power functions.

For convenience, we describe these transformations in a general result.

Proposition 5.1. 1° *Let $e(t)$ be a function holomorphic in a sector V_{θ_0} (for some $\theta_0 \in\,]0, \frac{\pi}{2}[$),*

$$V_{\theta_0} = \{t = re^{i\theta} \mid r > 0, |\theta| < \theta_0\}, \tag{5.5}$$

such that $e(t)$ decreases exponentially for $|t| \to \infty$ and is $O(|t|^a)$ for $t \to 0$ in V_δ, any $\delta < \theta_0$, for some $a \in \mathbb{R}$. Let f be the Mellin transform of e,

$$f(s) = (\mathcal{M}e)(s) := \int_0^\infty t^{s-1}e(t)\, dt, \tag{5.6}$$

for $\operatorname{Re} s > -a$. Then $f(s)$ is holomorphic for $\operatorname{Re} s > -a$ and $f(c+i\xi)$ is $O(e^{-\delta|\xi|})$ for $|\xi| \to \infty$, when $c > -a$ (uniformly for c in compact intervals of $]-a, \infty[$); and $e(t)$ is recovered from $f(s)$ by the formula

$$e(t) = \tfrac{1}{2\pi i} \int_{\operatorname{Re} s=c} t^{-s} f(s)\, ds. \tag{5.7}$$

2° *Moreover, the following properties a) and b) are equivalent:*

a) $e(t)$ has an asymptotic expansion for $t \to 0$,

$$e(t) \sim \sum_{j=0}^\infty \sum_{l=0}^{m_j} a_{j,l} t^{\beta_j} (\log t)^l, \quad \beta_j \nearrow +\infty,\ m_j \in \{0, 1, 2, \dots\}, \tag{5.8}$$

uniformly for $t \in V_\delta$, *for each* $\delta < \theta_0$.

b) $f(s)$ *is meromorphic on* $\mathbb{C}$ *with the singularity structure*

$$f(s) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} \frac{(-1)^l l! a_{j,l}}{(s+\beta_j)^{l+1}}, \tag{5.9}$$

and for each real C_1, C_2 *and each* $\delta < \theta_0$,

$$|f(s)| \le C(C_1, C_2, \delta) e^{-\delta |\operatorname{Im} s|}, \ |\operatorname{Im} s| \ge 1, \ C_1 \le \operatorname{Re} s \le C_2. \tag{5.10}$$

3° *Let* $r(\lambda)$ *take values in a Banach space, and be holomorphic in* $S_{\delta_0} = \{|\pi - \arg \lambda| < \delta_0\}$ *for some* $\delta_0 \in]\frac{\pi}{2}, \pi]$ *and meromorphic at* $\lambda = 0$ *(holomorphic for* $0 < |\lambda| < \varrho$*). Assume that as* $\lambda \to \infty$ *in* S_δ *(for* $\delta < \delta_0$*),* $\partial_\lambda^m r(\lambda)$ *is* $O(|\lambda|^{-1-\varepsilon})$ *for some* $\varepsilon > 0$ *(so that* $r(\lambda)$ *is* $O(|\lambda|^{m-1})$*). Let* θ_0 *and* θ *be such that* $]\theta - \theta_0, \theta + \theta_0[\subset]\pi - \delta_0, \frac{\pi}{2}[$, *let* $\mathcal{C} = \mathcal{C}_{\theta, r_0}$ *as in* (2.2) *with* $r_0 \in]0, \varrho[$, *and let*

$$e(t) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} r(\lambda)\, d\lambda, \quad f(s) = \Gamma(s) \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-s} r(\lambda)\, d\lambda, \tag{5.11}$$

for $t \in V_{\theta_0}$ *resp.* $\operatorname{Re} s > m - \varepsilon$. *Then* $e(t)$ *is exponentially decreasing for* $t \to \infty$ *in sectors* V_δ *with* $\delta < \theta_0$, *and is* $O(|t|^{-m})$ *for* $t \to 0$, *and* $f(s)$ *and* $e(t)$ *correspond to one another by* (5.6), (5.7).

Proof. 1°. Note first that replacing $e(t)$ by $t^b e(t)$ replaces $f(s)$ by $f(s+b)$, so we can assume that $a > 0$ and then consider $c \ge 0$. The function $f(s)$ is holomorphic for $\operatorname{Re} s \ge 0$ since the integrand $t^{s-1} e(t)$ is so and has an integrable majorant there.

By a change of variables $t = e^x$, we see that $f_1(\xi) = f(\mathrm{i}\xi)$ is the conjugate Fourier transform of $e_1(x) = e(e^x) \in L_2(\mathbb{R})$:

$$f_1(\xi) = f(\mathrm{i}\xi) = \int_0^\infty t^{\mathrm{i}\xi} e(t) \frac{dt}{t} = \int_{-\infty}^\infty e^{\mathrm{i}x\xi} e(e^x)\, dx = \int_{-\infty}^\infty e^{\mathrm{i}x\xi} e_1(x)\, dx,$$

so by Fourier's inversion formula,

$$e(t) = e_1(x) = \tfrac{1}{2\pi} \int_{-\infty}^\infty e^{-\mathrm{i}x\xi} f_1(\xi)\, d\xi = \tfrac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = 0} t^{-s} f(s)\, ds. \tag{5.12}$$

Similarly, $f(c + \mathrm{i}\xi)$ is the conjugate Fourier transform of $e(e^x) e^{xc}$ for $c > 0$.

The hypothesis on exponential decrease of $e(t)$ in the sectors V_δ allows us to shift the path of integration in (5.6) from $t \in \mathbb{R}_+$ to $t \in e^{\mathrm{i}\delta} \mathbb{R}_+$ for $|\delta| < \theta_0$ (corresponding to a shift to $x \in \mathbb{R} + i\delta$); this gives:

$$f(c + \mathrm{i}\xi) = \int_0^\infty (re^{\mathrm{i}\delta})^{c + \mathrm{i}\xi} e(re^{\mathrm{i}\delta}) \frac{dr}{r} = e^{-\delta\xi} \int_0^\infty r^{\mathrm{i}\xi} e(re^{i\delta})(re^{\mathrm{i}\delta})^c \frac{dr}{r} = e^{-\delta\xi} g(\delta, \xi, c),$$

where g is bounded as a function of $\xi \in \mathbb{R}$, locally uniformly in $c \ge 0$. Taking $\delta > 0$ for $\xi > 0$ and $\delta < 0$ for $\xi < 0$, we see that $f(c + \mathrm{i}\xi)$ decreases exponentially (like $e^{-\delta|\xi|}$) for

$|\xi| \to \infty$, in any vertical strip $\{s = c + i\xi \mid C_1 \le c \le C_2,\ \xi \in \mathbb{R}\}$ with $0 \le C_1 \le C_2$. Then we can also shift the integration path in (5.12) from $\operatorname{Re} s = 0$ to $\operatorname{Re} s = c$, $c \ge 0$. This shows 1°.

2°. Assume now in addition (5.8). Let us first write $f(s)$ as

$$f(s) = \int_0^1 t^{s-1} e(t)dt + \int_1^\infty t^{s-1} e(t)dt.$$

The second integral defines an entire function of s. The expansion (5.8) means that

$$e(t) = \sum_{j=0}^{N-1} \sum_{l=0}^{m_j} a_{j,l} t^{\beta_j} (\log t)^l + \varrho_N(t), \quad \varrho_N(t) = O(|t|^{\beta_N - \varepsilon}) \text{ for } t \to 0 \text{ in } V_\delta, \tag{5.13}$$

for $\varepsilon > 0$ and any positive integer N; we insert this in the first integral. Observe the formulas, valid for $\operatorname{Re} s > -\beta$,

$$\int_0^1 t^{s-1+\beta} (\log t)^l \, dt = \frac{(-1)^l l!}{(s+\beta)^l}, \quad \int_0^\infty t^{s-1+\beta} (\log t)^l \, e^{-t} \, dt = \partial_s^l \Gamma(s+\beta), \tag{5.14}$$

where the cases $l > 0$ follow from the cases $l = 0$ by application of ∂_s^l. The remainder $\varrho_N(t)$ in (5.13) gives a function holomorphic for $\operatorname{Re} s > -\beta_N + \varepsilon$, and for the powers of t we use (5.14); this shows (5.9).

To show the exponential decrease of $f(s)$ on general vertical strips, one can shift the contour in (5.6) and proceed much as in the proof of Proposition 2.9. Another method, that we record here since it may be useful for further estimates, is to insert the expansion $e^t = \sum_{\nu \ge 0} \frac{1}{\nu!} t^\nu$, that gives

$$e^t t^{\beta_j} (\log t)^l = \sum_{\nu=0}^{M-1} \tfrac{1}{\nu!} t^{\beta_j + \nu} (\log t)^l + O(t^{\beta_j + M - \varepsilon}),$$

for any $\varepsilon > 0$ and positive integer M. Then we can write

$$e(t) = e(t) e^t e^{-t} = \Big(\sum_{\beta_j + \nu < M} \sum_{l \le m_j} a_{j,l} \tfrac{1}{\nu!} t^{\beta_j + \nu} (\log t)^l \Big) e^{-t} + \tilde{\varrho}_M(t),$$
$$\text{with } \tilde{\varrho}_M(t) = O(|t|^{M-\varepsilon}) \text{ for } t \to 0 \text{ in } V_\delta,$$

where $\tilde{\varrho}_M(t)$ is exponentially decreasing for $|t| \to \infty$ in V_δ since the other terms are so, and hence

$$f(s) = \int_0^\infty t^{s-1} \Big(\sum_{\beta_j + \nu < M} \sum_{l \le m_j} a_{j,l} \tfrac{1}{\nu!} t^{\beta_j + \nu} (\log t)^l \Big) e^{-t} dt + \int_0^\infty t^{s-1} \tilde{\varrho}_M(t) \, dt. \tag{5.15}$$

The last integral defines a function that is holomorphic for $\operatorname{Re} s > -M + \varepsilon$ and exponentially decreasing (like $e^{-\delta |\operatorname{Im} s|}$) on strips $-M + \varepsilon < C_1 \le \operatorname{Re} s \le C_2$, by 1°. For the contributions from the first integral we use the second formula in (5.14) together with the fact that the gamma function $\Gamma(s)$ and its derivatives are $O(e^{(-\frac{\pi}{2}+\varepsilon')|\operatorname{Im} s|})$, any $\varepsilon' > 0$, for $|\operatorname{Im} s| \ge 1$, $-\infty < C_1 \le \operatorname{Re} s \le C_2 < \infty$, cf. e.g. [B, pp. 181–182]. This gives (5.10), completing the proof of a) $\Longrightarrow$ b).

Conversely, assume b). Then $e(t)$ is given by (5.7), and we obtain the expansion (5.8) by shifting the contour of integration past the poles of $f(s)$. The remainder after all terms up to and including the singularity $s = -\beta_N$ is given by an integral like (5.7) but with $c < -\beta_N$; it is $O(|t|^{\beta_N - \varepsilon})$.

3°. That $e(t)$ defined here is exponentially decreasing for $|t| \to \infty$ in V_δ, $\delta < \theta_0$, follows since $|e^{-\lambda t}| \le e^{-\gamma |t|}$ with $\gamma > 0$ on the integration curve. The estimate for $t \to 0$ follows since

$$\int_{\mathcal{C}} e^{-\lambda t} r(\lambda)\, d\lambda = (-t)^{-m} \int_{\mathcal{C}} (\partial_\lambda^m e^{-\lambda t}) r(\lambda)\, d\lambda = t^{-m} \int_{\mathcal{C}} e^{-\lambda t} \partial_\lambda^m r(\lambda)\, d\lambda$$

for $t \in V_\delta$, where $e^{-\lambda t}\partial_\lambda^m r(\lambda)$ has a fixed integrable majorant for $t \to 0$. The formula (5.6) for f is shown by a complex change of variables, where we replace t by u/λ for each λ; when $\arg \lambda \in]0, \frac{\pi}{2}[$, the ray $\mathbb{R}_+$ is transformed to a ray Λ_λ with argument $-\arg\lambda \in]-\frac{\pi}{2}, 0[$, and vice versa. The integral of $u^{s-1}e^{-u}$ on such a ray is again equal to $\Gamma(s)$, as noted above. Thus (recall that $r(\lambda)$ is $O(|\lambda|^{m-1})$)

$$\begin{aligned}\int_0^\infty t^{s-1} \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} r(\lambda)\, d\lambda dt &= \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \int_{\Lambda_\lambda} u^{s-1}\lambda^{-s} e^{-u} r(\lambda)\, du d\lambda \\ &= \Gamma(s) \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-s} r(\lambda)\, d\lambda. \qquad \Box\end{aligned}$$

Definition 5.2. For $e(t)$ and $f(s)$ as in Proposition 5.1 1°, the operator mapping f to e will be denoted $\mathcal{M}^{-1}$.

5.2 Exponential trace estimates.

Now we can show:

Theorem 5.3. *Let D_{ij} and D'_{ij} be as in (2.15) ff., of order d, and let as usual $D''_{11} = D'_{11}$, $D''_{22} = \sigma^* D'_{22}\sigma$. The exponential function traces have the following structure (cf. Definition 5.2):*

$$\begin{aligned}\operatorname{Tr}(D_{ii}e^{-t\Delta_i}) &= \operatorname{Tr}_+(D_{ii}e^{-t\widetilde{\Delta}_i}) \\ &\quad + \mathcal{M}^{-1}_{s\to t}\big[\tfrac14 (F_{\frac12}(s) - 1)\zeta(D''_i, A^2, s) + (-1)^i \tfrac14 F_{\frac12}(s)\eta(D''_i, A^2, s)\big] \\ &\quad + (-1)^i \tfrac14 \operatorname{Tr}(D''_i \Pi_0(A)) + \varepsilon_{ii}(t), \quad i = 1, 2,\end{aligned} \tag{5.16}$$

$$\begin{aligned}\operatorname{Tr}(D_{12}Pe^{-t\Delta_1}) &= \operatorname{Tr}_+(D_{12}Pe^{-t\widetilde{\Delta}_1}) + \mathcal{M}^{-1}_{s\to t}\big[\tfrac14 (F_{\frac12}(s-1) - 1)\eta(D'_{12}\sigma, A, 2s-1)\big] \\ &\quad + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(D'_{12}\sigma\Pi_0(A))t^{-\frac12} + \varepsilon_{12}(t),\end{aligned} \tag{5.17}$$

$$\begin{aligned}\operatorname{Tr}(D_{21}P^*e^{-t\Delta_2}) &= \operatorname{Tr}_+(D_{21}P^*e^{-t\widetilde{\Delta}_2}) + \mathcal{M}^{-1}_{s\to t}\big[\tfrac14 (F_{\frac12}(s-1) - 1)\eta(\sigma^* D'_{21}, A, 2s-1)\big] \\ &\quad + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\sigma^* D'_{21}\Pi_0(A))t^{-\frac12} + \varepsilon_{21}(t);\end{aligned} \tag{5.18}$$

here the $\varepsilon_{ij}(t)$ are $O(t^M)$ for $t \to 0$, any M, and all terms are holomorphic in $V_{\frac{\pi}{2}}$.

The Tr_+ terms have expansions $\sum_{j\ge 0} c_{j,+} t^{\frac{j-n-d}{2}}$ (with $c_{j,+} = 0$ for $j+d$ odd), and the $\mathcal{M}^{-1}$ terms have expansions $\sum_{j\ge 1}(c_j \log t + c'_j) t^{\frac{j-n-d}{2}}$. For (5.16), the c_j are zero for $j < n$; and for (5.17), (5.18), the c_j are 0 for $j < n-1$.

Proof. Consider (5.16) with $i = 1$ and write D_{11} as D. We want to translate the information in Theorem 3.4 to a statement on the corresponding exponential function.

To do this, we first observe that with $\mathcal{C} = \mathcal{C}_{\theta, r_0}$, $\theta \in \,]0, \frac{\pi}{2}[$ and r_0 satisfying (3.11), the functions valued in $\mathcal{L}(L_2(X, E_1))$

$$
\begin{aligned}
e^{-t\Delta_1}\Pi_0^\perp(\Delta_1) &= \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} e^{-\lambda t}(\Delta_1 - \lambda)^{-1}\Pi_0^\perp(\Delta_1)\, d\lambda \quad \text{and} \\
\Gamma(s) Z(\Delta_1, s) &= \Gamma(s)\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}(\Delta_1 - \lambda)^{-1}\Pi_0^\perp(\Delta_1)\, d\lambda,
\end{aligned}
\tag{5.19}
$$

correspond to one another as in Proposition 5.1 3°; for $(\Delta_1 - \lambda)^{-1}\Pi_0^\perp(\Delta_1)$ is holomorphic at $\lambda = 0$ and $O(|\lambda|^{-1})$ with λ-derivative $O(|\lambda|^{-2})$ for $\lambda \to \infty$ in S_δ, any $\delta < \pi$. So they are defined from one another by formulas (5.6), (5.7). Consider the traces of these operators composed with D (for $\operatorname{Re} s$ sufficiently large); they can also be written as integrals of traces of resolvent derivatives (of order $k > \frac{n+d}{2} - 1$), by integration by parts:

$$
\begin{aligned}
e_1(t) &= \operatorname{Tr} D e^{-t\Delta_1}\Pi_0^\perp(\Delta_1) = t^{-k}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} e^{-\lambda t}\operatorname{Tr} D\partial_\lambda^k(\Delta_1 - \lambda)^{-1}\Pi_0^\perp(\Delta_1)\, d\lambda, \\
f_1(s) &= \operatorname{Tr}\Gamma(s) D Z(\Delta_1, s) = \tfrac{\Gamma(s)}{(s-k)\dots(s-1)}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{k-s}\operatorname{Tr} D\partial_\lambda^k(\Delta_1 - \lambda)^{-1}\Pi_0^\perp(\Delta_1)\, d\lambda.
\end{aligned}
\tag{5.20}
$$

Since $\operatorname{Tr} D\partial_\lambda^k(\Delta_1 - \lambda)^{-1}\Pi_0^\perp(\Delta_1)$ is holomorphic at 0, and $O(|\lambda|^{-2})$ for λ going to infinity in S_δ when $k \ge 1 + \frac{n+d}{2}$ in view of (3.27), Proposition 5.1 3° shows that $e_1(t)$ is holomorphic and exponentially decreasing in a sector V_{θ_0} and $O(|t|^{-k})$ for $t \to 0$ there. Proposition 2.9 shows that $f_1(s)$ is exponentially decreasing for $|\operatorname{Im} s| \to \infty$, $\operatorname{Re} s > k$. Then we can take traces in the formulas (5.6) and (5.7) relating the functions in (5.19), and we find that e_1 and f_1 are related in the same way, with e_1 satifying the hypotheses of Proposition 5.1 1°, in short,

$$
e_1 = \mathcal{M}^{-1} f_1.
$$

One shows in a similar way that

$$
\begin{aligned}
e_2(t) &= \operatorname{Tr}_+ D e^{-t\widetilde{\Delta}_1}\Pi_0^\perp(\widetilde{\Delta}_1) = \tfrac{\mathrm{i}}{2\pi} t^{-k}\int_{\mathcal{C}} e^{-\lambda t}\operatorname{Tr}_+ D\partial_\lambda^k(\widetilde{\Delta}_1 - \lambda)^{-1}\Pi_0^\perp(\widetilde{\Delta}_1)\, d\lambda \quad \text{and} \\
f_2(s) &= \operatorname{Tr}_+ \Gamma(s) D Z(\widetilde{\Delta}_1, s) = \tfrac{\Gamma(s)}{(s-k)\dots(s-1)}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{k-s}\operatorname{Tr}_+ D\partial_\lambda^k(\widetilde{\Delta}_1 - \lambda)^{-1}\Pi_0^\perp(\widetilde{\Delta}_1)\, d\lambda
\end{aligned}
$$

satisfy

$$
e_2 = \mathcal{M}^{-1} f_2.
$$

Next, consider

$$
\begin{aligned}
f_3(s) &= \Gamma(s)\big[\tfrac14(F_{\frac12}(s)-1)\zeta(D',A^2,s)-\tfrac14 F_{\frac12}(s)\eta(D',A,2s)\big]\\
&= \operatorname{Tr}_{X^0}(\Gamma(s)G_{Z,\mathrm{e},s}+\Gamma(s)G_{Z,\mathrm{o},s}),
\end{aligned}
\tag{5.21}
$$

cf. Proposition 3.2. Setting

$$
\begin{aligned}
F_3(s) &= D'\Gamma(s)\operatorname{tr}_n(G_{Z,\mathrm{e},s}+G_{Z,\mathrm{o},s}) = D'\Gamma(s)\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}\lambda^{-s}\operatorname{tr}_n(G_{\mathrm{e},\lambda}+G_{\mathrm{o},\lambda})\,d\lambda,\\
E_3(t) &= D'\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}e^{-\lambda t}\operatorname{tr}_n(G_{\mathrm{e},\lambda}+G_{\mathrm{o},\lambda})\,d\lambda,
\end{aligned}
$$

we see that these (operator valued) functions are related to one another as in Proposition 5.1 3° (in view of (3.10), $\operatorname{tr}_n G_{\mathrm{e},\lambda}$ and $\operatorname{tr}_n G_{\mathrm{o},\lambda}$ are holomorphic at $\lambda=0$ and $O(|\lambda|^{-1})$ in $L_2(X',E_1')$ operator norm for $\lambda\to\infty$, with first derivatives $O(|\lambda|^{-2})$), so they correspond to one another by (5.6), (5.7). Here $\operatorname{Tr}_{X'}F_3(s)=f_3(s)$.

Again we can rewrite the integrals by integration by parts, to get integrals with derivatives $\partial_\lambda^k D'\operatorname{tr}_n(G_{\mathrm{e},\lambda}+G_{\mathrm{o},\lambda})$ that are of trace class on X' for sufficiently large k, with traces having suitable behavior at 0 and at ∞ (cf. (3.28)), so that we can take traces in the formulas (5.6), (5.7) relating F_3 and E_3. Then $f_3(s)$ corresponds to the function

$$
e_3(t)=\operatorname{Tr}_{X'}E_3(t)=\operatorname{Tr}_{X^0}\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal C}e^{-\lambda t}D'(G_{\mathrm{e},\lambda}+G_{\mathrm{o},\lambda})\,d\lambda
\tag{5.22}
$$

by

$$
e_3=\mathcal M^{-1}f_3.
$$

We can now write (3.25) as

$$
\begin{aligned}
f_1(s) &= f_2(s)+f_3(s)+f_4(s),\\
f_4(s) &= \tfrac1s[\operatorname{Tr}_+(D\Pi_0(\widetilde\Delta_1))-\operatorname{Tr}(D\Pi_0(\Delta_1))-\tfrac14\operatorname{Tr}(D'\Pi_0(A))]+h_1(s),
\end{aligned}
\tag{5.23}
$$

where $f_4=\mathcal M e_4$ for a function e_4 satisfying the hypotheses of Proposition 5.1 1° (namely, $e_4=e_1-e_2-e_3$). Here f_4 has one simple pole at 0, cf. (5.23), so an application of Proposition 5.1 gives

$$
e_4(t)=\operatorname{Tr}_+(D\Pi_0(\widetilde\Delta_1))-\operatorname{Tr}(D\Pi_0(\Delta_1))-\tfrac14\operatorname{Tr}(D'\Pi_0(A))+O(t^M),\quad\text{for } t\to0,\ \text{any } M.
\tag{5.24}
$$

Since

$$
e_1(t)=\operatorname{Tr}De^{-t\Delta_1}-\operatorname{Tr}D\Pi_0(\Delta_1),\quad e_2(t)=\operatorname{Tr}_+De^{-t\widetilde\Delta_1}-\operatorname{Tr}_+D\Pi_0(\widetilde\Delta_1),
\tag{5.25}
$$

an application of $\mathcal M^{-1}$ to the first line in (5.23) and insertion of the detailed formulas for $e_1,\dots,e_4$ gives (5.16) for $i=1$.

The proof for $i = 2$ is analogous, and there are similar proofs of the formulas (5.17) and (5.18) based on Theorem 4.2.

The information on the asymptotic expansions of the Tr_+ terms is well-known (it can be deduced from Lemma 2.5 by Proposition 5.1), and the information on the $\mathcal{M}^{-1}$ terms follows, by Proposition 5.1, from the expansions with simple poles in Lemma 2.5 combined with the effect of the multiplication by $F_{\frac12}(s)$ resp. $F_{\frac12}(s-1)$. $\square$

For D_{ij} equal to a morphism φ, the precise singularity structure of the functions in (3.25) is described in Corollary 2.7 for n even and in Corollary 2.8 for n odd; and that of (4.7) and (4.8) is described in Corollary 4.4. We then find immediately, by application of Proposition 5.1 (recalling that the constant term must be subtracted from the exponential trace as in (5.25)):

Corollary 5.4. *The exponential trace* $\mathrm{Tr}(\varphi e^{-t\Delta_i})$ *has the following behavior for* $t \to 0$.

For n even:

$$
\begin{aligned}
(5.26)\quad \mathrm{Tr}(\varphi e^{-t\Delta_i}) &\sim \sum_{k\ge 0} c_{2k,+}(\varphi,\widetilde\Delta_i)\, t^{k-\frac n2} + \sum_{0\le k<\frac n2} \gamma_{n-1-2k} c_{2k}(\varphi^0,A^2)\, t^{k-\frac{n-1}{2}} \\
&+ \sum_{k\ge\frac n2} \big[-\beta_{k-\frac n2} c_{2k}(\varphi^0,A^2)\log t + \beta_{k-\frac n2} c'_{2k}(\varphi^0,A^2) + (\beta'_{k-\frac n2} - \tfrac14) c_{2k}(\varphi^0,A^2)\big]\, t^{k-\frac{n-1}{2}} \\
&+ (-1)^i \tfrac14 \sum_{0\le k\ne \frac n2 -1} \frac{c_{2k+1}(\varphi^0 A, A^2)}{\sqrt\pi\,(\frac n2 - k - 1)}\, t^{k+1-\frac n2} \\
&+ (-1)^i \big[-\frac{c_{n-1}(\varphi^0 A, A^2)}{4\sqrt\pi}\log t + \frac{c'_{n-1}(\varphi^0 A, A^2)}{4\sqrt\pi} + \tfrac14 \mathrm{Tr}(\varphi^0 \Pi_0(A))\big].
\end{aligned}
$$

For n odd:

$$
\begin{aligned}
(5.27)\quad \mathrm{Tr}(\varphi e^{-t\Delta_i}) &\sim \sum_{k\ge 0} c_{2k,+}(\varphi,\widetilde\Delta_i)\, t^{k-\frac n2} \\
&+ \sum_{k\ge 0} \gamma_{n-1-2k} c_{2k}(\varphi^0,A^2)\, t^{k-\frac{n-1}{2}} + \sum_{m\ge 0} \varepsilon_m \zeta(\varphi^0, A^2, -m-\tfrac12)\, t^{m+\frac12} \\
&+ (-1)^i \big[\tfrac14 \sum_{k\ge 0} \frac{c_{2k+1}(\varphi^0 A, A^2)}{\sqrt\pi\,(\frac n2 - k - 1)}\, t^{k+1-\frac n2} + \tfrac14 \eta(\varphi^0, A, 0) + \tfrac14 \mathrm{Tr}(\varphi^0 \Pi_0(A))\big].
\end{aligned}
$$

[G2] shows an expansion with the $n+1$ first terms (up to and including the constant term) plus $O(t^{\frac38})$ in the case $\varphi = 1$, also for manifolds X that are not cylindrical near ∂X.

Corollary 5.5. *The associated exponential trace* $\mathrm{Tr}(\varphi P e^{-t\Delta_1})$ *has the following behavior for* $t \to 0$.

For n even:

$$
\begin{aligned}
(5.28)\quad \mathrm{Tr}(\varphi P e^{-t\Delta_1}) &\sim \sum_{k\ge 0} c_{2k+1,+}(\varphi P,\widetilde\Delta_1)\, t^{k-\frac n2} \\
&+ \sum_{0\le k<\frac n2 -1} \gamma_{n-3-2k} c_{2k+1}(\varphi^0\sigma A, A^2)\, t^{k-\frac{n-1}{2}} + \sum_{k\ge \frac n2 -1} \big[-\beta_{k+1-\frac n2} c_{2k+1}(\varphi^0\sigma A, A^2)\, t^{k-\frac{n-1}{2}}\log t \\
&+ (\beta_{k+1-\frac n2} c'_{2k+1}(\varphi^0\sigma A, A^2) + (\beta'_{k+1-\frac n2} - \tfrac14) c_{2k+1}(\varphi^0\sigma A, A^2))\, t^{k-\frac{n-1}{2}}\big] + \tfrac{1}{4\sqrt\pi}\mathrm{Tr}(\varphi^0\sigma\Pi_0(A))\, t^{-\frac12}.
\end{aligned}
$$

For n odd:

$$(5.29)\quad \operatorname{Tr}(\varphi P e^{-t\Delta_1}) \sim \sum_{k\geq 0} c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)\, t^{k-\frac{n}{2}} + \sum_{k\geq 0} \gamma_{n-3-2k} c_{2k+1}(\varphi^0\sigma A, A^2)\, t^{k-\frac{n-1}{2}} + \sum_{m\geq 0} \delta_m \eta(\varphi^0\sigma A, -2m)\, t^{m-\frac{1}{2}} + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\varphi^0\sigma\Pi_0(A))\, t^{-\frac{1}{2}}.$$

There are similar formulas for $\operatorname{Tr}(\varphi P^* e^{-t\Delta_2})$, *with* $\varphi^0\sigma$ *replaced by* $\sigma^*\varphi^0$.

The proof shows the advantage of working with the power functions, where the contributions from the boundary condition appear as simple multiplicative formulas involving the zeta and eta functions of A; this allows an exact analysis of the pole coefficients which can then be translated over to the coefficients in the heat expansions (by Proposition 5.1). It seems harder to get the formulas directly in the heat operator framework.

Also in these results we can replace $P_\geq$ by P_B as discussed in Section 3.3 and Corollary 4.6 ff.; this gives modifications of the constant term in (5.26)–(5.27) resp. the coefficient of $t^{-\frac{1}{2}}$ in (5.28)–(5.29), and there are special conclusions when P_B is selfadjoint.

Full expansions are shown for the non-cylindrical case in [GS], by quite different methods.

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