A WEAKLY POLYHOMOGENEOUS CALCULUS FOR PSEUDODIFFERENTIAL BOUNDARY PROBLEMS

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Dedicated to Ralph S. Phillips[1]

Abstract. In a joint work with R. Seeley, a calculus of weakly parametric pseudodifferential operators on closed manifolds was introduced and used to obtain complete asymptotic expansions of traces of resolvents and heat operators associated with the Atiyah-Patodi-Singer problem. The present paper establishes a generalization to pseudodifferential boundary operators, defining weakly polyhomogeneous singular Green operators, Poisson operators and trace operators associated with a manifold with boundary, as well as a suitable transmission condition for pseudodifferential operators. Full composition formulas are established for the calculus, which contains the resolvents of APS-type problems. The operators in the calculus have complete asymptotic trace expansions in the parameter (when of trace class), with polynomial and logarithmic terms.

Introduction

Parameter-dependent pseudodifferential operators are of interest in many contexts, in particular in the study of resolvents of given boundary value problems, and their application to the construction of complex powers of operators, and solutions of time-dependent problems. The study of the trace of such operator families leads to the determination of geometric invariants — a famous example is the index of an elliptic operator A obtained as the difference between two "heat traces" $\operatorname{Tr} e^{-tA^*A} - \operatorname{Tr} e^{-tAA^*}$. Related examples are zeta function expansions and eta function expansions of differential and pseudodifferential operators (ψdo's).

Parameter-dependent calculi are simplest when the parameter μ enters in the same way as the cotangent variables ξ, with joint polyhomogeneity in (ξ,μ) for $|(\xi,\mu)| \geq 1$, as considered e.g. in Shubin [S78]; we call such symbols *strongly polyhomogeneous*. But some problems necessitate weaker homogeneity requirements, valid for $|\xi| \geq 1$ only and supplied with suitable estimates for small ξ (large μ). A very general setup for ψdo's as well as for pseudodifferential boundary operators (ψdbo's) was worked out in the book [G86,96]; however, it is somewhat crude in the analysis of traces (giving expansions in

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[1]This paper is dedicated to Ralph S. Phillips who fostered my interest in the finer aspects of Functional Analysis — beginning with a Ph.D.-project in the sixties to interpret his abstractly defined "lateral boundary conditions", for which pseudodifferential operators turned out to be a key ingredient. I am grateful for all that I learned under his guidance. (Appeared in J. Funct. Anal. 184 (2001), 19–76.)

finitely many terms). A finer and more restricted analysis for ψdo's on closed manifolds, giving complete asymptotic expansions of traces, was established in a joint work with Seeley [GS95], with applications to the Atiyah-Patodi-Singer problem. The purpose of the present paper is to construct a similar calculus of ψdbo's on manifolds with boundary, leading to full asymptotic expansions in the parameter.

One of the main results is that operator families $\mathcal{A}(\mu) = P(\mu)_+ + G(\mu)$ of trace class belonging to this calculus have trace expansions:

$$\operatorname{Tr} \mathcal{A}(\mu) \sim \sum_{j=0}^{\infty} c_j \mu^{m-n-j} + \sum_{l=0}^{\infty} (c'_l \log \mu + c''_l) \mu^{d-l}, \tag{1}$$

for μ going to ∞ in sectors of $\mathbb{C}$ where the operator family is defined. A large effort is spent on developing full composition rules for the elements of the calculus, which will allow the treatment of quite general operators.

The present work began with a sketch made during the collaboration with Seeley that led to [GS95]. In the treatment of Dirac operators, we first thought it necessary to handle general μ-dependent ψdbo's, but then found a simple way to "reduce to the boundary" (transforming the trace problems to problems for μ-dependent ψdo's on the boundary) so that the general study was no longer necessary. The present author has continued the work over a long period. The resulting study presented here is concerned with a class of symbols that includes all the strongly polyhomogeneous ones, as well as μ-independent ψdo's *in the boundary.* However, the present calculus is not made to include compositions with the general μ-independent Poisson, trace and singular Green operators of the original Boutet de Monvel calculus [BM71] (except for particularly convenient cases; see Remark 6.11). This is due to the fact that the normal variable plays together with the other variables in very different ways in the μ-dependent and the μ-independent cases.

In this connection it should be mentioned that the special question of asymptotic trace expansions for operators $\mathcal{A}R(\mu)$ that are compositions of a μ-independent ψdbo $\mathcal{A}$ and a resolvent $R(\mu)$ of an elliptic *differential operator* problem, is addressed in a joint work with Schrohe [GSc99] by different methods, making a very accurate use of the structure of the symbols of differential operator resolvents as rational functions with well-behaved poles.

The present calculus allows obtaining trace expansions for general compositions of parameter-dependent operators, with general interior symbols. The special cases treated earlier in [GS95], [G99] depended on having the singular Green operators expressed as finite combinations $\sum_{j\le J} K_j(\mu)S_j(\mu)T_j(\mu)$ with strongly polyhomogeneous Poisson and trace operators K_j and T_j and weakly polyhomogeneous ψdo's S_j; this is not needed here.

For cases where the manifold has conical singularities or corners instead of a boundary, expansions like (1) have been obtained for elliptic resolvents by Gil [Gi98], Loya [L98].

Plan of the paper: In Section 1, we recall some known facts on ψdbo's without a parameter. In Section 2, the symbol-kernels and symbols of a basic type of parameter-dependent boundary operators (Poisson, trace and singular Green operators of class 0) are introduced. Section 3 accounts for how the polyhomogeneous symbols enter in the calculus. Then we give a fairly selfcontained explanation of the composition rules for these operators in Section 4, based on a formulation using the x_n-variable (the "real formulation"). The reader who is mainly interested in trace expansions of such operators can go directly to Section 7 from here. However, for a complete calculus including an interior ψdo and operators of positive class, we need to use the "complex formulation" (the Wiener-Hopf calculus in terms of the conormal variable ξ_n in the Fourier transformed setting); this is the subject of the next sections, where the presentation leans heavily on that of [G86,96] to save repetition. (Henceforth we refer only to [G96]; a reader with [G86] at hand can find most of the information there too.) Section 5 introduces the symbols of positive class and their mechanisms, and Section 6 defines interior ψdo's with the appropriate transmission condition, and sets up the full composition rules. In Section 7 we draw the conclusions on trace formulas, establishing expansions of the form (1).

1. Background and preliminaries

In order to explain the new calculus, we begin by recalling some elements of the established theories. In the calculus introduced by Boutet de Monvel [BM71], one considers operator systems (sometimes called Green operators) on $\overline{\mathbb{R}}^n_+ = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$:

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\overline{\mathbb{R}}^n_+)^N \\ \times \\ C^\infty(\mathbb{R}^{n-1})^M \end{matrix} \rightarrow \begin{matrix} C^\infty(\overline{\mathbb{R}}^n_+)^{N'} \\ \times \\ C^\infty(\mathbb{R}^{n-1})^{M'} \end{matrix} . \tag{1.1}$$

Here P is a ψdo on $\mathbb{R}^n$ satisfying the transmission condition at the boundary $\mathbb{R}^{n-1}$ of $\overline{\mathbb{R}}^n_+$ and P_+ is its truncation to $\overline{\mathbb{R}}^n_+$,

$$P_+ = r^+ P e^+, \tag{1.2}$$

where r^+ means restriction to $\mathbb{R}^n_+$ and e^+ means extension by 0 on $\mathbb{R}^n \setminus \mathbb{R}^n_+$; G is a singular Green operator (s.g.o.); T is a trace operator; K is a Poisson operator (in some texts called a potential operator); S is a ψdo on $\mathbb{R}^{n-1}$. More generally, the operators can be defined (by the help of local coordinates) to act on a manifold X with boundary X'.

In the calculus of such systems it is important to show that the composition of two Green operators $\mathcal{A}$ and $\mathcal{A}'$ as in (1.1) gives a Green operator:

$$\mathcal{A}'' = \mathcal{A}\mathcal{A}' = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \begin{pmatrix} P'_+ + G' & K' \\ T' & S' \end{pmatrix} = \begin{pmatrix} P''_+ + G'' & K'' \\ T'' & S'' \end{pmatrix}, \tag{1.3}$$

where (with $G'' = L(P, P') + G'''$)

$$\begin{aligned} &\text{(i)} && P'' = PP' \text{ is a } \psi\text{do with the transmission condition,} \\ &\text{(ii)} && L(P, P') = (PP')_+ - P_+P'_+ \text{ is an s.g.o.,} \\ &\text{(iii)} && G''' = P_+G' + GP'_+ + GG' + KT' \text{ is an s.g.o.,} \\ &\text{(iv)} && T'' = TP'_+ + TG' + ST' \text{ is a trace operator,} \\ &\text{(v)} && K'' = P_+K' + GK' + KS' \text{ is a Poisson operator,} \\ &\text{(vi)} && S'' = TK' + SS' \text{ is a } \psi\text{do on } \Gamma. \end{aligned} \tag{1.4}$$

In fact this involves a number of composition rules among the various operators. The point of the present work is to define a calculus of parameter-dependent operators respecting these rules and moreover allowing full asymptotic trace formulas.

Let us recall the structure of the original operator types G, T and K. They are defined by Fourier integral formulas from functions of the form

$$\tilde f(w,x_n,\xi'),\quad \tilde f(w,x_n,y_n,\xi'),\quad f(w,\xi',\xi_n),\quad f(w,\xi',\xi_n,\eta_n). \tag{1.5}$$

(The hypotheses imposed on these functions in order to define operators are explained further below.) In the applications, w represents a tangential variable $x'\in\mathbb{R}^{n-1}$ (or y' or (x',y')), $x_n,y_n\in\mathbb{R}_+$ are normal variables, and $\xi_n,\eta_n\in\mathbb{R}$ are (conormal) variables corresponding to x_n resp. y_n in the Fourier transformed setting. In the following, we include a parameter μ to save repetition of the formulas later.

As in [GK93], [G96], we use the notation: $\dot\le$ means "$\le$ a constant, independent of the space variable, times"; similarly $\dot\ge$ means "$\ge$ a constant times"; and $\doteq$ means that both $\dot\le$ and $\dot\ge$ hold. The constants vary from case to case. As usual we denote $(1+|\xi_1|^2+\dots|\xi_n|^2)^{\frac12}=\langle\xi\rangle$, and denote by $[\xi]$ a smooth positive function of ξ that coincides with $|\xi|$ for $|\xi|\ge\frac12$. We denote $\{0,1,2,\dots\}=\mathbb{N}$.

Recall that a pseudodifferential operator $P=\operatorname{OP}(p(x,\xi,\mu))$ is defined for each fixed μ from a symbol $p(x,\xi,\mu)$ by:

$$Pu(x)=\operatorname{OP}(p(x,\xi,\mu))u=\int_{\mathbb{R}^n}e^{ix\cdot\xi}p(x,\xi,\mu)\hat u(\xi)\,đ\xi=\int_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi}p(x,\xi,\mu)u(y)\,dyđ\xi,$$

where $\hat u(\xi)=(\mathcal{F}u)(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)\,dx$, and $đ\xi=(2\pi)^{-n}d\xi$; here the last expression allows symbols p depending also on y. For boundary problems, $\mathbb{R}^n$ is considered as $\mathbb{R}^{n-1}\times\mathbb{R}$ with coordinates (x',x_n) (or (y',y_n), (ξ',ξ_n), etc.); the ψdo definition applied in $\mathbb{R}^{n-1}$ is denoted OP'.

We first recall the "real formulation" of the boundary operators defined relative to $\overline{\mathbb{R}}^n_+$. A function $\tilde f(x',y',x_n,\xi',\mu)$ defines (for each fixed μ) a *Poisson operator* $K=\operatorname{OPK}(\tilde f)$, or a *trace operator of class* 0, $T=\operatorname{OPT}(\tilde f)$, by:

$$\begin{aligned}(Kv)(x)&=\operatorname{OPK}(\tilde f)v=\int_{\mathbb{R}^{2n-2}}e^{i(x'-y')\cdot\xi'}\tilde f(x',y',x_n,\xi',\mu)v(y')\,dy'đ\xi',\\ (Tu)(x')&=\operatorname{OPT}(\tilde f)u=\int_{\mathbb{R}^{2n-2}}e^{i(x'-y')\cdot\xi'}\int_0^\infty\tilde f(x',y',y_n,\xi',\mu)u(y)\,dy_ndy'đ\xi',\end{aligned}\tag{1.6}$$

for $v\in C_0^\infty(\mathbb{R}^{n-1})$ resp. $u\in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)=r^+C_0^\infty(\mathbb{R}^n)$. T is the adjoint of a Poisson operator, namely $\operatorname{OPK}(\overline{\tilde f}(y',x',x_n,\xi',\mu))$. A trace operator of class $r\ge0$ is an operator of the form

$$T_r=\textstyle\sum_{0\le j<r}S_j\gamma_j+T,\tag{1.7}$$

where T is as in (1.6), the γ_j are the standard trace operators $\gamma_ju=(D^j_{x_n}u)|_{x_n=0}$ and the S_j are ψdo's on $\mathbb{R}^{n-1}$; here T is well-defined on L_2 functions in contrast to the γ_j. (The general trace operators are also covered if we include compositions $TD^j_{x_n}$, $j\in\mathbb{N}$, since

$$S_j\gamma_j=S_j\gamma_j(\langle D'\rangle-iD_n)_+^{-j-1}(\langle D'\rangle-iD_n)^{j+1}=T_j'(\langle D'\rangle-iD_n)^{j+1},\tag{1.8}$$

where $T_j'=S_j\gamma_j(\langle D'\rangle-iD_n)_+^{-j-1}$ is a trace operator of class 0 by [GK93, (1.34) and Lemma 3.4].)

A function $\tilde f(x',y',x_n,y_n,\xi',\mu)$ defines for each μ a *singular Green operator of class* 0

$$(1.9)\quad (Gu)(x)=\operatorname{OPG}(\tilde f)u=\int_{\mathbb{R}^{2n-2}} e^{i(x'-y')\cdot\xi'}\int_0^\infty \tilde f(x',y',x_n,y_n,\xi',\mu)u(y)\,dy_n dy' d\!\!{}^-\xi',$$

for $u\in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$. More generally, a singular Green operator of class $r\ge 0$ is an operator of the form

$$(1.10)\qquad G_r=\textstyle\sum_{0\le j<r}K_j\gamma_j+G,$$

where the K_j are Poisson operators and G is as above.

The function $\tilde f$ in (1.6) resp. (1.9) is called the *symbol-kernel* of K or T, resp. G. When $\tilde k$ defines a Poisson operator, the function $k(w,\xi',\xi_n,\mu)$ obtained by Fourier transformation $\mathcal{F}_{x_n\to\xi_n}$ of $e^+\tilde k(w,x_n,\xi',\mu)$ is called the associated *Poisson symbol*. When $\tilde t$ defines a trace operator of class 0, the associated *trace symbol* $t(w,\xi',\xi_n,\mu)$ is obtained from $e^+\tilde t$ by co-Fourier transformation $\overline{\mathcal{F}}_{x_n\to\xi_n}$. Here $\mathcal{F}_{x_n\to\xi_n}u=\int e^{-ix_n\xi_n}u(x_n)\,dx_n$, $\overline{\mathcal{F}}_{x_n\to\xi_n}u=\int e^{+ix_n\xi_n}u(x_n)\,dx_n$. When $\tilde g(x',y',x_n,y_n,\xi',\mu)$ defines a singular Green operator of class 0, the associated *singular Green symbol* is $g(x',y',\xi',\xi_n,\eta_n,\mu)=\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde g$. We shall use the abbreviations $\mathcal{S}_+$ and $\mathcal{S}_{++}$ for the spaces $\mathcal{S}(\overline{\mathbb{R}}_+)$ and $\mathcal{S}(\overline{\mathbb{R}}^2_{++})$ (where $\overline{\mathbb{R}}^2_{++}=\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+$):

$$(1.11)\qquad \begin{aligned}\mathcal{S}_+&=\mathcal{S}(\overline{\mathbb{R}}_+),\text{ the restriction of }\mathcal{S}(\mathbb{R})\text{ to }\overline{\mathbb{R}}_+,\\ \mathcal{S}_{++}&=\mathcal{S}(\overline{\mathbb{R}}^2_{++}),\text{ the restriction of }\mathcal{S}(\mathbb{R}^2)\text{ to }\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+;\end{aligned}$$

the symbol-kernels are in these spaces as functions of x_n, resp. (x_n,y_n).

Next, we recall the "complex formulation", where the symbols are used. Here one uses the conventions (where $\mathbb{C}[\xi_n]$ stands for the space of polynomials in ξ_n):

$$(1.12)\qquad \begin{gathered}\mathcal{F}(e^+\mathcal{S}_+)=\mathcal{H}^+=\mathcal{H}^+_{-1},\quad \overline{\mathcal{F}}(e^+\mathcal{S}_+)=\mathcal{H}^-_{-1},\quad \overline{\mathcal{F}}(e^+\mathcal{S}_+)\dot+\mathbb{C}[\xi_n]=\mathcal{H}^-,\\ \mathcal{H}^+\dot+\mathcal{H}^-=\mathcal{H},\quad \mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\mathcal{S}_{++}=\mathcal{H}^+\hat\otimes\mathcal{H}^-_{-1},\\ \mathcal{F}e^+r^+\mathcal{F}^{-1}=h^+,\quad \overline{\mathcal{F}}e^+r^+\overline{\mathcal{F}}^{-1}=h^-_{-1},\quad I-h^+=h^-;\end{gathered}$$

the operators h^+, h^-_{-1} and h^- are projections in $\mathcal{H}$ with range $\mathcal{H}^+$, $\mathcal{H}^-_{-1}$ resp. $\mathcal{H}^-$. (Further details are found e.g. in [G96, Sect. 2.2].) The ψdbo's are defined from the symbols by suitably interpreted formulas:

(1.13)

$$\begin{aligned}(Kv)(x)&=\operatorname{OPK}(k)v=\int_{\mathbb{R}^{2n-1}}e^{i(x'-y')\cdot\xi'+ix_n\xi_n}k(x',y',\xi,\mu)v(y')\,dy'd\!\!{}^-\xi,\\ (Tu)(x')&=\operatorname{OPT}(t)u=\int_{\mathbb{R}^{2n}}e^{i(x'-y')\cdot\xi'-iy_n\xi_n}t(x',y',\xi,\mu)u(y)\,dy d\!\!{}^-\xi,\\ (Gu)(x)&=\operatorname{OPG}(g)u=\int_{\mathbb{R}^{2n+1}}e^{i(x'-y')\cdot\xi'+ix_n\xi_n-iy_n\eta_n}g(x',y',\xi,\eta_n,\mu)u(y)\,dy d\!\!{}^-\xi d\!\!{}^-\eta_n.\end{aligned}$$

The integrations in ξ_n are here often understood as "plus-integrals": $\int^+\varphi(\xi_n)\,d\xi_n$ is the usual integral when $\varphi\in L_1(\mathbb{R})$, and it equals $\int_{\mathcal{L}_+}\varphi(\xi_n)\,d\xi_n$ when φ extends to a meromorphic function on $\mathbb{C}_+=\{z\in\mathbb{C}\mid \operatorname{Im}z>0\}$; then $\mathcal{L}_+$ is taken as a contour in $\mathbb{C}_+$ around the poles in $\mathbb{C}_+$.

The standard requirements for symbol-kernels and symbols of trace and Poisson operators *without a parameter* (cf. [BM71], [G96, Ch. 1]) are as follows: A *trace or Poisson symbol-kernel* $\tilde f(w,x_n,\zeta)$ of degree m is a function in $C^\infty(\mathbb R^\nu\times\overline{\mathbb R}_+\times\mathbb R^{n-1})$ that is in $\mathcal S_+$ with respect to x_n and satisfies

$$\sup_{x_n\ge 0}|x_n^l\partial_{x_n}^{l'}\partial_w^\beta\partial_\zeta^\alpha\tilde f(w,x_n,\zeta)|\mathrel{\dot\le}\langle\zeta\rangle^{m+1-l+l'-|\alpha|}, \tag{1.14}$$

for all indices $\alpha\in\mathbb N^{n-1}$, $\beta\in\mathbb N^\nu$, $l,l'\in\mathbb N$. An equivalent statement is that

$$\|x_n^l\partial_{x_n}^{l'}\partial_w^\beta\partial_\zeta^\alpha\tilde f(w,x_n,\zeta)\|_{L_2(\mathbb R_+)}\mathrel{\dot\le}\langle\zeta\rangle^{m+\frac12-l+l'-|\alpha|}, \tag{1.15}$$

for all indices $\alpha\in\mathbb N^{n-1}$, $\beta\in\mathbb N^\nu$, $l,l'\in\mathbb N$.

The associated *Poisson symbol* $f(w,\zeta,\xi_n)=\mathcal F_{x_n\to\xi_n}e^+\tilde f$ satisfies (equivalently)

$$\|h^+\partial_{\xi_n}^l\xi_n^{l'}\partial_w^\beta\partial_\zeta^\alpha f(w,\zeta,\xi_n)\|_{L_2(\mathbb R)}\mathrel{\dot\le}\langle\zeta\rangle^{m+\frac12-l+l'-|\alpha|}; \tag{1.16}$$

for all indices $\alpha\in\mathbb N^{n-1}$, $\beta\in\mathbb N^\nu$, $l,l'\in\mathbb N$. These symbols extend to C^∞ functions of $\xi_n\in\overline{\mathbb C}_-$ that are holomorphic on $\mathbb C_-=\{z\in\mathbb C\mid\operatorname{Im}z<0\}$. It is useful to observe that the estimates (1.16) hold also on lines $\operatorname{Im}\xi_n=-c$ with $c\ge0$, uniformly in c (in view of the Paley-Wiener theorem).

Similarly, the associated *trace symbol* $f_c(w,\zeta,\xi_n)=\overline{\mathcal F}_{x_n\to\xi_n}e^+\tilde f$ satisfies (equivalently)

$$\|h^-_{-1}\partial_{\xi_n}^l\xi_n^{l'}\partial_w^\beta\partial_\zeta^\alpha f_c(w,\zeta,\xi_n)\|_{L_2(\mathbb R)}\mathrel{\dot\le}\langle\zeta\rangle^{m+\frac12-l+l'-|\alpha|}; \tag{1.17}$$

for all indices $\alpha\in\mathbb N^{n-1}$, $\beta\in\mathbb N^\nu$, $l,l'\in\mathbb N$. The symbols f_c extend smoothly to holomorphic functions on $\mathbb C_+$, and the estimates (1.17) hold on lines $\operatorname{Im}\xi_n=c$ with $c\ge0$, uniformly in c.

The estimates in (1.14)–(1.15) are in particular satisfied at each w by C^∞ functions $\tilde f(w,x_n,\zeta)$ that are in $\mathcal S_+$ with respect to x_n and have the following *quasi-homogeneity:*

$$\tilde f(w,\tfrac{x_n}{a},a\zeta)=a^{m+1}\tilde f(w,x_n,\zeta),\ \text{ for } a\ge1,\ |\zeta|\ge1,\ \text{ all } x_n\in\overline{\mathbb R}_+. \tag{1.18}$$

To see this, note that for $|\zeta|\ge1$,

$$|\tilde f(w,x_n,\zeta)|=|\zeta|^{m+1}|\tilde f(w,|\zeta|x_n,\zeta/|\zeta|)|\le|\zeta|^{m+1}\sup_{u_n\ge0,|\eta|=1}|\tilde f(w,u_n,\eta)|$$

and that $x_n^l\partial_{x_n}^{l'}\partial_w^\beta\partial_\zeta^\alpha\tilde f$ satisfies (1.18) with $m+1$ replaced by $m+1-l+l'-|\alpha|$.

Since

$$\mathcal F_{x_n\to\xi_n}e^+\tilde f(w,\tfrac{x_n}{a},a\zeta)=af(w,a\zeta,a\xi_n),$$

the corresponding *Poisson symbol* $f(w,\zeta,\xi_n)$ is *homogeneous* in (ζ,ξ_n) of degree m for $|\zeta|\ge1$:

$$f(w,a\zeta,a\xi_n)=a^mf(w,\zeta,\xi_n)\text{ for } a\ge1,\ |\zeta|\ge1. \tag{1.19}$$

The corresponding trace symbol f_c is likewise homogeneous of degree m.

Since the homogeneity degree of these symbols is unambiguously defined (in contrast to the quasi-homogeneity property (1.18) where one could also let a shift from x_n to bx_n be decisive), our definition of *degree* will always refer to that of *the symbol.* The reason we say that the functions satisfying (1.14)–(1.17) are of degree m is that they obey the same estimates as functions associated with homogeneous symbols of degree m. Accordingly, we shall denote by

$$\mathcal{S}^m(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\mathcal{S}_+),\quad \mathcal{S}^m(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\mathcal{H}^+),\ \text{resp. } \mathcal{S}^m(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\mathcal{H}^-_{-1}), \tag{1.20}$$

the space of functions $\tilde f(w,x_n,\xi')$, $f(w,\xi',\xi_n)$, resp. $f_c(w,\xi',\xi_n)$ satisfying (1.15), (1.16) resp. (1.17). Short indications of the spaces in (1.20) are $\mathcal{S}^m(\mathcal{S}_+)$, $\mathcal{S}^m(\mathcal{H}^+)$, resp. $\mathcal{S}^m(\mathcal{H}^-_{-1})$.

A function $\tilde f(w,x_n,\zeta)$ (and the associated function $f=\mathcal{F}_{x_n\to\xi_n}e^+\tilde f$) is said to be a *classical polyhomogeneous* Poisson symbol-kernel (resp. symbol) of degree m, when there is a sequence of symbols f_j homogeneous of degree $m-j$, and corresponding quasi-homogeneous symbol-kernels $\tilde f_j$, such that $r_J=f-\sum_{j<J}f_j$ resp. $\tilde r_J=\tilde f-\sum_{j<J}\tilde f_J$ satisfies the estimates (1.16) resp. (1.14)–(1.15) with m replaced by $m-J$, for any J. There are similar conventions for trace symbol-kernels and symbols.

Example 1.1. A simple and important example of a Poisson symbol-kernel is $\tilde f(x_n,\zeta)=e^{-x_n[\zeta]}$, which is quasi-homogeneous as in (1.18) with $m=-1$, and hence satisfies (1.14)–(1.15) and lies in $\mathcal{S}^{-1}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\mathcal{S}_+)$. (Recall that $\mathrm{OPK}(e^{-x_n\langle\xi'\rangle})$ is the solution operator for the Dirichlet problem $(1-\Delta)u(x)=0$ on $\mathbb{R}^n_+$, $u(x',0)=\psi(x')$.) The corresponding symbol $\mathcal{F}_{x_n\to\xi_n}e^+e^{-x_n[\zeta]}=([\zeta]+i\xi_n)^{-1}$ is homogeneous of degree -1 in (ζ,ξ_n) for $|\zeta|\ge 1$.

Besides the degree, there is also a convention of assigning *orders* to the operators and their symbols. Here, when a symbol-kernel $\tilde f(w,x_n,\zeta)$ is of degree m, the Poisson operator it defines is said to be of order $m+1$, whereas the trace operator it defines is said to be of order m. This convention that stems from [BM71] and is also used in [G96], is chosen so that the composition TK of a trace operator of order m_1 and a Poisson operator of order m_2 is a ψdo in $\mathbb{R}^{n-1}$ of order m_1+m_2. In the present paper we shall mainly speak of the degree, since there is a large amount of other indices to keep track of.

A *singular Green symbol-kernel* $\tilde f(w,x_n,y_n,\zeta)$ of degree m and class 0 is a function in $C^\infty(\mathbb{R}^\nu\times\overline{\mathbb{R}}^2_{++}\times\mathbb{R}^{n-1})$ that is in $\mathcal{S}_{++}$ (cf. (1.11)) with respect to (x_n,y_n) and satisfies:

$$\sup_{x_n,y_n\ge 0}|x_n^l\partial_{x_n}^{l'}y_n^k\partial_{y_n}^{k'}\partial_w^\beta\partial_\zeta^\alpha\tilde f(w,x_n,y_n,\zeta)|\,\dot\le\,\langle\zeta\rangle^{m+2-l+l'-k+k'-|\alpha|}, \tag{1.21}$$

for all indices $\alpha\in\mathbb{N}^{n-1}$, $\beta\in\mathbb{N}^\nu$, $l,l',k,k'\in\mathbb{N}$. Equivalently,

$$\|x_n^l\partial_{x_n}^{l'}y_n^k\partial_{y_n}^{k'}\partial_w^\beta\partial_\zeta^\alpha\tilde f(w,x_n,y_n,\zeta)\|_{L_2(\mathbb{R}^2_{++})}\,\dot\le\,\langle\zeta\rangle^{m+1-l+l'-k+k'-|\alpha|}, \tag{1.22}$$

for all indices $\alpha\in\mathbb{N}^{n-1}$, $\beta\in\mathbb{N}^\nu$, $l,l',k,k'\in\mathbb{N}$. (Note that we here have a Hilbert-Schmidt norm of the integral operator on $\mathbb{R}_+$ with this kernel.) There is a third, equivalent formulation in terms of the associated *symbol* $f(w,\zeta,\xi_n,\eta_n)=\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde f$; it should satisfy:

$$\|h^+_{\xi_n}h^-_{-1,\eta_n}\partial_{\xi_n}^l\xi_n^{l'}\partial_{\eta_n}^k\eta_n^{k'}\partial_w^\beta\partial_\zeta^\alpha f(w,\zeta,\xi_n,\eta_n)\|_{L_2(\mathbb{R}^2)}\,\dot\le\,\langle\zeta\rangle^{m+1-l+l'-k+k'-|\alpha|}, \tag{1.23}$$

for all indices $\alpha \in \mathbb{N}^{n-1}$, $\beta \in \mathbb{N}^\nu$, $l, l', k, k' \in \mathbb{N}$. These symbols extend smoothly to functions of $(\xi_n, \eta_n) \in \overline{\mathbb{C}}_- \times \overline{\mathbb{C}}_+$ that are holomorphic on $\mathbb{C}_- \times \mathbb{C}_+$, such that the estimates hold for $\operatorname{Im} \xi_n = -c_1$, $\operatorname{Im} \eta_n = c_2$, uniformly in $c_1, c_2 \geq 0$.

We use the symbol-kernels (the (x_n, y_n)-formulation) in the immediate treatment of boundary operators, since they are easy to explain in a selfcontained way. But when we get to compositions with (interior) ψdo's in Section 5 below, we need the symbols and the Fourier transformed spaces. From then on we make heavy use of the constructions in the book [G96].

The estimates in (1.21)–(1.22) are in particular satisfied, at each w, by C^∞ functions $\tilde f(w, x_n, y_n, \zeta)$ that are in $\mathcal{S}_{++}$ with respect to (x_n, y_n) and have the following *quasi-homogeneity:*

$$\tilde f(w, \tfrac{x_n}{a}, \tfrac{y_n}{a}, a\zeta) = a^{m+2} \tilde f(w, x_n, y_n, \zeta), \text{ for } a \geq 1,\ |\zeta| \geq 1,\ \text{all } (x_n, y_n) \in \overline{\mathbb{R}}^2_{++}. \tag{1.24}$$

Here the corresponding *symbol* $f(w, \zeta, \xi_n, \eta_n)$ is *homogeneous* in (ζ, ξ_n, η_n) of degree m for $|\zeta| \geq 1$:

$$f(w, a\zeta, a\xi_n, a\eta_n) = a^m f(w, \zeta, \xi_n, \eta_n), \text{ for } a \geq 1,\ |\zeta| \geq 1. \tag{1.25}$$

As in the case of Poisson and trace symbol-kernels, the *degree* of $\tilde f$ or f refers to the homogeneity degree of the *symbol* f (and, for polyhomogeneous symbols, of its principal part).

Again we shall use *the degree* as indexation for the symbol- and symbol-kernel families, both for the symbols that are homogeneous and for those that just have estimates (1.21)–(1.23) *like* those of homogeneous functions. Thus, we shall denote by

$$\mathcal{S}^m(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathcal{S}_{++}), \text{ resp. } \mathcal{S}^m(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathcal{H}^+ \hat\otimes \mathcal{H}^-_{-1}), \tag{1.26}$$

the space of functions $\tilde f(w, x_n, y_n, \xi')$, resp. $f(w, \xi', \xi_n, \eta_n)$ satisfying (1.22) resp. (1.23). For a short notation we write $\mathcal{S}^m(\mathcal{S}_{++})$ resp. $\mathcal{S}^m(\mathcal{H}^+ \hat\otimes \mathcal{H}^-_{-1})$. An s.g.o. of degree m is said to be *of order* $m+1$ (this fits well with composition rules).

Finally, recall the basic composition rule for ψdo's: When $P = \mathrm{OP}(p(x, \xi, s))$ and $Q = \mathrm{OP}(q(x, \xi, t))$ (with parameters s and t), then PQ has symbol $p \circ q$, where

$$p(x, \xi, s) \circ q(x, \xi, t) \sim \sum_{\alpha \in \mathbb{N}^n} \tfrac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha p(x, \xi, s) \partial_x^\alpha q(x, \xi, t). \tag{1.27}$$

We write $\circ'$ instead of $\circ$ when the rule is applied with respect to variables $x', \xi' \in \mathbb{R}^{n-1}$ only. In compositions with boundary operators, $\circ'$ is applied in the tangential variables and there are special rules (denoted $\circ_n$) for the normal variable. Details are given e.g. in [G96] and will be taken up in the following when relevant.

2. Parameter-dependent symbol-kernels

We shall now introduce symbol classes depending on a parameter μ. Here we denote $\langle (\xi, \mu) \rangle = \langle \xi, \mu \rangle$ and $[(\xi, \mu)] = [\xi, \mu]$.

Let Γ denote a sector in the complex plane ($\Gamma \subset \mathbb{C} \setminus \{0\}$). As the sector Γ we usually take either $\mathbb{R}_+$ or a sector of the form $\{ \mu \in \mathbb{C} \mid \mu \neq 0, \, |\arg \mu| < \theta_0 \}$, for some $\theta_0 > 0$; then $\Gamma^{-1} = \Gamma$.

For our definition of symbols of boundary operators it is convenient to generalize the usual ψdo symbol spaces, valued in $\mathbb{C}$, to symbols taking values in $L_2(\mathbb{R}_+)$, $L_\infty(\mathbb{R}_+)$, $L_2(\mathbb{R}^2_{++})$ and other Banach spaces B. Then we define $S^m(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, B)$ to be the space of C^∞ functions $p(w, \xi')$ from $(w, \xi') \in \mathbb{R}^\nu \times \mathbb{R}^{n-1}$ to B satisfying

$$\|\partial_w^\beta \partial_{\xi'}^\alpha p(w, \xi')\|_B \dot{\le} \langle \xi' \rangle^{m-|\alpha|}, \text{ for all } \alpha, \beta. \tag{2.1}$$

The symbol spaces $S^{m,d}$ introduced in [GS95] will now be generalized in this manner, including moreover a growth factor and a power of $[\xi', \mu]$:

Definition 2.1. *Let m and $\delta \in \mathbb{R}$, d and $s \in \mathbb{Z}$. Let B be a Banach space, e.g., $B = \mathbb{C}$, $L_p(\mathbb{R}_+)$ or $L_p(\mathbb{R}^2_{++})$, $p \in [1, \infty]$. The space $S^{m,0,0}_\delta(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, B)$ consists of the C^∞ functions $\tilde{f}$ from $(w, \xi', \mu) \in \mathbb{R}^\nu \times \mathbb{R}^{n-1} \times \Gamma$ to $\tilde{f}(w, \xi', \mu) \in B$ that satisfy, for all $\alpha \in \mathbb{N}^{n-1}, \beta \in \mathbb{N}^\nu, j \in \mathbb{N}$, with $\mu = \frac{1}{z}$,*

$$\langle z\xi' \rangle^{-\delta} \|\partial_w^\beta \partial_{\xi'}^\alpha \partial_{|z|}^j \tilde{f}(w, \xi', \tfrac{1}{z})\|_B \dot{\le} \langle \xi' \rangle^{m-|\alpha|+j}, \text{ for } \tfrac{1}{z} \in \Gamma, \tag{2.2}$$
$$\text{with uniform estimates for } |z| \le 1, \ \tfrac{1}{z} \text{ in closed subsectors of } \Gamma.$$

Moreover, we set

$$S^{m,d,s}_\delta(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, B) = \mu^d [\xi', \mu]^s S^{m,0,0}_\delta(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, B). \tag{2.3}$$

Here we write $\tilde{f}$ as $\tilde{f}(w, x_n, \xi', \mu)$ resp. $\tilde{f}(w, x_n, y_n, \xi', \mu)$ when B is a space of functions of x_n, resp. (x_n, y_n) (e.g., $B = L_{p,x_n}(\mathbb{R}_+)$ or $L_{p,x_n,y_n}(\mathbb{R}^2_{++})$).

For $s = 0$, $\delta = 0$ this is a generalization of the symbol space $S^{m,d}$ defined in [GS95] to Banach space valued functions. The functions are called *holomorphic in μ* if they are holomorphic in $\mu \in \mathring{\Gamma}$ for $|(\xi', \mu)| \ge \varepsilon$ (some $\varepsilon > 0$); this property is preserved in compositions. We write $\partial_{|z|}$ instead of ∂_z as in [GS95], since it is really the radial derivatives that are controlled by the estimates, and $|z|$ will be used as a real variable. In detail, the estimates to be satisfied by functions in $S^{m,d,s}_\delta(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, B)$ are, with $z = \frac{1}{\mu}$,

$$\langle z\xi' \rangle^{-\delta} \|\partial_w^\beta \partial_{\xi'}^\alpha \partial_{|z|}^j (z^d [\xi', \tfrac{1}{z}]^{-s} \tilde{f}(w, \xi', \tfrac{1}{z}))\|_B \dot{\le} \langle \xi' \rangle^{m-|\alpha|+j}, \text{ for } \tfrac{1}{z} \in \Gamma, \tag{2.4}$$

uniformly for $|z| \le 1$, $\frac{1}{z}$ in closed subsectors of Γ.

The powers of $[\xi', \frac{1}{z}]$ are included to accommodate strongly polyhomogeneous symbols in a convenient way; cf. (2.28) and Theorem 3.2 below. δ will usually be 0 when B equals $\mathbb{C}$, $L_2(\mathbb{R}^2_{++})$ or an L_∞ space, resp. a half-integer $\pm\frac{1}{2}$ when $B = L_2(\mathbb{R}_+)$. (The half-integer powers of $\langle z\xi' \rangle$ do not fit well into the calculus otherwise.) When $\delta = 0$ it is omitted from the notation, and when both s and δ are 0, we often use the simpler notation

$$S^{m,d,0}_0(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, B) = S^{m,d}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, B). \tag{2.5}$$

As usual, such function spaces are provided with the Fréchet topologies defined by the associated systems of seminorms, e.g.,

$$\sup_{w\in\mathbb{R}^\nu,\,\xi'\in\mathbb{R}^{n-1},\,z\in\Gamma',\,|z|\le 1}\langle\xi'\rangle^{-m+|\alpha|-j}\langle z\xi'\rangle^{-\delta}\|\partial_w^\beta\partial_{\xi'}^\alpha\partial_{|z|}^j(z^d[\xi',\tfrac1z]^{-s}\tilde f(w,\xi',\tfrac1z))\|_B.$$

Note that $[\xi',\frac1z]=|(\xi',\frac1z)|$ for $|z|\le 1$, and that

$$|(\xi',\tfrac1z)|=|(\xi',\mu)|=|\mu|\,|(\xi'/\mu,1)|=|\mu|\,\langle\xi'/\mu\rangle=|z|^{-1}\langle z\xi'\rangle,\text{ when }\mu=\tfrac1z. \tag{2.6}$$

We henceforth use the short notation κ for $[\xi',\mu]$ (as in [G96]), so from now on,

$$\kappa=[\xi',\tfrac1z]=[\xi',\mu],\text{ with }\mu=\tfrac1z. \tag{2.7}$$

It is convenient to have several choices of δ available; fortunately the various spaces have simple relations to each other:

Lemma 2.2. *One has for all $\alpha\in\mathbb{N}^{n-1}$, $j\in\mathbb{N}$, $\sigma\in\mathbb{R}$,*

$$|\partial_{\xi'}^\alpha\partial_z^j\langle z\xi'\rangle^\sigma|\mathrel{\dot\le}\langle\xi'\rangle^{j-|\alpha|}\langle z\xi'\rangle^{\sigma-j}, \tag{2.8}$$

Hence, for all m,d,s,δ:

$$S_\delta^{m,d,s}(\Gamma,B)=S_{\delta-1}^{m,d-1,s+1}(\Gamma,B). \tag{2.9}$$

Proof. Observe that

$$\partial_{|z|}^j\langle z\xi'\rangle^\sigma=\sum_{|\gamma|=j}C_\gamma\xi'^\gamma\partial_{\eta'}^\gamma\langle\eta'\rangle^\sigma|_{\eta'=|z|\xi'}=\sum_{|\gamma|=j}\xi'^\gamma f_{\sigma,\gamma}(|z|\xi'),$$

where $f_{\sigma,\gamma}(\eta')\in S^{\sigma-|\gamma|}=S^{\sigma-j}$. Then

$$\begin{aligned}\partial_{\xi'}^\alpha\partial_{|z|}^j\langle z\xi'\rangle^\sigma&=\sum_{|\gamma|=j}\partial_{\xi'}^\alpha(\xi'^\gamma f_{\sigma,\gamma}(|z|\xi'))=\sum_{|\gamma|=j}\sum_{\beta\le\alpha}c_\beta\partial_{\xi'}^\beta\xi'^\gamma\partial_{\xi'}^{\alpha-\beta}f_{\sigma,\gamma}(|z|\xi')\\&=\sum_{|\gamma|=j}\sum_{\beta\le\alpha,\beta\le\gamma}\xi'^{\gamma-\beta}|z|^{|\alpha-\beta|}f_{\sigma,\gamma,\alpha-\beta}(|z|\xi'),\end{aligned}$$

where $f_{\sigma,\gamma,\alpha-\beta}(\eta')\in S^{\sigma-j-|\alpha-\beta|}$. Hence

$$\begin{aligned}|\partial_{\xi'}^\alpha\partial_{|z|}^j\langle z\xi'\rangle^\sigma|&\mathrel{\dot\le}\sum_{|\gamma|=j}\sum_{\beta\le\alpha,\beta\le\gamma}|\xi'|^{|\gamma-\beta|}|z|^{|\alpha-\beta|}\langle z\xi'\rangle^{\sigma-j-|\alpha-\beta|}\\&\mathrel{\dot\le}\sum_{\beta\le\alpha,|\beta|\le j}|\xi'|^{j-|\beta|}\langle\xi'\rangle^{|\alpha-\beta|}\langle z\xi'\rangle^{\sigma-j}\mathrel{\dot\le}\langle\xi'\rangle^{j-|\alpha|}\langle z\xi'\rangle^{\sigma-j},\end{aligned}$$

where we have used that $|z|^k\langle z\xi'\rangle^{-k}=[\xi',\frac1z]^{-k}\mathrel{\dot\le}\langle\xi'\rangle^{-k}$ for $k\ge 0$, $|z|\le 1$.

To show the inclusion from left to right in (2.9), let $g(w,\xi',\mu)\in S^{m,d,s}_{\delta}(\Gamma,B)$. We can assume that $d=s=0$, since the general case is reduced to this by multiplication of g by $z^d\kappa^{-s}$. That $g(w,\xi',\mu)\in S^{m,0,0}_{\delta}$ means that

$$\|\partial_w^\beta\partial_{\xi'}^\alpha\partial_{|z|}^j g(w,\xi',\tfrac1z)\|_B \mathrel{\dot{\le}} \langle z\xi'\rangle^{\delta}\langle\xi'\rangle^{m+j-|\alpha|}$$

for all indices. Since $|z|^{-1}[\xi',\frac1z]^{-1}=\langle z\xi'\rangle^{-1}$, $g(w,\xi',\mu)\in S^{m,-1,1}_{\delta-1}$ means that

$$\|\partial_w^\beta\partial_{\xi'}^\alpha\partial_{|z|}^j(\langle z\xi'\rangle^{-1}g(w,\xi',\tfrac1z))\|_B \mathrel{\dot{\le}} \langle z\xi'\rangle^{\delta-1}\langle\xi'\rangle^{m+j-|\alpha|}$$

for all indices, so this is what we have to show. We can disregard w in the following. Using (2.8) with $\sigma=-1$, we have:

$$\partial_{\xi'}^\alpha\partial_{|z|}^j(\langle z\xi'\rangle^{-1}g(\xi',\tfrac1z))=\textstyle\sum_{\beta\le\alpha,\,k\le j}c_{\beta,k}[\partial_{\xi'}^\beta\partial_{|z|}^k\langle z\xi'\rangle^{-1}][\partial_{\xi'}^{\alpha-\beta}\partial_{|z|}^{j-k}g(\xi',\tfrac1z)].$$

Then we find, using the estimates of $\langle z\xi'\rangle^{-1}$:

$$\begin{aligned}\|\partial_{\xi'}^\alpha\partial_{|z|}^j(\langle z\xi'\rangle^{-1}g(\xi',\tfrac1z))\|_B&\le\textstyle\sum_{\beta\le\alpha,k\le j}|c_{\beta,k}\partial_{|z|}^k\partial_{\xi'}^\beta\langle z\xi'\rangle^{-1}|\|\partial_{\xi'}^{\alpha-\beta}\partial_{|z|}^{j-k}g(\xi',\tfrac1z)\|_B\\&\mathrel{\dot{\le}}\sum_{k\le j,\beta<\alpha}\langle\xi'\rangle^{k-|\beta|}\langle z\xi'\rangle^{-1}\langle z\xi'\rangle^{\delta}\langle\xi'\rangle^{m+j-k-|\alpha-\beta|}\mathrel{\dot{\le}}\langle z\xi'\rangle^{\delta-1}\langle\xi'\rangle^{m+j-|\alpha|};\end{aligned}$$

as was to be shown. The other inclusion uses (2.8) with $\sigma=1$ in a similar way. □

Example 2.3. To motivate the scaling in x_n in the following definition of parameter-dependent boundary symbols, consider the basic example $\tilde f(x_n,\xi',\mu)=e^{-x_n|(\xi',\mu)|}$ (with Fourier transform $\mathcal{F}_{x_n\to\xi_n}e^+\tilde f=(|(\xi',\mu)|+i\xi_n)^{-1}$); it is the symbol-kernel of the Poisson operator solving the Dirichlet problem $(-\Delta+\mu^2)u(x)=0$ on $\mathbb{R}^n_+$, $u(x',0)=\psi(x')$, for $\mu>0$. Setting $\mu=\frac1z$, we find for $\tilde f(x_n,\xi',\frac1z)=e^{x_nz^{-1}\langle z\xi\rangle}$, by use of the fact that

$$\sup\nolimits_{x_n\ge0}x_n^ke^{-\sigma x_n}=c_k\sigma^{-k},\quad c_k=k^ke^{-k}, \tag{2.10}$$

that for each j, $\sup_{x_n\ge0}|\partial_z^j\tilde f(x_n,\xi',\frac1z)|$ is $O(z^{-j})$ for each ξ'. This does not comply with the uniform estimates in $z\le1$ required in (2.2) (with $\delta=0$, $B=L_\infty(\mathbb{R}_+)$). But if we replace x_n by zu_n, we find using (2.8) that the resulting function $\tilde f_1(u_n,\xi',z)=\tilde f(zu_n,\xi',\frac1z)=e^{-u_n\langle z\xi'\rangle}$ has $\sup_{u_n\ge0}|\partial_z^je^{-u_n\langle z\xi'\rangle}|\mathrel{\dot{\le}}\langle\xi'\rangle^j$, $j\in\mathbb{N}$, which fits well with (2.2). (It is easy to check a few steps by hand calculation; a systematic proof covering this case is given below in Theorem 3.2, see Example 3.4.)

A reader wanting to bypass the complex formulation in Sections 5–6 can disregard the information on symbols below in Definition 2.4 3°–4°, (2.17), (2.19), etc.

Definition 2.4. *Let $m\in\mathbb{R}$, d and $s\in\mathbb{Z}$.*

1° *The space $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$ (briefly denoted $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_+)$) consists of the functions $\tilde f(w,x_n,\xi',\mu)$ in $C^\infty(\mathbb{R}^\nu\times\overline{\mathbb{R}}_+\times\mathbb{R}^{n-1}\times\Gamma)$ satisfying, for all $l,l'\in\mathbb{N}$,*

$$\begin{gathered}\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac1z)\in S^{m,d,s+1}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{\infty,u_n}(\mathbb{R}_+))\\ \text{(equivalently, } u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac1z)\in S^{m,d+l-l',s+1-l+l'}(\Gamma,L_{\infty,u_n}(\mathbb{R}_+))\text{)}.\end{gathered} \tag{2.11}$$

2° *The space* $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$ *(briefly denoted* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_{++})$*) consists of the functions* $\tilde f(w,x_n,y_n,\xi',\mu)$ *in* $C^\infty(\mathbb{R}^\nu\times\overline{\mathbb{R}}^2_{++}\times\mathbb{R}^{n-1}\times\Gamma)$ *satisfying, for all* $l,l',k,k'\in\mathbb{N}$,

$$\langle z\xi'\rangle^{l-l'+k-k'}u_n^l\partial_{u_n}^{l'}v_n^k\partial_{v_n}^{k'}\tilde f(w,|z|u_n,|z|v_n,\xi',\tfrac1z)\\ \in S^{m,d,s+2}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{\infty,u_n,v_n}(\mathbb{R}^2_{++})). \tag{2.12}$$

3° *The spaces obtained from* $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$ *by Fourier transformation, resp. co-Fourier transformation,*

$$\tilde f\mapsto f(w,\xi',\xi_n,\mu)=\mathcal{F}_{x_n\to\xi_n}e^+\tilde f,\ \text{resp. }\tilde f\mapsto f_c(w,\xi',\xi_n,\mu)=\overline{\mathcal{F}}_{x_n\to\xi_n}e^+\tilde f, \tag{2.13}$$

are denoted $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+)$ *resp.* $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^-_{-1})$ *(brief notation* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^+)$ *resp.* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^-_{-1})$ *).*

4° *The space obtained from* $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$ *by Fourier and co-Fourier transformation,*

$$\tilde f\mapsto f(w,\xi',\xi_n,\eta_n,\mu)=\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde f, \tag{2.14}$$

is denoted $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{-1})$ *(brief notation* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{-1})$*).*

Definition 2.5.

1° *The functions in* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_+)$ *are the* **Poisson symbol-kernels** *and* **trace symbol-kernels of class 0** *of degree* $m+d+s$*, in the parametrized calculus. The functions in* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^+)$ *are the* **Poisson symbols***, and the functions in* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^-_{-1})$ *are the* **trace symbols of class 0***, of degree* $m+d+s$*, in the parametrized calculus.*

2° *The functions in* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_{++})$ *resp.* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{-1})$ *are the* **singular Green symbol-kernels***, resp.* **singular Green symbols, of class** 0*, and degree* $m+d+s$*, in the parametrized calculus.*

There is some explanation for the $+1$ resp. $+2$ in the s-index in (2.11) resp. (2.12) in the following proposition, which shows how the indices in the $S^{m,d,s}$-estimates are modified when the Poisson, trace and singular Green symbol-kernels are replaced by symbols (Fourier transforms). The indices have the most natural values when we use L_∞ norms for the symbols, included in Section 5. As we shall see in Section 3, the degree indication $m+d+s$ is consistent with the homogeneity degree of the principal part of polyhomogeneous symbols in the calculus.

Proposition 2.6.

1° *A* C^∞ *function* $\tilde f(w,x_n,\xi',\mu)$ *belongs to* $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$ *if and only if, for all* $l,l'\in\mathbb{N}$,

$$\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac1z)\in S^{m,d+1,s}_{\frac12}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{2,u_n}(\mathbb{R}_+)). \tag{2.15}$$

Here (2.15) *can be replaced by*

$$\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac1z)\in S^{m,d,s+1}_{-\frac12}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{2,u_n}(\mathbb{R}_+)). \tag{2.16}$$

Moreover, the properties (2.15) are equivalent with the following properties of the Fourier transform or co-Fourier transform (2.13):

$$\langle z\xi'\rangle^{l-l'} h_{-1}\partial_{\zeta_n}^l \zeta_n^{l'} f(w,\xi',|\tfrac{1}{z}|\zeta_n,\tfrac{1}{z}) \in S^{m,d,s}_{\frac{1}{2}}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{2,\zeta_n}(\mathbb{R})). \tag{2.17}$$

2° A C^∞ function $\tilde f(w,x_n,y_n,\xi',\mu)$ belongs to $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$ if and only if, for all $l,l',k,k'\in\mathbb{N}$,

$$\begin{aligned}\langle z\xi'\rangle^{l-l'+k-k'} u_n^l\partial_{u_n}^{l'} v_n^k\partial_{v_n}^{k'}\tilde f(w,|z|u_n,|z|v_n,\xi',\tfrac{1}{z})&\\ \in S^{m,d+1,s+1}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{2,u_n,v_n}(\mathbb{R}^2_{++})).&\end{aligned} \tag{2.18}$$

The properties (2.18) are equivalent with the following properties of the Fourier and co-Fourier transform (2.14):

$$\begin{aligned}\langle z\xi'\rangle^{l-l'+k-k'} h_{-1,\zeta_n}h_{-1,\varrho_n}\partial_{\zeta_n}^l\zeta_n^{l'}\partial_{\varrho_n}^k\varrho_n^{k'} f(w,\xi',|\tfrac{1}{z}|\zeta_n,|\tfrac{1}{z}|\varrho_n,\tfrac{1}{z})&\\ \in S^{m,d-1,s+1}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,L_{2,\zeta_n,\varrho_n}(\mathbb{R}^2)).&\end{aligned} \tag{2.19}$$

Proof. For functions in $\mathcal{S}_+$ one has the well-known inequalities

$$\begin{aligned}\|\varphi(u_n)\|^2_{L_\infty(\mathbb{R}_+)} &\le 2\|\varphi(u_n)\|_{L_2(\mathbb{R}_+)}\|\partial_{u_n}\varphi(u_n)\|_{L_2(\mathbb{R}_+)},\\ \|\varphi\|^2_{L_2(\mathbb{R}_+)} &= \int_0^\infty \tfrac{1+\varepsilon^2u_n^2}{1+\varepsilon^2u_n^2}|\varphi(u_n)|^2\,du_n\\ &\le c(\varepsilon\|\varphi(u_n)\|^2_{L_\infty(\mathbb{R}_+)}+\varepsilon^{-1}\|u_n\varphi(u_n)\|^2_{L_\infty(\mathbb{R}_+)}),\end{aligned} \tag{2.20}$$

for all $\varepsilon>0$. In detail, the statement in (2.11) means that for all j,α,β,

$$\|\partial_w^\beta\partial_{\xi'}^\alpha\partial_{|z|}^j(z^d\kappa^{-s-1}\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac{1}{z}))\|_{L_\infty(\mathbb{R}_+)} \dot\le \langle\xi'\rangle^{m+j-|\alpha|}, \tag{2.21}$$

and the statement in (2.15) means that for all j,α,β,

$$\langle z\xi'\rangle^{-\frac{1}{2}}\|\partial_w^\beta\partial_{\xi'}^\alpha\partial_{|z|}^j(z^{d+1}\kappa^{-s}\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac{1}{z}))\|_{L_{2,u_n}(\mathbb{R}_+)} \dot\le \langle\xi'\rangle^{m+j-|\alpha|}, \tag{2.22}$$

valid uniformly for $|z|\le 1$, $\frac{1}{z}$ in closed subsectors of Γ. To pass from (2.22) to (2.21), we use the first inequality in (2.20). The basic step is the inequality in the case $d=s=j=0$, α and $\beta=0$:

$$\begin{aligned}\|\kappa^{-1}\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac{1}{z})\|^2_{L_\infty(\mathbb{R}_+)} &= \|z\langle z\xi'\rangle^{l-l'-1}u_n^l\partial_{u_n}^{l'}\tilde f\|^2_{L_\infty(\mathbb{R}_+)}\\ \le 2\|z\langle z\xi'\rangle^{l-l'-1}u_n^l\partial_{u_n}^{l'}\tilde f\|_{L_{2,u_n}(\mathbb{R}_+)}&\|z\langle z\xi'\rangle^{l-l'-1}\partial_{u_n}(u_n^l\partial_{u_n}^{l'}\tilde f)\|_{L_{2,u_n}(\mathbb{R}_+)}\\ \dot\le \langle z\xi'\rangle^{-\frac{1}{2}}\langle\xi'\rangle^m(\|z\langle z\xi'\rangle^{l-l'-1}u_n^{l-1}\partial_{u_n}^{l'}\tilde f\|_{L_2}&+\|z\langle z\xi'\rangle^{l-l'-1}u_n^l\partial_{u_n}^{l'+1}\tilde f\|_{L_2}) \dot\le \langle\xi'\rangle^{2m};\end{aligned}$$

the general estimates follow the same pattern.

To pass from (2.21) to (2.22), we use the second inequality in (2.20) with $\varepsilon = \langle z\xi'\rangle^{-1}$, to get the basic estimate:

$$\begin{aligned}(2.23)\qquad &\langle z\xi'\rangle^{-1}\|z\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac1z)\|^2_{L_{2,u_n}(\mathbb{R}_+)}\\ &= \|\kappa^{-1}\langle z\xi'\rangle^{l-l'+\frac12}u_n^l\partial_{u_n}^{l'}\tilde f(w,|z|u_n,\xi',\tfrac1z)\|^2_{L_{2,u_n}(\mathbb{R}_+)}\\ &\dot\le \|\kappa^{-1}\langle z\xi'\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}\tilde f\|^2_{L_\infty} + \|\kappa^{-1}\langle z\xi'\rangle^{l-l'+1}u_n^{l+1}\partial_{u_n}^{l'}\tilde f\|^2_{L_\infty} \dot\le \langle\xi'\rangle^{2m},\end{aligned}$$

and the other estimates follow the same pattern. This shows the equivalence of the set of conditions (2.11) with the set of conditions (2.15), and the equivalence with (2.16) follows immediately from (2.9).

For the Fourier transform, we note that the statement in (2.17) means that for all j,α,β,

$$(2.24)\quad \langle z\xi'\rangle^{-\frac12}\|h_{-1}\partial_w^\beta\partial_{\xi'}^\alpha\partial_{|z|}^j\big(z^d\kappa^{-s}\langle z\xi'\rangle^{l-l'}\zeta_n^l\partial_{\zeta_n}^{l'}f(w,\xi',|\tfrac1z|\zeta_n,\tfrac1z)\big)\|_{L_{2,\zeta_n}(\mathbb{R})} \dot\le \langle\xi'\rangle^{m+j-|\alpha|}$$

(uniformly for $|z|\le 1$, $\frac1z$ in closed subsectors of Γ). Now the Fourier transform of $e^+u_n^l\partial_{u_n}^{l'}\tilde f(|z|u_n)$ is $|\frac1z|h_{-1}(\partial_{\zeta_n}^l\zeta_n^{l'}f(|\frac1z|\zeta_n))$, having the factor $|\frac1z|$. This explains the shift from $d+1$ to d in the power of z, and with this taken into account, the passage between (2.15) and (2.17) follows immediately from the Parseval-Plancherel theorem. The same holds for the passage to (2.18), and we have shown 1°.

For 2°, we use the results of 1° with respect to both variables u_n and v_n. This shows that $S^{m,d,s+2}(\Gamma,L_\infty(\mathbb{R}^2_{++}))$ in (2.12) can be replaced by $S_1^{m,d+2,s}(\Gamma,L_{2,u_n,v_n}(\mathbb{R}^2_{++}))$. By Lemma 2.2, this equals $S^{m,d+1,s+1}(\Gamma,L_{2,u_n,v_n}(\mathbb{R}^2_{++}))$, so (2.12) is equivalent with (2.18). Equation (2.19) follows by the Parseval-Plancherel theorem, when we observe that the Fourier transformations remove a factor $|z|^2$. □

The function $e^{-x_n[\xi',\mu]}$ considered in Example 2.3 belongs to $\mathcal{S}^{0,0,-1}(\mathbb{R}_+,\mathcal{S}_+)$; cf. Example 3.4 below.

Remark 2.7. We cannot easily fit μ-independent symbols into this calculus, since the coordinate change $x_n=|z|u_n$ fits badly with their properties and does not give the desired z-boundedness of derived expressions. For example, $e^{-x_n[\xi']}=e^{-u_nz[\xi']}$ ($z\in\mathbb{R}_+$) satisfies: $\sup_{\mathbb{R}_+}|\partial_z^je^{-u_nz[\xi']}|=c_jz^{-j}$ (cf. (2.10)), which get more out of control for $z\to0$, the larger j is taken. (This does not exclude that compositions with this symbol can lead to operators in the calculus; see Remark 6.11.)

We denote

$$(2.25)\qquad \begin{aligned}\bigcap_{m\in\mathbb{R}}\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{K}) &= \mathcal{S}^{-\infty,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{K}),\\ \bigcup_{m\in\mathbb{R}}\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{K}) &= \mathcal{S}^{\infty,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{K}),\end{aligned}$$

for $\mathcal{K}=\mathcal{S}_+,\mathcal{H}^+,\mathcal{H}^-_{-1},\mathcal{S}_{++}$, or $\mathcal{H}^+\hat\otimes\mathcal{H}^-_{-1}$ (a similar convention is used for other scales of symbol spaces).

As in the case of ψdo symbols treated in [GS95], there are some straightforward rules for application of $\partial_{\xi'}^\alpha$ or ∂_w^β, multiplication by z^k or $[\xi',\frac1z]^r$, to these symbol spaces:

Lemma 2.8. *The following mappings are continuous:*

$$
\begin{aligned}
&\text{(i)} \quad \partial_w^\beta \partial_{\xi'}^\alpha \colon \mathcal{S}^{m,d,s} \to \mathcal{S}^{m-|\alpha|,d,s},\\
&\text{(ii)} \quad z^k \colon \mathcal{S}^{m,d,s} \overset{\sim}{\to} \mathcal{S}^{m,d-k,s},\\
&\text{(iii)} \quad [\xi', \tfrac{1}{z}]^r \colon \mathcal{S}^{m,d,s} \overset{\sim}{\to} \mathcal{S}^{m,d,s+r};
\end{aligned}
\tag{2.26}
$$

for $\alpha \in \mathbb{N}^{n-1}$, $\beta \in \mathbb{N}^\nu$, $k, r \in \mathbb{Z}$. Similar statements hold with $\mathcal{S}$ replaced by S or S_δ.

We also have the product rule:

Lemma 2.9. *Let $q(w, \xi', \mu) \in S^{m',d',s'}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathbb{C})$. When $\tilde{f}$ is in one of the spaces defined in Definition 2.1 or 2.4, the product $q\tilde{f}$ is in the space obtained by replacing m, d, s by $m + m', d + d', s + s'$.*

Proof. Follows easily by the Leibniz rule. □

We have in view of [GS95, Lemma 1.13] that for integer m (cf. also (2.5)),

$$
\begin{aligned}
&\langle \xi', \mu\rangle^m, |(\xi, \mu)|^m \text{ and } [\xi', \mu]^m \in S^{0,0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathbb{C})\\
&\qquad \subset \begin{cases} S^{m,0}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathbb{C}) + S^{0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathbb{C}) \text{ if } m \geq 0,\\ S^{m,0}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathbb{C}) \cap S^{0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathbb{C}) \text{ if } m \leq 0,\end{cases}
\end{aligned}
\tag{2.27}
$$

(The powers of $|(\xi', \mu)|$ have this property since the uniformity in the symbol estimates is required only for $\mu \geq 1$, where $|(\xi', \mu)|$ equals $[\xi', \mu]$ treated in [GS95].) In compositions of $S^{m,d}$ spaces with powers of $[\xi', \mu]$ one finds the consequential shifts in the indices; in particular when composing with a positive power one needs to keep track of a sum of two different spaces. It is in order to avoid this complicated accounting, that we have included compositions with powers of $[\xi', \mu]$ in the notation by the third upper index s.

From (2.27) and Lemma 2.9 follows that any of the spaces introduced above satisfies:

$$
\mathcal{S}^{m,d,s} \subset \begin{cases} \mathcal{S}^{m+s,d,0} + \mathcal{S}^{m,d+s,0} \text{ if } s \geq 0,\\ \mathcal{S}^{m+s,d,0} \cap \mathcal{S}^{m,d+s,0} \text{ if } s \leq 0.\end{cases}
\tag{2.28}
$$

Note that for k integer ≥ 0,

$$
\mathcal{S}^{m+k,d,s} \subset \mathcal{S}^{m,d,s+k} \cap \mathcal{S}^{m+k,d-k,s+k},
\tag{2.29}
$$

since $\tilde{f} \in \mathcal{S}^{m+k,d,s}$ implies $\kappa^{-k}\tilde{f} \in \mathcal{S}^{m,d,s} \cap \mathcal{S}^{m+k,d-k,s}$ by Lemma 2.9 and (2.27).

Note also that multiplication by one of the functions in (2.27) maps $\mathcal{S}^{m_1,d,s}$ into $\mathcal{S}^{m_1,d,s+m}$, which by (2.27) is contained in $\mathcal{S}^{m_1+m,d,s} + \mathcal{S}^{m_1,d+m,s}$ if $m \geq 0$, and in $\mathcal{S}^{m_1+m,d,s} \cap \mathcal{S}^{m_1,d+m,s}$ if $m \leq 0$.

These statements as well as (2.28), (2.29) likewise hold with $\mathcal{S}$ replaced by S or S_δ.

The operators $\partial_{x_n}^j$ and $x_n^{j'}$ act on the symbol-kernel spaces as follows:

Lemma 2.10.

1° *Let $\tilde{f} \in \mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_+)$. Then for $j, j' \in \mathbb{N}$,*

$$
x_n^j \partial_{x_n}^{j'} \tilde{f} \in \mathcal{S}^{m,d,s-j+j'}(\Gamma, \mathcal{S}_+),
\tag{2.30}
$$

and the mapping $x_n^j\partial_{x_n}^{j'}$ is continuous for these spaces.

2° *Let $\tilde f\in\mathcal S^{m,d,s}(\Gamma,\mathcal S_{++})$. Then for $i,i',j,j'\in\mathbb N$,*

$$x_n^i\partial_{x_n}^{i'}y_n^j\partial_{y_n}^{j'}\tilde f\in\mathcal S^{m,d,s-i+i'-j+j'}(\Gamma,\mathcal S_{++}),\tag{2.31}$$

and the mapping $x_n^i\partial_{x_n}^{i'}y_n^j\partial_{y_n}^{j'}$ is continuous for these spaces.

Proof. By (2.11),

$$\begin{aligned}u_n^l\partial_{u_n}^{l'}[x_n^j\partial_{x_n}^{j'}\tilde f(w,x_n,\xi',\tfrac1z)]_{x_n=|z|u_n}&=u_n^l\partial_{u_n}^{l'}u_n^j\partial_{u_n}^{j'}\tilde f(w,|z|u_n,\xi',\tfrac1z)|z|^{j-j'}\\&\in z^{j-j'}S^{m,d+l-l'+j-j',s-j+j'-l+l'}=\kappa^{j-j'}S^{m,d-l+l',s-l+l'};\end{aligned}$$

here we use that $\partial_{u_n}^{l'}(u_n^jg)=\sum_{l''\le l'}C_{l''}u_n^{j-l''}\partial_{u_n}^{l'-l''}g$, where l'' cancels out in the end result. This shows (2.30); the proof of 2° is similar. □

Symbols may appear as described in terms of asymptotic series. Here, when $\tilde f\in\mathcal S^{m,d,s}(\Gamma,\mathcal S_+)$ and $\tilde f_{m-j}\in\mathcal S^{m-j,d,s}(\Gamma,\mathcal S_+)$ for $j\in\mathbb N$, $\tilde f\sim\sum_{j\in\mathbb N}\tilde f_{m-j}$ means that for any $J\in\mathbb N$, $\tilde f-\sum_{j<J}\tilde f_{m-j}\in\mathcal S^{m-J,d,s}(\Gamma,\mathcal S_+)$. There is the usual fact that for a given sequence of symbols $\tilde f_{m-j}\in\mathcal S^{m-j,d,s}(\Gamma,\mathcal S_+)$, $j\in\mathbb N$, there *exists* (by a Borel-type construction) a symbol $\tilde f\in\mathcal S^{m,d,s}(\Gamma,\mathcal S_+)$ such that $\tilde f\sim\sum_{j\in\mathbb N}\tilde f_{m-j}$. Similar statements hold for the classes $\mathcal S^{m,d,s}(\Gamma,\mathcal S_{++})$.

Remark 2.11. One of the important properties of the parameter-dependent ψdo symbols introduced in [GS95] is a kind of Taylor expansion in z for $z\to0$ (this is an entirely different type of asymptotic expansion than the one defined above). It does not generalize to boundary operators in a simple way, because of the scaling in the normal variable in our estimates. What we can show is the following:

When $\tilde g\in\mathcal S^{m,d,0}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathcal S_{++})$ with symbol $g\in\mathcal S^{m,d,0}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathcal H^+\hat\otimes\mathcal H^-_{-1})$, then the limits

$$\tilde g_{(d,j)}(w,u_n,v_n,\xi')=\lim_{z\to0,z\in\mathbb R_+}\tfrac{1}{j!}\partial_z^j(z^{d+2}\tilde g(w,zu_n,zv_n,\xi',\tfrac1z))\tag{2.32}$$

exist for all j and belong to $S^{m+j}(\mathbb R^\nu\times\mathbb R^{n-1},L_2(\mathbb R^2_{++}))$, and $\mu^{-2}\tilde g(w,x_n,y_n,\xi',\mu)$ resp. $g(w,\xi',\xi_n,\eta_n,\mu)$ have the following expansions in decreasing powers of $\mu\in\mathbb R_+$:

$$\begin{aligned}&\mu^{-2}\tilde g(w,\tfrac1\mu u_n,\tfrac1\mu v_n,\xi',\mu)-\sum_{0\le j<N}\mu^{d-j}\tilde g_{(d,j)}(w,u_n,v_n,\xi')\\&\qquad\in S^{m+N,d-N,0}(\mathbb R^\nu,\mathbb R^{n-1},\Gamma,L_{2,u_n,v_n}(\mathbb R^2_{++})),\\&g(w,\xi',\mu\zeta_n,\mu\varrho_n,\mu)-\sum_{0\le j<N}\mu^{d-j}g_{(d,j)}(w,\xi',\zeta_n,\varrho_n)\\&\qquad\in S^{m+N,d-N,0}(\mathbb R^\nu,\mathbb R^{n-1},\Gamma,L_{2,\zeta_n,\varrho_n}(\mathbb R^2)),\end{aligned}\tag{2.33}$$

for any N; $g_{(d,j)}=\mathcal F_{u_n\to\zeta_n}\overline{\mathcal F}_{v_n\to\varrho_n}\tilde g_{(d,j)}$. (Similar expansions hold on the other rays in Γ.)

The result is of limited interest because of the scaling involved (and the proof will be left out). In fact, when g from (2.33) is inserted in the last line of (1.13), we get an expansion:

$$Gu = \sum_{0\le j<N} \mu^{d-j}\,\mathrm{OPG}(g_{(d,j)}(x',\xi',\tfrac{1}{\mu}\xi_n,\tfrac{1}{\mu}\eta_n))u + \text{ remainder},$$

where a certain μ-dependence is present in the coefficients of μ^{d-j} in the sum.

However, as we shall see later, more useful expansions are obtained after one takes the normal trace; see the end of Section 4, and Section 7.

3. Polyhomogeneous symbols

A symbol-kernel $\tilde f$ satisfying Definition 2.4 1° is said to be *polyhomogeneous* when there is a sequence $\tilde f_j$ of symbol-kernels in $\mathcal{S}^{m-j,d,s}(\Gamma,\mathcal{S}_+)$ that are quasi-homogeneous of degree $m-j+d+s$ such that $\tilde f-\sum_{j<J}\tilde f_j$ is in $\mathcal{S}^{m-J,d,s}(\Gamma,\mathcal{S}_+)$ for any J. Here quasi-homogeneity of degree m' means that:

$$\tilde f_j(w,\tfrac{x_n}{a},a\xi',a\mu)=a^{m'+1}\tilde f_j(w,x_n,\xi',\mu),\ \text{ for } a\ge 1,\ |\xi'|\ge 1,\ x_n\in\overline{\mathbb{R}}_+,\ \mu\in\Gamma; \tag{3.1}$$

it corresponds to *homogeneity of degree m' of the associated Poisson (or trace) symbol* $f_j=\mathcal{F}_{x_n\to\xi_n}e^+\tilde f_j$ (resp. $f_{j,c}=\overline{\mathcal{F}}_{x_n\to\xi_n}e^+\tilde f_j$):

$$f_j(w,a\xi',a\xi_n,a\mu)=a^{m'}f_j(w,\xi',\xi_n,\mu),\ \text{ for } a\ge 1,\ |\xi'|\ge 1,\ \xi_n\in\mathbb{R},\ \mu\in\Gamma. \tag{3.2}$$

A symbol-kernel $\tilde f$ satisfying Definition 2.4 2° is polyhomogeneous when there is a sequence of symbol-kernels $\tilde f_j\in\mathcal{S}^{m-j,d,s}(\Gamma,\mathcal{S}_{++})$, $j\in\mathbb{N}$, that are quasi-homogeneous of degree $m-j+d+s$ such that $\tilde f-\sum_{j<J}\tilde f_j$ is in $\mathcal{S}^{m-J,d,s}(\Gamma,\mathcal{S}_{++})$ for any J. Here quasi-homogeneity of degree m' means that:

$$\tilde f_j(w,\tfrac{x_n}{a},\tfrac{y_n}{a},a\xi',a\mu)=a^{m'+2}\tilde f_j(w,x_n,y_n,\xi',\mu),\quad \text{for } a\ge 1,\ |\xi'|\ge 1,\ (x_n,y_n)\in\overline{\mathbb{R}}^2_{++},\ \mu\in\Gamma; \tag{3.3}$$

this corresponds to *homogeneity of degree m' of the symbol* $f_j=\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}\tilde f_j$:

$$f_j(w,,a\xi',a\xi_n,a\eta_n,a\mu)=a^{m'}f_j(w,\xi',\xi_n,\eta_n,\mu),\quad \text{for } a\ge 1,\ |\xi'|\ge 1,\ (\xi_n,\eta_n)\in\mathbb{R}^2,\ \mu\in\Gamma. \tag{3.4}$$

The exclusion of small $|\xi'|$ is important here. In fact, when $\tilde f$ satisfies (1.18) (resp. (1.24)) with $\zeta=(\xi',\mu)$, we say that $\tilde f$ is *strongly* quasi-homogeneous, but when merely (3.1) (resp. (3.3)) holds, we say that $\tilde f$ is *weakly* quasi-homogeneous.

An important special case of functions satisfying Definition 2.4 1° are those obtained by taking classical quasi-homogeneous (or poly(quasi)homogeneous) Poisson or trace symbol-kernels $\tilde q(w,x_n,\zeta)$ with $\zeta\in\mathbb{R}^n$, and setting $\xi'=(\zeta_1,\dots,\zeta_{n-1})$, $\mu=\zeta_n$. We call such symbol-kernels *strongly polyhomogeneous.* Similarly, special cases of Definition 2.4 2° arise from classical singular Green symbol-kernels in one more cotangent variable. To verify this, we need a variant of [GS95, Lemma 1.3], for functions taking values in a Banach space B (e.g., $B=L_p(\mathbb{R}_+)$ or $L_p(\mathbb{R}^2_{++})$, $p=\infty$ or 2); we can include a growth factor $\langle\eta'\rangle^\delta$.

Lemma 3.1. *Let $v(w,\eta') \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, B)$ satisfy, for some $\delta \in \mathbb{R}$,*

$$\langle \eta' \rangle^{-\delta} \| \partial_{\eta'}^\alpha \partial_w^\beta v(w,\eta') \|_B \dot{\le} \langle \eta' \rangle^{-|\alpha|}, \tag{3.5}$$

uniformly in w, for all α and β. Then

$$\tilde f(w,\xi',\mu) = v(w,\xi'/\mu) \in S_\delta^{0,0,0}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, B), \tag{3.6}$$

i.e., $\tilde f$ satisfies (2.2) with $m = 0$, $\Gamma = \mathbb{R}_+$.

Proof. The proof follows that of [GS95, Lemma 1.3].

First we have, with $\eta' = \xi'/\mu = z\xi'$, $z = \frac{1}{\mu} \le 1$,

$$\begin{aligned} \langle z\xi' \rangle^{-\delta} \| \partial_{\xi'}^\alpha v(w,z\xi') \|_B &= \langle \eta' \rangle^{-\delta} z^{|\alpha|} \| \partial_{\eta'}^\alpha v(w,\eta') \|_B \\ &\dot{\le} z^{|\alpha|} (1 + |z\xi'|^2)^{-|\alpha|/2} = (z^{-2} + |\xi'|^2)^{-|\alpha|/2} \le (1+|\xi'|^2)^{-|\alpha|/2}, \end{aligned} \tag{3.7}$$

where we used that $z \le 1$ and $-|\alpha| \le 0$. This shows the desired estimates (2.2) for $j = 0$, $\beta = 0$. Derivatives in w are included since $\partial_w^\beta v$ is of the same type as v.

For $j > 0$ we observe that

$$\partial_z^j v(w,z\xi') = \sum_{|\gamma|=j} C_\gamma \xi'^\gamma \partial_{\eta'}^\gamma v(w,\eta')|_{\eta'=z\xi'}. \tag{3.8}$$

Here $\partial_{\eta'}^\gamma v(w,\eta')$ satisfies

$$\langle \eta' \rangle^{-\delta} \| \partial_{\eta'}^\alpha \partial_w^\beta \partial_{\eta'}^\gamma v(w,\eta') \|_B \dot{\le} \langle \eta' \rangle^{-|\alpha|-|\gamma|} \le \langle \eta' \rangle^{-|\alpha|},$$

so the first part of the proof shows that $\partial_{\eta'}^\gamma v(w,\eta')|_{\eta'=z\xi'}$ and its w-derivatives satisfy estimates as in (3.7):

$$\langle z\xi' \rangle^{-\delta} \| \partial_{\xi'}^\alpha [\partial_w^\beta \partial_{\eta'}^\gamma v(w,\eta')|_{\eta'=z\xi'}] \|_B \dot{\le} \langle \xi' \rangle^{-|\alpha|}.$$

By multiplication by ξ'^γ ($|\gamma| = j$) and insertion in (3.8) we then find that

$$\langle z\xi' \rangle^{-\delta} \| \partial_{\xi'}^\alpha \partial_w^\beta \partial_z^j v(w,z\xi') \|_B \dot{\le} \langle \xi' \rangle^{j-|\alpha|},$$

as required. □

Theorem 3.2. *Let $m \in \mathbb{Z}$.*

1° Let $\tilde q(w,x_n,\zeta) \in C^\infty(\mathbb{R}^\nu \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n)$ be in $\mathcal{S}_+$ with respect to x_n, uniformly in w, and satisfy (1.18) (i.e., it is quasi-homogeneous of degree m). Set $\tilde f(w,x_n,\xi',\mu) = \tilde q(w,x_n,(\xi',\mu))$. Then

$$\tilde f(w,x_n,\xi',\mu) \in \mathcal{S}^{0,0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathcal{S}_+). \tag{3.9}$$

2° Let $\tilde q(w,x_n,y_n,\zeta) \in C^\infty(\mathbb{R}^\nu \times \overline{\mathbb{R}}^2_{++} \times \mathbb{R}^n)$ be in $\mathcal{S}_{++}$ with respect to (x_n,y_n), uniformly in w, and satisfy (1.24) (i.e., it is quasi-homogeneous of degree m). Set $\tilde f(w,x_n,y_n,\xi',\mu) = \tilde q(w,x_n,y_n,(\xi',\mu))$. Then

$$\tilde f(w,x_n,y_n,\xi',\mu) \in \mathcal{S}^{0,0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathcal{S}_{++}). \tag{3.10}$$

3° *Similar statements hold more generally when $\tilde q(w,x_n,\zeta)$ (resp. $\tilde q(w,x_n,y_n,\zeta)$) is a classical polyhomogeneous Poisson or trace (resp. singular Green) symbol-kernel of degree m; the expansion of $\tilde q$ in quasi-homogeneous terms then corresponds to the expansion of $\tilde f$ in quasi-homogeneous terms.*

It suffices for these statements to have $\tilde q$ defined only for $\zeta_n \ge 0$ and quasi-homogeneous in (x_n,ζ) (resp. (x_n,y_n,ζ)) there, for $|\zeta|\ge 1$.

Proof. 1°. It was shown after (1.18) that $\tilde q$ satisfies

$$\|x_n^l\partial_{x_n}^k\partial_w^\beta\partial_\zeta^\alpha \tilde q(w,x_n,\zeta)\|_{L_\infty(\mathbb{R}_+)} \dot\le \langle\zeta\rangle^{m-l+k-|\alpha|+1}, \tag{3.11}$$

for all indices.

Consider first the case $m=-1$. We have from (3.11) in this case that $\tilde f(w,u_n,\eta',1)=\tilde q(w,u_n,(\eta',1))$ satisfies

$$\|\partial_{\eta'}^\alpha\partial_w^\beta \tilde f(w,u_n,\eta',1)\|_{L_\infty(\mathbb{R}_+)} \dot\le \langle\eta'\rangle^{-|\alpha|},\ \text{all } \alpha,\beta, \tag{3.12}$$

uniformly in w. By the quasi-homogeneity (cf. (1.18)), we have for $|\xi'|\ge 1$:

$$\tilde f(w,zu_n,\xi',\tfrac1z)=\tilde f(w,u_n,z\xi',1),\ \text{for } z\le 1. \tag{3.13}$$

Applying Lemma 3.1 with $\delta=0$ and $B=L_\infty(\mathbb{R}_+)$ to $\tilde f(w,u_n,z\xi',1)$, we find that $\tilde f(w,u_n,z\xi',1)\in S^{0,0,0}(\mathbb{R}_+,L_\infty(\mathbb{R}_+))$, so by (3.13), (2.11) holds for $l=l'=0$ with $m=d=0$, $s=-1$.

Now let l and l' be arbitrary. Consider

$$\tilde f^{(l,l')}(w,zu_n,\xi',\tfrac1z)\equiv u_n^l\partial_{u_n}^{l'}\tilde f(w,zu_n,\xi',\tfrac1z)=u_n^l\partial_{u_n}^{l'}q(w,u_n,(z\xi',1)).$$

Here $\langle\zeta\rangle^{l-l'}u_n^l\partial_{u_n}^{l'}q(w,u_n,\zeta)$ is again C^∞ and quasi-homogeneous of degree -1, so it follows as above that $\langle z\xi'\rangle^{l-l'}\tilde f^{(l,l')}\in S^{0,0,0}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\mathbb{R}_+,L_\infty(\mathbb{R}_+))$. Thus $\tilde f$ satisfies all the requirements for belonging to $\mathcal{S}^{0,0,-1}(\mathbb{R}_+,\mathcal{S}_+)$.

When m is arbitrary, we write $\tilde f=[\xi',\mu]^{m+1}([\xi',\mu]^{-m-1}\tilde f)$. Here $[\xi',\mu]^{-m-1}\tilde f$ is of the kind we have just treated, so (3.9) follows by Lemma 2.8 and (2.27). This ends the proof of 1°.

For (3.10), we depart from (1.21) in the case $m=-2$. This gives (for $\mu\ge 1$ where $[\zeta]=|\zeta|$):

$$\|\partial_{\eta'}^\alpha\partial_w^\beta \tilde f(w,u_n,v_n,\eta',1)\|_{L_\infty(\mathbb{R}^2_{++})} \dot\le \langle\eta'\rangle^{-|\alpha|},\ \text{all } \alpha,\beta, \tag{3.14}$$

uniformly in w. By the quasi-homogeneity (1.24), we have for $|\xi'|\ge 1$, since $m=-2$:

$$\tilde f(w,zu_n,zv_n,\xi',\tfrac1z)=\tilde f(w,u_n,v_n,z\xi',1),\ \text{for } z\le 1. \tag{3.15}$$

By (3.14), we can apply Lemma 3.1 with $\delta=0$ and $B=L_\infty(\mathbb{R}^2_{++})$, finding that $\tilde f(w,u_n,v_n,z\xi',1)\in S^{0,0,0}(\mathbb{R}_+,L_\infty(\mathbb{R}^2_{++}))$. Thus, in view of (3.15),

$$\tilde f(w,zu_n,zv_n,\xi',\tfrac1z)\in S^{0,0,0}(\mathbb{R}_+,L_\infty(\mathbb{R}^2_{++})),$$

so (2.12) holds for $l=l'=k=k'=0$ with $m=d=0$, $s=-2$.

Other values of l,l',k,k' are included as in the proof of 1°. Then the result is established for $m=-2$, and it extends to general m as in 1°.

3° follows by combining this with remainder estimates as in the proof of [GS95, Th. 1.16] (on the remainder after N terms, each of the relevant norms is $O(\langle\xi',\mu\rangle^{-N+C})$ for some C). □

By Fourier transformation, we get from Proposition 2.6:

Corollary 3.3. *Let $m \in \mathbb{Z}$.*

1° Let $q(w,\zeta,\xi_n) \in C^\infty(\mathbb{R}^\nu \times \overline{\mathbb{R}}^n_+ \times \mathbb{R})$ be in $\mathcal{H}^+$ with respect to ξ_n, uniformly in w, and satisfy (1.19) (i.e., it is homogeneous of degree m in (ζ,ξ_n) for $|\zeta| \ge 1$). Set $f(w,\xi',\xi_n,\mu) = q(w,(\xi',\mu),\xi_n)$. Then

$$f(w,\xi',\xi_n,\mu) \in \mathcal{S}^{0,0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathcal{H}^+). \tag{3.16}$$

There is a similar statement with $\mathcal{H}^+$ replaced by $\mathcal{H}^-_{-1}$.

2° Let $q(w,\zeta,\xi_n,\eta_n) \in C^\infty(\mathbb{R}^\nu \times \overline{\mathbb{R}}^n_+ \times \mathbb{R}^2)$ be in $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$ with respect to (ξ_n,η_n), uniformly in w, and satisfy (1.25) (i.e., it is homogeneous of degree m in (ζ,ξ_n,η_n) for $|\zeta| \ge 1$). Set $f(w,\xi',\xi_n,\eta_n,\mu) = q(w,(\xi',\mu),\xi_n,\eta_n)$.

Then $f(w,\xi',\xi_n,\eta_n,\mu) \in \mathcal{S}^{0,0,m}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$.

3° Similar statements hold more generally when $q(w,\zeta,\xi_n)$ (resp. $q(w,\zeta,\xi_n,\eta_n)$) is a classical polyhomogeneous Poisson or trace (resp. singular Green) symbol of degree m; the expansion of q in homogeneous terms then corresponds to the expansion of f in homogeneous terms.

It suffices for these statements to have q defined only for $\zeta_n \ge 0$ and homogeneous in (ζ,ξ_n) (resp. (ζ,ξ_n,η_n)) there, for $|\zeta| \ge 1$.

The theorem and its corollary could also be proved by use of any of the other auxiliary systems of spaces $S^{m,d,s}_\delta(\Gamma,B)$ in (2.15)–(2.19).

Example 3.4. The above theorem applies to $e^{-x_n[\zeta]}$, showing that

$$e^{-x_n[\xi',\mu]} \in \mathcal{S}^{0,0,-1}(\mathbb{R}_+,\mathcal{S}_+) \subset \mathcal{S}^{-1,0,0}(\mathbb{R}_+,\mathcal{S}_+) \cap \mathcal{S}^{0,-1,0}(\mathbb{R}_+,\mathcal{S}_+), \tag{3.17}$$

since it is strongly quasi-homogeneous of degree -1 (recall Example 1.1 and (2.27)). Note that also $e^{-x_n|(\xi',\mu)|} \in \mathcal{S}^{0,0,-1}(\mathbb{R}_+,\mathcal{S}_+)$. Similarly, the corollary applies to $([\zeta] \pm i\xi_n)^{-1}$, showing that $([\xi',\mu] \pm i\xi_n)^{-1}$ are in $\mathcal{S}^{0,0,-1}(\mathbb{R}_+,\mathcal{H}^\pm_{-1})$, respectively. On the other hand, $e^{-x_n[\xi']}$ and $([\xi'] \pm i\xi_n)^{-1}$ do not fit into this framework.

Example 3.5. Let $a^0(x',\xi)$ be an $N\times N$ matrix that is the principal symbol (at $x_n = 0$) of a uniformly elliptic differential operator of order $m \ge 1$, such that $\det(a^0(x',\xi) - \lambda) \ne 0$ for $\lambda \in \overline{\mathbb{R}}_-$, $|(\xi,\lambda)| \ne 0$. Let $p^0(x',\xi,\mu) = (a^0(x',\xi) + \mu^m I)^{-1}$, and let $\tilde{p}^0(x',x_n,\xi',\mu) = \mathcal{F}^{-1}_{\xi_n\to x_n} p^0(x',\xi,\mu)$. Then there is a sector Γ around $\mathbb{R}_+$ such that $a^0(x',\xi) + \mu^m I$ is invertible for $\mu \in \Gamma$, and

$$r^+\tilde{p}^0(x',x_n,\xi',\mu) \text{ and } r^+\tilde{p}^0(x',-x_n,\xi',\mu) \in \mathcal{S}^{0,0,-m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+) \otimes \mathcal{L}(\mathbb{C}^N \times \mathbb{C}^N). \tag{3.18}$$

To see this, consider a ray $\{\mu = re^{i\theta} \mid r > 0\}$ in Γ and let $\zeta = (\zeta',\zeta_n) = (\xi',r)$. The hypotheses imply that the symbol $p^0(x',\zeta,\xi_n) = a^0(x',\xi',\xi_n) + \zeta_n^m I$ has the properties of a standard elliptic symbol of order m with $(n+1)$-dimensional cotangent variable (ζ,ξ_n) (for $\zeta_n \ge 0$); it is polynomial in (ζ,ξ_n), and its inverse $p^0(x',\zeta,\xi_n)$ is a rational matrix-function of (ζ,ξ_n). For $\tilde{p}^0(x',x_n,\zeta) = \mathcal{F}^{-1}_{\xi_n\to x_n} p^0(x',\zeta,\xi_n)$ it is a basic and elementary fact in the calculus of [BM71] that $r^+\tilde{p}^0(x',x_n,\zeta)$ and $r^+p^0(x',-x_n,\zeta)$ are quasi-homogeneous

Poisson symbol-kernels of degree $-m$. Then we get (3.18) by application of Theorem 3.2 1°. The analyticity in μ follows from the analyticity of p^0.

For the full parametrix symbol $p(x',\xi,\mu)$ of $a(x',\xi)+\mu^m I$, where a is a differential operator symbol with principal part a^0 as above, one finds from Theorem 3.2 3° that $r^+\tilde p(x',x_n,\xi',\mu)$ and $r^+\tilde p(x',-x_n,\xi',\mu)$ belong to the spaces in (3.18); they are strongly polyhomogeneous. In fact, the quasi-homogeneity of each term extends smoothly to all $\xi'\in\mathbb{R}^{n-1}$, $\mu\in\Gamma\cup\{0\}$ with $|\xi',\mu|\ge 1$.

Similar statements hold for $\partial_\mu^k\tilde p(x',x_n,\xi',\mu)$, with $-m$ replaced by $-m-k$.

Parametrices of elliptic symbols p that also depend on x_n are included by consideration of their Taylor expansions in x_n at $x_n=0$.

4. Composition of boundary operators of class 0

We now begin to study compositions. For the boundary operators with symbol-kernels as in Section 2 together with ψdo's on the boundary, this can be carried out quite simply by integration in the x_n-variable (the "real formulation") and standard ψdo-rules for the x'-variables; we show this in the following. For compositions involving an interior ψdo we have to use the somewhat heavier "complex formulation" in terms of $\mathcal{H}^\pm$ spaces (in the Fourier transformed variable ξ_n) and their projections; this is the subject of Section 5.

Proposition 4.1. *Let*

$$\begin{aligned}\tilde g(w,x_n,y_n,\xi',\mu)&\in\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++}),\\ \tilde t(w,x_n,\xi',\mu)\text{ and }\tilde k(w,x_n,\xi',\mu)&\in\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{+}),\\ q(w,\xi',\mu)&\in S^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathbb{C}),\end{aligned}\tag{4.1}$$

and let $\tilde g'$, $\tilde t'$, $\tilde k'$ and q' be given similarly with m, d, s, ν and w replaced by m', d', s', ν' and w'. Define

$$m''=m+m',\quad d''=d+d',\quad s''=s+s',\quad \nu''=\nu+\nu'.\tag{4.2}$$

Then

(4.3)

1° $q\circ_n q'=q(w,\xi',\mu)q'(w',\xi',\mu)\in S^{m'',d'',s''}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$,

2° $q\circ_n\tilde t'=q(w,\xi',\mu)\tilde t'(w',x_n,\xi,\mu)\in\mathcal{S}^{m'',d'',s''}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$,

3° $\tilde k\circ_n q'=\tilde k(w,x_n,\xi',\mu)q'(w',\xi',\mu)\in\mathcal{S}^{m'',d'',s''}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$,

4° $\tilde k\circ_n\tilde t'=\tilde k(w,x_n,\xi',\mu)\tilde t'(w',y_n,\xi',\mu)\in\mathcal{S}^{m'',d'',s''}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$,

5° $\tilde t\circ_n\tilde k'=\int_0^\infty\tilde t(w,x_n,\xi',\mu)\tilde k'(w',x_n,\xi',\mu)\,dx_n\in S^{m'',d'',s''+1}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$,

6° $\tilde t\circ_n\tilde g'=\int_0^\infty\tilde t(w,x_n,\xi',\mu)\tilde g'(w',x_n,y_n,\xi',\mu)\,dx_n\in\mathcal{S}^{m'',d'',s''+1}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$,

7° $\tilde g\circ_n\tilde k'=\int_0^\infty\tilde g(w,x_n,y_n,\xi',\mu)\tilde k'(w',y_n,\xi',\mu)\,dy_n\in\mathcal{S}^{m'',d'',s''+1}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$,

8° $\tilde g\circ_n\tilde g'=\int_0^\infty\tilde g(w,x_n,z_n,\xi',\mu)\tilde g'(w',z_n,y_n,\xi',\mu)\,dz_n$
$\in\mathcal{S}^{m'',d'',s''+1}(\mathbb{R}^{\nu''}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$.

Proof. The composition rules in this list are seen from the defining formulas (1.6), (1.9) together with the fact that $\circ_n$-composition with q or q' is just multiplication.

The results in 1°–3° then follow from Lemma 2.9 (in fact 1° was shown already in [GS95]). 4° is a straightforward generalization, using that $\|\tilde k(|z|u_n)\tilde t'(|z|v_n)\|_{L_\infty} \le \|\tilde k(|z|u_n)\|_{L_\infty}\|\tilde t'(|z|v_n)\|_{L_\infty}$ and that the indices $s+1$ and $s'+1$ in (2.11) add up to $s''+2$ in (2.12).

Now consider 5°. We have from Proposition 2.6 that $\tilde t \in S^{m,d+1,s}_{\frac12}(\Gamma, L_{2,u_n})$ and $\tilde k \in S^{m,d,s+1}_{-\frac12}(\Gamma, L_{2,u_n})$. This gives on each ray $\mu = re^{i\theta}$:

$$|z^{d+d'}\kappa^{-s-s'-1}\int_0^\infty \tilde t(x_n)\tilde k'(x_n)\,dx_n| = |\int_0^\infty z^{d+1}\kappa^{-s}\tilde t(|z|u_n)z^{d'}\kappa^{-s'-1}\tilde k'(|z|u_n)\,du_n|$$
$$\le \|z^{d+1}\kappa^{-s}\tilde t\|_{L_{2,u_n}}\|z^{d'}\kappa^{-s'-1}\tilde k\|_{L_{2,u_n}} \dot\le \langle\xi'\rangle^{m+m'},$$

since $\langle z\xi'\rangle^{-\frac12}$ and $\langle z\xi'\rangle^{\frac12}$ cancel out. A similar pattern is found for all the derivatives in w, w', ξ' and z, when we note that

$$\partial^\beta_{(w,w')}\partial^\alpha_{\xi'}\partial^j_{|z|}(z^{d+d'}\kappa^{-s-s'-1}\int_0^\infty \tilde t(x_n)\tilde k(x_n)\,dx_n)$$
$$= \partial^\beta_{(w,w')}\partial^\alpha_{\xi'}\partial^j_{|z|}(\int_0^\infty z^{d+1}\kappa^{-s}\tilde t(|z|u_n)z^{d'}\kappa^{-s'-1}\tilde k(|z|u_n)\,du_n) \tag{4.4}$$

and apply the Leibniz formula inside the integral; this shows 5°.

For the remaining statements in (4.3), the proofs are very similar. We may in fact assume $d = d' = s = s' = 0$, since the general case is reduced to this by replacing the symbol-kernel by its product with $z^d[\xi',\mu]^{-s}$, etc. For 6°, we note that $\tilde t_1 = \tilde t \circ_n \tilde g'$ satisfies:

$$|z\kappa^{-1}\tilde t_1(y_n)| = |z\kappa^{-1}\int_0^\infty \tilde t(x_n)\tilde g'(x_n,y_n)\,dx_n|$$
$$= |\int_0^\infty z\tilde t(|z|u_n)z\kappa^{-1}\tilde g'(|z|u_n,y_n)\,du_n| \le \|z\tilde t(|z|u_n)\|_{L_{2,u_n}}\|z\kappa^{-1}\tilde g'(|z|u_n,y_n)\|_{L_{2,u_n}}, \tag{4.5}$$

and hence

$$\langle z\xi'\rangle^{-\frac12}\|z\kappa^{-1}\tilde t_1(|z|v_n)\|_{L_{2,v_n}}$$
$$\le \langle z\xi'\rangle^{-\frac12}\|z\tilde t(|z|u_n)\|_{L_{2,u_n}}\|z\kappa^{-1}\tilde g'(|z|u_n,|z|v_n)\|_{L_{2,u_n,v_n}} \dot\le \langle\xi'\rangle^m\langle\xi'\rangle^{m'}, \tag{4.6}$$

since $\tilde t \in S^{m,1,0}_{\frac12}(\Gamma, L_{2,u_n})$, $\tilde g \in S^{m,1,1}(\Gamma, L_{2,u_n,v_n})$. This shows the basic estimate for 6°; derivatives are treated by use of the Leibniz formula. 7° and 8° are shown similarly; it is used that the application of $x_n^l\partial_{x_n}^{l'}$ or $y_n^l\partial_{y_n}^{l'}$ only hits one of the factors. □

Remark 4.2. The symbols resp. symbol-kernels of the operators resulting from composition in all variables follow the usual ψdo composition rules with respect to the tangential variables: If w and w' are replaced by x' resp. $y' \in \mathbb{R}^{n-1}$ in Proposition 4.1, the resulting symbols are described by the same formulas as there, now depending on (x',y'). This is reduced to symbols depending on x' by the usual reduction for ψdo's, as given in [GS95, Th. 1.18]. If the given symbols $\tilde f$ and $\tilde f'$ depend on x', one changes the right hand factor to y'-form, performs $\circ_n$ and reduces to x'-form afterwards; this results in expansions

$$\tilde f \circ \tilde f' \sim \sum_{\alpha\in\mathbb{N}^{n-1}} \frac{(-i)^{|\alpha|}}{\alpha!}\partial^\alpha_{\xi'}\tilde f \circ_n \partial^\alpha_{x'}\tilde f'. \tag{4.7}$$

Here is another useful rule:

Lemma 4.3. *Let* $\tilde{k}(w, x_n, \xi', \mu) \in \mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_+)$ *and define*

$$\tilde{g}(w, x_n, y_n, \xi', \mu) = \tilde{k}(w, x_n + y_n, \xi', \mu). \tag{4.8}$$

Then $\tilde{g}$ *is a singular Green symbol-kernel in* $\mathcal{S}^{m,d,s-1}(\Gamma, \mathcal{S}_{++})$.

Proof. Consider a ray $\mu = re^{i\theta}$; we can assume that $\theta = 0$; moreover we can reduce to the case $d = s = 0$, so $\tilde{k} \in \mathcal{S}^{m,0,0}$. Now by the change of variables $w_n = u_n + v_n$, $z_n = u_n - v_n$,

$$\begin{aligned} \int_0^\infty \int_0^\infty |\tilde{k}(z(u_n + v_n))|^2 \, du_n dv_n &= \int_0^\infty \int_{-w_n}^{w_n} |k(zw_n)|^2 \, \tfrac{1}{2} dw_n dz_n \\ &= \int_0^\infty w_n |\tilde{k}(zw_n)|^2 \, dw_n \le \|\tilde{k}(zw_n)\|_{L_{2,w_n}} \|w_n \tilde{k}(zw_n)\|_{L_{2,w_n}} \\ &\dot{\le} z^{-1} \langle z\xi\rangle^{\frac{1}{2}} \langle \xi'\rangle^m z^{-1} \langle z\xi'\rangle^{-1} \langle z\xi'\rangle^{\frac{1}{2}} \langle \xi'\rangle^m = z^{-2} \langle \xi'\rangle^{2m}, \end{aligned} \tag{4.9}$$

by (2.15). With similar estimates of $u_n^l \partial_{u_n}^{l'} v_n^j \partial_{v_n}^{j'} \tilde{k}(u_n + v_n)$, we find that $\tilde{g} \in S^{m,1,0}(\Gamma, L_2(\mathbb{R}^2_{++}))$. In view of (2.18), this shows the basic step in the proof that $\tilde{g} \in \mathcal{S}^{m,0,-1}(\Gamma, \mathcal{S}_{++})$, and in a similar way the desired estimates are found for all the derivatives in w, ξ' and z. □

Example 4.4. Consider $P = \mathrm{OP}(p)$, where p is the parametrix symbol of $a(x', \xi) + \mu^m$ as in Example 3.5. The singular Green operator $G^+(P) = r^+ P e^- J$, where J is the reflection $J\colon u(x', x_n) \mapsto u(x', -x_n)$, has the symbol-kernel

$$\tilde{g}^+(p)(x', x_n, y_n, \xi', \mu) = r^+ \tilde{p}(x', x_n + y_n, \xi', \mu), \tag{4.10}$$

cf. [G84] or [G96]. It can be regarded in several ways. For one thing, $r^+ \tilde{p}(x', x_n, \xi', \mu)$ is a strongly polyhomogeneous Poisson symbol-kernel of degree $-m$ and therefore belongs to $\mathcal{S}^{0,0,-m}(\Gamma, \mathcal{S}_+)$ by Theorem 3.2 1°, as noted in Example 3.5. Then Lemma 4.3 shows that $\tilde{g}^+(p) \in \mathcal{S}^{0,0,-m-1}(\Gamma, \mathcal{S}_{++})$. On the other hand, $\tilde{g}^+(p)$ is a strongly polyhomogeneous singular Green symbol-kernel and is of degree $-m - 1$, so Theorem 3.2 2° gives that $\tilde{g}^+(p) \in \mathcal{S}^{0,0,-m-1}(\Gamma, \mathcal{S}_{++})$, which is consistent with the first information.

To illustrate this with a simple case, let $a(x, \xi) = |\xi|^2$, then for $\mu \ge 1$,

$$p(x, \xi, \mu) = \frac{1}{|\xi|^2 + \mu^2} = \frac{1}{2\kappa}\Big(\frac{1}{\kappa + i\xi_n} + \frac{1}{\kappa - i\xi_n}\Big). \tag{4.11}$$

Thus

$$r^+ \tilde{p}(x_n, \xi', \mu) = (2\kappa)^{-1} e^{-x_n \kappa}, \quad \tilde{g}^+(p)(x_n, y_n, \xi', \mu) = (2\kappa)^{-1} e^{-(x_n + y_n)\kappa}, \tag{4.12}$$

which is in $\mathcal{S}^{0,0,-3}(\Gamma, \mathcal{S}_{++})$ by Theorem 3.2 2°. Since $e^{-x_n\kappa} \in \mathcal{S}^{0,0,-1}$ (Example 3.4), the composition rule (4.3) 4° likewise gives $\tilde{g}^+(p) \in \mathcal{S}^{0,0,-3}$, which is also what Lemma 4.3 gives.

The usefulness of the boundary operator calculus is closely connected with the following rule, showing that the *normal trace*

$$\mathrm{tr}_n \, \tilde{g} = \int_0^\infty \tilde{g}(w, x_n, x_n, \xi', \mu) \, dx_n \tag{4.13}$$

of a singular Green symbol-kernel is a ψdo symbol with the right type of parameter-dependence.

Theorem 4.5. *Let $\tilde g$ be a singular Green symbol-kernel*

$$\tilde g(w,x_n,y_n,\xi',\mu)\in\mathcal{S}^{m,d,s-1}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++}). \tag{4.14}$$

Then the normal trace of $\tilde g$ is a ψdo symbol satisfying

$$\operatorname{tr}_n\tilde g(w,\xi',\mu)=\int_0^\infty\tilde g(w,x_n,x_n,\xi',\mu)\,dx_n\in S^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathbb{C}). \tag{4.15}$$

Proof. Reduce to the case $d=s=0$, $z\in\mathbb{R}_+$, so that (2.18) holds with $S^{m,1,0}(\Gamma,L_2(\mathbb{R}^2_{++}))$. We use the method of the proof of [G96, Th. 3.3.9]. With ε to be chosen further below, we estimate

$$\begin{aligned}|\operatorname{tr}_n\tilde g(w,\xi',\mu)|^2&=\Big(\int_0^\infty\tfrac{\varepsilon+x_n}{\varepsilon+x_n}|\tilde g(w,x_n,x_n,\xi',\mu)|dx_n\Big)^2\\&\le c\varepsilon^{-1}\int_0^\infty(\varepsilon^2+x_n^2)|\tilde g(x_n,x_n)|^2dx_n\\&\le c\int_0^\infty\int_0^\infty\big(\varepsilon\partial_{y_n}|\tilde g(x_n,y_n)|^2+\varepsilon^{-1}\partial_{y_n}|y_n\tilde g(x_n,y_n)|^2\big)\,dx_ndy_n\\&\le 2c(\varepsilon\|\tilde g\|\,\|\partial_{y_n}\tilde g\|+\varepsilon^{-1}\|y_n\tilde g\|\,\|\partial_{y_n}y_n\tilde g\|),\end{aligned} \tag{4.16}$$

with norms in $L_{2,x_n,y_n}(\mathbb{R}^2_{++})$. (A trace estimate as in [G96, (A.54)] was used to extend the integration to y_n.) This gives *with norms in* $L_{2,u_n,v_n}(\mathbb{R}^2_{++})$:

$$\begin{aligned}|\operatorname{tr}_n\tilde g|^2&\,\dot\le\,\varepsilon z^2\|\tilde g(zu_n,zv_n)\|\,\|z^{-1}\partial_{v_n}\tilde g(zu_n,zv_n)\|\\&\quad+\varepsilon^{-1}z^2\|zv_n\tilde g(zu_n,zv_n)\|\,\|\partial_{v_n}v_n\tilde g(zu_n,zv_n)\|\\&\,\dot\le\,\varepsilon z^2z^{-1}\langle\xi'\rangle^mz^{-2}\langle z\xi'\rangle^{-1}\langle\xi'\rangle^m+\varepsilon^{-1}z^2\langle z\xi'\rangle\langle\xi'\rangle^mz^{-1}\langle\xi'\rangle^m\doteq\langle\xi'\rangle^{2m},\end{aligned}$$

where we set $\varepsilon=z\langle z\xi'\rangle$. This is the basic estimate for $\operatorname{tr}_n\tilde g\in S^{m,0,0}(\Gamma,\mathbb{C})$. Derivatives are treated similarly. □

When G is the singular Green operator with symbol-kernel $\tilde g(x',x_n,y_n,\xi',\mu)$, the pseudodifferential operator with symbol $(\operatorname{tr}_n\tilde g)(x',\xi',\mu)$ will be denoted $\operatorname{tr}_n G$. When $m<1-n$, it has a continuous kernel

$$\begin{aligned}\mathcal{K}_{\operatorname{tr}_nG}(x',y',\mu)&=\int_{\mathbb{R}^{n-1}}e^{i(x'-y')\cdot\xi'}(\operatorname{tr}_n\tilde g)(x',\xi',\mu)\,đ\xi'\\&=\int_{\mathbb{R}^n_+}e^{i(x'-y')\cdot\xi'}\tilde g(x',x_n,x_n,\xi',\mu)\,dx_nđ\xi'.\end{aligned} \tag{4.17}$$

When $\operatorname{tr}_n\tilde g\in S^{m,d,0}(\Gamma,\mathbb{C})$ and is holomorphic in μ, the symbol multiplied by z^d has a Taylor expansion in z at $z=0$, by [GS95, Th. 1.12], i.e. an expansion for $\mu\to\infty$:

Corollary 4.6. *Let $\tilde g(x',x_n,y_n,\xi',\mu)\in\mathcal{S}^{m,d,s-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$, holomorphic in μ, and define $\operatorname{tr}_n\tilde g\in S^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$ according to Theorem 4.5.*

If $s=0$, there is an expansion for $\mu\to\infty$ on rays in Γ:

$$(\operatorname{tr}_n\tilde g)(x',\xi',\mu)-\sum_{0\le k<N}\mu^{d-k}h_k(x',\xi')\in S^{m+N,d-N}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C}),\tag{4.18}$$

for any N; here

$$h_k(x',\xi')=\tfrac{1}{k!}\partial_z^k(z^d(\operatorname{tr}_n\tilde g)(x',\xi',\tfrac{1}{z}))|_{z=0}\in S^{m+k}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}).\tag{4.19}$$

For $s\neq0$, similar expansions are found after application of (2.28).

We note in particular that there is a full expansion of the kernel of $\operatorname{tr}_n G$ for operators with $m=-\infty$ (as in [GS95, Prop. 1.21]):

Corollary 4.7. *Let $\tilde g(x',x_n,y_n,\xi',\mu)\in\mathcal{S}^{-\infty,d,s-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$, holomorphic in μ, so that $\operatorname{tr}_n\tilde g\in S^{-\infty,d+s,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$ by Theorem 4.5 and* (2.28). *Then the kernel $\mathcal{K}_{\operatorname{tr}_n G}(x',y',\mu)$ has an asymptotic expansion*

$$\mathcal{K}_{\operatorname{tr}_n G}(x',y',\mu)\sim\sum_{k\ge0}\mu^{d+s-k}\mathcal{K}_{h_k}(x',y'),\tag{4.20}$$

where $\mathcal{K}_{h_k}\in C^\infty(\mathbb{R}^{2n-2})$, and $\mathcal{K}_{\operatorname{tr}_n G}-\sum_{k<N}\mu^{d+s-k}\mathcal{K}_{h_k}$ is in $C^\infty(\mathbb{R}^{2n-2}\times\Gamma)$, holomorphic in $\mu\in\mathring{\Gamma}$ for $|\mu|\ge c$ and $O(\mu^{d+s-N})$ in closed subsectors of Γ, for all N.

In particular, if $\tilde g$ vanishes for x' outside a compact set, then G is trace class, and the trace has an expansion for $\mu\to\infty$ in closed subsectors of Γ:

$$\operatorname{Tr}G\sim\sum_{k\ge0}c_k\mu^{d+s-k}.\tag{4.21}$$

Proof. (4.20) is a straightforward consequence of Corollary 4.6, and (4.21) then follows since the trace equals $\int_{\mathbb{R}^{n-1}}\mathcal{K}_{\operatorname{tr}_n g}(x',x',\mu)\,dx'$. □

We continue the study of asymptotic trace expansions for singular Green operators in Section 7 below (which can be read directly after this).

In the next sections we extend the calculus to include operators of class >0 as well as interior ψdo's.

5. Boundary symbols of all classes

For the treatment of compositions with ψdo's on the interior we need to use the *symbol* calculus, working with the $x_n\to\xi_n$ Fourier transformed versions of the symbol-kernels. This is preferable because the ψdo's act multiplicatively and are estimated by sup-norms in the ξ_n-formulation, whereas they act like convolutions in the x_n-formulation with more complicated estimates. Moreover, we need to include symbols of positive class (and we include a notation for negative class too).

A complete parameter-dependent calculus was worked out in the book [G96], with full explanations of the complex machinery (the use of (1.12)). We here need a somewhat specialized version. To save space, we shall present it while referring as much as possible to the explanations in the book. Some results here are based directly on results in the book, so a reader who wants to see full arguments would have to look up some details there anyway; therefore we shall not strive to make this particular part of the paper selfcontained. We refer to [G96, Sect. 2.2] for a detailed explanation of the spaces $\mathcal{H}$, $\mathcal{H}_d$, $\mathcal{H}^+$, $\mathcal{H}^-$, $\mathcal{H}_d^-$ and the associated projections h_d, h^+, h^-, h_d^-, etc. in $\mathcal{H}$.

The definitions of the symbol spaces for trace operators are extended to include symbols of general class by defining (somewhat similarly to [G96, Def. 2.3.2]):

Definition 5.1. *Let $m \in \mathbb{R}$, d, s and $r \in \mathbb{Z}$. Let $\mathcal{K}$ stand for either of the spaces $\mathcal{H}_{r-1}, \mathcal{H}_{r-1}^-$ or $\mathcal{H}^+$.*

The space $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{K})$ (often abbreviated to $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{K})$) consists of the functions $f(w, \xi', \xi_n, \mu) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n \times \Gamma)$, lying in $\mathcal{K}$ with respect to ξ_n, such that when f is written in the form

$$f(w, \xi', \xi_n, \mu) = \sum_{0 \le j \le r-1} s_j(w, \xi', \mu) \xi_n^j + f'(w, \xi', \xi_n, \mu) \tag{5.1}$$

with $f' = h_{-1,\xi_n} f$, then

$$s_j(w, \xi', \mu) \in S^{m,d,s-j}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathbb{C}), \tag{5.2}$$

and f' satisfies, with $z = \frac{1}{\mu}$,

$$\langle z\xi' \rangle^{l-l'} h_{-1} \partial_{\zeta_n}^l \zeta_n^{l'} f'(w, \xi', |\tfrac{1}{z}|\zeta_n, \tfrac{1}{z}) \in S_{\frac{1}{2}}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, L_{2,\zeta_n}(\mathbb{R})), \tag{5.3}$$

for all $l, l' \in \mathbb{N}$.

If f moreover has an asymptotic expansion

$$f \sim \sum_{l \in \mathbb{N}} f_{m-l}, \tag{5.4}$$

where $f - \sum_{l<M} f_{m-l} \in \mathcal{S}^{m-M,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{K})$ for any $M \in \mathbb{N}$, and the symbols f_{m-l} are homogeneous of degree $m + d - l$ in (ξ, μ) on the set where $|\xi'| \ge 1$:

$$f_{m-l}(w, t\xi, t\mu) = t^{m+d-l} f_{m-l}(w, \xi, \mu) \text{ for } |\xi'| \ge 1,\ t \ge 1, \tag{5.5}$$

we say that f is (weakly) polyhomogeneous in $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{K})$.

The spaces $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}_{r-1}^-)$ are the spaces of (parameter-dependent) **trace symbols** *of degree $m + d + s$ and class r.*

In view of Proposition 2.6 1°, this generalizes Definitions 2.4 3° and 2.5 1°, which correspond to the cases $\mathcal{H}_{-1}^-$ and $\mathcal{H}^+$. In the decomposition (5.1), the sum over j is empty when $r \le 0$; this holds when $\mathcal{K} \subset \mathcal{H}_{-1}$ (e.g., $\mathcal{K} = \mathcal{H}_{-1}^-$ or $\mathcal{H}^+$). Note that f' is in the space with r replaced by $\min\{r, 0\}$; it is called "the part of f of class 0" or "the $\mathcal{H}_{-1}$-part." The sum over $0 \le j < r$ is called "the polynomial part." A formulation in terms of L_∞ norms is given in Lemma 5.5. The L_2-formulation is used in Lemma 5.4.

There is also an extended definition of singular Green symbols, parallel to [G96, Def. 2.3.7]:

Definition 5.2. *Let $m \in \mathbb{R}$, d, s and $r \in \mathbb{Z}$. Let $\mathcal{K}$ stand for either of the spaces $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1}$ or $\mathcal{H}_{-1}\hat{\otimes}\mathcal{H}_{r-1}$.*

The space $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{K})$ (often abbreviated to $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{K})$) consists of the functions $f(w, \xi', \xi_n, \eta_n, \mu) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^{n+1} \times \Gamma)$, lying in $\mathcal{K}$ with respect to (ξ_n, η_n), such that when f is written in the form

$$f(w, \xi', \xi_n, \eta_n, \mu) = \sum_{0\le j\le r-1} k_j(w, \xi', \xi_n, \mu)\eta_n^j + f'(w, \xi', \xi_n, \eta_n, \mu) \tag{5.6}$$

with $f' = h_{-1,\eta_n} f$, then $k_j \in \mathcal{S}^{m,d,s-j}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}^+)$ for each j, and f' satisfies the conditions, with $z = 1/\mu$,

$$\begin{aligned}\langle z\xi'\rangle^{l-l'+k-k'} h_{-1,\zeta_n} h_{-1,\varrho_n} \partial^l_{\zeta_n} \zeta_n^{l'} \partial^k_{\varrho_n} \varrho_n^{k'} f'(w, \xi', |\tfrac{1}{z}|\zeta_n, |\tfrac{1}{z}|\varrho_n, \tfrac{1}{z}) \\ \in S^{m,d-1,s+1}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, L_{2,\zeta_n,\varrho_n}(\mathbb{R}^2)),\end{aligned} \tag{5.7}$$

for all $l, l', k, k' \in \mathbb{N}$.

If f moreover has an asymptotic expansion

$$f \sim \sum_{l\in\mathbb{N}} f_{m-l}, \tag{5.8}$$

where $f - \sum_{l<M} f_{m-l} \in \mathcal{S}^{m-M,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{K})$ for any $M \in \mathbb{N}$, and the symbols f_{m-l} are homogeneous of degree $m + d + s - l$ in $(\xi', \xi_n, \eta_n, \mu)$ when $|\xi'| \ge 1$:

$$f_{m-l}(w, t\xi', t\xi_n, t\eta_n, t\mu) = t^{m+d+s-l} f_{m-l}(w, \xi', \xi_n, \eta_n, \mu) \text{ for } t \text{ and } |\xi'| \ge 1, \tag{5.9}$$

then f is said to be (weakly) polyhomogeneous, of degree $m + d + s$.

The spaces $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{r-1})$ are the spaces of (parameter-dependent) **singular Green symbols** *of degree $m + d + s$ and class r.*

For $\mathcal{K} = \mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$ this is the space defined in Definitions 2.4 4° and 2.5 2°, cf. Proposition 2.6 2°.

Remark 5.3. In applications of the decompositions in a polynomial part and an $\mathcal{H}_{-1}$-part it is convenient to observe the following rule, once and for all:

$$h_{-1}[\partial^l_{\xi_n} \xi_n^{l'} h_{-1}(\partial^k_{\xi_n} \xi_n^{k'} f(\xi_n))] = h_{-1}[\partial^l_{\xi_n} \xi_n^{l'} \partial^k_{\xi_n} \xi_n^{k'} f(\xi_n)], \text{ for } f \in \mathcal{H},\ l, l', k, k' \in \mathbb{N}. \tag{5.10}$$

The most intuitively understandable proof of this is perhaps to use that the inverse Fourier transform of the space $\mathcal{H}$ is the distribution space

$$\dot{\mathcal{S}}(\mathbb{R}) = e^+\mathcal{S}(\overline{\mathbb{R}}_+) \dotplus e^-\mathcal{S}(\overline{\mathbb{R}}_-) \dotplus \mathbb{C}[\delta'], \tag{5.11}$$

where $\mathbb{C}[\delta']$ is the space of " polynomials" $\sum c_k \delta^{(k)}$, $\delta^{(k)} = D^k_{x_n}\delta$ (cf. [G96, Prop. 2.2.2]). On this space, h_{-1} corresponds to the projection of $\dot{\mathcal{S}}(\mathbb{R})$ onto $e^+\mathcal{S}(\overline{\mathbb{R}}_+) \dotplus e^-\mathcal{S}(\overline{\mathbb{R}}_-)$ that removes the component in $\mathbb{C}[\delta']$. This projection can also be identified with $e_M r_M$, which restricts the distribution to $M = \mathbb{R}_+ \cup \mathbb{R}_-$ and extends the result back to $\mathbb{R}$ as a function; it gives a function in $e^+\mathcal{S}(\overline{\mathbb{R}}_+)\dot{+}e^-\mathcal{S}(\overline{\mathbb{R}}_-)$. Recall that $\partial_{\xi_n}^l\xi_n^{l'}$ on $\mathcal{H}$ corresponds to $i^{l-l'}x_n^l\partial_{x_n}^{l'}$ on $\dot{\mathcal{S}}$. Now clearly

$$e_M r_M x_n^l\partial_{x_n}^{l'} e_M r_M x_n^k\partial_{x_n}^{k'}\tilde f = e_M r_M x_n^l\partial_{x_n}^{l'} x_n^k\partial_{x_n}^{k'}\tilde f \text{ for } \tilde f\in\dot{\mathcal{S}}, \tag{5.12}$$

and this implies (5.10) by Fourier transformation $\mathcal{F}_{x_n\to\xi_n}$. Similar rules hold with h_{-1} replaced by h^+ or h^-_{-1}, since they correspond to application of e^+r^+ resp. e^-r^- in the inverse Fourier transformed situation.

We also observe that when the functions depend on further parameters x',ξ',μ, say, then

$$h_{-1,\xi_n}\partial^\gamma_{x',\xi',\mu}f(x',\xi',\mu,\xi_n) = \partial^\gamma_{x',\xi',\mu}h_{-1,\xi_n}f(x',\xi',\mu,\xi_n), \tag{5.13}$$

since this holds in the inverse Fourier transformed situation:

$$e_M r_M\partial^\gamma_{x',\xi',\mu}\tilde f(x',\xi',\mu,x_n) = \partial^\gamma_{x',\xi',\mu}e_M r_M\tilde f(x',\xi',\mu,x_n); \tag{5.14}$$

there are similar rules for h^+ and h^-_{-1}. Multiplication by functions of x', ξ' and μ likewise commute with the projections.

In preparation for the general composition rules, we establish some mapping properties for the boundary symbols.

Lemma 5.4. *Let $m\in\mathbb{R}$, d, s and $r\in\mathbb{Z}$.*

1° The mappings h_{-1}, h^+, h^-_{-1} and h^- are continuous from $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}_{r-1})$ to the space $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{K})$ with, respectively, $\mathcal{K}=\mathcal{H}_{\min\{r,0\}-1},\mathcal{H}^+,\mathcal{H}^-_{-1}$ and $\mathcal{H}^-_{\max\{r,0\}-1}$.

2° In the expansion (cf. [G96, Sect. 2.2]*)*

$$f\sim\sum_{-\infty<j<r}s_j(w,\xi',\mu)\xi_n^j, \tag{5.15}$$

the mapping $f\mapsto s_j(w,\xi',\mu)$ is continuous:

$$\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}_{r-1})\to S^{m,d,s-j}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathbb{C}), \tag{5.16}$$

for $j\in\mathbb{Z}$, $j<r$.

3° The mapping $f\mapsto\partial^l_{\xi_n}\xi_n^{l'}f$ is continuous from $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}_{r-1})$ to $\mathcal{S}^{m,d,s-l+l'}(\mathbb{R}^\nu\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}_{r-l+l'-1})$ for $l,l'\in\mathbb{N}$.

Proof. 1°. The statement for h_{-1} follows from Definition 5.1 (when $r\le 0$, h_{-1} acts like the identity). The statements for h^+ and h^-_{-1} are immediate consequences, since h^+ and h^-_{-1} are (complementing) L_2-orthogonal projections in $\mathcal{H}_{-1}$, with

$$\|h^+f\|=\|h^+h_{-1}f\|\le\|h_{-1}f\|,\quad \|h^-_{-1}f\|=\|h^-_{-1}h_{-1}f\|\le\|h_{-1}f\|. \tag{5.17}$$

The statement for h^- follows since it preserves the polynomial part (the sum over $0\le j<r$) and acts on $\mathcal{H}_{-1}$ like h^-_{-1}.

2°. For $j\ge 0$, the statement follows from the definition. For $j=-1-k$, $k\in\mathbb N$, we use the estimate [G96, (2.2.48)]:

$$(5.18)\quad \begin{aligned} |s_{-1-k}|^2 &\dot\le \|h_{-1}(\xi_n^k f)\|_{L_{2,\xi_n}}\|h_{-1}(\xi_n^{k+1}f)\|_{L_{2,\xi_n}}\\ &= |\tfrac1z|\|h_{-1}(\zeta_n^k z^{-k} f(|\tfrac1z|\zeta_n))\|_{L_{2,\zeta_n}}\|h_{-1}(\zeta_n^{k+1}z^{-k-1}f(|\tfrac1z|\zeta_n))\|_{L_{2,\zeta_n}}.\end{aligned}$$

We can let $d=s=0$. Then (5.18) gives e.g. for $f\in\mathcal S^{m,0,0}$, since $\zeta_n^k f(|\frac1z|\zeta_n)\in S_{\frac12}^{m,-k,k}$:

$$(5.19)\quad \begin{aligned} &|\kappa^{-k-1}s_{-1-k}(w,\xi',\mu)|^2\\ &\dot\le |z|^{-1}\kappa^{-1}\|h_{-1}(\zeta_n^k z^{-k}\kappa^{-k}f(|\tfrac1z|\zeta_n))\|_{L_{2,\zeta_n}}\|h_{-1}(\zeta_n^{k+1}z^{-k-1}\kappa^{-k-1}f(|\tfrac1z|\zeta_n))\|_{L_{2,\zeta_n}}\\ &\qquad\dot\le |z|^{-1}\kappa^{-1}\langle z\xi'\rangle^{\frac12}\langle\xi'\rangle^m\langle z\xi'\rangle^{\frac12}\langle\xi'\rangle^m=\langle\xi'\rangle^{2m}.\end{aligned}$$

With similar estimates for derivatives, we find that $s_{-1-k}\in S^{m,0,-k-1}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathbb C)$, as was to be shown.

3°. Clearly, $\partial_{\xi_n}^l\xi_n^{l'}f$ is in $\mathcal H_{r-l+l'-1}$ with respect to ξ_n. For $h_{-1}\partial_{\xi_n}^l\xi_n^{l'}f=h_{-1}\partial_{\xi_n}^l\xi_n^{l'}h_{-1}f$ (cf. (5.10)), the proof of symbol estimates goes as in Lemma 2.10. It remains to consider the polynomial part $p^{l,l'}$ of $\partial_{\xi_n}^l\xi_n^{l'}f$. It consists of those terms in $\partial_{\xi_n}^l\xi_n^{l'}\sum_{j<r}s_j\xi_n^j$ (cf. (5.6)) for which $j-l+l'\ge 0$, so it equals

$$p^{l,l'}(w,\xi',\xi_n,\mu)=\sum_{l-l'\le j<r}s_j(w,\xi',\mu)c_{j,l,l'}\xi_n^{j-l+l'}=\sum_{0\le j'<r-l+l'}s'_{j'}(w,\xi',\mu)\xi_n^{j'},$$
$$\text{with } c_{j,l,l'}=\tbinom{j+l'}{l},\quad s'_{j'}=s_{j'+l-l'}c_{j'-l+l',l,l'}.$$

Here we observe, for $f\in\mathcal S^{m,d,s}$, that $s_{j'+l-l'}\in S^{m,d,s-(j'+l-l')}(\Gamma,\mathbb C)$ by 2°, and this is the space required for terms in the polynomial part. This shows 3°.

The continuity assertions are verified by inspection. □

We use Lemma 5.4 2° in the following proof that the spaces defined in Definition 5.1 can also be described in terms of L_∞ norms:

Lemma 5.5. *The space $\mathcal S^{m,d,s}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathcal K)$ defined in Definition (5.1) may equivalently be defined by a formulation where the conditions (5.3) are replaced by the conditions*

$$(5.20)\qquad \langle z\xi'\rangle^{l-l'}h_{-1}\partial_{\zeta_n}^l\zeta_n^{l'}f(w,\xi',|\tfrac1z|\zeta_n,\tfrac1z)\in S^{m,d,s}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,L_\infty(\mathbb R))$$

(valid for all $l,l'\in\mathbb N$), and everything else is unchanged.

Proof. For the terms s_j in (5.1) there is nothing to prove, so we can assume that $\mathcal K=\mathcal H_{r-1}^+$, $\mathcal H_{r-1}$ or $\mathcal H_{r-1}^-$ with $r\le 0$. Then $f=h_{-1}f$, $\partial_{\xi_n}f=h_{-1}\partial_{\xi_n}f$. We reduce to the case $d=s=0$ by replacing f by $z^d\kappa^{-s}f$. When f satisfies (5.3), then $f\in S_{\frac12}^{m,0,0}(\Gamma,L_{2,\zeta_n})$ and $\partial_{\zeta_n}f\in S_{\frac12}^{m,1,-1}(\Gamma,L_{2,\zeta_n})$ imply

$$\begin{aligned}\|h_{-1}[f(x',\xi',|\tfrac1z|\zeta_n,\tfrac1z)]\|_{L_\infty}^2&=\|f(x',\xi',|\tfrac1z|\zeta_n,\tfrac1z)\|_{L_\infty}^2\\ &\dot\le\|f(x',\xi',|\tfrac1z|\zeta_n,\tfrac1z)\|_{L_{2,\zeta_n}}\|\partial_{\zeta_n}f(x',\xi',|\tfrac1z|\zeta_n,\tfrac1z)\|_{L_{2,\zeta_n}}\\ &\dot\le\langle z\xi'\rangle^{\frac12}\langle\xi'\rangle^m\langle z\xi'\rangle^{\frac12}\langle z\xi'\rangle^{-1}\langle\xi'\rangle^m=\langle\xi'\rangle^{2m},\end{aligned}$$

showing the basic estimate for the assertion. A similar pattern is found for the derived functions. For the opposite direction, we note that $\zeta_n h_{-1} f = |z|\xi_n h_{-1} f = |z| s_{-1} + h_{-1}(\zeta_n f)$, where s_{-1} is the term in the expansion (5.15) of f. Then (recall (2.20))

$$\begin{aligned}\|h_{-1} f(|\tfrac{1}{z}|\zeta_n)\|_{L_{2,\zeta_n}} &= \|f(|\tfrac{1}{z}|\zeta_n)\|_{L_{2,\zeta_n}} \le \varepsilon \|f(|\tfrac{1}{z}|\zeta_n)\|_{L_\infty} + \varepsilon^{-1}\|\zeta_n f(|\tfrac{1}{z}|\zeta_n)\|_{L_\infty} \\ &\dot\le \varepsilon\langle \xi'\rangle^m + \varepsilon^{-1}(|z||s_{-1}| + \|h_{-1}[\zeta_n f(|\tfrac{1}{z}|\zeta_n)]\|_{L_\infty}) \\ &\dot\le \varepsilon\langle \xi'\rangle^m + \varepsilon^{-1}(|z|\kappa + \langle z\xi'\rangle)\langle \xi'\rangle^m \doteq \langle z\xi'\rangle^{\frac12}\langle \xi'\rangle^m,\end{aligned}$$

where we set $\varepsilon = \langle z\xi'\rangle^{\frac12}$. This is the basic step, and the derived functions are estimated in a similar way. □

For singular Green symbols, there is a result similar to Lemma 5.4:

Lemma 5.6. *Let $m \in \mathbb{R}$, d, s and $r \in \mathbb{Z}$.*

1° *The mappings h_{-1,η_n}, $h^+_{\xi_n}$, h^-_{-1,η_n} and $h^-_{\eta_n}$ are continuous from $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1})$ to the space $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{K})$ with, respectively, $\mathcal{K} = \mathcal{H}_{-1}\hat\otimes\mathcal{H}_{\min\{r,0\}-1}$, $\mathcal{H}^+\hat\otimes\mathcal{H}_{r-1}$, $\mathcal{H}_{-1}\hat\otimes\mathcal{H}^-_{-1}$ and $\mathcal{H}_{-1}\hat\otimes\mathcal{H}^-_{\max\{r,0\}-1}$.*

2° *In the expansion*

$$f \sim \sum_{-\infty<j<r} k_j(w,\xi',\xi_n,\mu)\xi_n^j, \tag{5.21}$$

the mapping $f \mapsto k_j(w,\xi',\xi_n,\mu)$ is continuous from $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1})$ to $\mathcal{S}^{m,d,s-j}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}_{-1})$ for $j \in \mathbb{Z}$, $j < r$.

3° *The mapping $f \mapsto h_{-1,\xi_n}(\partial^l_{\xi_n}\xi_n^{l'}\partial^k_{\eta_n}\eta_n^{k'} f)$ is continuous from $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1})$ to $\mathcal{S}^{m,d,s-l+l'-k+k'}(\Gamma, \mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-k+k'-1})$, for l, l', k, $k' \in \mathbb{N}$.*

Proof. Let us just account for 2°; the other proofs are very much like those of Lemma 5.4. The statement follows from the definition when $j \ge 0$. When $j < 0$, we estimate as follows for $j = -1-l$, $l \in \mathbb{N}$, by (5.18) applied with respect to the second variable $\eta_n = |\frac{1}{z}|_n$, setting $f' = h_{-1,\varrho_n} f$ and taking $d = s = 0$:

$$\begin{aligned}&\int_{\mathbb{R}} |k_{-1-l}(w,\xi',|\tfrac{1}{z}|\zeta_n,\mu)|^2\, đ\zeta_n \\ &\quad\dot\le |z|^{-1}\int_{\mathbb{R}} \|h_{-1,\varrho_n}(\varrho_n^l z^{-l} f')\|_{L_{2,\varrho_n}} \|h_{-1,\varrho_n}(\varrho_n^{l+1} z^{-l-1} f')\|_{L_{2,\varrho_n}}\, đ\zeta_n \\ &\quad\dot\le |z|^{-2l-2}\|h_{-1,\varrho_n}(\varrho_n^l f')\|_{L_{2,\zeta_n,\varrho_n}} \|h_{-1,\varrho_n}(\varrho_n^{l+1} f')\|_{L_{2,\zeta_n,\varrho_n}} \\ &\quad\dot\le |z|^{-2l-2} z\kappa\langle z\xi'\rangle^l\langle\xi'\rangle^m z\kappa\langle z\xi'\rangle^{l+1}\langle\xi'\rangle^m = \kappa^{2l+2}\langle z\xi'\rangle\langle\xi'\rangle^{2m},\end{aligned} \tag{5.22}$$

in view of (2.19). With similar estimates for derivatives, this shows that $k_j \in \mathcal{S}^{m,0,-j}(\Gamma,\mathcal{H}_{-1})$ as asserted. □

We can then also replace L_2 norms by L_∞ norms on $\mathbb{R}^2$:

Lemma 5.7. *The space $\mathcal{S}^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, \mathcal{K})$ defined in Definition* (5.2) *may equivalently be defined by a formulation where the conditions* (5.7) *are replaced by the conditions*

$$\langle z\xi' \rangle^{l-l'+k-k'} h_{-1,\zeta_n} h_{-1,\varrho_n} \partial^l_{\zeta_n} \zeta_n^{l'} \partial^k_{\varrho_n} \varrho_n^{k'} f'(w, \xi', |\tfrac{1}{z}|\zeta_n, |\tfrac{1}{z}|\varrho_n, \tfrac{1}{z}) \in S^{m,d,s}(\mathbb{R}^\nu \times \mathbb{R}^{n-1}, \Gamma, L_\infty(\mathbb{R}^2)) \tag{5.23}$$

(valid for all $l, l', k, k' \in \mathbb{N}$), and everything else is unchanged.

Proof. The proof is very similar to that of Lemma 5.5. □

Remark 5.8. Also in the parameter-dependent case, the symbols valued in $\mathcal{H}^+$, $\mathcal{H}^-_{-1}$ resp. $\mathcal{H}^+ \hat{\otimes} \mathcal{H}^-_{-1}$ extend smoothly to $\xi_n \in \overline{\mathbb{C}}_-$, $\xi_n \in \overline{\mathbb{C}}_+$ resp. $(\xi_n, \eta_n) \in \overline{\mathbb{C}}_- \times \overline{\mathbb{C}}_+$ with uniformity of estimates (on the lines parallel to $\mathbb{R}$) and holomorphy on the interior.

6. Compositions with interior pseudodifferential operators

As ψdo symbols $p(x, \xi, \mu)$ on $\mathbb{R}^n$ we shall here primarily take the strongly polyhomogeneous ones, where μ follows the rules for cotangent variables, on a par with the cotangent variables ξ. For these symbols we take as transmission condition simply the uniform version of the usual two-sided transmission condition for polyhomogeneous symbols in one more variable. The resulting condition can then be formulated also for more general ψdo symbols in such a way that it is preserved under multiplication, and we include such symbols in the calculus.

First consider strongly polyhomogeneous ψdo's:

Definition 6.1. *Let $p(x, \xi, \mu) \in S^{0,0,m}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$ be strongly polyhomogeneous of degree $m \in \mathbb{Z}$ on $\mathbb{R}^n$. We say that p satisfies the* **uniform transmission condition** *at $x_n = 0$, when p and its derivatives at $x_n = 0$ are in $\mathcal{H}$ as functions of ξ_n such that, with the notation $\zeta = (\xi, \mu)$, $\zeta' = (\xi', \mu)$, the decompositions*

$$\xi_n^{l'} p(x', 0, \zeta) = \sum_{-l' \le k \le m} s_k(x', \zeta') \xi_n^{k+l'} + h_{-1}[\xi_n^{l'} p(x', 0, \zeta)] \tag{6.1}$$

hold with polynomials s_k of degree $m - k$ in ζ' (the coefficients bounded with bounded derivatives in x'), and for all indices,

$$|h_{-1}[\partial^l_{\xi_n} \xi_n^{l'} \partial^\beta_x \partial^\alpha_{\zeta'} p(x', 0, \zeta)]| \mathrel{\dot{\le}} \langle \zeta' \rangle^{m-l+l'-|\alpha|+1} \langle \zeta \rangle^{-1}, \tag{6.2}$$

uniformly for μ in closed subsectors of Γ, $|\mu| \ge 1$.

The space of symbols satifying these conditions is denoted $S^{0,0,m}_{\mathrm{sphg,ut}}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$.

That (6.1)–(6.2) together form an appropriate uniform version of the (two-sided) transmission condition for symbols with $n+1$ cotangent variables is accounted for in [GH90], see also [G96, Sect. 1.2]. The symbols are called *holomorphic in μ* when they are holomorphic in $\mu \in \mathring{\Gamma}$ for $|(\xi, \mu)| \ge \varepsilon$, some $\varepsilon > 0$.

Lemma 6.2.

1° *In Definition* 6.1, (6.2) *may equivalently be replaced by either*

$$\|h_{-1}[\partial_{\xi_n}^l \xi_n^{l'} \partial_x^\beta \partial_{\zeta'}^\alpha p(x',0,\zeta)]\|_{L_{\infty,\xi_n}(\mathbb{R})} \dot{\le} \langle\zeta'\rangle^{m-l+l'-|\alpha|} \text{ (for all indices)}, \tag{6.3}$$

or

$$\|h_{-1}[\partial_{\xi_n}^l \xi_n^{l'} \partial_x^\beta \partial_{\zeta'}^\alpha p(x',0,\zeta)]\|_{L_{2,\xi_n}(\mathbb{R})} \dot{\le} \langle\zeta'\rangle^{m-l+l'-|\alpha|+\frac12} \text{ (for all indices)}. \tag{6.4}$$

2° *It follows that when* $p(x,\xi,\mu) \in S^{0,0,m}_{\mathrm{sphg,ut}}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$, *then* $h_{-1}p$ *belongs to the symbol space*

$$h_{-1}p(x',0,\xi,\mu) \in \mathcal{S}^{0,0,m}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}_{-1}). \tag{6.5}$$

Moreover, the projections h^+p *and* $h^-_{-1}p$ *belong to this space with* $\mathcal{H}_{-1}$ *replaced by* $\mathcal{H}^+$ *resp.* $\mathcal{H}^-_{-1}$. *Similar statements hold with* p *replaced by* $\partial_{x_n}^k \partial_\mu^j p$ *and* m *replaced by* $m-j$.

Proof. 1° is proved by a variant of the proof of Lemma 5.5: Clearly, (6.2) implies (6.3). Conversely, if (6.1) and (6.3) hold, note that

$$\xi_n h_{-1}[\xi_n^{l'} p] = h_{-1}[\xi_n h_{-1}(\xi_n^{l'} p)] + s_{-l'-1} = h_{-1}[\xi_n^{l'+1} p] + s_{-l'-1}, \tag{6.6}$$

cf. (5.10), where the polynomial $s_{-l'-1}$ is 0 if $m+l'+1<0$ and is $O(\langle\zeta'\rangle^{m+l'+1})$ otherwise. Then by (6.3),

$$|(\langle\zeta'\rangle + |\xi_n|) h_{-1}[\xi_n^{l'} p]| \dot{\le} \langle\zeta'\rangle^{m+l'+1},$$

showing (6.2) for $l, \alpha, \beta = 0$. Derivatives in ξ_n, x', ζ' are easily included (for the latter, cf. (5.13)).

(6.2) implies (6.4) simply by integration. For the converse direction we use that

$$\sup_{\xi_n} |h_{-1}[\xi_n^{l'} p]|^2 \dot{\le} \|h_{-1}[\xi_n^{l'} p]\|_{L_2} \|\partial_{\xi_n} h_{-1}[\xi_n^{l'} p]\|_{L_2} \dot{\le} \langle\zeta'\rangle^{m+l'+\frac12} \langle\zeta'\rangle^{m+l'-\frac12} = (\langle\zeta'\rangle^{m+l'})^2$$

by (6.4). This show the estimates in (6.3), which together with (6.1) suffice to assure (6.2), as we have already seen. This shows 1°.

2°. μ is considered on rays in Γ; we can assume that the ray is $\mathbb{R}_+$. Denote $h_{-1}p$ by p' for brevity. Observe that $\tilde p' = \mathcal{F}^{-1}_{\xi_n \to x_n} p'$ by restriction to $\mathbb{R}_\pm$ defines the functions $\tilde p'_\pm = r^\pm \tilde p'$ that are (quasi-)polyhomogeneous of degree m in (x_n, ζ') with

$$\tilde p'_+(x', x_n, \zeta') \in \mathcal{S}^m(\mathbb{R}^{n-1} \times \mathbb{R}^n, \mathcal{S}_+), \quad \tilde p'_-(x', -x_n, \zeta') \in \mathcal{S}^m(\mathbb{R}^{n-1} \times \mathbb{R}^n, \mathcal{S}_+)$$

(recall (1.20)). Then Theorem 3.2 applies to show that

$$\tilde p'_+(x', x_n, \xi', \mu) \text{ and } \tilde p'_-(x', -x_n, \xi', \mu) \in \mathcal{S}^{0,0,m}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathcal{S}_+). \tag{6.7}$$

Since $\mathcal{H}_{-1} = \mathcal{H}^+ \dot{+} \mathcal{H}^-_{-1}$, a Fourier transformation carries this over into the information that $p' = \mathcal{F}_{x_n \to \xi_n}(e^+ p'_+ + e^- p'_-) \in \mathcal{S}^{0,0,m}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \mathbb{R}_+, \mathcal{H}_{-1})$.

We can similarly apply Theorem 3.2 to $\partial_{x_n}^k \partial_\mu^j p$, which satisfies Definition 6.1 with m replaced by $m-j$. □

For 2° above, one could also have appealed directly to a variant of Theorem 3.2 pertaining to L_∞ norms in the Fourier transformed setting.

We get a slightly more general set-up by imitating the above results on h_{-1}-projections in a *definition* of the transmission condition for not necessarily strongly polyhomogeneous symbols:

Definition 6.3. *Let d and $m \in \mathbb{Z}$.*

A ψdo symbol $q(x,\xi,\mu) \in S^{0,d,m}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$ will be said to satisfy the **uniform transmission condition** *at $x_n = 0$ when $p = \mu^{-d} q$ and its derivatives at $x_n = 0$ are in $\mathcal{H}$ with respect to ξ_n, such that the decompositions*

$$\xi_n^{l'} p(x',0,\xi',\xi_n,\mu) = \sum_{-l' \le k \le m} s_k(x',\xi',\mu)\xi_n^{k+l'} + h_{-1}[\xi_n^{l'} p(x',0,\xi',\xi_n,\mu)] \tag{6.8}$$

hold with polynomials s_k of degree $m-k$ in (ξ',μ) (the coefficients bounded with bounded derivatives in x'), and there are similar decompositions of all derivatives

$$\begin{aligned} &\partial_{\xi_n}^l \xi_n^{l'} \partial_x^\beta \partial_{\xi'}^\alpha p(x',0,\zeta) \\ &\quad = \sum_{-l' \le k \le m-l-|\alpha|} s_{k,\alpha,\beta,l}(x',\xi',\mu)\xi_n^{k+l'} + h_{-1}[\partial_{\xi_n}^l \xi_n^{l'} \partial_x^\beta \partial_{\xi'}^\alpha p(x',0,\xi',\xi_n,\mu)] \end{aligned} \tag{6.9}$$

with polynomials $s_{k,\alpha,\beta,l}$ of degree $m-k-|\alpha|-l$ in (ξ',μ); moreover the h_{-1}-projections are assumed to satisfy

$$h_{-1}\partial_{\xi_n}^l \xi_n^{l'} \partial_x^\beta \partial_{\xi'}^\alpha p(x',0,\xi,\mu) \in \mathcal{S}^{0,0,m-|\alpha|-l+l'}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma, \mathcal{H}_{-1}). \tag{6.10}$$

Then (by Lemma 5.4) the h^+ and h_{-1}^- projection of $\partial_{\xi_n}^l \xi_n^{l'} \partial_x^\beta \partial_{\xi'}^\alpha p$ belong to these spaces with $\mathcal{H}_{-1}$ replaced by $\mathcal{H}^+$ resp. $\mathcal{H}_{-1}^-$.

The space of such symbols q will be denoted $S_{\rm ut}^{0,d,m}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$. We say that q is "special μ-dependent", if $\partial_\mu^j p$ has all these properties with m replaced by $m-j$, for any $j \in \mathbb{N}$.

For a useful calculus, we must show that the product of two symbols satisfying the uniform transmission condition likewise satisfies the condition.

Theorem 6.4. *When $p(x,\xi,\mu) \in S_{\rm ut}^{0,d,m}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$ and $p'(x,\xi,\mu) \in S_{\rm ut}^{0,d',m'}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C})$, then the product $p\,p'$ as well as the composition $p \circ p'$ satisfy*

$$p\,p' \text{ and } p \circ p' \in S_{\rm ut}^{0,d+d',m+m'}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma, \mathbb{C}). \tag{6.11}$$

Here, if p and p' are special μ-dependent, then so are pp' and $p \circ p'$.

Proof. We know from the simple product rules that $p\,p' \in S^{0,d+d',m+m'}(\Gamma,\mathbb{C})$, so it is the transmission condition that has to be checked.

Assume first that $d = d' = 0$, and denote temporarily $p(x',0,\xi,\mu)$ and $p'(x',0,\xi,\mu)$ by p and p'. Let us use L_∞ norms. The most important step is to estimate $h_{-1}(\xi_n^k pp')$. Here we have that

$$\begin{aligned} \xi_n^k pp' &= \big(\xi_n^k p\big)p' = \Big(\sum_{-k \le j \le m} s_j \xi_n^{j+k} + h_{-1}(\xi_n^k p)\Big)\Big(\sum_{0 \le j' \le m'} s'_{j'} \xi_n^{j'} + h_{-1}(p')\Big) \\ &= I + II + III + IV, \\ I &= \sum_{-k \le j \le m} s_j \xi_n^{j+k} \sum_{0 \le j' \le m'} s'_{j'} \xi_n^{j'}, \quad II = \sum_{-k \le j \le m} s_j \xi_n^{j+k} h_{-1} p', \\ III &= \sum_{0 \le j' \le m'} s'_{j'} \xi_n^{j'} h_{-1}(\xi_n^k p), \quad IV = h_{-1}(\xi_n^k p) \cdot h_{-1} p', \end{aligned} \tag{6.12}$$

where the s_j and $s'_{j'}$ are polynomials in (ξ',μ) of degree $m-j$ resp. $m'-j'$, with symbols in $S^{0,0,m-j}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$ resp. $S^{0,0,m'-j'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$. The h_{-1}-terms satisfy the respective versions of (6.10). Now I is a polynomial in (ξ,μ) of degree $m+m'+k$, where the coefficient $\tilde s_l$ of ξ_n^{l+k} is a sum of products $s_js'_{j'}$ with $j+j'=l$. It follows from the elementary product rule Lemma 2.9 that $\tilde s_l\in S^{0,0,m+m'-l}(\Gamma,\mathbb{C})$. II contributes a polynomial of this kind plus $h_{-1}(\sum_{-k\le j\le m}s_j\xi_n^{j+k}h_{-1}p')$, which satisfies

$$\sup_{\xi_n}|h_{-1}(\sum_{-k\le j\le m}s_j\xi_n^{j+k}h_{-1}p')|\,\dot\le\,\kappa^{m+m'+k},$$

by (6.10) and an elementary product rule. III contributes in a similar way. Finally IV satisfies

$$|h_{-1}(\xi_n^kp)\cdot h_{-1}p'|\,\dot\le\,\kappa^{m+m'+k}.$$

When we add the contributions, we get the sum of a polynomial $\sum_{-k\le l\le m+m'}s''_l\xi_n^{l+k}$ with $s''_l\in S^{0,0,m+m'-l}(\Gamma,\mathbb{C})$ and a h_{-1}-term satisfying

$$\sup_{\zeta_n}|\kappa^{-m-m'-k}z^{-k}h_{-1}(\zeta_n^kp\,p')|\,\dot\le\,1;$$

the basic estimate required for (6.10). This sets the general pattern that is found in the treatment of derivatives as well as Taylor coefficients in x_n at $x_n=0$, showing altogether that $p\,p'$ is as asserted.

When d and d' have general values, we apply the study to z^dp and $z^{d'}p$, finding that $z^{d+d'}p\,p'$ is in the space with second index zero. Finally, the statement on $p\circ p'$ follows from the general composition formula (1.27) on $\mathbb{R}^n$. $\square$

As elliptic elements in the calculus we have at least the symbols in $S^{0,0,m}_{\mathrm{sphg,ut}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma,\mathbb{C})$ with invertible, uniformly estimated principal part (as in [G96]); they have parametrices of the same kind. (Whether this extends to elements of $S^{0,0,m}_{\mathrm{ut}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma,\mathbb{C})$ has not yet been cleared up.)

The heart of the treatment of compositions of boundary operators with interior ψdo's lies in the following versions of [G96, Lemmas 2.6.2–3]:

Lemma 6.5. *Let $m,m'\in\mathbb{R}$ and $d,d',s,s',\tilde m,r\in\mathbb{Z}$.*

Let $q(x',\xi',\mu)\in S^{m',d',s'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$ and let $p\in S^{0,d',\tilde m}_{\mathrm{ut}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma,\mathbb{C})$. The following mappings are continuous from $\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}_{r-1})$:

$$\begin{aligned}&1^\circ\quad f\mapsto f\cdot q\in\mathcal{S}^{m+m',d+d',s+s'}(\Gamma,\mathcal{H}_{r-1}),\\&2^\circ\quad f\mapsto f\cdot p(x',0,\xi,\mu)\in\mathcal{S}^{m,d+d',s+\tilde m}(\Gamma,\mathcal{H}_{r+\tilde m-1}).\end{aligned}\tag{6.13}$$

The mapping property in (6.13) 1° likewise holds with $\mathcal{H}_{r-1}$ replaced by $\mathcal{H}^-_{r-1}$ or $\mathcal{H}^+$ in the initial and final symbol space. For the mapping in (6.13) 2° composed with h_{-1}, h^+, h^-_{-1} resp. h^-, there are similar results with $\mathcal{H}_{r+\tilde m-1}$ replaced by $\mathcal{H}_{\min\{r+\tilde m,0\}-1}$, $\mathcal{H}^+$, $\mathcal{H}^-_{-1}$, resp. $\mathcal{H}^-_{\max\{r+\tilde m,0\}-1}$.

Proof. 1° is a simple product rule (using the Leibniz formula), similar to Lemma 2.9. Now consider 2°. We can assume $d=d'=s=s'=0$ and write $p(x',\xi,\mu)$ instead of

$p(x',0,\xi,\mu)$. For f and p we have decompositions, by (5.1)–(5.3) and (6.10)ff.,

$$\begin{aligned} f &= \sum_{0\le j<r-1} s_j(x',\xi',\mu)\xi_n^j + h_{-1}f, \\ \xi_n^k p &= \sum_{-k\le j\le \tilde m} s'_j(x',\xi',\mu)\xi_n^{j+k} + h_{-1}[\xi_n^k p], \end{aligned} \tag{6.14}$$

where $s_j \in S^{m,0,-j}(\Gamma,\mathbb{C})$, $s'_j \in S^{0,0,\tilde m-j}(\Gamma,\mathbb{C})$, $h_{-1}f \in S_{\frac12}^{m,0,0}(\Gamma,\mathcal{H}_{-1})$, and $h_{-1}\xi_n^k p \in \mathcal{S}^{0,0,\tilde m+k}(\Gamma,\mathcal{H}_{-1})$. In the study of $f\cdot p$, we typically have to consider

$$\xi_n^k f p = \big(\sum_{0\le j<r} s_j\xi_n^j + h_{-1}f\big)\big(\sum_{-k\le j'\le\tilde m} s'_{j'}\xi_n^{j'+k} + h_{-1}[\xi_n^k p]\big). \tag{6.15}$$

The product of the two polynomials is a polynomial $\sum_{-k\le l<r+\tilde m} s''_l\xi_n^{l+k}$ with $s''_l \in S^{m,0,\tilde m-l}(\Gamma,\mathbb{C})$. For $(h_{-1}f)\cdot\sum_{-k\le j\le\tilde m} s'_j\xi_n^{j+k}$ and for $\sum_{0\le j<r-1} s_j\xi_n^j\cdot h_{-1}[\xi_n^k p]$, we can use Lemma 5.4 3° for the multiplication by ξ_n^{j+k} and rule 1° above for the multiplication by the s_j or s'_j, to see that they lie in $\mathcal{S}^{m,0,\tilde m}(\Gamma,\mathcal{H}_{\tilde m+r-1})$. Finally, $(h_{-1}f)\cdot h_{-1}[\xi_n^k p]$ lies in $\mathcal{H}_{-1}$ and is estimated by use of the L_∞ norms defining the spaces; derivatives are treated by use of the Leibniz formula.

For the last statement, we use that $\mathcal{H}^-_{r-1}\subset\mathcal{H}_{r-1}$, $\mathcal{H}^+\subset\mathcal{H}$, and that h^- and h^+ act on the spaces as described in Lemma 5.4 1° (based on L_2 norms). □

The proof is slightly simpler than the one in [G96] since our spaces are fully characterized by L_∞ norms here.

A similar proof (straightforward when L_∞ norms are used) shows:

Lemma 6.6. *Let $m,m'\in\mathbb{R}$, $d,d',s,s',\tilde m,r\in\mathbb{Z}$.*

Let $q(x',\xi',\mu)\in S^{m',d',s'}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$ and let $p\in S^{0,d',\tilde m}_{\rm ut}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma,\mathbb{C})$. The following mappings are continuous from $\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1})$:

$$\begin{aligned} &1^\circ \quad f\mapsto f\cdot q\in\mathcal{S}^{m+m',d+d',s+s'}(\Gamma,\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1}),\\ &2^\circ \quad f\mapsto f\cdot p(x',0,\xi',\eta_n,\mu)\in\mathcal{S}^{m,d+d',s+\tilde m}(\Gamma,\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r+\tilde m-1}),\\ &3^\circ \quad f\mapsto h_{-1,\xi_n}[p(x',0,\xi',\xi_n,\mu)\cdot f]\in\mathcal{S}^{m,d+d',s+\tilde m}(\Gamma,\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1}). \end{aligned} \tag{6.16}$$

The mapping property in (6.16) 1° likewise holds with $\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1}$ replaced by $\mathcal{H}^+\hat\otimes\mathcal{H}^-_{r-1}$ in the initial and final symbol space. For the mappings in (6.16) 2° and 3° composed with $h^+_{\xi_n}h^-_{\eta_n}$, there are similar results with $\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r+\tilde m-1}$ resp. $\mathcal{H}_{-1}\hat\otimes\mathcal{H}_{r-1}$ replaced by $\mathcal{H}^+\hat\otimes\mathcal{H}^-_{\max\{r+\tilde m,0\}-1}$ resp. $\mathcal{H}^+\hat\otimes\mathcal{H}^-_{\max\{r,0\}-1}$.

We now have all the ingredients for a theorem on $\circ_n$-compositions (compositions with respect to the normal variable x_n):

Theorem 6.7. *Let $m,m'\in\mathbb{R}$, $s,s',d,d',r,r'\in\mathbb{Z}$. Let*

$$\begin{aligned} &\text{(i)} \quad p(x',x_n,\xi,\mu)\in S^{0,d,m}_{\rm ut}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma,\mathbb{C}), \quad \text{with } m\in\mathbb{Z},\\ &\text{(ii)} \quad g(x',\xi',\xi_n,\eta_n,\mu)\in\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{r-1}),\\ &\text{(iii)} \quad t(x',\xi,\mu)\in\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^-_{r-1}),\\ &\text{(iv)} \quad k(x',\xi,\mu)\in\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+),\\ &\text{(v)} \quad q(x',\xi',\mu)\in S^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C}), \end{aligned} \tag{6.17}$$

and let p', g', t', k' and q' be given similarly with symbols in the spaces with m, d, s and r replaced by m', d', s' and r'. Define

$$m''=m+m',\quad d''=d+d',\quad s''=s+s',\quad r''=\max\{r+m',0\}. \tag{6.18}$$

Then the $\circ_n$-compositions give rise to Green operators whose symbols are determined by the following formulas if p and p' are independent of x_n:

(6.19)

$$\begin{aligned}
&1^\circ \quad p_+\circ_n k' = h^+_{\xi_n}[p(x',\xi,\mu)k'(x',\xi,\mu)]\in \mathcal{S}^{m',d'',m+s'}(\Gamma,\mathcal{H}^+),\\
&2^\circ \quad g\circ_n k' = \int^+ g(x',\xi,\eta_n,\mu)k'(x',\xi',\eta_n,\mu)đ\eta_n \in \mathcal{S}^{m'',d'',s''+1}(\Gamma,\mathcal{H}^+),\\
&3^\circ \quad k\circ_n q' = k(x',\xi,\mu)q'(x',\xi',\mu)\in \mathcal{S}^{m'',d'',s''}(\Gamma,\mathcal{H}^+),\\
&4^\circ \quad t\circ_n p'_+ = h^-_{\xi_n}[t(x',\xi,\mu)p'(x',\xi,\mu)]\in \mathcal{S}^{m,d'',s+m'}(\Gamma,\mathcal{H}^-_{r''-1}),\\
&5^\circ \quad t\circ_n g' = \int^+ t(x',\xi,\mu)g'(x',\xi,\eta_n,\mu)\,đ\xi_n \in \mathcal{S}^{m'',d'',s''+1}(\Gamma,\mathcal{H}^-_{r'-1}),\\
&6^\circ \quad q\circ_n t' = q(x',\xi',\mu)t'(x',\xi,\mu)\in \mathcal{S}^{m'',d'',s''}(\Gamma,\mathcal{H}^-_{r'-1}),\\
&7^\circ \quad t\circ_n k' = \int^+ t(x',\xi,\mu)k'(x',\xi,\mu)\,đ\xi_n \in S^{m'',d'',s''+1}(\Gamma,\mathbb{C}),\\
&8^\circ \quad q\circ_n q' = q(x',\xi',\mu)q'(x',\xi',\mu)\in S^{m'',d'',s''}(\Gamma,\mathbb{C}),
\end{aligned}$$

$$\begin{aligned}
&9^\circ \quad p_+\circ_n g' = h^+_{\xi_n}[p(x',\xi,\mu)g'(x',\xi,\eta_n,\mu)]\in \mathcal{S}^{m',d'',m+s'}(\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{r'-1}),\\
&10^\circ \quad g\circ_n p'_+ = h^-_{\eta_n}[g(x',\xi,\eta_n,\mu)p'(x',\xi',\eta_n,\mu)]\\
&\qquad\qquad \in \mathcal{S}^{m,d'',s+m'}(\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{r''-1}),\\
&11^\circ \quad g\circ_n g' = \int^+ g(x',\xi',\xi_n,\zeta_n,\mu)g'(x',\xi,\zeta_n,\eta_n,\mu)đ\zeta_n\\
&\qquad\qquad \in \mathcal{S}^{m'',d'',s''+1}(\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{r'-1}),\\
&12^\circ \quad k\circ_n t' = k(x',\xi',\xi_n,\mu)t'(x',\xi',\eta_n,\mu)\in \mathcal{S}^{m'',d'',s''}(\Gamma,\mathcal{H}^+\hat\otimes\mathcal{H}^-_{r'-1}).
\end{aligned}$$

When p and p' depend on x_n, compositions are obtained by applying the above formulas to the Taylor expansions for p and p' in x_n (such as $p(x',x_n,\xi,\mu)\sim\sum_{j\in\mathbb{N}}\frac{1}{j!}x_n^j\partial_{x_n}^j p(x',0,\xi,\mu)$), resulting in asymptotic expansions of symbols as in [G96, Th. 2.7.2], *lying in the spaces indicated in the right-hand side of* (6.19).

Proof. We recall from the explanation in [G96] (or earlier sources) that the plus-integral $\int^+$ (applied to functions in $\mathcal{H}$) vanishes on $\mathcal{H}^-$ and for $f\in\mathcal{H}^+$ gives

$$\int^+ f(\xi_n)\,đ\xi_n = \lim_{x_n\to 0+}\mathcal{F}^{-1}_{\xi_n\to x_n}f = is_{-1}, \tag{6.20}$$

where s_{-1} is the first coefficient in the expansion $f(\xi_n)\sim\sum_{j\le -1}s_j\xi_n^j$ for $|\xi_n|\to\infty$, cf. e.g. [G96, (2.2.44)]. When $f\in\mathcal{H}_{-2}$, $\int^+ f\,đ\xi_n$ is just the ordinary integral $\int_{\mathbb{R}} f\,đ\xi_n$.

All cases except 1°, 4°, 9° and 10° have already been treated in Section 4 when the symbols have class 0; here $\int^+ aa'\,đ\xi_n$ corresponds to the $\mathbb{R}_+$-integral of the product of the symbol-kernels $\mathcal{F}^{-1}_{\xi_n\to x_n}a\overline{\mathcal{F}}^{-1}_{\xi_n\to x_n}a'$. For these rules it remains to treat contributions from polynomials in ξ_n. Here 3° and 6° are easily achieved by use of Lemma 5.4 3° and Lemma 6.5 1°. For 7° we have that when $t=\sum_{0\le j<r}s_j(x',\xi',\mu)\xi_n^j$, and k' has the expansion $k'\sim\sum_{-\infty<j'<0}s'_{j'}(x',\xi',\mu)\xi_n^{j'}$ as in (5.15), then

(6.21)
$$t\circ_n k'=\int^+\sum_{0\le j<r}s_j(x',\xi',\mu)\xi_n^j k'(x',\xi,\mu)\,đ\xi_n=i\sum_{0\le j<r}s_j(x',\xi',\mu)s'_{-j-1}(x',\xi',\mu),$$

which lies in $S^{m,d,s-j}\cdot S^{m',d',s'-(-j-1)}\subset S^{m+m',d+d',s+s'+1}$ by Lemma 5.4 2°. The contributions from polynomials to 2° and 5° are similarly behaved.

1° and 4° follow from Lemma 6.5. 9° and 10° follow from Lemma 6.6.

This part of the proof is similar to that of [G96, Th. 2.6.1] for x_n-independent ψdo symbols. The x_n-dependent symbols are then treated by a generalization of the arguments in [G96, Sect. 2.7]. The detailed formulas for the asymptotic expansions resulting from the Taylor expansions are given there; we use that multiplication by powers of x_n corresponds to application of powers of $\pm D_{\xi_n}$, which, by Lemma 5.4 3° and Lemma 5.6 3°, just lowers the s-index. □

One can get slightly more general statements concerning classes by taking *the class of* p_++g *resp.* p'_++g' into account (it can be lower than that of the individual terms), getting results as in [G96, Rem. 2.8.5]. This is important for the study of best possible mapping properties of elliptic systems, see details in [G90].

In the rules above, one can of course replace x' by (x',y') or y'.

With this theorem and Theorem 6.4, we have dealt with all occurring compositions except the leftover term in a composition of two truncated ψdo's:

$$L(P,Q)=(PQ)_+-P_+Q_+. \tag{6.22}$$

Here we have as in [G96, Th. 2.6.14]:

Theorem 6.8. *Let p and p' and the indices be as in Theorem 6.7.*

1° *Let p and p' be independent of x_n, and write*

$$p'=\sum_{0\le j\le m'}s'_j(x',\xi',\mu)\xi_n^j+h_{-1}p',\quad \mathcal{F}^{-1}_{\xi_n\to w_n}p(x',\xi,\mu)=\tilde p(x',w_n,\xi',\mu). \tag{6.23}$$

Then $L(p,p')$ is a singular Green boundary symbol operator with symbol

$$L(p,p')(x',\xi,\eta_n,\mu)=\sum_{0\le j<m'}k_j(x',\xi,\mu)\eta_n^j+g^+(p)\circ_n g^-(p'), \tag{6.24}$$

where

$$\begin{aligned}
k_j(x',\xi,\mu)&=-ih^+\sum_{j+1\le l\le m'}p(x',\xi,\mu)s'_l(x',\xi',\mu)\xi_n^{l-1-j},\\
g^+(p)(x',\xi,\eta_n,\mu)&=\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde g^+(p),\\
g^-(p)(x',\xi,\eta_n,\mu)&=\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n}e^+_{x_n}e^+_{y_n}\tilde g^-(p),\ \text{with}\\
\tilde g^+(p)(x',x_n,y_n,\xi',\mu)&=r^+\tilde p(x',x_n+y_n,\xi',\mu),\\
\tilde g^-(p)(x',x_n,y_n,\xi',\mu)&=r^-\tilde p(x',x_n+y_n,\xi',\mu);
\end{aligned} \tag{6.25}$$

$g^+(p)$ and $g^-(p')$ depend only on h_+p resp. $h^-_{-1}p'$. Here

$$
\begin{aligned}
k_j &\in \mathcal{S}^{0,d'',m''-j-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+),\\
g^+(p) &\in \mathcal{S}^{0,d,m-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}),\\
g^-(p') &\in \mathcal{S}^{0,d',m'-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}),
\end{aligned}
\tag{6.26}
$$

and finally, with (6.16),

$$
L(p,p') \in \mathcal{S}^{0,d'',m''-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{\max\{m',0\}-1}). \tag{6.27}
$$

2° *When p and p' depend on x_n, $L(p,p')$ is obtained by applying the above formula to the Taylor expansions for p and p' in x_n; this results in an asymptotic expansion as in* [G96, Th. 2.7.5], *lying in $\mathcal{S}^{0,d'',m''-1}(\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{\max\{m',0\}-1})$.*

Proof. 1° The formulas (6.24)–(6.25) are shown in [G96] (and date back to [G84]). Since

$$
h^+p \in \mathcal{S}^{0,d,m}(\Gamma,\mathcal{H}^+),\quad h^-_{-1}p' \in \mathcal{S}^{0,d',m'}(\Gamma,\mathcal{H}^-_{-1}),
$$

by (6.10)ff., we get from Lemma 4.3:

$$
g^+(p) \in \mathcal{S}^{0,d,m-1}(\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}),\quad g^-(p') \in \mathcal{S}^{0,d',m'-1}(\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}),
$$

and then (6.19 11°) shows that

$$
g^+(p)\circ_n g^-(p') \in \mathcal{S}^{0,d'',m''-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}). \tag{6.28}
$$

For k_j, we note that by (6.10)ff., $h^+[\xi_n^{l-1-j}p] = h^+[\xi_n^{l-1-j}h^+p]$ is in $\mathcal{S}^{0,d,m+l-j-1}(\Gamma,\mathcal{H}^+)$, so that the product rule (6.13 3°) gives $k_j \in \mathcal{S}^{0,d'',m''-j-1}(\Gamma,\mathcal{H}^+)$. Then the sum over j in (6.24) is in $\mathcal{S}^{0,d'',m''-1}(\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{m'-1})$ when $m'>0$ and vanishes when $m'\le 0$. Taken together with (6.28), this shows (6.27).

2° When p and p' depend on x_n, we insert their Taylor series and proceed as in the proof of [G96, Th. 2.7.5]. □

For composition formulas with respect to all variables we also have to apply $\circ'$ (cf. (1.27)ff.); this works fine in a similar way as in Section 4, Remark 4.2. (One changes the right factor into y'-form so that the $\circ'$-composition is a simple product in (x',y')-form; then one reduces to x'-form by the usual formulas — e.g. as in [GS95, Th. 1.18] or [G96, Th. 2.1.18].) For ψdo's depending on x_n, there is moreover a Taylor expansion in x_n at $x_n=0$ to deal with. The whole procedure goes as in [G96, Th. 2.7.6], which generalizes to the present case, with resulting symbol classes as in Theorems 6.4, 6.7 and 6.8:

Theorem 6.9. *The composition* (1.3) *of two μ-dependent Green operators $\mathcal{A}$ and $\mathcal{A}'$ in the present calculus gives an operator $\mathcal{A}''$ belonging to the calculus. More precisely, when the symbols of the entries in* (1.3) *are as in Theorem* 6.7, *the resulting symbols of the operators in* (1.4) *are in the spaces listed in Theorems* 6.4, 6.7 *and* 6.8. *Formulas and notation for the compositions are as in* [G96, Th. 2.7.6].

There is one more calculation rule that plays a role in the analysis of trace formulas: The passage from a singular Green operator to its *normal trace* $\operatorname{tr}_n G$. This was defined for operators of class 0 in (4.13) in terms of symbol-kernels, and the formula carries over to a very similar formula in terms of symbols: $\operatorname{tr}_n \tilde g$ satisfies

$$\operatorname{tr}_n \tilde g = \int^+ g(w,\xi',\xi_n,\xi_n,\mu)\, đ\xi_n; \text{ also denoted } \operatorname{tr}_n g. \tag{6.29}$$

In fact (omitting w, ξ' and μ, that just enter as parameters), since $\tilde g(x_n,y_n)\in\mathcal S(\overline{\mathbb R}^2_{++})$, it is the sum of a series of terms $\tilde k_j(x_n)\tilde t_j(y_n)$ with $\tilde k_j,\tilde t_j\in\mathcal S(\overline{\mathbb R}_+)$, converging in $\mathcal S(\overline{\mathbb R}^2_{++})$, and there is a corresponding statement for $g(\xi_n,\eta_n)$:

$$\begin{aligned}\tilde g(x_n,y_n)&=\sum_{j\in\mathbb N}\tilde k_j(x_n)\tilde t_j(y_n) \text{ converging in } \mathcal S(\overline{\mathbb R}^2_{++}),\ \text{with } \tilde k_j,\tilde t_j\in\mathcal S(\overline{\mathbb R}_+),\\ g(\xi_n,\eta_n)&=\sum_{j\in\mathbb N}k_j(\xi_n)t_j(\eta_n) \text{ converging in } \mathcal H^+\hat\otimes\mathcal H^-_{-1},\ \text{with } k_j\in\mathcal H^+, t_j\in\mathcal H^-_{-1}.\end{aligned}\tag{6.30}$$

For each term we have, by Fourier transformation,

$$\operatorname{tr}_n(\tilde k_j(x_n)\tilde t_j(y_n))=\int_0^\infty \tilde k_j(x_n)\tilde t_j(x_n)\,dx_n=\int_{\mathbb R}\mathcal F\tilde k_j\overline{\mathcal F\bar{\tilde t}_j}\,đ\xi_n=\int^+ k_j(\xi_n)t_j(\xi_n)\,đ\xi_n;$$

then the result for g follows by summation.

The concept is generalized to operators of positive class by *defining*

$$\operatorname{tr}_n g=\int^+ g(w,\xi',\xi_n,\xi_n,\mu)\,đ\xi_n, \tag{6.31}$$

for an arbitrary singular Green symbol $g(w,\xi',\xi_n,\eta_n,\mu)$; again the operator $\operatorname{OP}'(\operatorname{tr}_n g)$ will be denoted $\operatorname{tr}_n G$. ($\operatorname{tr}_n g$ was denoted $\tilde{\tilde g}$ in [G86,92,96] and used with the present notation in [GS96].)

Now Theorem 4.5 is generalized straightforwardly:

Theorem 6.10. *Let $g(w,\xi',\xi_n,\eta_n,\mu)\in\mathcal S^{m,d,s-1}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathcal H^+\hat\otimes\mathcal H^-_{r-1})$. Then the normal trace of g is a ψdo symbol satisfying*

$$\operatorname{tr}_n g(w,\xi',\mu)=\int^+ g(w,\xi',\xi_n,\xi_n,\mu)\,đ\xi_n\in S^{m,d,s}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathbb C). \tag{6.32}$$

Proof. By definition (cf. (5.6)), g has the form

$$g(w,\xi',\xi_n,\eta_n,\mu)=\sum_{0\le j\le r-1}k_j(w,\xi',\xi_n,\mu)\eta_n^j+g'(w,\xi',\xi_n,\eta_n,\mu) \tag{6.33}$$

where $k_j\in\mathcal S^{m,d,s-1-j}(\mathbb R^\nu\times\mathbb R^{n-1},\Gamma,\mathcal H^+)$ for each j and $g'=h_{-1,\eta_n}g$ is of class 0. Symbol-kernels of class 0 has already been treated in Theorem 4.5, which gives that $\operatorname{tr}_n g'\in S^{m,d,s}(\Gamma,\mathbb C)$. We can now add that

$$\int^+ k_j(w,\xi',\xi_n,\mu)\xi_n^j\,đ\xi_n=is_{-1-j}(k_j)(w,\xi',\mu)$$

as in (6.21); this belongs to $S^{m,d,s-1-j-(-1-j)}(\Gamma,\mathbb C)=S^{m,d,s}(\Gamma,\mathbb C)$ by Lemma 5.4 2°. □

Remark 6.11. As indicated in the introduction and Remark 2.7, it is difficult to include general μ-independent Poisson symbols (or trace or s.g.o. symbols) in the calculus. This does not mean that there cannot be useful composition results in special cases. In [GSc99] it is shown how compositions of a μ-independent trace operator with a μ-dependent Poisson operator stemming from an elliptic differential operator resolvent, gives a ψdo on the boundary belonging to the weakly polyhomogeneous calculus. For a more general μ-dependent Poisson symbol, say $k(x',\xi',\xi_n,\mu)\in\mathcal S^{0,0,0}(\mathbb R^{n-1}\times\mathbb R^{n-1},\Gamma,\mathcal H^+)$, one can show using Remark 5.8 that

$$k(x',\xi',-i[\xi'],\mu)\in S^{0,0,0}(\mathbb R^{n-1}\times\mathbb R^{n-1},\Gamma,\mathbb C). \tag{6.34}$$

This can be used to define compositions $f_l\circ_n k$ with trace symbols f_l that are Laguerre functions $f_l=([\xi']+i\xi_n)^l([\xi']-i\xi_n)^{-(l+1)}$ (for $l\ge 0$), by residue calculus, showing that the resulting symbol is in $S^{0,0,0}(\mathbb R^{n-1}\times\mathbb R^{n-1},\Gamma,\mathbb C)$. However, when a general μ-independent trace symbol is expanded in Laguerre functions (cf. e.g. [G96, (2.2.10), (2.3.30)]), extra conditions on the expansion coefficients will be needed to get convergence of the series resulting from performing the composition termwise. We expect to return to the subject elsewhere.

7. Asymptotic trace expansions

The operators can be defined to act in smooth vector bundles over manifolds with boundary by use of partitions of unity and local trivializations in the same way as in [GK93], [G96]; we can take bundles over compact manifolds, or noncompact manifolds of the type called admissible there (they include exterior domains in $\mathbb R^n$ and $\overline{\mathbb R}^n_+$).

Observe in particular the effect of cut-off functions: Let

$$\begin{gathered}\zeta(x',x_n)\in C^\infty(\overline{\mathbb R}^n_+)\text{ with }D^\alpha\zeta\text{ bounded for all }\alpha,\\ \zeta=0\text{ for }x_n\le\delta_1(x'),\ \zeta=1\text{ for }x_n\ge\delta_2(x'),\ 0<\delta_1(x')<\delta_2(x').\end{gathered} \tag{7.1}$$

Then $\zeta(x)K$, $T\zeta(x)$, $\zeta(x)G$ and $G\zeta(x)$ can be written with arbitrary high powers of x_n as a factor (as in [G96, (2.4.34)ff.]); this replaces the symbol class $\mathcal S^{m,d,s}$ by $\mathcal S^{m,d,s-2N}\subset\mathcal S^{m-N,d,s-N}\cap\mathcal S^{m,d-N,s-N}$ for any large N in view of Lemma 2.10, so in fact the resulting symbol is in $\mathcal S^{-\infty,-\infty,-\infty}$. We have shown:

Lemma 7.1. *Let $\zeta(x)$ be as in (7.1), and let T, K and G be operators in our calculus. Then $\zeta(x)K$, $T\zeta(x)$, $\zeta(x)G$ and $G\zeta(x)$ have symbols in $\mathcal S^{-\infty,-\infty,-\infty}$.*

We now consider operators defined on a smooth compact n-dimensional manifold X with boundary X'. For singular Green operators in the general symbol classes considered in [G86,96] one has, by the proof of [G86, Th. 3.3.11] ([G96, Th. 3.3.10]), also taken up in [G92, Appendix]:

Proposition 7.2. *Let G be a μ-dependent polyhomogeneous singular Green operator of order m and class 0, and of regularity $\nu\in\frac12\mathbb Z$ in its dependence on the parameter $\mu\in\Gamma$, as defined in* [G86]*. Let ν' be the largest integer $<\nu$. If $m<-n$, G is trace class and the trace has an expansion in μ for $\mu\to\infty$ on rays in Γ:*

$$\operatorname{Tr}G\sim c_0\mu^{m+n-1}+c_1\mu^{m+n-2}+\cdots+c_{n-1+\nu'}\mu^{m-\nu'}+O(\mu^{m-\nu+\frac14}). \tag{7.2}$$

We shall now show that we have full asymptotic expansions for the singular Green operators in the present (more restricted) calculus.

Theorem 7.3. *Let G be a μ-dependent weakly polyhomogeneous s.g.o. of class 0 on X. Assume that the symbol-kernel of G is holomorphic in μ and lies in $\mathcal{S}^{m,d,-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$ (in local coordinates), so that the normal trace belongs to $S^{m,d,0}(\Gamma,\mathbb{C})$ in local coordinates on X'. Assume moreover that the homogeneous terms in the normal trace lying in $S^{m-j,d,0}$ with $m-j\ge -n$ belong to $S^{m',d',0}$ for some $m'\le -n$, $d'\in\mathbb{Z}$. Then G is trace class, and the trace of G has an asymptotic expansion*

$$\operatorname{Tr}G\sim\sum_{j\ge 0}c_j\mu^{m+d+n-1-j}+\sum_{l\ge 0}(c_l'\log\mu+c_l'')\mu^{d-l}\tag{7.3}$$

for $\mu\to\infty$ in closed subsectors of Γ. The coefficients c_j and c_l' are determined from the homogeneous terms in the symbol.

Proof. S.g.o.s on compact manifolds with symbols of degree $<-n$ (order $<1-n$) are trace class (since they are of order $-\infty$ on interior patches, and their kernels at the boundary are continuous in (x',y') valued in $\mathcal{S}_{++}$). We have as for the operators studied in [G96] (cf. e.g. Th. 4.2.11 there) that the contributions to the trace from coordinate patches near X' are of the form $\operatorname{Tr}_{X'}\operatorname{OP}'(\operatorname{tr}_n g)$; the trace of a ψdo on X'. Since $\operatorname{tr}_n g$ is a weakly polyhomogeneous ψdo symbol in $S^{m,d,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C})$, holomorphic in μ, we can apply [GS95, Th. 2.1] to $\operatorname{OP}'(\operatorname{tr}_n g)$, which gives an expansion as in (7.3) for each localized component.

For interior patches with positive distance from X', the contributions to the trace are $O(\mu^{-N})$ for any N, in view of Lemma 7.1 and Corollary 4.7. □

Remark 7.4. The expansion (7.2) can be useful to see whether some log-terms in (7.3) vanish. As a typical case, consider an s.g.o. G with symbol-kernel $\tilde g\in\mathcal{S}^{0,0,-d-1}(\Gamma,\mathcal{S}_{++})$ for some $d\ge n$ (in local coordinates near the boundary) so that $\operatorname{tr}_n\tilde g\in S^{0,0,-d}(\Gamma,\mathbb{C})\subset S^{-d,0,0}(\Gamma,\mathbb{C})\cap S^{0,-d,0}(\Gamma,\mathbb{C})$ (recall (2.28)). Both spaces are important; the former assures trace class and the latter gives the best d-exponent in the application of Theorem 7.3, which shows that $\operatorname{Tr}G\sim\sum_j c_j\mu^{n-d-1-j}+\sum_l(c_l'\log\mu+c_l'')\mu^{-d-l}$. If G is the s.g.o. term in a resolvent as in [G96, Th. 3.3.2], the regularity number is a half-integer $\nu\in[\frac12,d]$, and (7.2) is an expansion $\operatorname{Tr}G=\tilde c_0\mu^{n-d-1}+\dots+\tilde c_{n-1-\nu'}\mu^{-d-\nu'}+O(\mu^{-d-\nu+\frac14})$. This gives the additional information on the preceding expansion, that the log-coefficients c_l' vanish for $l\le\nu'$.

We now also get trace expansions for systems as in (1.1) (Green operators) with μ-dependent weakly polyhomogeneous entries, acting in in smooth vector bundles E and F of dimensions N resp. M over X resp. X'. When $\mathcal{A}$ is trace class, the trace equals $\operatorname{Tr}_X(P_++G)+\operatorname{Tr}_{X'}S$. For S one has the results for ψdo's on a closed manifold X' established in [GS95], so we need not consider it further. For P_++G one has:

Corollary 7.5. *Let P_+ and G be μ-dependent weakly polyhomogeneous operators defined in a smooth vector bundle E of dimension N over a smooth compact n-dimensional manifold X with boundary X', such that in local trivializations, the symbols p and g satisfy:*

$$\begin{gathered}p(x,\xi,\mu)\in S^{m,d,0}_{\mathrm{ut}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma,\mathbb{C})\otimes\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N),\\ \tilde g(x',x_n,y_n,\xi',\mu)\in\mathcal{S}^{m,d,-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})\otimes\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)\end{gathered}\tag{7.4}$$

and are holomorphic in μ. Assume moreover that the homogeneous j'th terms of p and $\operatorname{tr}_n \tilde{g}$ (in $S^{m-j,d,0}$ spaces) with $m-j \ge -n$ belong to $S^{m',d',0}$ resp. $S^{m''+1,d'',0}$ for some $m', m'' < -n$, $d', d'' \in \mathbb{Z}$. Then $P_+ + G$ is trace class, and the trace has an asymptotic expansion in μ:

$$\operatorname{Tr}(P_+ + G) \sim \sum_{j\ge 0} c_j \mu^{m+d+n-j} + \sum_{l\ge 0} (c'_l \log \mu + c''_l)\mu^{d-l} \tag{7.5}$$

for $\mu \to \infty$ in closed subsectors of Γ. The coefficients c_j and c'_l are determined from the homogeneous symbols, and c_0 depends only on P.

Proof. By [GS95, Th. 2.1], the diagonal kernel $\mathcal{K}_P(x,x,\mu)$ of P (on an open manifold extending X) is continuous and has a pointwise expansion as in (7.5), so by integrating its fiber trace over X we find an expansion of $\operatorname{Tr}_X P_+$ of the form (7.5). A similar expansion is obtained when we add the contribution (7.3) from G to this. □

Remark 7.6. For operator families $\mathcal{A}(\mu)$ defined over noncompact manifolds with boundary, admissible with admissible vector bundles as defined in [GK93], [G96], the corollary extends immediately to the operators $\chi\mathcal{A}(\mu)$ obtained by composition with a morphism χ of compact support.

The problems treated in [GS95], [GS96] and [G99] belong to this calculus, in fact to a more restricted class, since the interior ψdo is a *differential operator resolvent* and the occurring s.g.o.s are *finite* sums of products $K(\mu)S(\mu)T(\mu)$ with $K(\mu)$ and $T(\mu)$ strongly polyhomogeneous.

Example 7.7. Consider the resolvent $R(\mu)$ of the realization P_T of $P = -\Delta + P_1$ on $\overline{\mathbb{R}}^n_+$ with the boundary condition $Tu = 0$; $Tu = \partial_{x_n} u(x',0) - (1-\Delta_{x'})^{\frac12} u(x',0)$, and P_1 a d.o. of order 1. Here $R(\mu) = (P_T + \mu^2)^{-1}$ solves the problem (with $f \in L_2(\mathbb{R}^n_+)$, $u \in H^2(\mathbb{R}^n_+)$)

$$(P+\mu^2)u = f \text{ on } \mathbb{R}^n_+, \quad \partial_{x_n} u(x',0) - \operatorname{OP}'(\langle \xi' \rangle)u(x',0) = 0. \tag{7.6}$$

We let $\mu \ge 1$; it could also be taken complex. One finds that $R(\mu) = Q(\mu)_+ + G(\mu) = \operatorname{OP}(q)_+ + \operatorname{OPG}(\tilde{g})$, where $q(x,\xi,\mu) \in S^{0,0,-2}_{\mathrm{sphg,ut}}(\mathbb{R}_+,\mathbb{C})$ with principal part $[\xi,\mu]^{-2}$ and $\tilde{g}(x', x_n, y_n, \xi', \mu) \in \mathcal{S}^{0,0,-3}(\mathbb{R}_+, \mathcal{S}_{++})$ with principal part $\frac{c([\xi',\mu]-\langle\xi'\rangle)}{[\xi',\mu]([\xi',\mu]+\langle\xi'\rangle)} e^{-[\xi',\mu](x_n+y_n)}$. By the composition rules in Theorem 6.7 9° and 10° and Theorem 6.8, $R(\mu)^k = Q(\mu)^k_+ + G^{(k)}(\mu) = \operatorname{OP}(q^{\circ k})_+ + \operatorname{OPG}(\tilde{g}^{(k)})$, where $q^{\circ k} \in S^{0,0,-2k}_{\mathrm{sphg,ut}}$ and $\tilde{g}^{(k)} \in \mathcal{S}^{0,0,-2k-1}(\mathbb{R}_+, \mathcal{S}_{++})$. Let $\chi \in C^\infty_{(0)}(\overline{\mathbb{R}}^n_+)$ and let $k > \frac{n}{2}$. It is well-known that $\operatorname{Tr}(\chi Q^k_+)$ has an expansion $\sum_{j\ge 0} a_j \mu^{n-2k-j}$ (with $a_j = 0$ for j odd). For $G^{(k)}$, the normal trace of the symbol-kernel is in $S^{0,0,-2k}(\mathbb{R}_+,\mathbb{C}) \subset S^{-2k,0,0}(\mathbb{R}_+,\mathbb{C}) \cap S^{0,-2k,0}(\mathbb{R}_+,\mathbb{C})$, so by Theorem 7.3, $\operatorname{Tr}(\chi G^{(k)}(\mu)) \sim \sum_{j\ge 0} b_j \mu^{n-1-2k-j} + \sum_{l \ge 0} (b'_l \log\mu + b''_l)\mu^{-2k-l}$. The theory of [G96] likewise applies, and here $G^{(k)}$ is of regularity 1, so we see that $b'_0 = 0$. To this we add the expansion of $\operatorname{Tr} \chi Q^k(\mu)_+$, obtaining an expansion

$$\operatorname{Tr} \chi(Q^k(\mu)_+ + G^{(k)}(\mu)) \sim \sum_{j \ge 0} c_j \mu^{n-2k-j} + \sum_{l\ge 1} c'_l \mu^{-2k-l} \log \mu. \tag{7.7}$$

There is a corresponding heat trace expansion, obtained from $\operatorname{Tr}\chi(Q^k(\mu)_+ + G^{(k)}(\mu))$ by the transition formulas explained in detail e.g. in [GS96]:

$$\operatorname{Tr}(\chi e^{-tP_T}) \sim \sum_{j\geq 0} \tilde{c}_j t^{\frac{j-n}{2}} + \sum_{l\geq 1} \tilde{c}'_l t^{\frac{l}{2}} \log t. \tag{7.8}$$

One can replace $P+\mu^2$ by $(D_{x_n}^4 - \Delta_{x'}^2 + \mu^4)/(-\Delta+\mu^2)$ in (7.6) to get a case where the interior operator differs from those in [G99].

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