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SPECTRAL BOUNDARY CONDITIONS FOR GENERALIZATIONS OF LAPLACE AND DIRAC OPERATORS

Gerd Grubb

Copenhagen Univ. Math. Dept., Universitetsparken 5,
DK-2100 Copenhagen, Denmark. E-mail grubb@math.ku.dk

Abstract. Spectral boundary conditions for Laplace-type operators on a compact manifold X with boundary are partly Dirichlet, partly (oblique) Neumann conditions, where the partitioning is provided by a pseudodifferential projection; they have an interest in string and brane theory. Relying on pseudodifferential methods, we give sufficient conditions for the existence of the associated resolvent and heat operator, and show asymptotic expansions of their traces in powers and power-log terms, allowing a smearing function φ. The leading log-coefficient is identified as a non-commutative residue, which vanishes when $\varphi = 1$.

The study has new consequences for well-posed (spectral) boundary problems for first-order, Dirac-like elliptic operators (generalizing the Atiyah-Patodi-Singer problem). It is found e.g. that the zeta function is always regular at zero. In the selfadjoint case, there is a stability of the zeta function value and the eta function regularity at zero, under perturbations of the boundary projection of order $-\dim X$.

[1]Introduction.

Spectral boundary conditions (involving a pseudodifferential projection Π on the boundary) were first employed by Atiyah, Patodi and Singer [APS75] in their seminal work on Dirac operators D on manifolds X with boundary, introducing the eta-invariant of the tangential part of D as an extremely interesting new geometric object. For such a Dirac realization D_Π, the operator $D_\Pi{}^*D_\Pi$ (the square $D_\Pi{}^2$ in the selfadjoint case) is a Laplace operator with a boundary condition of the form

$$\Pi\gamma_0 u = 0, \quad (I-\Pi)(\gamma_1 u + B\gamma_0 u) = 0; \tag{0.1}$$

here $\gamma_j u = (\frac{\partial}{\partial n})^j u|_{\partial X}$, and B denotes a first-order operator on ∂X (in the case derived from D_Π it is the tangential part of D). In this work we present a new analysis of Laplace-type operators with boundary conditions (0.1), under quite general choices of Π and B, showing existence of heat trace expansions and meromorphic zeta functions, and analyzing the leading logarithmic and nonlocal terms. The methods developed here moreover lead to new results on the corresponding questions for Dirac operator problems.

In a physics context, spectral boundary conditions are used in studies of axial anomalies (see e.g. Hortacsu, Rothe and Schroer [HRS80], Ninomiya and Tan [NT85], Niemi and Semenoff [NS86], Forgacs, O'Raifertaigh and Wipf [FOW87], see also Eguchi, Gilkey and

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Hanson [EGH80]), in quantum cosmology (see e.g. D'Eath and Esposito [EE91]), and in the theory of the Aharonov-Bohm effect (see e.g. Beneventano, De Francia and Santangelo [BFS99]). For a long time, applications of spectral boundary conditions were limited to fields of half-integer spin whose dynamics are governed by first-order operators of Dirac type. It is natural (and even required by supersymmetry arguments) to extend this scheme to integer spin fields, and, therefore, to operators of Laplace type. Indeed, a first step in this direction has been taken in Vassilevich [V01], [V02], where spectral boundary conditions are formulated for bosonic strings to describe some collective states of open strings and Dirichlet branes.

Local cases of (0.1), where Π is a projection morphism and B is a differential operator, have been treated earlier (see e.g. Avramidi and Esposito [AE99] and its references); such cases have a mathematical foundation in Greiner [Gre71], Grubb [G74], Gilkey and Smith [GiS83], whereas the general global case is a new subject.

Establishing heat trace asymptotics is very important in quantum field theory since they are related to ultra violet divergences and quantum anomalies (see e.g. the surveys of Avramidi [A02] and Fursaev [F02]). A vanishing of the leading logarithmic term means that the standard definition of a functional determinant through the zeta function derivative at 0 is applicable, and the road is open to renormalization of the effective action.

Overview of the contents: The problems for Laplace-type operators P are considered in Sections 1–4. In Section 2 we give sufficient conditions for the existence of the resolvent $(P_T-\lambda)^{-1}$ in an angular region, with an explicit formula (Th. 2.10), and we use this to show asymptotic expansions of traces $\operatorname{Tr}(F(P_T-\lambda)^{-m})$ (for sufficiently large m) in powers and log-powers of λ (Th. 2.13); here F is an arbitrary differential operator. In Section 3 we show corresponding expansions of the heat trace $\operatorname{Tr}(Fe^{-tP_T})$ in powers and log-powers of t (Cor. 3.1), and establish meromorphic extensions of zeta functions $\zeta(F,P_T,s)=\operatorname{Tr}(FP_T^{-s})$ with simple and double real poles in s. (These constructions do not need a differential operator square root of P as discussed in [V01].)

The leading logarithmic term and nonlocal term are analyzed in Section 4 when F is a morphism or a first-order operator. This is done by determining which part of the resolvent actually contributes to these values (Th. 4.1). It follows that the log-coefficient identifies with a certain non-commutative residue, vanishing e.g. when $F=I$ (Th. 4.2, 4.5). Then the zeta function is regular at 0, and the value at zero can be set in relation to an eta-invariant associated with Π (Th. 4.9, Def. 4.10). Special results are also obtained when F is a first-order operator and certain symmetry properties hold.

In Section 5 we draw some conclusions for operators ${D_\Pi}^*D_\Pi$ defined from a Dirac-type operator D (of the form $\sigma(\partial_{x_n}+A+\text{perturb.})$ near ∂X with A selfadjoint elliptic on ∂X) together with a pseudodifferential orthogonal projection Π on ∂X defining the boundary condition $\Pi\gamma_0 u=0$. Atiyah, Patodi and Singer [APS75] considered such problems with Π equal to the nonnegative eigenprojection $\Pi_{\geq}(A)$, and increasingly general choices have been treated through the years: Douglas and Wojciechowski [DW91], Müller [M94], Dai and Freed [DF94], Grubb and Seeley [GS95], [GS96] gave results on perturbations $\Pi_{\geq}(A)+\mathcal{S}$ by finite rank operators $\mathcal{S}$; [W99] allowed smoothing operators $\mathcal{S}$. Brüning and Lesch [BL99] introduced a special class of other projections $\Pi(\theta)$, and [G99], [G01′] included all projections satisfying the well-posedness condition of Seeley [S69″].

The methods of the present study lead to new results both for zeta and eta functions. The zeta function $\zeta(D_\Pi^*D_\Pi,s)=\operatorname{Tr}((D_\Pi^*D_\Pi)^{-s})$ is shown to be regular at $s=0$ for *any*

well-posed Π such that the principal parts of Π and A^2 commute (Cor. 5.3). This was known previously for perturbations of $\Pi_{\geq}(A)$ and $\Pi(\theta)$ of order $-\dim X$ (cf. [G01′]); the new result includes in particular the perturbations of $\Pi_{\geq}(A)$ and $\Pi(\theta)$ of order -1. The value $\zeta(D_\Pi^* D_\Pi, 0)$ is determined from Π and $\ker D_\Pi$ modulo local terms (Cor. 5.4).

Restricting the attention to cases with selfadjointness at ∂X, assuming

$$\sigma A = -A\sigma, \quad \sigma^2 = -I, \quad \Pi = -\sigma\Pi^\perp\sigma, \tag{0.2}$$

we get new results for the eta function $\eta(D_\Pi, s) = \operatorname{Tr}(D(D_\Pi^* D_\Pi)^{-\frac{s+1}{2}})$: It has at most a simple pole at $s = 0$ for general well-posed Π (Cor. 5.8); this includes perturbations of $\Pi_{\geq}(A)$ and $\Pi(\theta)$ of order -1, where it was previously known for order $-\dim X$ in the selfadjoint product case. Moreover, the residue at $s = 0$ is locally determined, and stable under perturbations of Π of order $\leq -\dim X$ (Th. 5.9). In particular, in the selfadjoint product case, the vanishing of the simple pole, shown for special cases in [DW91], [M94], [W99], [BL99], is stable under perturbations of order $-\dim X$. (For the last result, see also Lei [L02].) There are similar stability results for the value of zeta at 0 (Th. 5.7).

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1. Boundary conditions with projections.

Consider a second-order strongly elliptic differential operator P acting on the sections of an N-dimensional C^∞ vector bundle E over a compact C^∞ n-dimensional manifold X with boundary $X' = \partial X$.

X is provided with a volume element and E with a hermitian metric defining a Hilbert space structure on the sections, $L_2(E)$. We denote $E|_{X'} = E'$. A neighborhood of X' in X has the form $X_c = X' \times [0, c[$ (the points denoted $x = (x', x_n)$), and there E is isomorphic to the pull-back of E'. On X_c, there is a smooth volume element $v(x)dx'dx_n$, and $v(x', 0)dx'$ is the volume element on X' (defining $L_2(E')$).

We assume that P is principally selfadjoint, i.e., $P - P^*$ is of order ≤ 1. Moreover, to have simple ingredients to work with, we assume that P is of the following form near X':

Assumption 1.1. *On X_c, P has the form*

$$P = -\partial_{x_n}^2 + P' + x_n P_2 + P_1, \tag{1.1}$$

where P' is an elliptic selfadjoint nonnegative second-order differential operator in E' (independent of x_n) and the P_j are differential operators of order j in $E|_{X_c}$.

Let Π_1 be a classical pseudodifferential operator (ψdo) in E' of order 0 with $\Pi_1^2 = \Pi_1$, i.e., a projection operator, and denote the complementing projection by Π_2:

$$\Pi_2 = I - \Pi_1. \tag{1.2}$$

Let B be a first-order differential or pseudodifferential operator in E'. Then we consider the boundary condition for P:

$$\Pi_1\gamma_0 u = 0, \quad \Pi_2(\gamma_1 u + B\gamma_0 u) = 0, \tag{1.3}$$

where the notation $\gamma_j u = (\partial_{x_n}^j u)|_{X'}$ is used. In short, $Tu = 0$, where

$$T = \{\Pi_1\gamma_0, \Pi_2(\gamma_1 + B\gamma_0)\}. \tag{1.4}$$

We shall study the resolvent and the heat operator defined from P under this boundary condition, when suitable parameter-ellipticity conditions are satisfied. More precisely, with $H^s(E)$ denoting the Sobolev space of order s (with norm $\|u\|_s$), we define the realization P_T in $L_2(E)$ determined by the boundary condition (1.3) as the operator acting like P and with domain

$$D(P_T) = \{u \in H^2(E) \mid Tu = 0\};$$

then we want to construct the resolvent $(P_T - \lambda)^{-1}$ and the heat operator e^{-tP_T} and analyze their trace properties. In particular, we want to show a heat trace expansion

$$\operatorname{Tr} e^{-tP_T} \sim \sum_{-n\le k<0} a_k t^{\frac{k}{2}} + \sum_{k\ge 0}(-a_k' \log t + a_k'')t^{\frac{k}{2}}. \tag{1.5}$$

Such expansions are known to hold for normal differential boundary conditions (Seeley [S69], [S69′], Greiner [Gre71]) without the logarithmic terms, and for pseudo-normal ψdo boundary conditions [G99], but the condition (1.3) is in general not of these types.

Example 1.2. Assumption 1.1 holds if $P = D^*D$ for an operator type including Dirac operators, namely a first-order elliptic differential operator D from E to another N-dimensional bundle E_1 over X such that

$$D = \sigma(\partial_{x_n} + A_1) \ \text{ with } \ A_1 = A + x_n A_{11} + A_{10} \ \text{ on } X_c, \tag{1.6}$$

as considered in numerous works, originating in Atiyah, Patodi and Singer [APS75]; here σ is a unitary morphism from E' to E_1', A is a selfadjoint first-order elliptic operator in E' independent of x_n, and the A_{1j} are x_n-dependent differential operators in E' of order j. Then we can take $P' = A^2$. However, for the study in the following one does not need to have a differential operator "square root" of P'. (And, if P is derived from (1.6) near X', the factorization need not extend to all of X.) See Vassilevich [V01] for a discussion of how one can find such D when P is a Laplacian.

When $P = D^*D$, one can for example take as Π_1 the orthogonal projection $\Pi_\ge(A)$ onto the nonnegative eigenspace of A and let $B = A_1|_{x_n=0}$, also denoted $A_1(0)$. Then P_T equals $D_\ge^* D_\ge$, where $D_\ge$ is the realization of D under the boundary condition $\Pi_\ge(A)\gamma_0 u = 0$, and (1.5) is known from [GS95] (also with certain finite rank perturbations of $\Pi_\ge(A)$). Boundary conditions (1.3) with such choices of Π_1 are often called *spectral boundary conditions*.

The paper [G99] allows much more general realizations D_{Π_1} of D, where $\Pi_\ge(A)$ is replaced by a ψdo projection Π_1 that is "well-posed" with respect to D. Well-posedness (introduced by Seeley [S69″]) means that the principal symbol π_1^0 at each $(x',\xi') \in S^*(X')$ maps $N_+(x',\xi')$ bijectively onto the range of $\pi_1^0(x',\xi')$ in $\mathbb{C}^N$, where $N_+(x',\xi')$ is the space of boundary values of null-solutions:

$$N_+(x',\xi') = \{\, z(0) \in \mathbb{C}^N \mid d^0(x',0,\xi',D_{x_n})z(x_n) = 0,\ z \in L_2(\mathbb{R}_+)^N \,\},$$

d^0 denoting the principal symbol of D. As shown in [S69″], the closed-range operator defining the boundary condition may always be taken to be a projection; it may even be taken orthogonal. When Π_1 is orthogonal, then $(D_{\Pi_1})^* D_{\Pi_1} = P_T$ with $B = A_1(0)$ in (1.3)–(1.4), and (1.5) holds. (Further details in Section 5.)

In [V01], Vassilevich points to the need for considering other choices of B — and not just $B = 0$ but moreover cases unrelated to A_1 and having complex coefficients, where P_T is not selfadjoint. He inquires about heat kernel results for such cases, and that is precisely what we want to develop here.

The conditions in (1.3) can be defined without reference to a factorization as in Example 1.2; Π_1 can be a general pseudodifferential projection in E' unrelated to P'.

For the case where Π_1 is a local projection (a projection morphism in E' ranging in a subbundle F_0), the question of heat trace expansions is covered by Greiner [Gre71], see also [G74], [G96].

Since P is strongly elliptic, it satisfies the Gårding inequality, so we can assume (possibly after addition of a constant) that

$$\operatorname{Re}(Pu, u) \geq c_0 \|u\|_1^2, \text{ for } u \in C_0^\infty(X^\circ), \tag{1.7}$$

with $c_0 > 0$. Then the Dirichlet realization P_{D} (the realization of P in $L_2(E)$ with domain $D(P_{\mathrm{D}}) = \{u \in H^2(E) \mid \gamma_0 u = 0\}$) is invertible. We shall use the following notation for regions in $\mathbb{C}$:

$$\Gamma_\theta = \{\, \mu \in \mathbb{C} \setminus \{0\} \mid |\arg \mu| < \theta \,\}, \quad \Gamma = \Gamma_{\frac{\pi}{2}}, \quad \Gamma_{\theta,r} = \{\, \mu \in \Gamma_\theta \mid |\mu| > r \,\}. \tag{1.8}$$

Since the principal symbol of P is positive selfadjoint, the spectrum of P_{D} is for any $\delta > 0$ contained in a set

$$\Sigma_{\delta,R} = \Gamma_\delta \cup \{|\lambda| \leq R(\delta)\} \tag{1.9}$$

for some $R = R(\delta)$, in addition to being contained in $\{\operatorname{Re} \lambda > 0\}$.

We denote $(P_{\mathrm{D}} - \lambda)^{-1} = R_{\mathrm{D}}(\lambda)$, the resolvent of the Dirichlet problem, i.e. the solution operator for the semi-homogeneous problem

$$(P - \lambda)u = f \text{ in } X, \quad \gamma_0 u = 0 \text{ on } X'; \tag{1.10}$$

R_{D} maps $H^s(E)$ continuously into $H^{s+2}(E)$ for $s > -\frac{3}{2}$, when $\lambda \notin \Sigma_{\delta,R}$. The other semi-homogeneous Dirichlet problem

$$(P - \lambda)u = 0 \text{ in } X, \quad \gamma_0 u = \varphi \text{ on } X', \tag{1.11}$$

is likewise uniquely solvable for $\lambda \notin \Sigma_{\delta,R}$; the solution operator will be denoted $K_{\mathrm{D}}(\lambda)$. This is an elementary Poisson operator (in the notation of Boutet de Monvel [BM71]), mapping $H^s(E')$ continuously into $H^{s+\frac{1}{2}}(E)$ for all $s \in \mathbb{R}$.

We have Green's formula

$$(Pu, v)_X - (u, P^* v)_X = (\gamma_1 u + \sigma_0 \gamma_0 u, \gamma_0 v)_{X'} - (\gamma_0 u, \gamma_1 v)_{X'}, \tag{1.12}$$

with a certain morphism σ_0 in E'.

2. Resolvent constructions.

Our way to construct the heat operator for P_T goes via the resolvent, which is particularly well suited to calculations in the pseudodifferential framework, since the spectral parameter λ (or, rather, the square root $\mu = \sqrt{-\lambda}$) enters to some extent like a cotangent variable. In the following, we shall freely use the notation and results of Grubb and Seeley [GS95], Grubb [G01]. To save space, we do not repeat many details here but refer to these papers or to the perhaps simpler resumé of the needed parts of the calculus in [G02, Sect. 2]. Let us just recall the definition of the symbol space $S^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Lambda)$ (denoted $S^{m,d,s}(\Lambda)$ for short), where Λ is a sector of $\mathbb{C}\setminus\{0\}$ and $m,d,s\in\mathbb{Z}$:

A C^∞ function $p(x',\xi',\mu)$ lies in $S^{m,0,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Lambda)$ when, for every closed subsector Λ',

$$\partial^j_{|z|}p(x',\xi',1/z)\in S^{m+j}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}) \text{ uniformly for } |z|\le 1, 1/z\in\Lambda';$$

here $S^k(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ is the usual symbol space of functions $q(x',\xi')$ with

$$|\partial^\beta_{x'}\partial^\alpha_{\xi'}q(x',\xi')|\le C_{\alpha,\beta}\langle\xi'\rangle^{k-|\alpha|}$$

for all $\alpha,\beta\in\mathbb{N}^{n-1}$; we write $\langle x\rangle=(1+|x|^2)^{\frac12}$ and $\mathbb{N}=\{0,1,2,\dots\}$. Moreover,

$$S^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Lambda)=\mu^d(|\mu|^2+|\xi'|^2)^{s/2}S^{m,0,0}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Lambda).$$

In the applications, we often need p and its derivatives to be holomorphic in $\mu\in\Lambda^\circ$ for $|\mu|+|\xi'|\ge\varepsilon>0$; such symbols will just be said to be holomorphic (in μ).

The space denoted $S^{m,d}(\Lambda)$ in [GS95] is the space of holomorphic symbols in $S^{m,d,0}(\Lambda)$. The third upper index s was added in [G01] for convenience; one has in view of [GS95, Lemma 1.13] that

$$\begin{aligned} &S^{m,d,s}(\Lambda)\subset S^{m+s,d,0}(\Lambda)\cap S^{m,d+s,0}(\Lambda) \text{ for } s\le 0,\\ &S^{m,d,s}(\Lambda)\subset S^{m+s,d,0}(\Lambda)+S^{m,d+s,0}(\Lambda) \text{ for } s\ge 0,\end{aligned}\tag{2.1}$$

and the s-index saves us from keeping track of a lot of sums and intersections.

The ψdo with symbol $p(x',\xi',\mu)$ is defined by the usual formula:

$$\mathrm{OP}'(p)\colon v(x')\mapsto\int_{\mathbb{R}^{2(n-1)}}e^{i(x'-y')\cdot\xi'}p(x',\xi',\mu)v(y')\,dy'đ\xi',$$

where $đ\xi'$ stands for $(2\pi)^{1-n}d\xi'$; the analogous definition on $\mathbb{R}^n$ is indicated by OP. Ψdo's in bundles over manifolds are defined by use of local trivializations.

The symbols, we consider, moreover have expansions in homogeneous terms, $p\sim\sum_{j\in\mathbb{N}}p_{m-j}$ with $p_{m-j}\in S^{m-j,d,s}(\Lambda)$, homogeneous in (ξ',μ) of degree $m-j+d+s$. In the general, so-called weakly polyhomogeneous case, the homogeneity takes place for $|\xi'|\ge1$, but if it extends to $|\xi'|+|\mu|\ge1$ (in such a way that the symbol behaves as a standard classical symbol in the non-parametrized calculus with an extra cotangent variable $|\mu|$ entering on a par with ξ' in the estimates), the symbol is called strongly polyhomogeneous.

The composition rules for these spaces are straightforward: When $p_i\in S^{m_i,d_i,s_i}(\Lambda)$ for $i=1,2$, then p_1p_2 and $p_1\circ p_2\in S^{m_1+m_2,d_1+d_2,s_1+s_2}(\Lambda)$; the latter is the symbol of the composed operator $\mathrm{OP}'(p_1)\,\mathrm{OP}'(p_2)$, satisfying

$$(p_1\circ p_2)(x',\xi',\mu)\sim\sum_{\alpha\in\mathbb{N}^{n-1}}\frac{(-i)^{|\alpha|}}{\alpha!}\partial^\alpha_{\xi'}p_1(x',\xi',\mu)\partial^\alpha_{x'}p_2(x',\xi',\mu).\tag{2.2}$$

Before discussing the construction of the resolvent, we shall introduce some auxiliary pseudodifferential operators on X'. When $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$, we write $\mu = (-\lambda)^{\frac{1}{2}}$, defined such that $\mu \in \Gamma$ (cf. (1.8)).

Definition 2.1. *The operator $\mathfrak{A}(\lambda)$ is defined for $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$ by*

$$\mathfrak{A}(\lambda) = (P' - \lambda)^{\frac{1}{2}};$$

it is a ψdo in E' of order 1.

As a function of $\mu = (-\lambda)^{\frac{1}{2}} \in \Gamma$, $\mathfrak{A}$ is a strongly polyhomogeneous ψdo with symbol in $S^{0,0,1}(\Gamma)^{N\times N}$ in local trivializations, its principal symbol being equal to

$$\mathfrak{a}^0(x', \xi', \mu) = ({p'}^0(x', \xi') + \mu^2)^{\frac{1}{2}}. \tag{2.3}$$

This follows essentially from Seeley [S69], since $p'(x', \xi') + e^{2i\theta}t^2$ is a classical elliptic symbol of order 2 with respect to the cotangent variables (ξ', t). Moreover, $\mathfrak{A}(-\mu^2)$ is invertible for $\mu \in \Gamma$, and parameter-elliptic (as defined in [G96]), and $\mathfrak{A}^{-1}$ has symbol in $S^{0,0,-1}(\Gamma)^{N\times N}$ in local trivializations, with principal part $(\mathfrak{a}^0)^{-1} = ({p'}^0(x', \xi') + \mu^2)^{-\frac{1}{2}}$. The symbols are holomorphic in μ. We observe furthermore that since $\partial_\lambda^r \mathfrak{A} = c_r \mathfrak{A}^{1-2r}$, $\partial_\lambda^r \mathfrak{A}$ has symbol in $S^{0,0,1-2r}(\Gamma)^{N\times N}$ for $r \in \mathbb{N}$.

The indication by an upper index $N\times N$ means that the symbols are $N\times N$-matrix valued. The statement "has symbol in $S^{m,d,s}(\Gamma)^{N\times N}$ in local trivializations" will be written briefly as: "$\in \mathrm{OP}'\, S^{m,d,s}(\Gamma)$".

In the case considered in Example 1.2, $\mathfrak{A}(\lambda)$ is the operator called A_λ in [GS96] and [G02], A_μ in [GS95], and $\mathfrak{A}(0) = |A|$.

Definition 2.2. *The Dirichlet-to-Neumann operator A_{DN} is defined for $\lambda \notin \Sigma_{\delta,R}$ by:*

$$A_{\mathrm{DN}}(\lambda) = \gamma_1 K_{\mathrm{D}}(\lambda); \tag{2.4}$$

cf. (1.11) *ff.*

The operator A_{DN} is a ψdo of order 1 for each λ; this is a well-known fact in the calculus of pseudodifferential boundary problems (cf. Boutet de Monvel [BM71], Grubb [G96]). We observe moreover:

Lemma 2.3. *The Dirichlet-to-Neumann operator $A_{\mathrm{DN}}(\lambda)$ is a strongly polyhomogeneous ψdo in E' of order 1, which is principally equal to $-\mathfrak{A}(\lambda)$, i.e.,*

$$A_{\mathrm{DN}}(\lambda) = -\mathfrak{A}(\lambda) + A'_{\mathrm{DN}}(\lambda), \tag{2.5}$$

where $A_{\mathrm{DN}}(-\mu^2) \in \mathrm{OP}'\, S^{0,0,1}(\Gamma)$ and $A'_{\mathrm{DN}}(-\mu^2) \in \mathrm{OP}'\, S^{0,0,0}(\Gamma)$, with holomorphic symbols.

Proof. Clearly, $A_{\mathrm{DN}}(-\mu^2)$ is strongly polyhomogeneous of order 1, in the terminology of [GS95], [G01], since its symbol can be found from the corresponding calculation in the case where $\mu = e^{i\theta}t$ is replaced by $e^{i\theta}\partial_{x_{n+1}}$. Then its symbol is in $S^{0,0,1}(\Gamma)^{N\times N}$. Formula (2.5) is derived from the fact that the principal part of $P + \mu^2$ at $x_n = 0$ equals the principal part of $-\partial_{x_n}^2 + P' + \mu^2$. For the latter operator considered on $X' \times \overline{\mathbb{R}}_+$, the Poisson

operator solving the semi-homogeneous Dirichlet problem as in (1.11) is the mapping $\varphi(x') \mapsto z(x', x_n) = e^{-x_n \mathfrak{A}(\lambda)}\varphi$ (as in [G02, Prop. 2.11] with A^2 replaced by P'); application of ∂_{x_n} followed by restriction to $x_n = 0$ gives the mapping $\varphi \mapsto -\mathfrak{A}(\lambda)\varphi$. Then $A_{\mathrm{DN}}(\lambda)$ and $-\mathfrak{A}(\lambda)$ are principally equal, so their difference A'_{DN} is strongly polyhomogeneous of order 0, hence has symbol in $S^{0,0,0}(\Gamma)^{N\times N}$. $\square$

In particular, $A_{\mathrm{DN}}(-\mu^2)$ is parameter-elliptic, hence invertible for large enough μ, the inverse having symbol in $S^{0,0,-1}(\Gamma)^{N\times N}$ (see Proposition 2.8 below). Note that $\partial_\lambda^r A_{\mathrm{DN}}(\lambda)$ is strongly polyhomogeneous of degree $1-2r$, hence lies in $\mathrm{OP}'\, S^{0,0,1-2r}(\Gamma)$.

In our construction of the resolvent below, we need to be able to commute A_{DN} and Π_1 with an error having symbol in $S^{0,0,0}$. For this we introduce

Assumption 2.4. *The principal symbols of* Π_1 *and* P' *commute.*

This holds of course if Π_1 commutes with P' (as in Example 1.2 with $P' = A^2$, $\Pi_1 = \Pi_\geq$ or $\Pi_<$); it holds for general choices of Π_1 if P' has scalar principal symbol.

Proposition 2.5. *Under Assumption* 2.4, $[A_{\mathrm{DN}}, \Pi_1] = A_{\mathrm{DN}}\Pi_1 - \Pi_1 A_{\mathrm{DN}}$ *and* $[\mathfrak{A}, \Pi_1] = \mathfrak{A}\Pi_1 - \Pi_1\mathfrak{A}$ *are in* $\mathrm{OP}'\, S^{0,0,0}(\Gamma)$, *with holomorphic symbols. Moreover, for any* $r \geq 0$, *the* r*'th* λ*-derivatives are in* $\mathrm{OP}'\, S^{0,0,-2r}(\Gamma)$.

Proof. The main effort lies in the treatment of the case $r = 0$. Note first that since A'_{DN} and Π_1 are in $\mathrm{OP}'\, S^{0,0,0}$, so is their commutator $[A'_{\mathrm{DN}}, \Pi_1]$, so in view of (2.5), what we have to show is that $[\mathfrak{A}, \Pi_1]$ is in $\mathrm{OP}'\, S^{0,0,0}$. Denote $P' + \mu^2 = \overline{P}$, with symbol $p'(x', \xi') + \mu^2$ in local coordinates. The powers of $\overline{P}$ are defined for low values of s by

$$\overline{P}^s = \tfrac{i}{2\pi} \int_{\mathcal{C}} \varrho^s (\overline{P} - \varrho)^{-1}\, d\varrho,$$

where $\mathcal{C}$ is a curve in $\mathbb{C} \setminus \overline{\mathbb{R}}_-$ encircling the spectrum of $\overline{P}$; we let it begin with a ray with angle δ and end with a ray with angle $-\delta$, for some $\delta \in]0, \frac{\pi}{2}[$. (The location of $(r + \mu^2)^{\frac{1}{2}}$ when $r \in \mathbb{R}_+$ is discussed in Remark 2.11 below.) Then since $[(\overline{P} - \varrho)^{-1}, \Pi_1] = (\overline{P} - \varrho)^{-1}[\Pi_1, P'](\overline{P} - \varrho)^{-1}$,

$$[\overline{P}^s, \Pi_1] = \tfrac{i}{2\pi} \int_{\mathcal{C}} \varrho^s (\overline{P} - \varrho)^{-1}[\Pi_1, P'](\overline{P} - \varrho)^{-1}\, d\varrho.$$

Here, by the commutativity of the principal symbols, $[\Pi_1, P'] = M$ is a first-order ψdo (independent of ϱ and μ). The integral makes good sense for $s < 1$ (converges in the norm of operators from $H^1(E')$ to $L_2(E')$), so we can write, since $\overline{P}^{\frac{1}{2}} = \mathfrak{A}$,

$$[\mathfrak{A}, \Pi_1] = \tfrac{i}{2\pi} \int_{\mathcal{C}} \varrho^{\frac{1}{2}} (\overline{P} - \varrho)^{-1} M (\overline{P} - \varrho)^{-1}\, d\varrho. \tag{2.6}$$

The symbol of $[\mathfrak{A}, \Pi_1]$ in a local coordinate system can be found (modulo smoothing terms) from this formula. It is represented by a series of terms obtained from (2.6) by insertion of the symbol expansions of the involved operators $(\overline{P} - \varrho)^{-1}$ and M and applications of the composition rule (2.2). We have that the symbol of $(\overline{P} - \varrho)^{-1}$ is an $(N \times N)$-matrix $r(x', \xi', \mu, \varrho)$ whose entries r_{ij} are series of strongly homogeneous functions of $(\xi', \mu, \varrho^{\frac{1}{2}})$,

whereas the symbol $m(x',\xi')$ of M is a matrix whose elements m_{ij} are series of functions homogeneous in ξ' outside a neighborhhod of 0. The composition will give rise to integrals of products

$$\partial^{\gamma}_{x',\xi'} r_{ij}\, \partial^{\gamma'}_{x',\xi'} m_{jk}\, \partial^{\gamma''}_{x',\xi'} r_{kl};$$

there are finitely many contributions to each degree of homogeneity. The important thing is that in each homogeneous contribution, we can take the factor coming from m_{jk} outside the integral sign, since it does not depend on ϱ (nor on μ). What is left is a completely homogeneous integrand, which after integration gives a strongly homogeneous function of (ξ',μ). The factors coming from m are of degree ≤ 1, hence lie in $S^{1,0,0}$, and the contributions from the integration are of degree ≤ -1, hence lie in $S^{0,0,-1}$ (moreover, ξ'-differentiation of order α lowers the former to $S^{1-|\alpha|,0,0}$ and the latter to $S^{0,0,-1-|\alpha|}$). The full contributions are then in $S^{1,0,0}\cdot S^{0,0,-1}\subset S^{0,0,0}$, and can be collected in a series of terms of falling degrees. Thus $[\mathfrak{A},\Pi_1]$ has a symbol series in $S^{0,0,0}$.

There is still the question of whether the remainder $\mathcal{R}$, the difference between the operator defined from $(\overline{P}-\varrho)^{-1}M(\overline{P}-\varrho)^{-1}$ and an operator S defined from a superposition of the homogeneous terms, also has the right kind of symbol. It takes a certain effort to prove this. We know à priori, since $\mathfrak{A}$ has symbol in $S^{0,0,1}$ and Π_1 has symbol in $S^{0,0,0}$, that $[\mathfrak{A},\Pi_1]$ has symbol in $S^{0,0,1}\subset S^{1,0,0}+S^{0,1,0}$. The remainder $\mathcal{R}$ will be of order $-\infty$ in this class, hence have symbol in $S^{-\infty,1,0}$, and we have to show that this can be reduced to $S^{-\infty,0,0}$.

Recall that an operator $\mathcal{R}$ with symbol in $S^{-\infty,1,0}$ has a kernel expansion

$$K(\mathcal{R})(x',y',\mu)\sim\sum_{j\ge 0}K_j(x',y')\mu^{1-j},$$

with $K(\mathcal{R})-\sum_{j<J}K_j\mu^{1-j}=O(\langle\mu\rangle^{1-J})$ for all J, where the $K_j(x',y')$ are C^∞ kernels, by [GS95, Prop. 1.21]. With $\mathcal{R}_0$ denoting the operator with kernel K_0, we can write $\mathcal{R}=\mu\mathcal{R}_0+\mathcal{R}'$ where $\mathcal{R}'$ has the kernel expansion $\sum_{j\ge1}K_j\mu^{1-j}$, hence symbol in $S^{-\infty,0,0}$, so we are through if we show that $\mathcal{R}_0=0$.

Let μ run on the ray $\mathbb{R}_+$, and let $\varepsilon\in]0,\frac12[$. Then

$$\overline{P}^{-\varepsilon}[\mathfrak{A},\Pi_1]=\tfrac{i}{2\pi}\int_{\mathcal{C}}\varrho^{\frac12}(\overline{P}-\varrho)^{-1}\overline{P}^{-\varepsilon}M(\overline{P}-\varrho)^{-1}\,d\varrho.$$

Since $\overline{P}=P'+\mu^2$, $\|\overline{P}^s f\|_{L_2}\simeq\|f\|_{H^{2s,\mu}}$ (uniformly in μ for $\mu\ge1$), where $H^{t,\mu}$ is the Sobolev space defined by localization from the case $H^{t,\mu}(\mathbb{R}^{n-1})$ with norm $\|u\|_{H^{t,\mu}}=\|\langle(\xi',\mu)\rangle^t\hat{u}(\xi')\|_{L_2}$, $t\in\mathbb{R}$. From the resolvent estimates

$$\|\varrho(\overline{P}-\varrho)^{-1}f\|_{L_2}\le C_1\|f\|_{L_2},\quad \|(\overline{P}-\varrho)^{-1}\|_{H^{2,\mu}}\simeq\|\overline{P}(\overline{P}-\varrho)^{-1}f\|_{L_2}\le C_2\|f\|_{L_2}, \tag{2.7}$$

that are valid for $\varrho\in\mathcal{C}$, then follow by interpolation that

$$|\varrho|^{\frac12+\varepsilon}\|(\overline{P}-\varrho)^{-1}f\|_{H^{1-2\varepsilon,\mu}}\le C_3\|f\|_{L_2}. \tag{2.8}$$

We also have that

$$\|\overline{P}^{-\varepsilon}Mv\|_{L_2}\le C_4\|Mv\|_{H^{-2\varepsilon,\mu}}\le C_5\|Mv\|_{H^{-2\varepsilon}}\le C_6\|v\|_{H^{1-2\varepsilon}}\le C_7\|v\|_{H^{1-2\varepsilon,\mu}}, \tag{2.9}$$

using that $1-2\varepsilon\ge 0$. Thus

$$\begin{aligned}(2.10)\quad &\|\overline{P}^{-\varepsilon}[\mathfrak{A},\Pi_1]\|_{\mathcal{L}(L_2)}\\ &\le C\int_{\mathcal{C}}|\varrho|^{\frac12}\|(\overline{P}-\varrho)^{-1}\|_{\mathcal{L}(L_2)}\|\overline{P}^{-\varepsilon}M\|_{\mathcal{L}(H^{1-2\varepsilon,\mu},L_2)}\|\|(\overline{P}-\varrho)^{-1}\|_{\mathcal{L}(L_2,H^{1-2\varepsilon,\mu})}\,|d\varrho|\\ &\le C'\int_{\mathcal{C}}|\varrho|^{\frac12-1-\frac12-\varepsilon}\,|d\varrho|\le C'',\end{aligned}$$

where we applied (2.7) to the first factor, (2.9) to the middle factor and (2.8) to the last factor. We have that $[\mathfrak{A},\Pi_1]=S+\mu\mathcal{R}_0+\mathcal{R}'$, where $S+\mathcal{R}'$ is bounded in L_2 uniformly in μ for $\mu\ge 1$, hence $\overline{P}^{-\varepsilon}(S+\mathcal{R}')$ is à fortiori so. Thus, by (2.10), $\|\mu\mathcal{R}_0f\|_{H^{-2\varepsilon,\mu}}$ is bounded in μ for $\mu\ge 1$, for any f. But if $\mathcal{R}_0f$ is a nonzero function, $\mu\|\mathcal{R}_0f\|_{H^{-2\varepsilon,\mu}}$ cannot be bounded for $\mu\to\infty$ (recall that $\varepsilon<\frac12$). Thus $\mathcal{R}_0=0$, and the proof for the case $r=0$ is complete.

For $r\ge 1$, we observe that $\partial_\lambda^r\mathfrak{A}=c_r\mathfrak{A}^{1-2r}$ and that

$$\begin{aligned}[\Pi_1,\mathfrak{A}^{-1}]&=\mathfrak{A}^{-1}\mathfrak{A}\Pi_1\mathfrak{A}^{-1}-\mathfrak{A}^{-1}\Pi_1\mathfrak{A}\mathfrak{A}^{-1}=\mathfrak{A}^{-1}[\mathfrak{A},\Pi_1]\mathfrak{A}^{-1}\in \mathrm{OP}'\,S^{0,0,-2};\\ [\Pi_1,\mathfrak{A}^{-k}]&=\Pi_1\mathfrak{A}^{-1}\mathfrak{A}^{-k+1}-\mathfrak{A}^{-1}\Pi_1\mathfrak{A}^{-k+1}+\mathfrak{A}^{-1}\Pi_1\mathfrak{A}^{-k+1}-\cdots-\mathfrak{A}^{-k}\Pi_1\\ &=\sum_{0\le j\le k-1}\mathfrak{A}^{-j}[\Pi_1,\mathfrak{A}^{-1}]\mathfrak{A}^{-k+1+j}\in \mathrm{OP}'\,S^{0,0,-1-k}\end{aligned}\tag{2.11}$$

by the result for $r=0$ and the composition rules. The first calculation shows the assertion on $\partial_\lambda^r[\Pi_1,\mathfrak{A}]$ for $r=1$; the second calculation shows it for general $r\ge 2$, when we take $k=1-2r$.

The result now likewise follows for $\partial_\lambda^r[\Pi_1,A_{\mathrm{DN}}]$, when we use that $\partial_\lambda^rA_{\mathrm{DN}}=-\partial_\lambda^r\mathfrak{A}+\partial_\lambda^rA'_{\mathrm{DN}}$, where $\partial_\lambda^rA'_{\mathrm{DN}}\in \mathrm{OP}'\,S^{0,0,-2r}$ since it is strongly polyhomogeneous of degree $-2r$. $\square$

The resolvent $(P_T-\lambda)^{-1}$ is the solution operator for the problem

$$\begin{aligned}(P-\lambda)u&=f\ \text{ on } X,\\ \Pi_1\gamma_0u&=0\ \text{ on } X',\\ \Pi_2(\gamma_1u+B\gamma_0u)&=0\ \text{ on } X',\end{aligned}\tag{2.12}$$

where λ runs in a suitable subset of $\mathbb{C}$. The problem of constructing the resolvent will be transformed by some auxiliary constructions:

First, we can extend P to a strongly elliptic principally selfadjoint differential operator $\widetilde{P}$ on an n-dimensional compact manifold $\widetilde{X}$ in which X is smoothly imbedded, modifying the definition of $R(\delta)$ in (1.9) such that also the spectrum of $\widetilde{P}$ is contained in $\Sigma_{\delta,R}$. Assume in the following that $\lambda\notin\Sigma_{\delta,R}$. Let $Q(\lambda)$ be the inverse of $\widetilde{P}-\lambda$ and let $Q_+=r^+Qe^+$ be its truncation to X (r^+ is restriction from $\widetilde{X}$ to X, e^+ is extension by zero on $\widetilde{X}\setminus X$). It is well-known that $R_{\mathrm{D}}(\lambda)$ and $K_{\mathrm{D}}(\lambda)$ (cf. (1.10), (1.11)) are connected by the formula

$$R_{\mathrm{D}}=Q_+-K_{\mathrm{D}}\gamma_0Q_+,\tag{2.13}$$

for $\lambda \notin \Sigma_{\delta,R}$. Let

$$v = R_{\mathrm{D}} f, \quad z = u - v, \quad \psi = -\Pi_2 \gamma_1 v, \tag{2.14}$$

then (2.12) may be replaced by the problem for z:

$$\begin{aligned} (P-\lambda)z &= 0 \text{ on } X, \\ \Pi_1 \gamma_0 z &= 0 \text{ on } X', \\ \Pi_2(\gamma_1 z + B\gamma_0 z) &= \psi \text{ on } X'. \end{aligned} \tag{2.15}$$

Here one can let $f \in L_2(E)$ so that the solution u is sought in $H^2(E)$; then $v \in H^2(E)$, $z \in H^2(E)$, and $\gamma_j v$ and $\gamma_j z$ are in $H^{\frac{3}{2}-j}(E')$, and so are $\Pi_i \gamma_j v$ and $\Pi_i \gamma_j z$, since the projections Π_j are bounded in $H^s(E')$ for all $s \in \mathbb{R}$. One could also carry out the whole calculation for C^∞ sections while keeping track of the orders of the involved operators.

We now make a reduction of (2.15) using A_{DN}. When $z \in H^2(E)$ with $(P-\lambda)z = 0$, z is in 1-1 correspondence with $\gamma_0 z$ running through the space $H^{\frac{3}{2}}(E')$:

$$z = K_{\mathrm{D}} \gamma_0 z, \tag{2.16}$$

in view of the unique solvability of the Dirichlet problem. Denote $\gamma_0 z = \varphi$; then we can replace $\gamma_0 z$ by φ in (2.15). Moreover, $\gamma_1 z = A_{\mathrm{DN}}\varphi$. So, with $z = K_{\mathrm{D}}\varphi$ (assuring the validity of the first line in (2.15)), we arrive at the problem for φ:

$$\begin{aligned} \Pi_1 \varphi &= 0, \\ \Pi_2(A_{\mathrm{DN}} + B)\varphi &= \psi. \end{aligned} \tag{2.17}$$

Since the mapping $\gamma_1 \colon D(P_{\mathrm{D}}) \to H^{\frac{1}{2}}(E')$ is surjective, $\gamma_1 v$ runs through all of $H^{\frac{1}{2}}(E')$ when f runs through $L_2(E)$. So the reduced problem (2.17) must be solved for all $\psi \in \Pi_2(H^{\frac{1}{2}}(E'))$. When f is given in $L_2(E)$ and v and ψ are defined by (2.14), then, if $\varphi \in H^{\frac{3}{2}}(E')$ solves (2.17) and we set $z = K_{\mathrm{D}}\varphi$, $u = v + z$, we find that $u \in H^2(E)$ solves (2.12).

Lemma 2.6. *For $\lambda \notin \Sigma_{\delta,R}$, define*

$$\begin{aligned} S(\lambda) &= A_{\mathrm{DN}} + [A_{\mathrm{DN}}, \Pi_1] + \Pi_2 B \Pi_2, \\ S'(\lambda) &= A_{\mathrm{DN}} - [A_{\mathrm{DN}}, \Pi_1]. \end{aligned} \tag{2.18}$$

When ψ runs through $\Pi_2 H^{\frac{1}{2}}(E')$, the problem (2.17) is uniquely solvable with $\varphi \in H^{\frac{3}{2}}(E')$ for those λ for which $S(\lambda)$ and $S'(\lambda)$ are invertible from $H^{\frac{3}{2}}(E')$ to $H^{\frac{1}{2}}(E')$. When this holds, the solution is

$$\varphi = S(\lambda)^{-1}\psi. \tag{2.19}$$

Uniqueness of solution holds when merely $S(\lambda)$ is injective.

Proof. Assume first that φ is a solution of (2.17). By the first equation, $\varphi = \Pi_2\varphi$, so we can rewrite the second equation as:

$$A_{\mathrm{DN}}\Pi_2\varphi + [\Pi_2, A_{\mathrm{DN}}]\varphi + \Pi_2 B\Pi_2\varphi = \psi. \tag{2.20}$$

Now we compose the first equation of (2.17) with A_{DN} and add it to (2.20); this gives that φ satisfies

$$(A_{\mathrm{DN}} + [\Pi_2, A_{\mathrm{DN}}] + \Pi_2 B\Pi_2)\varphi = \psi.$$

Since $[\Pi_2, A_{\mathrm{DN}}] = [A_{\mathrm{DN}}, \Pi_1]$, the operator in the left-hand side equals $S(\lambda)$ defined in (2.18), so φ satisfies

$$S(\lambda)\varphi = \psi. \tag{2.21}$$

When $S(\lambda)$ is injective, there is at most one solution φ of (2.21), hence of (2.17); when $S(\lambda)$ is bijective, the only possibility is that it equals $S(\lambda)^{-1}\psi$.

Now assume that also $S'(\lambda)$ is invertible, and, for a given ψ with $\Pi_1\psi = 0$, define φ by (2.19). Then of course (2.21) holds, and we shall show that φ solves (2.17). Since $\Pi_1\Pi_2 = 0$ and $\Pi_1^2 = \Pi_1$, an application of Π_1 to (2.21) gives:

$$\begin{aligned} 0 &= \Pi_1 S\varphi = \Pi_1(A_{\mathrm{DN}}\varphi + [A_{\mathrm{DN}}, \Pi_1]\varphi + \Pi_2 B\Pi_2\varphi) \\ &= \Pi_1 A_{\mathrm{DN}}\varphi + \Pi_1 A_{\mathrm{DN}}\Pi_1\varphi - \Pi_1^2 A_{\mathrm{DN}}\varphi = \Pi_1 A_{\mathrm{DN}}\Pi_1\varphi \\ &= (A_{\mathrm{DN}} - [A_{\mathrm{DN}}, \Pi_1])\Pi_1\varphi = S'(\lambda)\Pi_1\varphi. \end{aligned} \tag{2.22}$$

It follows that $\Pi_1\varphi = 0$, so φ satisfies the first equation in (2.17). The second equation is then retrieved from this and (2.21): Since $\Pi_2\varphi = \varphi$,

$$\begin{aligned} \psi &= (A_{\mathrm{DN}} + [A_{\mathrm{DN}}, \Pi_1] + \Pi_2 B\Pi_2)\varphi \\ &= A_{\mathrm{DN}}\Pi_2\varphi + [\Pi_2, A_{\mathrm{DN}}]\varphi + \Pi_2 B\varphi = \Pi_2(A_{\mathrm{DN}} + B)\varphi. \quad \square \end{aligned} \tag{2.23}$$

When Assumption 2.4 holds, the commutator terms in (2.18) have symbol in $S^{0,0,0}$. Then we will show that $S'(\lambda)$ can be inverted for large $|\lambda|$ within the weakly polyhomogeneous calculus, and that the same holds for $S(\lambda)$ when the following assumption is satisfied:

Assumption 2.7. *There is a $\theta \in]0, \frac{\pi}{2}]$ such that, with $b^h(x', \xi')$ and $\pi_i^h(x', \xi')$ denoting the strictly homogeneous principal symbols of B and the Π_i,*

$$\mathfrak{a}^0 - \pi_2^h b^h \pi_2^h \text{ is invertible for } \xi' \in \mathbb{R}^n, \mu \in \Gamma_\theta \cup \{0\} \text{ with } (\xi', \mu) \neq (0, 0). \tag{2.24}$$

This will allow construction of a resolvent family for large λ with $|\arg(-\lambda)| < 2\theta$. If, moreover, $\theta > \frac{\pi}{4}$, we can also construct a heat operator family. Note that when B is a differential operator, its symbol is a polynomial, so b^h equals the usual principal symbol b^0. For Π_2, the strictly homogeneous principal symbol π_2^h is in general not continuous at $\xi' = 0$, but when it is multiplied by b^h, which is $O(|\xi'|)$, we get a continuous function at $\xi' = 0$ (taking the value 0 there).

Proposition 2.8. *Let Assumption* 2.4 *hold.*

(i) *For each* $\theta' \in]0, \frac{\pi}{2}[$ *there is an* $r(\theta') \geq 0$ *such that* $A_{\mathrm{DN}}(-\mu^2)$ *and* $S'(-\mu^2)$ *are invertible for* $\mu \in \Gamma_{\theta',r}$.

(ii) *Let moreover Assumption* 2.7 *hold. Then for each* $\theta' \in]0, \theta[$ *there is an* $r(\theta') \geq 0$ *such that* $S(-\mu^2)$ *is invertible for* $\mu \in \Gamma_{\theta',r}$.

Here the operator families $A_{\mathrm{DN}}(-\mu^2)^{-1}$ *and* $S'(-\mu^2)^{-1}$ *belong to* $\mathrm{OP}'\, S^{0,0,-1}(\Gamma)$, *and* $S(-\mu^2)^{-1}$ *belongs to* $\mathrm{OP}'\, S^{0,0,-1}(\Gamma_\theta)$, *with holomorphic symbols. For each* $r \in \mathbb{N}$, ∂_λ^r *map them into operators in* $\mathrm{OP}'\, S^{0,0,-1-2r}(\Gamma_\theta)$.

Proof. Let us go directly to the proof of (ii), the statements in (i) are proved by easier variants. We have on one hand that $S(-\mu^2)$ is composed of operators of the type considered in [G96], with the μ-dependent factors of regularity ∞ and the Π_i of regularity 0, B of regularity ∞ resp. 1 if it is a differential resp. pseudodifferential operator. Removing the commutator term $[A_{\mathrm{DN}}, \Pi_1]$ from (2.20) for a moment, we have an operator

$$S''(-\mu^2) = A_{\mathrm{DN}} + \Pi_2 B \Pi_2 = -\mathfrak{A} + A'_{\mathrm{DN}} + \Pi_2 B \Pi_2, \tag{2.25}$$

which is of regularity 1 (since $\Pi_2 B \Pi_2$ is so), so that the invertibility of the strictly homogeneous principal symbol $-\mathfrak{a}^0 + \pi_2^h b^h \pi_2^h$ assures parameter-ellipticity in the sense of [G96], cf. Prop. 2.1.12 there. Then $S''(-\mu^2)$ is invertible for $\mu \in \Gamma_{\theta',r}$ with a sufficiently large r, the inverse being continuous from $H^{s,\mu}(E)$ to $H^{s+1,\mu}(E)$ for $s \in \mathbb{R}$, by [G96, Th. 3.2.11]. Since the commutator term has L_2-norm bounded in μ by Proposition 2.5, we find by a Neumann series argument that $S(-\mu^2)$ itself is invertible for large r with an inverse that is bounded from $L_2(E)$ to $H^{1,\mu}(E)$.

Now $\mathfrak{A}^{-1}S$ is likewise invertible for the considered μ, and lies in $\mathrm{OP}'\, S^{0,0,0}(\Gamma_\theta)$ since $\mathfrak{A}^{-1}$ lies in $\mathrm{OP}'\, S^{0,0,-1}(\Gamma)$. Then the "spectral invariance theorem" [G99, Th. 6.5], applied to $\mathfrak{A}^{-1}S$, shows that its inverse $S^{-1}\mathfrak{A}$ belongs to our weakly polyhomogeneous calculus and lies in $\mathrm{OP}'\, S^{0,0,0}(\Gamma_\theta)$. It follows that $S^{-1} = (S^{-1}\mathfrak{A})\mathfrak{A}^{-1} \in \mathrm{OP}'\, S^{0,0,-1}(\Gamma_\theta)$. Since holomorphy is preserved under composition, the resulting symbols are holomorphic. The statements on λ-derivatives follows by successive applications of the formula $\partial_\lambda S^{-1} = -S^{-1}\partial_\lambda S\, S^{-1}$, using that $\partial_\lambda S = \partial_\lambda A_{\mathrm{DN}} + \partial_\lambda [A_{\mathrm{DN}}, \Pi_1]$, with properties described in Lemma 2.3 ff. and Proposition 2.4.

Similar proofs work for the other operators without the complication due to the presence of B (so any $\theta' \in]0, \frac{\pi}{2}[$ is allowed there). $\square$

Remark 2.9. It should be noted that the proof of Lemma 2.6 only shows the necessity of unique solvability of (2.21) for ψ in the range of Π_2, so that Assumption 2.7 (assuring solvability for general ψ) may seem too strong. However, when Assumption 2.4 holds, solvability of (2.21) for $\psi \in \Pi_2 H^{\frac{1}{2}}(E')$, $\mu = re^{i\theta_0}$, $r \geq r_0$ (some $r_0 \geq 0$, $|\theta_0| < \theta$), implies solvability for all $\psi \in H^{\frac{1}{2}}(E')$, $r \geq r_1$ with some $r_1 \geq r_0$:

Uniqueness of course holds. Existence is assured as follows: Let $\psi \in H^{\frac{1}{2}}(E')$ and write $\psi = \Pi_1\psi + \Pi_2\psi$. Define

$$\widetilde{S}(\lambda)\psi = A_{\mathrm{DN}}^{-1}\Pi_1\psi + S^{-1}\Pi_2\psi,$$

for r so large that also A_{DN} is invertible. Then

$$\begin{aligned} S\widetilde{S}\psi &= SA_{\mathrm{DN}}^{-1}\Pi_1\psi + \Pi_2\psi = \psi + [A_{\mathrm{DN}}, \Pi_1]A_{\mathrm{DN}}^{-1}\Pi_1\psi + \Pi_2 B \Pi_2 A_{\mathrm{DN}}^{-1}\Pi_1\psi \\ &= \psi + [A_{\mathrm{DN}}, \Pi_1]A_{\mathrm{DN}}^{-1}\Pi_1\psi + \Pi_2 B \Pi_2 [A_{\mathrm{DN}}^{-1}, \Pi_1]\psi = (I + \widetilde{S}_1)\psi. \end{aligned}$$

where $\widetilde{S}_1 = [A_{\mathrm{DN}}, \Pi_1]A_{\mathrm{DN}}^{-1}\Pi_1 + \Pi_2 B\Pi_2 A_{\mathrm{DN}}^{-1}[\Pi_1, A_{\mathrm{DN}}]A_{\mathrm{DN}}^{-1}$ has symbol in $S^{0,0,-1}(\Gamma)$. So, like the operators treated in the proof of Proposition 2.8, $I + \widetilde{S}_1$ is invertible for large enough μ on the ray, and $\varphi = \widetilde{S}(I + \widetilde{S}_1)^{-1}\psi$ solves the equation $S\varphi = \psi$.

We can now describe the resolvent. Here we shall use the terminology of weakly polyhomogeneous pseudodifferential boundary operators worked out in [G01] (the relevant parts summed up in [G02, Sect. 2]), extending the calculus of Boutet de Monvel [BM71]. One can get quite far with linear combinations of compositions of elementary operators as in [GS95], [G99], but when the expressions get increasingly complicated, it seem advantageous to use the systematic calculus. We shall here just recall the basic definitions in the case where X, X' are replaced by $\overline{\mathbb{R}}^n_+$, $\mathbb{R}^{n-1}$.

One considers Poisson operators K (mapping functions on $\mathbb{R}^n_+$ to functions on $\mathbb{R}^{n-1}$), trace operators T of class 0 (mapping functions on $\mathbb{R}^{n-1}$ to functions on $\mathbb{R}^n_+$) and singular Green operators G of class 0 (mapping functions on $\mathbb{R}^n_+$ to functions on $\mathbb{R}^n_+$), of the form

$$\begin{aligned} K &= \mathrm{OPK}(\tilde{k})\colon v(x') \mapsto \int_{\mathbb{R}^{2(n-1)}} e^{i(x'-y')\cdot\xi'}\tilde{k}(x', x_n, \xi', \mu)v(y')\,dy'đ\xi', \\ T &= \mathrm{OPT}(\tilde{t})\colon u(x) \mapsto \int_{\mathbb{R}^{2(n-1)}}\int_0^\infty e^{i(x'-y')\cdot\xi'}\tilde{t}(x', x_n, \xi', \mu)u(y', x_n)\,dx_n dy'đ\xi', \\ G &= \mathrm{OPG}(\tilde{g})\colon u(x) \mapsto \int_{\mathbb{R}^{2(n-1)}}\int_0^\infty e^{i(x'-y')\cdot\xi'}\tilde{g}(x', x_n, y_n, \xi', \mu)u(y)\,dy đ\xi'. \end{aligned} \tag{2.26}$$

Note that the usual ψdo definition is used with respect to the x'-variable. In fact, we can view OPK, OPT and OPG as $\mathrm{OPK}_n\,\mathrm{OP}'$, $\mathrm{OPT}_n\,\mathrm{OP}'$ resp. $\mathrm{OPG}_n\,\mathrm{OP}'$, where OPK_n etc. stand for the application of (2.26) with respect to x_n-variables alone.

The functions $\tilde{k}$, $\tilde{t}$ and $\tilde{g}$ are called the *symbol-kernels* of K, T, resp. G. There is also a "complex formulation", where e.g. the symbol-kernel $\tilde{k}(x', x_n, \xi', \mu)$ is replaced by the *symbol* $k(x', \xi', \xi_n, \mu) = \mathcal{F}_{x_n\to\xi_n}\tilde{k}$, and $\mathcal{F}^{-1}_{\xi_n\to x_n}$ is included in the definition of the operator ($\mathcal{F}$ denotes Fourier transformation).

We say that $\tilde{k} \in \mathcal{S}^{m,d,s}(\Lambda, \mathcal{S}_+)$, resp. $\tilde{g} \in \mathcal{S}^{m,d,s}(\Lambda, \mathcal{S}_{++})$, when

$$\begin{aligned} &\sup_{u_n\in\mathbb{R}_+} |\partial^j_{|z|}(z^d\kappa^{-s-1}\langle z\xi'\rangle^{l-l'}u_n^l\partial^{l'}_{u_n}\tilde{k}(x', |z|u_n, \xi', 1/z))| \le C\langle\xi'\rangle^{m+j}, \text{ resp.} \\ &\sup_{u_n, v_n\in\mathbb{R}_+} |\partial^j_{|z|}(z^d\kappa^{-s-2}\langle z\xi'\rangle^{l-l'+k-k'}u_n^l\partial^{l'}_{u_n}v_n^k\partial^{k'}_{v_n}\tilde{g}(x', |z|u_n, |z|v_n, \xi', 1/z))| \le C\langle\xi'\rangle^{m+j}, \end{aligned} \tag{2.27}$$

for all indices, uniformly for $|z| \le 1$, $\frac{1}{z}$ in closed subsectors of Λ, with similar estimates for the derivatives $\partial^\alpha_{\xi'}\partial^\beta_{x'}$ with m replaced by $m - |\alpha|$. Here $\kappa = (|\mu|^2 + |\xi'|^2)^{\frac{1}{2}} = (|\frac{1}{z}|^2 + |\xi'|^2)^{\frac{1}{2}}$. $\tilde{t}$ is estimated like $\tilde{k}$.

Again the symbol-kernels are said to be holomorphic (in μ), when they and their derivatives are holomorphic for $\mu \in \Lambda^\circ$, $|\mu| + |\xi'| \ge \varepsilon > 0$. The motivation for the scaling $x_n = |z|u_n$ is explained in [G01], where complete and satisfactory composition rules are worked out. The operators are defined on X, X' by standard localization methods.

We recall one further operation, that of taking the normal trace tr_n: When G is a singular Green operator as above, the normal trace $\mathrm{tr}_n\,G$ is the ψdo on $\mathbb{R}^{n-1}$ with symbol

$$(\mathrm{tr}_n\,\tilde{g})(x', \xi', \mu) = \int_0^\infty \tilde{g}(x', x_n, x_n, \xi', \mu)\,dx_n \tag{2.28}$$

(called $\tilde{\tilde{g}}$ in [G96]). Here the symbol map tr_n acts as follows:

$$\operatorname{tr}_n\colon \mathcal{S}^{m,d,s-1}(\Lambda,\mathcal{S}_{++})\to S^{m,d,s}(\Lambda). \tag{2.29}$$

For operators of trace class,

$$\operatorname{Tr}_{\mathbb{R}^n_+} G=\operatorname{Tr}_{\mathbb{R}^{n-1}}(\operatorname{tr}_n G) \tag{2.30}$$

(if G has the kernel $K(x,y,\mu)$ then $\operatorname{tr}_n G$ has the kernel $\int_0^\infty K(x',x_n,y',x_n,\mu)\,dx_n$), and there is a similar rule for the operators carried over to the manifold situation, when the symbol-kernel of G is supported in X_c and the volume element on X_c is taken of the form $v(x')dx'dx_n$:

$$\operatorname{Tr}_X G=\operatorname{Tr}_{X'}(\operatorname{tr}_n G). \tag{2.31}$$

One has that $G-\chi G\chi$ is smoothing and $O(|\mu|^{-M})$ for $\mu\to\infty$ in closed subsectors of Λ, all M, when $\chi\in C_0^\infty(\,]-c,c[\,)$ and is 1 near $x_n=0$ (cf. e.g. [G01, Lemma 7.1]). Thus in the trace expansion calculations for singular Green operators on X, we can replace G by $\chi G\chi$ and use (2.31) to reduce to a calculation for a ψdo on X'; here [GS95, Th. 2.1] can be applied.

We shall express the fact that "the symbol-kernel is in $\mathcal{S}^{m,d,s}(\Gamma_\theta,\mathcal{S}_+)^{N\times N}$ resp. $\mathcal{S}^{m,d,s}(\Gamma_\theta,\mathcal{S}_{++})^{N\times N}$ in local trivializations" more briefly by saying that the operator lies in $\operatorname{OPK}\mathcal{S}^{m,d,s}(\Gamma_\theta,\mathcal{S}_+)$, $\operatorname{OPT}\mathcal{S}^{m,d,s}(\Gamma_\theta,\mathcal{S}_+)$ resp. $\operatorname{OPG}\mathcal{S}^{m,d,s}(\Gamma_\theta,\mathcal{S}_{++})$.

The symbol-kernels we consider, moreover have expansions in appropriately quasi-homogeneous terms, e.g., $\tilde g\sim\sum_{j\in\mathbb{N}}\tilde g_{m-j}$ with $\tilde g_{m-j}\in\mathcal{S}^{m-j,d,s}(\Gamma_\theta,\mathcal{S}_{++})$. There is the same distinction between weakly polyhomogeneous symbol-kernels (with homogeneity for $|\xi'|\ge 1$) and strongly polyhomogeneous symbol-kernels (with homogeneity for $|\xi'|+|\mu|\ge 1$, etc.) as for ψdo symbols.

The following elementary examples are basic in our calculations:

(1) $K_D(-\mu^2)$ is a strongly polyhomogeneous Poisson operator in $\operatorname{OPK}\mathcal{S}^{0,0,-1}(\Gamma,\mathcal{S}_+)$.
(2) $\gamma_0 Q(-\mu^2)_+$ and $\gamma_1 Q(-\mu^2)_+$ are strongly polyhomogeneous trace operators of class 0 in $\operatorname{OPT}\mathcal{S}^{0,0,-2}(\Gamma,\mathcal{S}_+)$ resp. $\operatorname{OPT}\mathcal{S}^{0,0,-1}(\Gamma,\mathcal{S}_+)$.

As mentioned in the proof of Lemma 2.3, K_{D} is principally the same as the operator $K_{\mathfrak{A}}\colon v\mapsto e^{-x_n\mathfrak{A}}v$, for $x_n\in X_c$ (cf. [G02, Prop. 2.11]). We also have (by [G02, (1.17), (4.14)] with A^2 replaced by P') that $\gamma_0 Q_+$ and $\gamma_1 Q_+$ *act principally,* for functions supported in X_c, *like the operators* $\frac12\mathfrak{A}^{-1}T_{\mathfrak{A}}$ *resp.* $\frac12 T_{\mathfrak{A}}$, where

$$T_{\mathfrak{A}}u=\int_0^\infty e^{-x_n\mathfrak{A}}u(x',x_n)\,dx_n. \tag{2.32}$$

Simple examples of singular Green operators are $K_{\mathrm{D}}\gamma_0 Q_+$ (whose negtive is the singular Green part of R_{D}, cf. (2.13)) and $K_{\mathrm{D}}\gamma_1 Q_+$; they are strongly polyhomogeneous and belong to $\operatorname{OPG}\mathcal{S}^{0,0,-3}(\Gamma,\mathcal{S}_{++})$ resp. $\operatorname{OPG}\mathcal{S}^{0,0,-2}(\Gamma,\mathcal{S}_{++})$. $2K_{\mathrm{D}}\gamma_1 Q_+$ is (on X_c) principally equal to $G_{\mathfrak{A}}$, which acts as follows:

$$G_{\mathfrak{A}}u=\int_0^\infty e^{-(x_n+y_n)\mathfrak{A}}u(x',y_n)\,dy_n. \tag{2.33}$$

For the latter, tr_n is easy to determine by functional calculus:

$$\operatorname{tr}_n G_{\mathfrak{A}}=\int_0^\infty e^{-2x_n\mathfrak{A}}\,dx_n=(2\mathfrak{A})^{-1}; \tag{2.34}$$

this kind of calculation plays a role in our analysis of trace coefficients in Section 4.

Theorem 2.10. *Let Assumptions* 1.1, 2.4 *and* 2.7 *hold. Then for each* $\theta' \in]0,\theta[$ *there is an* $r = r(\theta') \geq 0$ *such that for* $\mu \in \Gamma_{\theta',r}$, $P_T + \mu^2 = P_T - \lambda$ *is a bijection from* $D(P_T)$ *to* $L_2(E)$ *with inverse* $(P_T - \lambda)^{-1} = R_T(\lambda)$ *of the form*

$$\begin{aligned} R_T(\lambda) &= Q(\lambda)_+ + G(\lambda), \\ G(\lambda) &= -K_{\mathrm{D}}(\lambda)\gamma_0 Q(\lambda)_+ + K_{\mathrm{D}}(\lambda)[S_0(\lambda)\gamma_0 + S_1(\lambda)\gamma_1]Q(\lambda)_+; \end{aligned} \tag{2.35}$$

here S_0 *and* S_1 *(given in* (2.36) *below) are weakly polyhomogeneous* ψ*do's in* E' *lying in* $\mathrm{OP}'\, S^{0,0,0}(\Gamma_\theta)$ *resp.* $\mathrm{OP}'\, S^{0,0,-1}(\Gamma_\theta)$*, and hence* G *is a singular Green operator of class* 0 *in* $\mathrm{OPG}\,\mathcal{S}^{0,0,-3}(\Gamma_\theta, \mathcal{S}_{++})$.

Proof. For a $\theta' \in]0,\theta[$, take $r = r(\theta')$ such that A_{DN} is well-defined and the operators $S(\lambda)$ and $S'(\lambda)$ are invertible for $\lambda = -\mu^2$, $\mu \in \Gamma_{\theta',r}$. Then (2.17) is solved uniquely by (2.19), and it remains to draw the conclusions for the original problem (2.12). In view of (2.13), (2.14) and (2.19), the solution is:

$$\begin{aligned} u = v + z &= R_{\mathrm{D}} f + K_{\mathrm{D}}\varphi \\ &= R_{\mathrm{D}} f - K_{\mathrm{D}} S^{-1}\Pi_2\gamma_1 R_{\mathrm{D}} f \\ &= Q_+ f - K_{\mathrm{D}}\gamma_0 Q_+ f - K_{\mathrm{D}} S^{-1}\Pi_2\gamma_1 Q_+ f + K_{\mathrm{D}} S^{-1}\Pi_2 A_{\mathrm{DN}}\gamma_0 Q_+ f. \end{aligned}$$

This shows (2.35) with

$$S_0 = S^{-1}\Pi_2 A_{\mathrm{DN}}, \quad S_1 = -S^{-1}\Pi_2; \tag{2.36}$$

they lie in $\mathrm{OP}'\, S^{0,0,0}(\Gamma_\theta)$ resp. $\mathrm{OP}'\, S^{0,0,-1}(\Gamma_\theta)$ by the rules of calculus. The statement on G now follows from the information given before the theorem on K_{D}, $\gamma_0 Q_+$ and $\gamma_1 Q_+$, and the composition rules. □

Remark 2.11. Let us give some sufficient conditions for the validity of Assumption 2.7. Consider, in a local trivialization, a point (x', ξ') with $|\xi'| = 1$ (the result is carried over to general $\xi' \neq 0$ by homogeneity, and for $\xi' = 0$ the assumption is trivially satisfied). Let

$$0 < \lambda_1(x', \xi') \leq \lambda_2(x', \xi') \leq \cdots \leq \lambda_N(x', \xi')$$

be the eigenvalues of the matrix $p'^0(x', \xi')$, associated with the orthonormal system of eigenvectors $e_1(x', \xi'), \dots, e_N(x', \xi')$ in $\mathbb{C}^N$, and denote

$$a_i(x', \xi') = \sqrt{\lambda_i(x', \xi')}$$

(all equal to $\sqrt{c}$, when $p'^0 = cI$). Denote $\Gamma_{\theta,\pm} = \{\mu \in \Gamma_\theta \mid \operatorname{Im}\mu \in \overline{\mathbb{R}}_\pm\}$. When μ runs through $\Gamma_{\theta,\pm} \cup \{0\}$, then $(a_1^2 + \mu^2)^{\frac{1}{2}}$ runs through a convex subset $V_{a_1,\pm}$ of $\Gamma_{\theta,\pm}$ lying to the right of a curve $C_{a_1,\pm}$ passing through a_1 on the real axis.

Since $a_j \geq a_1$ for $j \geq 1$, we also have that $(a_j^2 + \mu^2)^{\frac{1}{2}}$ lies in $V_{a_1,\pm}$ when $\mu \in \Gamma_{\theta,\pm} \cup \{0\}$. We denote $V_{a_1,+} \cup V_{a_1,-} = V_{a_1}$. It is important that although V_{a_1} is not in general convex, the $V_{a_1,\pm}$ are so.

Let $\mu \in \Gamma_{\theta,+} \cup \{0\}$. Then for general $v \in \mathbb{C}^N$ with norm 1, decomposed as $v = c_1 e_1 + \cdots + c_N e_N$ with $|c_1|^2 + \cdots + |c_N|^2 = 1$,

$$\begin{aligned}\mathfrak{a}^0 v \cdot \bar{v} &= (p'^0 + \mu^2)^{\frac{1}{2}} v \cdot \bar{v} \\ &= \sum_{i,j=1}^N (a_i^2 + \mu^2)^{\frac{1}{2}} c_i e_i \cdot \overline{c_j e_j} = \sum_{i=1}^N (a_i^2 + \mu^2)^{\frac{1}{2}} |c_i|^2 \in V_{a_1,+},\end{aligned}$$

since $V_{a_1,+}$ is convex. There is a similar argument for $\mu \in \Gamma_{\theta,-}$, showing altogether that

$$(p'^0 + \mu^2)^{\frac{1}{2}} v \cdot \bar{v} \in V_{a_1}, \text{ when } \mu \in \Gamma_\theta \cup \{0\},\ |v| = 1. \tag{2.37}$$

Now (2.24) is obtained, if at each (x', ξ') with $|\xi'| = 1$,

$$|((p'^0 + \mu^2)^{\frac{1}{2}} - \pi_2^h b^h \pi_2^h) v \cdot \bar{v}| \geq \delta |v|^2, \quad v \in \mathbb{C}^N, \tag{2.38}$$

for some $\delta > 0$. In view of (2.37), this holds if $\pi_2^h b^h \pi_2^h v \cdot \bar{v}$ lies in a set in $\mathbb{C}$ with distance δ from V_{a_1} when $|v| = 1$.

We list some special cases where this holds; here we assume that π_2^h is an orthogonal projection.

(1) Let b^h be the principal symbol of a scalar first-order differential operator with real coefficients. Then b^h is purely imaginary and

$$\pi_2^h b^h \pi_2^h v \cdot \bar{v} = b^h |\pi_2^h v|^2 \in i\mathbb{R}, \tag{2.39}$$

which certainly has positive distance from V_{a_1}. More generally, we can take b^h such that $b^h v \cdot \bar{v}$ ranges in the sectors around $i\mathbb{R}$ consisting of complex numbers with argument in $]\theta_1, \frac{\pi}{2} - \theta_1[$ or $]\pi + \theta_1, \frac{3\pi}{2} - \theta_1[$ for some $\theta_1 \in]\theta, \frac{\pi}{2}]$ (allowing also pseudodifferential choices).

(2) Let $b^h = ib_1$, where b_1 is the principal symbol of a scalar first-order differential operator with real coefficients. Then b^h is real, and the real number $\pi_2^h b^h \pi_2^h v \cdot \bar{v} = b^h |\pi_2^h v|^2$ has positive distance from V_{a_1} for $|v| = 1$ if

$$|b^h| < a_1. \tag{2.40}$$

More generally, we can take b^h real selfadjoint with numerical range in $[-a_1 + \varepsilon, +\infty[$ for some $\varepsilon \in]0, a_1[$, i.e.,

$$b^h v \cdot \bar{v} \geq -a_1 + \varepsilon \text{ for } |v| = 1. \tag{2.41}$$

When B is a differential operator, this in fact requires that $|b^h v \cdot \bar{v}| \leq a_1 - \varepsilon$, since the symbol b^h is odd in ξ'. This case seems to have an interest in brane theory according to [V01].

(3) Let $r_1 = \operatorname{dist}(V_{a_1}, 0)$. Then it suffices that the norm of b^h is $< r_1$.

Since R_T is of order -2 and X is compact, the powers R_T^m are trace-class when $m > \frac{n}{2}$. They can be studied by composition or by differentiation, in view of the fact that $\partial_\lambda^{m-1}(P_T - \lambda)^{-1} = (m-1)!(P_T - \lambda)^{-m}$.

Corollary 2.12. *Under the hypotheses of Theorem* 2.10*, the resolvent powers* R_T^m *have the structure, for any* $m \geq 1$*:*

$$R_T^m = (Q^m)_+ + G^{(m)} = \tfrac{1}{(m-1)!}\partial_\lambda^{m-1} R_T = \tfrac{1}{(m-1)!}(\partial_\lambda^{m-1} Q)_+ + \tfrac{1}{(m-1)!}\partial_\lambda^{m-1} G, \tag{2.42}$$

where $G^{(m)} = \frac{1}{(m-1)!}\partial_\lambda^{(m-1)} G$ *is a singular Green operator of class* 0 *lying in* $\mathrm{OPG}\,\mathcal{S}^{0,0,-2m-1}(\Gamma_\theta, \mathcal{S}_{++})$.

Proof. One proof consists of applying the rules of calculus ([G01], [G02]) to the compositions $(Q_+ + G)\dots(Q_+ + G)$. Another proof is to use the exact formula we found in Theorem 2.10, combined with the fact that all the factors have the property that a differentiation with respect to λ lowers the s-index by 2. □

Theorem 2.13. *Assumptions as in Theorem* 2.10.

(i) *Let* φ *be a morphism in* E *and let* $m > \frac{n}{2}$*. Then* $\varphi R_T^m(\lambda)$ *is trace-class and the trace has an expansion for* $|\lambda| \to \infty$ *with* $\arg\lambda \in]\pi - 2\theta, \pi + 2\theta[$ *(uniformly in closed subsectors):*

$$\mathrm{Tr}\big(\varphi R_T^m(\lambda)\big) \sim \sum_{-n \leq k < 0} \tilde{a}_k(\varphi)(-\lambda)^{-\frac{k}{2}-m} + \sum_{k \geq 0}\big(\tilde{a}_k'(\varphi)\log(-\lambda) + \tilde{a}_k''(\varphi)\big)(-\lambda)^{-\frac{k}{2}-m}. \tag{2.43}$$

The coefficients $\tilde{a}_k$ *and* $\tilde{a}_k'$ *are locally determined.*

(ii) *Let* F *be a differential operator in* E *of order* m' *and let* $m > \frac{n+m'}{2}$*. Then* $FR_T^m(\lambda)$ *is trace-class and the trace has an expansion for* $|\lambda| \to \infty$ *with* $\arg\lambda \in]\pi - 2\theta, \pi + 2\theta[$ *(uniformly in closed subsectors):*

(2.35)
$$\mathrm{Tr}\big(FR_T^m(\lambda)\big) \sim \sum_{-n \leq k < 0} \tilde{a}_k(F)(-\lambda)^{\frac{m'-k}{2}-m} + \sum_{k \geq 0}\big(\tilde{a}_k'(F)\log(-\lambda) + \tilde{a}_k''(F)\big)(-\lambda)^{\frac{m'-k}{2}-m},$$

with locally determined coefficients $\tilde{a}_k$ *and* $\tilde{a}_k'$*. If* m' *is odd,* $\tilde{a}_{-n} = 0$*.*

Here, if F *is tangential (differentiates only with respect to* x'*) on* X_c*, the log-coefficients* $\tilde{a}_k'$ *with* $0 \leq k < m'$ *vanish, and the* $\tilde{a}_k''$ *with* $0 \leq k < m'$ *are locally determined.*

Proof. (i) is the special case of (ii) where $m' = 0$, so we can treat them at the same time. It is well-known that the "interior" ψdo term FQ_+^m produces a series of powers $\sum_{k=-n}^{\infty} \tilde{c}_k(-\lambda)^{\frac{m'-k}{2}-m}$ for $\lambda \to \infty$ in closed subsectors of $\mathbb{C} \setminus \mathbb{R}_+$, with coefficients determined from the successive homogeneous terms in the symbol; here the terms with $k-n-m'$ odd vanish since they are produced by terms that are odd in ξ.

Now consider the singular Green term $FG^{(m)}$. By Corollary 2.12 and the composition rules it lies in $\mathrm{OPG}\,\mathcal{S}^{0,0,m'-2m-1}(\Gamma_\theta)$ in general, and in $\mathrm{OPG}\,\mathcal{S}^{m',0,-2m-1}(\Gamma_\theta)$ if F is tangential. With a cut-off function χ as after (2.31), we have that $\mathrm{Tr}(FG^{(m)} - \chi FG^{(m)}\chi)$ is $O(|\mu|^{-M})$ for $\mu \to \infty$ in closed subsectors of Γ_θ, all M; this difference does not contribute to the trace expansion, so it suffices to treat $\chi FG^{(m)}\chi$. This operator can be considered as a singular Green operator on $X_+^0 = X' \times \mathbb{R}_+$, with the same symbol-kernel estimates as indicated above. Now we take the normal trace, obtaining an operator

(2.45)
$$\mathcal{S} = \mathrm{tr}_n(\chi FG^{(m)}\chi) \in \mathrm{OP}'\,S^{0,0,m'-2m}(\Gamma_\theta) \subset \mathrm{OP}'(S^{m'-2m,0,0}(\Gamma_\theta) \cap S^{0,m'-2m,0}(\Gamma_\theta))$$

by (2.29) and (2.1). If F is tangential, the last indication is replaced by $\mathrm{OP}'(S^{m'-2m,0,0}(\Gamma_\theta)\cap S^{m',-2m,0}(\Gamma_\theta))$.

Here we apply [GS95, Th. 1.2]. It is an important point in that theorem that one gets a sum of two expansions

$$\sum_{j\geq 0} c_j\mu^{M_0-j}+\sum_{k\geq 0}(c'_k\log\mu+c''_k)\mu^{M_1-k}, \tag{2.46}$$

where M_0 equals the dimension (here $n-1$) plus the order of the operator (here $m'-2m$), and M_1 equals the lowest d-index associated with the operator. The coefficients c_j and c'_k are local (each stems from a specific homogeneous term in the symbol), whereas the c''_k are global in the sense that they depend on the full structure. So in the application to $\mathcal{S}$, the first series begins with a term $c\mu^{n-1+m'-2m}$, and the second series begins with a term $(c'\log\mu+c'')\mu^{m'-2m}$ for general F, $(c'\log\mu+c'')\mu^{-2m}$ for tangential F. Adding the expansions and inserting $\mu=(-\lambda)^{\frac12}$, we find the expansion (2.35) ff. after an adjustment of the indexation. □

Remark 2.14. One has in particular that when all the occurring operators are strongly polyhomogeneous, there is only the first expansion in (2.46), no logarithmic or global terms. This applies to the special case where Π_1 is a morphism in E' and B is a differential operator; then

$$\mathrm{Tr}\big(FR_T^m(\lambda)\big)\sim\sum_{k\geq -n}\tilde a_k(F)(-\lambda)^{\frac{m'-k}{2}-m}. \tag{2.47}$$

3. Heat operators and power operators.

The heat operator associated with P_T is the solution operator $u_0(x)\mapsto u(x,t)=e^{-tP_T}u_0$ for the problem

$$\begin{aligned}\partial_t u+Pu&=0 \text{ on } X\times\mathbb{R}_+,\\ \Pi_1\gamma_0 u&=0 \text{ on } X'\times\mathbb{R}_+,\\ \Pi_2(\gamma_1 u+B\gamma_0 u)&=0 \text{ on } X'\times\mathbb{R}_+,\\ u|_{t=0}&=u_0 \text{ on } X.\end{aligned} \tag{3.1}$$

When $\theta>\frac{\pi}{4}$ in the above constructions, the heat operator can be defined from the resolvent powers or derivatives (recall (2.42)) by the formula

$$e^{-tP_T}=t^{-m}\tfrac{i}{2\pi}\int_{\mathcal{C}'}e^{-t\lambda}\partial_\lambda^m(P_T-\lambda)^{-1}\,d\lambda; \tag{3.2}$$

here $\mathcal{C}'$ is a positively oriented curve in $\mathbb{C}$ going around the spectrum (like the boundary of $\Sigma_{\delta,R}$ in (1.9) with $\delta\in\,]0,\frac{\pi}{2}[$; one can take $\delta=\pi-2\theta'$ for a $\theta'\in\,]\frac{\pi}{4},\theta[$).

One could construct the heat operator directly instead of passing via the resolvent as we did above; one advantage of our approach is that we can compose our λ-dependent operators pointwise in λ, whereas calculations with respect to the time-variable t need convolutions. (The passage from the λ-framework to the t-framework is essentially an

inverse Laplace transformation; here products are turned into convolutions, as is usual for such integral transforms.)

As shown e.g. in [GS96] (or see Sect. 2 of [G97]), the transition formula likewise applies to the trace expansions, carrying the expansions in Theorem 2.13 over to heat trace expansions with logarithms. In the resulting statement, we repeat our hypotheses for the convenience of the reader:

Corollary 3.1. *Let P_T be the realization of P defined by the boundary condition* (1.3)*; let Assumptions* 1.1 *and* 2.4 *hold, and let Assumption* 2.7 *hold with $\theta > \frac{\pi}{4}$. Then the heat operator e^{-tP_T} is well-defined and its trace has the asymptotic expansion for $t \to 0+$, for any morphism φ in E:*

$$\operatorname{Tr}\big(\varphi e^{-tP_T}\big) \sim \sum_{-n\le k<0} a_k(\varphi) t^{\frac{k}{2}} + \sum_{k\ge 0}\big(-a_k'(\varphi)\log t + a_k''(\varphi)\big)t^{\frac{k}{2}}. \tag{3.4}$$

The coefficients a_k and a_k' are locally determined.

Moreover, when F is a differential operator in E of order m' there is a trace expansion for $t \to 0+$:

$$\operatorname{Tr}\big(Fe^{-tP_T}\big) \sim \sum_{-n\le k<0} a_k(F) t^{\frac{k-m'}{2}} + \sum_{k\ge 0}\big(-a_k'(F)\log t + a_k''(F)\big)t^{\frac{k-m'}{2}}, \tag{3.5}$$

the coefficients a_k and a_k' being locally determined. If m' is odd, $a_{-n} = 0$.

Here, if F is tangential on X_c, the log-coefficients a_k' with $0 \le k < m'$ vanish, and the a_k'' with $0 \le k < m'$ are locally determined.

The coefficients a_k, a_k', a_k'' are related to the coefficients $\tilde a_k, \tilde a_k', \tilde a_k''$ in Theorem 2.13 *by universal nonzero proportionality factors; in particular,*

$$a_0'(F) = \tilde a_0'(F) \text{ and } a_0''(F) = \tilde a_0''(F). \tag{3.6}$$

Observe that as in Remark 2.14, the expansion simplifies to

$$\operatorname{Tr}\big(Fe^{-tP_T}\big) \sim \sum_{k\ge -n} a_k(F) t^{\frac{k-m'}{2}}, \tag{3.7}$$

when Π_1 is a morphism and B is a differential operator.

The power function FP_T^{-s} (defined as 0 on the nullspace of P_T) is derived from the resolvent by the formula

$$P_T^{-s} = \tfrac{i}{2\pi(s-1)\dots(s-m)} \int_{\mathcal C} \lambda^{m-s}\partial_\lambda^m (P_T-\lambda)^{-1}\, d\lambda, \tag{3.8}$$

where $\mathcal C$ is a curve in $\mathbb C \setminus \overline{\mathbb R}_-$ around the nonzero spectrum; here we do not need the extra hypothesis $\theta > \frac{\pi}{4}$. (Further details on transition formulas are found e.g. in [GS96] or [G97].)

Then one can also deduce from Theorem 2.13 that there is the following pole structure of the zeta function $\zeta(F,P_T,s)=\operatorname{Tr}(FP_T^{-s})$ for $s\in\mathbb{C}$:

$$\Gamma(s)\operatorname{Tr}\big(\varphi P_T^{-s}\big)\sim\sum_{-n\le k<0}\frac{a_k(\varphi)}{s+\frac{k}{2}}-\frac{\operatorname{Tr}\varphi\Pi_0(P_T)}{s}+\sum_{k\ge0}\Big(\frac{a_k'(\varphi)}{(s+\frac{k}{2})^2}+\frac{a_k''(\varphi)}{s+\frac{k}{2}}\Big),\tag{3.9}$$

$$\Gamma(s)\operatorname{Tr}\big(FP_T^{-s}\big)\sim\sum_{-n\le k<0}\frac{a_k(F)}{s+\frac{k-m'}{2}}-\frac{\operatorname{Tr}F\Pi_0(P_T)}{s}+\sum_{k\ge0}\Big(\frac{a_k'(F)}{(s+\frac{k-m'}{2})^2}+\frac{a_k''(F)}{s+\frac{k-m'}{2}}\Big);\tag{3.10}$$

here a_k, a_k' and a_k'' are derived from $\tilde a_k$, $\tilde a_k'$ and $\tilde a_k''$ by *the same* universal formulas as in Corollary 3.1. In particular, (3.6) holds.

Consider the case where F is a first-order operator D_1 with the structure $D_1=\psi(\partial_{x_n}+B_1)$ on X' for some morphism ψ. Here we get the generalized eta-function $\operatorname{Tr}\big(D_1P_T^{-s}\big)$, which has the pole structure

$$\Gamma(s)\operatorname{Tr}\big(D_1P_T^{-s}\big)\sim\sum_{-n\le k<0}\frac{a_k(D_1)}{s+\frac{k-1}{2}}-\frac{\operatorname{Tr}D_1\Pi_0(P_T)}{s}+\sum_{k\ge0}\Big(\frac{a_k'(D_1)}{(s+\frac{k-1}{2})^2}+\frac{a_k''(D_1)}{s+\frac{k-1}{2}}\Big).\tag{3.11}$$

With $s=\frac{s'+1}{2}$ this takes the more customary form

$$\operatorname{Tr}\big(D_1P_T^{-\frac{s'+1}{2}}\big)\sim\frac{1}{\Gamma(\frac{s'+1}{2})}\Big(\sum_{-n\le k<0}\frac{2a_k(D_1)}{s'+k}-\frac{2\operatorname{Tr}D_1\Pi_0(P_T)}{s'+1}+\sum_{k\ge0}\Big(\frac{4a_k'(D_1)}{(s'+k)^2}+\frac{2a_k''(D_1)}{s'+k}\Big)\Big).\tag{3.12}$$

Note that since $\Gamma(\frac{s'+1}{2})^{-1}$ is regular at $s'=0$, this function in general has a double pole at $s'=0$. Moreover, $a_{-n}(D_1)=0$ since the principal interior symbol is odd in ξ.

4. The first log-term and nonlocal term.

The hypotheses of Theorem 2.10 are assumed throughout this section.

It is important to investigate whether the first log-term a_0' in (3.4), resp. $\tilde a_0'$ in (2.43) vanishes, equivalently (cf. (3.9)) whether the zeta function $\zeta(\varphi,P_T,s)$ is regular at $s=0$. The values of $\tilde a_0''$ and a_0'' are of great interest too, and similar questions can be asked with φ replaced by F, in particular for (3.12).

In view of (3.6), we can interchange $\tilde a_0'(F)$ and $\tilde a_0''(F)$ freely with $a_0'(F)$ resp. $a_0''(F)$.

The coefficient a_0' is known to vanish in the Atiyah-Patodi-Singer case described in Example 1.2, where $\Pi_1=\Pi_{\ge}$, $B=A_1(0)$ and $\varphi=I$, by a slightly tricky argument ([G92] proved it using [APS75]; see also [GS96, pf. of Cor. 2.3]). It also vanishes in cases of local boundary conditions, as mentioned in Remark 2.14.

As indicated in the proof of Theorem 2.13, the first power where log-terms and global coefficients appear is determined from the possible values of d, the second upper index in the symbol spaces $S^{m,d,0}$ that the normal trace of the singular Green operator belongs to. Consider first the case where $F=\varphi$, a morphism in E (taken to be constant in x_n on X_c). Then Theorem 2.13 tells us that the index d can be taken equal to $-2m$. We shall in the following strive to isolate the part of $G^{(m)}$ that contributes nontrivially to the first log-term and global term, in a way that gives information on the value.

Theorem 4.1. *Let φ be a morphism in E, independent of x_n on X_c (its restriction to X' likewise denoted φ). Consider $\chi\varphi G^{(m)}\chi$ (cf. Theorem 2.13) as an operator on $X_+^0 = X'\times\mathbb{R}_+$. With the notation introduced around (2.32), we have that*

$$\begin{aligned}\chi\varphi G^{(m)}\chi &= \tfrac{\partial_\lambda^{m-1}}{(m-1)!}\varphi\Pi_2 K_{\mathfrak{A}}\mathfrak{A}^{-1}T_{\mathfrak{A}} + \mathcal{G}_1 + \mathcal{G}_2,\\ \operatorname{tr}_n(\chi\varphi G^{(m)}\chi) &= \tfrac{\partial_\lambda^{m-1}}{(m-1)!}\tfrac12\varphi\Pi_2\mathfrak{A}^{-2} + \mathcal{S}_1 + \mathcal{S}_2,\end{aligned}\tag{4.1}$$

where $\mathcal{G}_1$ is a strongly polyhomogeneous singular Green operator of degree $-2m-1$, $\mathcal{G}_2\in \operatorname{OPG}\mathcal{S}^{1,0,-2-2m}(\Gamma_\theta,\mathcal{S}_{++})$, $\mathcal{S}_1$ is a strongly polyhomogeneous ψdo in E' of degree $-2m$, and $\mathcal{S}_2\in\operatorname{OP}'S^{1,0,-1-2m}(\Gamma_\theta)$.

Proof. By Theorem 2.10 and (2.42), we have that

$$\chi\varphi G^{(m)}\chi = -\tfrac{\partial_\lambda^{m-1}}{(m-1)!}\chi\varphi K_{\mathrm{D}}\gamma_0 Q_+\chi + \tfrac{\partial_\lambda^{m-1}}{(m-1)!}\chi\varphi K_{\mathrm{D}}[S_0\gamma_0 + S_1\gamma_1]Q_+\chi.$$

The first term is a $\mathcal{G}_1$. For the second term, we first note that, since $(1-\chi)\partial_\lambda^r K_{\mathfrak{A}}$ and $\partial_\lambda^r T_{\mathfrak{A}}(1-\chi)$ are smoothing and $O(|\mu|^{-M})$ in closed subsectors of Γ for all M (cf. e.g. [G01, Lemma 7.1]),

$$\begin{aligned}\partial_\lambda^r\chi K_{\mathrm{D}} - \partial_\lambda^r K_{\mathfrak{A}} &\in \operatorname{OPK}\mathcal{S}^{0,0,-2-2r}(\Gamma,\mathcal{S}_+),\\ \partial_\lambda^r\gamma_0 Q_+\chi - \partial_\lambda^r\tfrac12\mathfrak{A}^{-1}T_{\mathfrak{A}} &\in \operatorname{OPT}\mathcal{S}^{0,0,-3-2r}(\Gamma,\mathcal{S}_+),\\ \partial_\lambda^r\gamma_1 Q_+\chi - \partial_\lambda^r\tfrac12 T_{\mathfrak{A}} &\in \operatorname{OPT}\mathcal{S}^{0,0,-2-2r}(\Gamma,\mathcal{S}_+),\end{aligned}\tag{4.2}$$

in view of the information around (2.32)–(2.34). Thus, using the formulas for S_0, S_1 and S and replacing A_{DN} by $-\mathfrak{A}$,

$$\begin{aligned}\chi\varphi G^{(m)}\chi &= \tfrac{\partial_\lambda^{m-1}}{(m-1)!}\varphi K_{\mathfrak{A}}S^{-1}\Pi_2[(-\mathfrak{A})\tfrac12\mathfrak{A}^{-1} - \tfrac12]T_{\mathfrak{A}} + \mathcal{G}_1 + \mathcal{G}_2',\\ &= -\tfrac{\partial_\lambda^{m-1}}{(m-1)!}\varphi K_{\mathfrak{A}}(-\mathfrak{A}+\Pi_2 B\Pi_2)^{-1}\Pi_2 T_{\mathfrak{A}} + \mathcal{G}_1 + \mathcal{G}_2'',\end{aligned}\tag{4.3}$$

where $\mathcal{G}_2'$ and $\mathcal{G}_2''$ are in $\operatorname{OPG}\mathcal{S}^{0,0,-2-2m}(\Gamma_\theta,\mathcal{S}_{++})$. We have furthermore:

$$(-\mathfrak{A}+\Pi_2 B\Pi_2)^{-1}\Pi_2 = -\mathfrak{A}^{-1}\Pi_2 + (-\mathfrak{A}+\Pi_2 B\Pi_2)^{-1}\Pi_2 B\Pi_2\mathfrak{A}^{-1}\Pi_2.\tag{4.4}$$

Here ∂_λ^{m-1} of $\mathfrak{A}^{-1}\Pi_2$ is in $\operatorname{OP}'S^{0,0,1-2m}(\Gamma)$. Since $\partial_\lambda^r(-\mathfrak{A}+\Pi_2 B\Pi_2)^{-1}$ is in $\operatorname{OP}'S^{0,0,-1-2r}(\Gamma_\theta)$, $\Pi_2 B\Pi_2\in\operatorname{OP}'S^{1,0,0}(\Gamma)$ and $\partial_\lambda^r\mathfrak{A}^{-1}\Pi_2\in\operatorname{OP}'S^{0,0,-1-2r}(\Gamma)$, ∂_λ^{m-1} of the last term is in $\operatorname{OP}'S^{1,0,-2m}(\Gamma_\theta)$. Thus, if we use Proposition 2.5 to replace $\partial_\lambda^r\mathfrak{A}^{-1}\Pi_2$ by $\partial_\lambda^r\Pi_2\mathfrak{A}^{-1}$, we can write

$$\chi\varphi G^{(m)}\chi = \tfrac{\partial_\lambda^{m-1}}{(m-1)!}\varphi K_{\mathfrak{A}}\Pi_2\mathfrak{A}^{-1}T_{\mathfrak{A}} + \mathcal{G}_1 + \mathcal{G}_2'',\tag{4.5}$$

with $\mathcal{G}_2'''$ like $\mathcal{G}_2$ in the theorem. Note that B has disappeared from the main term!

It will be convenient to do one more commutation, placing Π_2 in front of the Poisson operator $K_{\mathfrak{A}}$, which likewise gives an error like $\mathcal{G}_2$. For this we use that $e^{-x_n\mathfrak{A}} = \operatorname{OPK}_n((\mathfrak{A}+i\xi_n)^{-1})$ in the complex formulation. Here $[(\mathfrak{A}+i\xi_n)^{-1},\Pi_2] =$

$(\mathfrak{A}+i\xi_n)^{-1}[\Pi_2,\mathfrak{A}](\mathfrak{A}+i\xi_n)^{-1}$, which by application of OPK_n gives a Poisson operator in OPK $\mathcal{S}^{0,0,-2}(\Gamma_\theta,\mathcal{S}_+)$ (with $(m-1)$'st derivative in OPK $\mathcal{S}^{0,0,-2m}(\Gamma_\theta,\mathcal{S}_+)$), so that we can write

$$\chi\varphi G^{(m)}\chi = \tfrac{\partial_\lambda^{m-1}}{(m-1)!}\varphi\Pi_2 K_{\mathfrak{A}}\mathfrak{A}^{-1}T_{\mathfrak{A}} + \mathcal{G}_1 + \mathcal{G}_2^{(4)},$$

with $\mathcal{G}_2^{(4)}$ like $\mathcal{G}_2$. This shows the first formula in (4.1).

By functional calculus as in (2.34), we have that $\mathrm{tr}_n(K_{\mathfrak{A}}(\mathfrak{A})^{-1}T_{\mathfrak{A}}) = \frac12\mathfrak{A}^{-2}$. Then also $\mathrm{tr}_n(\varphi\Pi_2K_{\mathfrak{A}}(\mathfrak{A})^{-1}T_{\mathfrak{A}}) = \frac12\varphi\Pi_2\mathfrak{A}^{-2}$, showing the second formula in (4.1); the symbol properties follow from (2.29). (Placing the ψdo $\varphi\Pi_2$ in front of the singular Green operator $K_{\mathfrak{A}}(\mathfrak{A})^{-1}T_{\mathfrak{A}}$, and taking it outside tr_n, is justified by the point of view where OPG $=$ $\mathrm{OPG}_n\,\mathrm{OP}'$.) $\square$

The last commutation in the proof is not needed when $\varphi = I$, for then we can instead use circular permutation in the treatment of $\frac{\partial_\lambda^{m-1}}{(m-1)!}K_{\mathfrak{A}}\Pi_2\mathfrak{A}^{-1}T_{\mathfrak{A}}$:

$$\mathrm{Tr}_{X_+^0}(\tfrac{\partial_\lambda^{m-1}}{(m-1)!}K_{\mathfrak{A}}\Pi_2\mathfrak{A}^{-1}T_{\mathfrak{A}}) = \mathrm{Tr}_{X'}(\tfrac{\partial_\lambda^{m-1}}{(m-1)!}\Pi_2\mathfrak{A}^{-1}T_{\mathfrak{A}}K_{\mathfrak{A}}) = \tfrac12\,\mathrm{Tr}_{X'}(\tfrac{\partial_\lambda^{m-1}}{(m-1)!}\Pi_2\mathfrak{A}^{-2}), \tag{4.6}$$

since $T_{\mathfrak{A}}K_{\mathfrak{A}} = (2\mathfrak{A})^{-1}$.

Observe that

$$\varphi\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}\mathfrak{A}^{-2} = \varphi\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-1}; \tag{4.7}$$

it lies in $\mathrm{OP}'\,S^{0,0,-2m}(\Gamma)$. Let us list the three trace expansions connected with this operator family (which follow from [GS95, Th. 2.7]):

$$\begin{aligned}
\mathrm{Tr}_{X'}(\tfrac12\varphi\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-1}) &\sim \sum_{k\ge n-1}\tilde c_k(\varphi)(-\lambda)^{-\frac k2-m}\\
&\quad+\sum_{l\ge0}\big(\tilde c_{2l}'(\varphi)\log(-\lambda)+\tilde c_{2l}''(\varphi)\big)(-\lambda)^{-l-m},\\
\mathrm{Tr}_{X'}(\tfrac12\varphi\Pi_2e^{-tP'}) &\sim \sum_{k\ge1-n}c_k(\varphi)t^{\frac k2}+\sum_{l\ge0}\big(-c_{2l}'(\varphi)\log t+c_{2l}''(\varphi)\big)t^l,\\
\Gamma(s)\,\mathrm{Tr}_{X'}(\tfrac12\varphi\Pi_2(P')^{-s}) &\sim \sum_{k\ge1-n}\frac{c_k(\varphi)}{s+\frac k2}-\frac{\mathrm{Tr}[\frac12\varphi\Pi_2\Pi_0(P')]}{s}\sum_{l\ge0}\Big(\frac{c_{2l}'(\varphi)}{(s+l)^2}+\frac{c_{2l}''(\varphi)}{s+l}\Big);
\end{aligned} \tag{4.8}$$

again with $c_0'(\varphi)=\tilde c_0'(\varphi)$, $c_0''(\varphi)=\tilde c_0''(\varphi)$.

Theorem 4.2. *Let φ be as in Theorem 4.1. In a comparison of (2.43) with the trace expansion of $\frac12\varphi\Pi_2\frac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-1}$ in (4.8), we have that*

$$\begin{aligned}
\tilde a_0'(\varphi) &= \tilde c_0'(\varphi),\\
\tilde a_0''(\varphi) &= \tilde c_0''(\varphi) + \textit{ local contributions},
\end{aligned} \tag{4.9}$$

where the local contributions are defined from P (at X'), φ and the strictly homogeneous terms in the symbol of Π_2 of orders $\{0,-1,\dots,1-n\}$.

In particular, $\tilde a_0'(\varphi)$ equals the residue $c_0'(\varphi)$ at 0 of the zeta function $\zeta(\frac12\varphi\Pi_2, P', s) = \operatorname{Tr}_{X'}(\frac12\varphi\Pi_2(P')^{-s})$, also identifiable with the non-commutative residue of $\frac12\varphi\Pi_2$ times $\frac12$:

$$\tilde a_0'(\varphi) = a_0'(\varphi) = c_0'(\varphi) = \operatorname{Res}_{s=0}\operatorname{Tr}(\tfrac12\varphi\Pi_2(P')^{-s}) = \tfrac14\operatorname{res}(\varphi\Pi_2). \tag{4.10}$$

Moreover, when $\theta > \frac{\pi}{4}$, $a_0'(\varphi)$ in (3.4) equals the coefficient $c_0'(\varphi)$ of $\log t$ in the trace expansion of $\frac12\varphi\Pi_2 e^{-tP'}$ in (4.8).

Proof. As noted in Theorem 2.13, the part $\varphi(Q^m)_+$ of φR_T^m contributes only a local power expansion, and $\varphi G^{(m)}$ contributes the same expansion as $\chi\varphi G^{(m)}\chi$, so it suffices to study $\operatorname{Tr}_{X'}\operatorname{tr}_n(\chi\varphi G^{(m)}\chi)$, where we can use (4.1). Here the strongly polyhomogeneous part $\mathcal S_1$ gives an expansion with purely local power terms (as in (2.47) with $k \ge 1-n$ and $m' = 0$), and the term $\mathcal S_2$ with symbol in $S^{1,0,-1-2m}(\Gamma_\theta) \subset S^{-2m,0,0}(\Gamma_\theta) \cap S^{1,-1-2m,0}(\Gamma_\theta)$ gives an expansion as in (2.46) with $M_0 = n-2m$ and $M_1 = -1-2m$. So the non-local and log-contributions from $\mathcal S_1 + \mathcal S_2$ begin with a term $(c'\log(-\lambda) + c'')(-\lambda)^{-\frac12 - m}$, implying (4.9).

The identification of $c_0'(\varphi)$ with $\frac12$ times the noncommutative residue $\operatorname{res}(\frac12\varphi\Pi_2)$ refers to the work of Wodzicski [W84], [W84′], where it was shown that for a classical integer-order ψdo C in E', the following formula:

$$\operatorname{res}(C) = \operatorname{ord} Q\ \operatorname{Res}_{s=0}\operatorname{Tr}(CQ^{-s}) = \int_{X'}\int_{|\xi'|=1}(\operatorname{tr}_{E'} c(x',\xi'))_{1-n}\, d\sigma(\xi') \tag{4.11}$$

can be given a sense (the trace Tr is defined as a meromorphic extension from large $\operatorname{Re} s$ to $s \in \mathbb C$, Q is an auxiliary invertible elliptic ψdo of positive order, $\operatorname{tr}_{E'}$ is the fiber trace, and subscript $1-n$ indicates the term of degree $1-n = -\dim X'$ in the symbol); the functional res vanishes on commutators.

If P' is invertible, the last equality sign in (4.10) follows directly by taking $Q = P'$. If P' is not invertible but has a (necessarily finite dimensional) nullspace $V_0(P')$, we use that $(P')^{-s}$ is defined to be 0 on that nullspace, and that a replacement of P' by the invertible operator

$$Q = P' + \Pi_0(P') \tag{4.12}$$

in (4.8) leaves the coefficient $c_0'(\varphi)$ invariant (whereas $c_0''(\varphi)$ is changed). □

These formulas have consequences for the original Atiyah-Patodi-Singer problem, that we take up in Section 5. Let us also observe:

Theorem 4.3. *Let D_1 be a first-order differential operator on X, of the form $D_1 = \psi(\partial_{x_n} + B_1)$ on X_c, where B_1 is tangential and ψ is a morphism independent of x_n. Then*

$$\begin{aligned}\chi D_1 G^{(m)}\chi &= -\psi\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}K_{\mathfrak A}T_{\mathfrak A} + \mathcal G_1' + \mathcal G_2',\\ \operatorname{tr}_n(\chi D_1 G^{(m)}\chi) &= -\tfrac12\psi\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}\mathfrak A^{-1} + \mathcal S_1' + \mathcal S_2',\end{aligned} \tag{4.13}$$

where $\mathcal G_1'$ is a strongly polyhomogeneous singular Green operator of degree $-2m$, $\mathcal G_2' \in \operatorname{OPG}\mathcal S^{1,0,-1-2m}(\Gamma_\theta, \mathcal S_{++})$, $\mathcal S_1'$ is a strongly polyhomogeneous ψdo in E' of degree $-2m+1$

and $\mathcal{S}_2 \in \mathrm{OP}'\, S^{1,0,-2m}(\Gamma_\theta)$. *Here* $-\frac{1}{2}\psi\Pi_2 \frac{\partial_\lambda^{m-1}}{(m-1)!}\mathfrak{A}^{-1} = -\frac{1}{2}\psi\Pi_2 \frac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-\frac{1}{2}}$ *has a trace expansion:*

$$\begin{aligned} -\operatorname{Tr}_{X'}\big(\tfrac{1}{2}\psi\Pi_2 \tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-\frac{1}{2}}\big) \sim &\sum_{k\geq n-1} \tilde{d}_k(\psi)(-\lambda)^{\frac{1-k}{2}-m} \\ &+ \sum_{k\geq 0}\big(\tilde{d}'_k(\psi)\log(-\lambda) + \tilde{d}''_k(\psi)\big)(-\lambda)^{\frac{1-k}{2}-m}, \end{aligned} \tag{4.14}$$

where the $\tilde{d}'_k$ *and* $\tilde{d}''_k$ *are zero for* k *odd. Hence in a comparison of* (2.35) *for* $F = D_1$ *with* (4.13), *we have that*

$$\begin{aligned} \tilde{a}'_0(D_1) &= \tilde{d}'_0(\psi), \\ \tilde{a}''_0(D_1) &= \tilde{d}''_0(\psi) + \text{ local contributions}, \end{aligned} \tag{4.15}$$

where the local contributions are defined from P *and* D_1 *(at* X'*),* ψ *and the strictly homogeneous terms in the symbol of* Π_2 *of orders* $\{0,-1,\dots,1-n\}$.

(The same formulas hold for $a'_0(D_1)$ *resp.* $a''_0(D_1)$*.)*

The coefficient $\tilde{d}'_0(\psi)$ *is proportional to the noncommutative residue of* $\psi\Pi_2$*:*

$$\tilde{d}'_0(\psi) = \alpha \operatorname{res}(\psi\Pi_2), \tag{4.16}$$

for some nonzero constant α *depending only on* m *and the dimension.*

Proof. (4.13) follows from (4.1), when we use that the tangential operator B_1 lifts the first upper index in our symbol classes by 1, $\partial_{x_n} e^{-x_n\mathfrak{A}} = -\mathfrak{A}e^{-x_n\mathfrak{A}}$, and $[D_1,\chi]$ is supported away from $x_n = 0$. Now (4.14) holds by application of [GS95, Th. 2.1] (with $\mu^2 = -\lambda$ as usual); the more precise information that the log-terms and nonlocal terms only occur for even k — which is not needed for our main purposes here — follows from [GH02] (or from an analysis as in [G02, Th. 5.2]). Since $\mathcal{S}'_1$ produces only local power terms, and the logarithmic and global terms for $\mathcal{S}'_2$ begin with the power $-m$, the result (4.15) on the zero'th coefficients follows.

For (4.16) one applies an analysis as in [G02, proof of Th. 5.2] to (4.14). We have that

$$\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-\frac{1}{2}} = c_m(P'-\lambda)^{-\frac{1}{2}-m} = (-\lambda)^{-\frac{1}{2}-m}(\varrho P'+1)^{-\frac{1}{2}-m}, \quad \varrho = -\lambda^{-1}. \tag{4.17}$$

Now

$$\psi\Pi_2(\varrho P'+1)^{-\frac{1}{2}-m} \sim \sum_{j\geq 0}\tbinom{-\frac{1}{2}-m}{j}\varrho^j\psi\Pi_2(P')^j \tag{4.18}$$

similarly to [G02, (5.6)], and the subsequent analysis there gives that the first log-term comes from the term with $j=0$ and equals

$$c\int_{X'}\int_{|\xi'|=1}(\operatorname{tr}[\psi(x')\Pi_2(x',\xi')])_{1-n}\,d\sigma(\xi') = c'\operatorname{res}(\psi\Pi_2), \tag{4.19}$$

with nonzero universal constants c and c'. □

The result has an interest for the analysis of the eta function associated with the APS-problem, which we take up in Section 5.

Remark 4.4. Let us comment on what is meant by "local contributions" in (4.9) and (4.15). It is taken in a rather strict sense, based on the derivation of asymptotic formulas in [GS95, Th. 2.1] (also recalled in [G02, Th. 2.10]). The local contributions at a certain index k come from the homogeneous symbol terms with the degree that matches the index.

Since $\varphi R_T^m = \varphi Q_+^m + \varphi G^{(m)}$ is of order $-2m$, local contributions at $k=0$ (in (4.9)) come from terms of homogeneity degree $-2m-n$ in (ξ,μ) in the symbol of φQ^m and terms of homogeneity degree $-2m-n+1$ in (ξ',μ) in the symbol of $\operatorname{tr}_n \varphi G^{(m)}$, in local trivializations.

Since $D_1 R_T^m = D_1 Q_+^m + D_1 G^{(m)}$ is of order $-2m+1$, local contributions at $k=0$ (in (4.15)) come from terms of homogeneity degree $-2m+1-n$ in (ξ,μ) in the symbol of $D_1 Q^m$ and terms of homogeneity degree $-2m-n+2$ in (ξ',μ) in the symbol of $\operatorname{tr}_n D_1 G^{(m)}$, in local trivializations.

These contributions can be traced back to the symbols of φ and P, resp. ψ, P and D_1 at $x_n=0$, and the x_n-derivatives up to order n at $x_n=0$ of the symbol of P (resp. P and D_1), together with the first n homogeneous terms (down to order $1-n$) in the symbol of Π_2 (or Π_1).

In many cases one can show that the first log-coefficient vanishes and get some information on the nonlocal content of the coefficient behind it.

We first observe:

Theorem 4.5. *We always have that $a_0'(I)=0$. Moreover, $a_0'(\varphi)=0$ if $\varphi\Pi_2$ is a projection, and $a_0'(D_1)=0$ if $\psi\Pi_2$ is a projection.*

Proof. This follows from the fact, observed already by Wodzicki in [W84], [W84′], that when Π is a ψdo projection, then $\operatorname{res}(\Pi)=0$. The three statements in the theorem follow by applying this to the formulas shown above:

$$\begin{aligned} a_0'(I) &= c_0'(I) = \tfrac{1}{4}\operatorname{res}(\Pi_2),\\ a_0'(\varphi) &= c_0'(\varphi) = \tfrac{1}{4}\operatorname{res}(\varphi\Pi_2),\\ a_0'(D_1) &= d_0'(\psi) = \alpha\operatorname{res}(\psi\Pi_2); \end{aligned} \tag{4.20}$$

they give 0 when the operator to which res is applied is a projection. □

Other systematic results can be obtained when Π_1 is a spectral projection or a suitable perturbation of such an operator.

Definition 4.6. *A pseudodifferential orthogonal projection Π in $L_2(E')$ will be said to be a spectral projection associated with C, when C is a selfadjoint first-order elliptic classical ψdo in E' and Π satisfies*

$$\Pi = \Pi_>(C) + \Pi_{V_0'}, \tag{4.21}$$

where $\Pi_>(C)$ is the orthogonal projection onto the positive eigenpace of C and $\Pi_{V_0'}$ is the orthogonal projection onto a subspace V_0' of $V_0(C)$ (the nullspace of C). We denote $V_0(C)\ominus V_0' = V_0''$.

It is shown below in Proposition 4.8 that for any orthogonal ψdo projection Π there exists an invertible C such that $\Pi=\Pi_>(C)$.

A particular result is the following:

Theorem 4.7. *Let*

$$\Pi_1 = \Pi_>(C) + \mathcal{S}, \tag{4.22}$$

where C is a selfadjoint elliptic **differential** *operator of order 1 and $\mathcal{S}$ is a ψdo of order $\le -n$. If n is odd, then* $\operatorname{res}(\varphi\Pi_2)$ *and* $\operatorname{res}(\psi\Pi_2)$ *are zero and hence $\tilde{a}_0'(\varphi)$ and $\tilde{a}_0'(D_1)$ in* (4.9), (4.15) *are zero.*

Proof. We have that $\varphi\Pi_2 = \varphi\Pi_\le(C) - \varphi\mathcal{S}$, where

$$\Pi_\le(C) = \tfrac{|C|-C}{2|C'|} + \Pi_0(C) = \tfrac{1}{2}(I - \tfrac{C}{|C'|} + \Pi_0(C)), \text{ with } C' = C + \Pi_0(C). \tag{4.23}$$

Here $\frac{1}{2}\varphi$, $\varphi\Pi_0$ and $\varphi\mathcal{S}$ have residue 0, since they have no $1-n$-degree term in the symbol. The symbol of $\frac{C}{|C'|}$ has even-odd parity (the terms of even degree of homogeneity order are odd in ξ' and vice versa; more on such symbols e.g. in [G02, Sect. 5]), and so does the symbol composed with φ. Thus, when the interior dimension n is odd, the term of order $1-n$ in the symbol is odd in ξ', so the integration with respect to ξ' in (4.11) gives zero. □

Before considering the more delicate results on the term a_0'', we include some words about ψdo projections.

Proposition 4.8.
(i) *When Π is a ψdo projection in $L_2(E')$, then*

$$\Pi_{\mathrm{ort}} = \Pi\Pi^*[\Pi\Pi^* + (I-\Pi^*)(I-\Pi)]^{-1} \tag{4.24}$$

is an orthogonal ψdo projection with the same range. Moreover,

$$R = \Pi + (I - \Pi_{\mathrm{ort}})(I-\Pi) \tag{4.25}$$

is an invertible elliptic zero-order ψdo (with inverse $\Pi_{\mathrm{ort}} + (I-\Pi)(I-\Pi_{\mathrm{ort}})$) such that

$$\Pi_{\mathrm{ort}} = R\Pi R^{-1}. \tag{4.26}$$

(ii) *Let Π be an orthogonal ψdo projection in $L_2(E')$. There exists a selfadjoint invertible elliptic ψdo C of order 1 in E' such that $\Pi = \Pi_>(C)$.*

Proof. (i). The formula (4.24) is known from Birman and Solomyak [BS82], details of verification can also be found in Booss-Bavnbek and Wojciechowski [BW93, Lemma 12.8]. The statements on R are easily checked.

(ii). If the principal symbol π^0 equals the identity or 0, we are in a trivial case, so let us assume that $\pi^0 \neq I$ and 0; then Π and $\Pi^\perp$ both have infinite dimensional range. Let C_1 be a selfadjoint positive first-order elliptic ψdo with scalar principal symbol $c_1^0(x',\xi')$ (e.g. $= |\xi'|$). Let

$$C' = \Pi C_1 \Pi - \Pi^\perp C_1 \Pi^\perp.$$

C' is a ψdo of order 1 with principal symbol

$$c'^0 = \pi^0 c_1^0 \pi^0 - (I-\pi^0)c_1^0(I-\pi^0) = (2\pi^0 - I)c_1^0,$$

which is invertible since $2\pi^0 - I$ and c_1^0 are so. Clearly, C' is selfadjoint, and $\Pi C_1 \Pi \geq 0$, $\Pi^\perp C_1 \Pi^\perp \geq 0$ in view of the positivity of C_1. Moreover, Π commutes with C'.

Since C' is selfadjoint elliptic, it has a spectral decomposition in smooth orthogonal finite dimensional eigenspaces V_k with mutually distinct eigenvalues λ_k, $k \in \mathbb{Z}$, such that $\lambda_k < 0$ for $k < 0$, $\lambda_k > 0$ for $k > 0$, $\lambda_0 = 0$ (V_0 may be 0). The positive resp. negative eigenspace of C' is $V_> = \oplus_{k>0} V_k$ resp. $V_< = \oplus_{k<0} V_k$. The eigenspaces are invariant under Π: For $u_k \in V_k$,

$$C'\Pi u_k = \Pi C' u_k = \Pi \lambda_k u_k = \lambda_k \Pi u_k;$$

hence $\Pi u_k \in V_k$. Now if $u_k \in V_k \setminus \{0\}$ with $k < 0$, then Πu_k must be zero, for otherwise

$$(C'\Pi u_k, \Pi u_k) = \lambda_k \|\Pi u_k\|^2 < 0,$$

in contrast with

$$(C'\Pi u_k, \Pi u_k) = (\Pi C_1 \Pi \Pi u_k, \Pi u_k) = (C_1 \Pi u_k, \Pi u_k) \geq c\|\Pi u_k\|^2 > 0;$$

here $c > 0$ is the lower bound of C_1. Thus $V_< \subset R(\Pi^\perp)$. Similarly, $V_> \subset R(\Pi)$.

Finally, let Π_{0+} be the orthogonal projection onto ΠV_0, let Π_{0-} be the orthogonal projection onto $\Pi^\perp V_0$ (note that $V_0 = \Pi V_0 \oplus \Pi^\perp V_0$ since $\Pi V_0 \subset V_0$), and set

$$C = C' + \Pi_{0+} - \Pi_{0-};$$

it is injective. Then $V_>(C) = V_>(C') \oplus \Pi V_0 \subset R(\Pi)$ and $V_<(C) = V_<(C') \oplus \Pi^\perp V_0 \subset R(\Pi^\perp)$, so since they are complementing subspaces, they equal $R(\Pi)$ resp. $R(\Pi^\perp)$, and C is as asserted. $\square$

Parts of the above proof details are given in Brüning and Lesch [BL99, Lemma 2.6]. They can be used to show the fact that $\operatorname{res}(\Pi) = 0$ for any ψdo projection, that we used above in Theorem 4.5. In fact, with the notation of the proposition, we have since res vanishes on commutators,

$$\begin{aligned}(4.27)\qquad \operatorname{res}(\Pi) &= \operatorname{res}(R^{-1}\Pi_{\mathrm{ort}}R) = \operatorname{res}(\Pi_{\mathrm{ort}}) = \operatorname{res}(\Pi_>(C)) = \tfrac{1}{2}\operatorname{res}(I + C|C|^{-1}) \\ &= \tfrac{1}{2}\operatorname{res}(C|C|^{-1}) = \tfrac{1}{2}\operatorname{Res}_{s=0}\operatorname{Tr}(C|C|^{-s-1}) = \tfrac{1}{2}\operatorname{Res}_{s=0}\eta(C,s) = 0,\end{aligned}$$

where the last equality follows from the vanishing of the eta residue of C shown by Atiyah, Patodi and Singer [APS76] (odd dimensions) and Gilkey [Gi81]. (The relation between the vanishing of the noncommutative residue on projections, and the vanishing of eta residues, enters also in [W84].)

Theorem 4.9. *Let Π_1 be an orthogonal pseudodifferential projection, and let C be a first-order selfadjoint elliptic ψdo such that $\Pi_1 = \Pi_>(C) + \Pi_{V_0'}$ as in Definition 4.6. Then*

$$(4.28)\qquad \Pi_2(P' - \lambda)^{-m} - \Pi_2(C^2 - \lambda)^{-m} \in \mathrm{OP}'\, S^{2,0,-2m-2}(\Gamma).$$

The power function $\frac{1}{2}\Pi_2(C^2)^{-s}$ corresponding to $\frac{1}{2}\Pi_2(C^2 - \lambda)^{-m}$ (cf. (3.8)) satisfies

$$(4.29)\qquad \begin{aligned}\tfrac{1}{2}\Pi_2(C^2)^{-s} &= \tfrac{1}{4}((C^2)^{-s} - C|C|^{-2s-1}),\\ \operatorname{Tr}[\tfrac{1}{2}\Pi_2(C^2)^{-s}] &= \tfrac{1}{4}\zeta(C^2,s) - \tfrac{1}{4}\eta(C,2s).\end{aligned}$$

In particular, in (4.8),

$$c_0''(I) = -\tfrac{1}{4}(\eta(C,0) + \dim V_0' - \dim V_0'') + \text{ local contributions.} \tag{4.30}$$

It follows that when Π_1 *is the projection entering in the construction in Theorems* 2.10 *and* 2.13, *then*

$$\tilde{a}_0''(I) = -\tfrac{1}{4}(\eta(C,0) + \dim V_0' - \dim V_0'') + \text{ local contributions.} \tag{4.31}$$

Proof. It follows from [GS95] (adapted to the present notation) that $(C^2 - \lambda)^{-m} \in \mathrm{OP}'\, S^{0,0,-2m}(\Gamma)$, and that

$$\begin{aligned} \Pi_2 (P' - \lambda)^{-m} - \Pi_2 (C^2 - \lambda)^{-m} \\ = \Pi_2 \tfrac{\partial_\lambda^{m-1}}{(m-1)!}[(P'-\lambda)^{-1}(C^2 - P')(C^2-\lambda)^{-1}] \in \mathrm{OP}'\, S^{2,0,-2m-2}(\Gamma), \end{aligned} \tag{4.32}$$

so this difference contributes no log-terms or nonlocal terms at the powers $(-\lambda)^{-m}$ and $(-\lambda)^{-m-\frac{1}{2}}$. (This reflects the fact that in the residue construction, one can replace the auxiliary operator P' by C^2.) Now in (4.29), the first line follows from (4.23) when we recall that the powers are defined to be zero on $V_0(C)$; the second line follows by taking the trace. Then since $\mathrm{Tr}[\frac{1}{2}\Pi_2\Pi_0(C)] = \frac{1}{2}\dim V_0''$ and $\zeta(C^2,0) = -\dim V_0(C) +$ a local coefficient, we have at $s=0$:

$$\begin{aligned} c_0''(I) - \tfrac{1}{2}\dim V_0'' &= \mathrm{Tr}[\tfrac{1}{2}\Pi_2 (C^2)^{-s}]_{s=0} = \tfrac{1}{4}\zeta(C^2,0) - \tfrac{1}{4}\eta(C,0)) \\ &= -\tfrac{1}{4}\dim V_0(C) - \tfrac{1}{4}\eta(C,0) + \text{ a local coefficient,} \end{aligned}$$

which implies (4.30) since $-\frac{1}{4}\dim V_0(C) + \frac{1}{2}\dim V_0'' = -\frac{1}{4}(\dim V_0' - \dim V_0'')$. Finally, (4.31) follows in view of (4.9). □

We can give a name to the "nonlocal part of $a_0''(I)$" appearing in this way:

Definition 4.10. *In the situation of Definition* 4.6, *we define the associated eta-invariant* $\eta_{C,V_0'}$ *by:*

$$\eta_{C,V_0'} = \eta(C,0) + \dim V_0' - \dim V_0''. \tag{4.33}$$

Note in particular that

$$\eta_{C,V_0'} = \eta(C,0) + \dim V_0(C), \quad \text{if } \Pi_1 = \Pi_{\geq}(C), \tag{4.34}$$

$$\eta_{C,V_0'} = \eta(C,0), \quad \text{if } \dim V_0' = \tfrac{1}{2}\dim V_0(C). \tag{4.35}$$

Note also that since $\zeta(P_T,0) = a_0''(I) - \dim V_0(P_T)$, we have in the situation of Theorem 4.9 that

$$\zeta(P_T,0) = -\tfrac{1}{4}\eta_{C,V_0'} - \dim V_0(P_T) + \text{ local contributions.} \tag{4.36}$$

Under special circumstances, we can show that $\tilde{c}_0''$ and $\tilde{d}_0''$ are purely local:

Theorem 4.11. *Let Π_1 be an orthogonal pseudodifferential projection (so that $\Pi_2 = \Pi_1^\perp$), and assume that there exists a unitary morphism σ such that*

$$\sigma^2 = -I, \quad \sigma P' = P'\sigma, \quad \Pi_1^\perp = -\sigma\Pi_1\sigma. \tag{4.37}$$

Then

$$\begin{aligned} \operatorname{Tr}_{X'}(\tfrac12\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-1}) &= \tfrac14\operatorname{Tr}_{X'}(\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-1}),\\ -\operatorname{Tr}_{X'}(\tfrac12\sigma\Pi_2\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-\frac12}) &= -\tfrac14\operatorname{Tr}_{X'}(\sigma\tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-\frac12}). \end{aligned} \tag{4.38}$$

Thus in (4.8), $\tilde c_0''(I)$ $(= c_0''(I))$ *is locally determined (from the symbol of P'), and in* (4.14) *with $\psi = \sigma$,*

$$\tilde d_0'(\sigma) = 0, \quad \tilde d_0''(\sigma) \text{ is locally determined} \tag{4.39}$$

(from the symbol of P' and σ).

It follows that in the situation of Theorems 4.2 *and* 4.3 *with $D_1 = \sigma(\partial_{x_n} + B_1)$,*

$$\begin{gathered} \tilde a_0'(D_1) = a_0'(D_1) = 0;\\ \tilde a_0''(I), a_0''(I) \text{ and } \tilde a_0''(D_1) \text{ are locally determined} \end{gathered} \tag{4.40}$$

(depending only on finitely many homogeneous terms in the symbols of P and T, resp. P, T and D_1).

Proof. Introduce the shorter notation

$$R_{m,1} = \tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-1}, \quad R_{m,2} = \tfrac{\partial_\lambda^{m-1}}{(m-1)!}(P'-\lambda)^{-\frac12};$$

note that $\sigma R_{m,i} = R_{m,i}\sigma$ by (4.37). Then we have for the traces on X', using moreover that the trace is invariant under circular permutation:

$$\begin{aligned} \operatorname{Tr}(\Pi_2 R_{m,1}) &= -\operatorname{Tr}(\sigma\Pi_1\sigma R_{m,1}) = -\operatorname{Tr}(\Pi_1 R_{m,1}\sigma^2)\\ &= \operatorname{Tr}((1-\Pi_2)R_{m,1}) = \operatorname{Tr}(R_{m,1}) - \operatorname{Tr}(\Pi_2 R_{m,1}). \end{aligned} \tag{4.41}$$

It follows that

$$\operatorname{Tr}(\Pi_2 R_{m,1}) = \tfrac12\operatorname{Tr}(R_{m,1}). \tag{4.42}$$

Similarly,

$$\begin{aligned} \operatorname{Tr}(\sigma\Pi_2 R_{m,2}) &= -\operatorname{Tr}(\sigma^2\Pi_1\sigma R_{m,2}) = \operatorname{Tr}(\Pi_1\sigma R_{m,2})\\ &= \operatorname{Tr}((1-\Pi_2)\sigma R_{m,2}) = \operatorname{Tr}(\sigma R_{m,2}) - \operatorname{Tr}(\sigma\Pi_2 R_{m,2}), \end{aligned} \tag{4.43}$$

implying

$$\operatorname{Tr}(\sigma\Pi_2 R_{m,2}) = \tfrac12\operatorname{Tr}(\sigma R_{m,2}). \tag{4.44}$$

This shows (4.38). It is classically known that $\operatorname{Tr}(R_{m,1})$ has an expansion in powers with local coefficients; this shows the assertion on $\tilde c_0''(I) = c_0''(I)$. For $i = 2$, $\operatorname{Tr}(\sigma R_{m,2})$ has an expansion in powers of $(-\lambda)$ with only local coefficients, since $\sigma R_{m,2}$ is strongly polyhomogeneous, cf. [GS95] or [G02, Th. 2.10]. This implies the assertions on $\tilde d_0'(\sigma)$ and $\tilde d_0''(\sigma)$.

The last assertion now follows from (4.9) and (4.15). □

The proof also shows that $\tilde c_0'(I) = \tilde a_0'(I) = 0$, but we know that already from Theorem 4.5.

5. Consequences for the APS problem.

The preceding results have interesting new consequences for the realizations of first-order operators in Example 1.2, which we now consider in detail. Let D satisfy (1.6) and let Π be a well-posed orthogonal ψdo projection for D. Then in view of Green's formula:

$$(Du, v)_X - (u, D^*v)_X = -(\sigma\gamma_0 u, \gamma_0 v)_{X'}, \text{ for } u \in C^\infty(E_1), v \in C^\infty(E_2), \tag{5.1}$$

the adjoint $(D_\Pi)^*$ is the realization of D^* (of the form $(-\partial_{x_n} + A + x_n A_{21} + A_{20})\sigma^*$ on X_c) defined by the boundary condition $\Pi^\perp\sigma^*\gamma_0 v = 0$ (associated with the well-posed projection $\Pi' = \sigma\Pi^\perp\sigma^*$ for D^*). It follows that D^*D is of the form (1.1) with $P' = A^2$, and that ${D_\Pi}^*D_\Pi$ is the realization of D^*D defined by the boundary condition

$$\Pi\gamma_0 u = 0, \quad \Pi^\perp(\gamma_1 u + A_1(0)\gamma_0 u) = 0. \tag{5.2}$$

Thus ${D_\Pi}^*D_\Pi$ is of the type P_T considered in Sections 1–4, with $P = D^*D$, $P' = A^2$, $\Pi_1 = \Pi$ and $B = A_1(0)$. Note that the symbol considered in Assumption 2.7 is here

$$((a^0)^2 + \mu^2)^{\frac{1}{2}} - (I - \pi^h)a^0(I - \pi^h). \tag{5.3}$$

When the principal symbols of Π and A^2 commute, Assumption 2.7 is essentially equivalent with well-posedness. More precisely, we have:

Lemma 5.1. *Let Π be an orthogonal ψdo projection in $L_2(E_1')$ and let P_T be the realization of D^*D under the boundary condition* (5.2)*, and assume that the principal symbols of Π and A^2 commute.*

1° *When Π is well-posed for D, then Assumption* 2.7 *holds for $\{P, T\}$ with $\theta = \frac{\pi}{2}$.*

2° *If Assumption* 2.7 *holds for $\{P, T\}$ with some $\theta > 0$ and $\pi^0(x', \xi')$ has rank $N/2$, then Π is well-posed for D.*

Proof. 1°. Fix x', $|\xi'| \geq 1$, and consider the model realization $d^0_{\pi^0}$ (defined for the ordinary differential operator $d^0 = \sigma(x')(\partial_{x_n} + a^0(x', \xi'))$ in $L_2(\mathbb{R}_+, \mathbb{C}^N)$ by the boundary condition $\pi^0(x', \xi')u(0) = 0$), and the model realization $p^0_{t^0}$ (defined similarly from principal symbols). The well-posedness assures that $d^0_{\pi^0}$ is injective, hence $p^0_{t^0} = (d^0_{\pi^0})^* d^0_{\pi^0}$ is selfadjoint positive, as an unbounded operator in $L_2(\mathbb{R}_+, \mathbb{C}^N)$. It follows that $p^0_{t^0} - \lambda$ is bijective from its domain to $L_2(\mathbb{R}_+, \mathbb{C}^N)$, for all $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$. Using that π^0 commutes with $(a^0)^2$, we can carry out the calculations in the proof of Lemma 2.6 for the model problem (without commutation error terms), which allows us to conclude that the equation in $\mathbb{C}^N$:

$$[((a^0)^2 - \lambda)^{\frac{1}{2}} - (I - \pi^0)a^0(I - \pi^0)]\varphi = \psi, \tag{5.4}$$

is uniquely solvable for $\psi \in R(I - \pi^0)$. Moreover, the calculations in Remark 2.9 on the model level extend the solvability of (5.4) to all $\psi \in \mathbb{C}^N$. The invertibility property extends readily to the strictly homogeneous symbols for $\xi' \neq 0$, it is obvious for $\xi' = 0$ with $\lambda \neq 0$.

2°. Assumption 2.7 gives for $\mu = 0$, $|\xi'| \geq 1$, that $p^0_{t^0} = (d^0_{\pi^0})^* d^0_{\pi^0}$ is bijective. This implies injectiveness of $d^0_{\pi^0}$, i.e., injective ellipticity of $\{d^0, \pi^0\gamma_0\}$. Then well-posedness holds exactly when π^0 has rank $N/2$. (One may consult [G99, p. 55].)

Example 5.2. When Π is taken as $\Pi_{\geq} + \mathcal{S}$ with $\mathcal{S}$ of order -1 (cf. Example 1.2), π^0 commutes with a^0 itself, and we see directly that Assumption 2.7 holds simply because $-(I-\pi^h)a^0(I-\pi^h) \geq 0$. — For the projections $\Pi(\theta) = P(\theta)$ introduced by Brüning and Lesch in [BL99], Assumption 2.7 is also directly verifiable, since the conditions of [BL99] assure that $-(I-\pi^h)a^0(I-\pi^h) - c|a^0| \geq 0$ for some $c > -1$. Here $\Pi(\theta)$ commutes with A^2. Again, perturbations of order -1 are allowed.

Thus the results of Section 2–4 apply to ${D_\Pi}^* D_\Pi$ with $\Gamma_\theta = \Gamma$. So there are expansions (2.43), (3.4) and (3.9) for

$$\operatorname{Tr}(\varphi({D_\Pi}^* D_\Pi - \lambda)^{-m}), \quad \operatorname{Tr}(\varphi e^{-t{D_\Pi}^* D_\Pi}),$$
$$\text{and } \Gamma(s)\operatorname{Tr}(\varphi({D_\Pi}^* D_\Pi)^{-s}) = \Gamma(s)\zeta(\varphi, {D_\Pi}^* D_\Pi, s),$$

and, with the choice $F = \varrho D$ (ϱ a morphism from E_2 to E_1), there are expansions as in (2.35), (3.5) and (3.11)–(3.12) for

$$\operatorname{Tr}(\varrho D({D_\Pi}^* D_\Pi - \lambda)^{-m}), \quad \operatorname{Tr}(\varrho D e^{-t{D_\Pi}^* D_\Pi}), \quad \Gamma(s)\operatorname{Tr}(\varrho D({D_\Pi}^* D_\Pi)^{-s}),$$
$$\text{and } \Gamma(\tfrac{s+1}{2})\operatorname{Tr}(\varrho D({D_\Pi}^* D_\Pi)^{-\frac{s+1}{2}}) = \Gamma(\tfrac{s+1}{2})\eta(\varrho, D_\Pi, s).$$

Such expansions were shown in [G99] by a different procedure where D was regarded as part of a first-order system of the double size.

We get new results by drawing some consequences for the coefficients at $k = 0$ from Section 4.

Before doing this, let us also briefly look at $D_\Pi {D_\Pi}^*$. It is easily checked that $\sigma^* D D^* \sigma$ is of the form (1.1), and that $\sigma^* D_\Pi {D_\Pi}^* \sigma$ is the realization of it with boundary condition

$$\Pi^\perp \gamma_0 u = 0, \quad \Pi(\gamma_1 u - (A + A_{20}(0))\gamma_0 u) = 0. \tag{5.5}$$

In the consideration of trace formulas for $D_\Pi {D_\Pi}^*$, a composition to the left with σ and to the right with σ^* leaves the formulas corresponding to (2.43), (3.4) and (3.9) unchanged if $\varphi = I$.

Theorems 4.5 and 4.7 imply immediately:

Corollary 5.3. *Let $P_T = D_\Pi^* D_\Pi$, where D is as in Example 1.2, Π is well-posed for D, and the principal symbols of Π and A^2 commute.*
(i) *For the expansions* (2.43), (3.4), (3.9), *related to the zeta function,*

$$\tilde{a}_0'(I) = a_0'(I) = 0. \tag{5.6}$$

Moreover,

$$\tilde{a}_0'(\varphi) = a_0'(\varphi) = \tfrac{1}{4}\operatorname{res}(\varphi\Pi^\perp); \tag{5.7}$$

it is zero in the following cases (a) *and* (b)*:*

(a) *$\varphi\Pi^\perp$ is a projection,*
(b) *n is odd and $\Pi^\perp = \Pi_>(C) + \mathcal{S}$ for some first-order selfadjoint elliptic* **differential** *operator C of order 1, $\mathcal{S}$ of order $-n$.*

(ii) *For the expansions* (2.35), (3.5), (3.11)–(3.12) *with* $F = \varrho D$, *related to the eta function,*

$$\tilde{a}_0'(\varrho D) = a_0'(\varrho D) = \alpha \operatorname{res}(\varrho\sigma\Pi^\perp); \tag{5.8}$$

it is zero if $\varrho\sigma\Pi^\perp$ *is a projection, or if* (b) *holds.*

Note that (5.6) means that the zeta function of $D_\Pi^* D_\Pi$ is regular at zero. Since the hypotheses assuring this (once D is taken of the form (1.6)), are entirely concerned with principal symbols, we have in particular: *The regularity of the zeta function at* $s = 0$ *is preserved under perturbations of* Π *of order* -1. When $[\pi^0, (a^0)^2] = 0$, this is a far better result than that of [G01′], where it was shown for perturbations of order $-n$.

We also have from Theorem 4.9 and the following considerations:

Corollary 5.4. *Assumptions as in Corollary* 5.3. *Let* Π *be a spectral projection as in Definition* 4.6, *with the notation introduced there and in Definition* 4.10. *Then in the expansions* (2.43), (3.4), (3.9),

$$a_0''(I) = -\tfrac{1}{4}\eta_{C,V_0'} + \text{ local contributions}, \tag{5.9}$$

$$\zeta(D_\Pi^* D_\Pi, 0) = -\tfrac{1}{4}\eta_{C,V_0'} - \dim V_0(D_\Pi) + \text{ local contributions}. \tag{5.10}$$

There is a similar result for $D_\Pi D_\Pi^*$; here Π is replaced by $\Pi^\perp = \Pi_>(-C) + \Pi_{V_0''}$ in view of the remarks above on $D_\Pi D_\Pi^*$. So in this case, Theorem 4.9 gives:

$$\begin{aligned} a_0''(I)(D_\Pi D_\Pi^*) &= -\tfrac{1}{4}(-\eta(C,0) + \dim V_0'' - \dim V_0') + \text{ local contributions} \\ &= \tfrac{1}{4}\eta_{C,V_0'} + \text{ local contributions}, \end{aligned} \tag{5.11}$$

Observe moreover that since

$$\operatorname{index} D_\Pi = \operatorname{Tr} e^{-tD_\Pi^* D_\Pi} - \operatorname{Tr} e^{-tD_\Pi D_\Pi^*} = a_0''(I)(D_\Pi^* D_\Pi) - a_0''(I)(D_\Pi D_\Pi^*), \tag{5.12}$$

we find:

Corollary 5.5. *In the situation of Corollary* 5.4,

$$\operatorname{index} D_\Pi = -\tfrac{1}{2}\eta_{C,V_0'} + \text{ local contributions}. \tag{5.13}$$

For the case where $\Pi = \Pi_\geq(A)$ (cf. (4.34)), (5.13) is known from [APS75], and (5.9) is known from [G92]; for $\Pi = \Pi_>(A) + \Pi_{V_0'}$ in the product case, cf. [GS96, Cor. 3.7]. We believe that it is an interesting new result that for rather general projections, the non-locality depends only on the projection, not the interior operator, in this sense.

Now we turn to cases with selfadjointness properties. We are here both interested in truly selfadjoint product cases and in nonproduct cases where D is principally selfadjoint at X'. Along with D we consider the operator of product type D_0 defined by

$$D_0 = \sigma(\partial_{x_n} + A) \text{ on } X_c, \text{ so that } D = D_0 + \sigma(x_n A_{11} + A_{10}). \tag{5.14}$$

In addition to the requirements that σ be unitary and A be selfadjoint, we now assume that $E_1 = E_2$ and that D_0 is formally selfadjoint on X_c when this is provided with the "product" volume element $v(x',0)dx'dx_n$; this means that

$$\sigma^2 = -I, \quad \sigma A = -A\sigma. \tag{5.15}$$

D_0 can always be extended to an elliptic operator on X (e.g. by use of D); let us denote the extension D_0 also. If the extension is selfadjoint, we call this a *selfadjoint product case*.

When Π is an orthogonal projection in $L_2(E_1') = L_2(E_2')$, it is well-posed for D if and only if it is so for D_0. For D_0 in selfadjoint product cases, some choices of Π will lead to selfadjoint realizations $D_{0,\Pi}$, namely (in view of (5.1)) those for which

$$\Pi = -\sigma\Pi^{\perp}\sigma. \tag{5.16}$$

The properties (5.15) and (5.16) imply (4.37) with $P' = A^2$, $\Pi_1 = \Pi$, so we can apply Theorem 4.11 to $D_\Pi^* D_\Pi$ (and $D_{0,\Pi}^2$).

As pointed out in the appendix A.1 of Douglas and Wojciechowski [DW91], it follows from Ch. 17 (by Palais and Seeley) of Palais [P65] that when (5.15) holds and n is odd, there exists a subspace L of $V_0(A)$ such that $\sigma L \perp L$ and $V_0(A) = L \oplus \sigma L$. Müller showed in [M94] (cf. (1.5)ff. and Prop. 4.26 there) that such L can be found in any dimension. Denoting the orthogonal projection onto L by Π_L, we have that

$$\Pi_+ = \Pi_>(A) + \Pi_L \tag{5.17}$$

satisfies (5.16). The projections $\Pi(\theta)$ introduced by Brüning and Lesch [BL99] likewise satisfy (5.16).

We here conclude from Theorem 4.11:

Corollary 5.6. *In addition to the assumptions of Corollary* 5.3, *assume that* $E_1 = E_2$ *and that* (5.15), (5.16) *hold. Then in* (2.43), (3.4), (3.9) *for* $P_T = D_\Pi^* D_\Pi$, $\tilde{a}_0''(I)$ $(= a_0''(I))$ *is locally determined. Equivalently, the value of* $\zeta(D_\Pi^* D_\Pi, s)$ *at zero satisfies:*

$$\zeta(D_\Pi^* D_\Pi, 0) = -\dim V_0(D_\Pi) + \text{ local contributions.} \tag{5.18}$$

We use this to show for the zeta function:

Theorem 5.7. *In addition to the hypotheses of Corollary* 5.6, *assume that*

$$\Pi = \overline{\Pi} + \mathcal{S}, \tag{5.19}$$

where $\overline{\Pi}$ *is a fixed well-posed projection satisfying* (5.16) *and* $\mathcal{S}$ *is of order* $\le -n$. *(*$\overline{\Pi}$ *can in particular be taken as* Π_+ *in* (5.17) *or* $\Pi(\theta)$ *from* [BL99].*)*

Then the $\tilde{a}_0''$*-terms (and* a_0''*-terms) in* (2.43), (3.4), (3.9) *for* $D_{\overline{\Pi}}$ *and* D_Π *are the same,*

$$\tilde{a}_0''(I)(D_\Pi^* D_\Pi) = \tilde{a}_0''(I)(D_{\overline{\Pi}}^* D_{\overline{\Pi}}), \tag{5.20}$$

so

$$\zeta(D_\Pi^* D_\Pi, 0) + \dim V_0(D_\Pi) = \zeta(D_{\overline{\Pi}}^* D_{\overline{\Pi}}, 0) + \dim V_0(D_{\overline{\Pi}}), \tag{5.21}$$

and in particular,

$$\zeta(D_\Pi^* D_\Pi, 0) = \zeta(D_{\overline{\Pi}}^* D_{\overline{\Pi}}, 0) \pmod{\mathbb{Z}}. \tag{5.22}$$

Proof. We shall combine the fact that $\tilde{a}_0''(I)$ is locally determined with order considerations. Let

$$R_{\overline{T}}^m = (D_{\overline{\Pi}}^* D_{\overline{\Pi}} - \lambda)^{-m}, \quad R_T^m = (D_\Pi^* D_\Pi - \lambda)^{-m}. \tag{5.23}$$

Note that they have the same pseudodifferential part $(D^*D - \lambda)_+^{-m}$, so their difference $R_{\overline{T}}^m - R_T^m$ is a singular Green operator. It is shown in [G01′, proof of Th. 1] that when $\mathcal{S}$ is of order $-n$, the ψdo $\operatorname{tr}_n(R_{\overline{T}}^m - R_T^m)$ on X' has symbol in $S^{-m-n,-m,0} \cap S^{-n,-2m,0}$. The total order is $-n-2m$, so the highest degree of the homogeneous terms in the symbol is $-n-2m$. As noted in Remark 4.4, the local contribution to the terms with index $k=0$ in the trace expansion of this difference comes from homogeneous terms of degree $1-n-2m$ (recall that $\dim X' = n-1$), so since the terms consist purely of local contributions, they must vanish. This shows (5.20), and the other consequences are immediate. □

In a language frequently used in this connection, the statement means that $\zeta(D_\Pi^* D_\Pi, 0) + \dim V_0(D_\Pi)$ *is constant on the Grassmannian of ψdo projections satisfying* (5.16) *and differing from* $\overline{\Pi}$ *by a term of order* $\leq -\dim X$.

When D equals D_0 in a selfadjoint product case, Wojciechowski shows a result like this in [W99, Sect. 3] for perturbations of Π_+ of order $-\infty$, assuming that D_{0,Π_+} and $D_{0,\Pi}$ are invertible. The non-invertible case is treated by Y. Lee in the appendix of [PW02]; he shows moreover that $\zeta(D_{0,\Pi_+}^2, 0) + \dim V_0(D_{0,\Pi_+}) = 0$, so we conclude that $\zeta(D_{0,\Pi}^2, 0) + \dim V_0(D_{0,\Pi}) = 0$ when $\Pi = \Pi_+ + \mathcal{S}$.

We can also discuss the eta function $\eta(D_\Pi, s) = \operatorname{Tr}(D(D_\Pi^* D_\Pi)^{-\frac{s+1}{2}})$, extended meromorphically as in (3.12). First we conclude from Theorem 4.11:

Corollary 5.8. *Assumptions of Corollary* 5.6. *In* (2.35), (3.5), (3.11)–(3.12) *for* $P_T = D_\Pi^* D_\Pi$, $D_1 = D$, *one has that* $\tilde{a}_0'(D) = a_0'(D) = 0$, *and* $\tilde{a}_0''(D)$ $(= a_0''(D))$ *is locally determined.*

In other words, the double pole of $\eta(D_\Pi, s)$ *at* 0 *vanishes and the residue at* 0 *is locally determined.*

(It may be observed that when (5.15) holds, the entries in the second line of (4.38) vanish identically, since $\operatorname{Tr}_{X'}(\sigma\partial_\lambda^{m-1}(P'-\lambda)^{-\frac{1}{2}}) = 0$.)

It is remarkable here that the hypotheses, besides (5.15)–(5.16), only contain requirements on principal symbols (the well-posedness of Π for D and the commutativity of the principal symbols of Π and A^2). So the result implies in particular that *the vanishing of the double pole of the eta function is invariant under perturbations of* Π *of order* -1 (respecting (5.16)). Earlier results have dealt with perturbations of order $-\infty$ [W99] or order $-n$ [G01′].

Now consider the simple pole of $\eta(D_\Pi, s)$ at 0. Here we can generalize the result of Wojciechowski [W99] on the regularity of the eta function after a perturbation of order $-\infty$, to perturbations of order $-n$ of general $\overline{\Pi}$:

Theorem 5.9. *Assumptions of Theorem* 5.7.

In (2.35), (3.5), (3.11)–(3.12) *with* $D_1 = D$, *the* $\tilde{a}_0''$*-terms (and* a_0''*-terms) for* $D_{\overline{\Pi}}^* D_{\overline{\Pi}}$ *and* $D_\Pi^* D_\Pi$ *are the same:*

$$\tilde{a}_0''(D)(D_\Pi^* D_\Pi) = \tilde{a}_0''(D)(D_{\overline{\Pi}}^* D_{\overline{\Pi}}); \tag{5.24}$$

in other words, $\operatorname{Res}_{s=0} \eta(D_\Pi, s) = \operatorname{Res}_{s=0} \eta(D_{\overline{\Pi}}, s)$.

In particular, if $\tilde{a}_0''(D)(D_{\overline{\Pi}}^* D_{\overline{\Pi}}) = 0$ *(this holds for* Π_+ *and for certain* $\Pi(\theta)$ *if* D *equals* D_0 *in a selfadjoint product case), then* $\tilde{a}_0''(D)(D_\Pi^* D_\Pi) = 0$, *i.e., the eta function* $\eta(D_\Pi, s)$ *is regular at* 0.

Proof. As in the proof of Theorem 5.7, we combine the fact that $\tilde{a}_0''(D)$ is locally determined with order considerations. Consider $DR_{\overline{T}}^m$ and DR_T^m, cf. (5.23). Since they have the same pseudodifferential part $D(D^*D - \lambda)_+^{-m}$, their difference $DR_{\overline{T}}^m - DR_T^m$ is a singular Green operator. It is shown in [G01′, proof of Th. 1] that when $\mathcal{S}$ is of order $-n$, the ψdo $\operatorname{tr}_n(DR_{\overline{T}}^m - DR_T^m)$ on X' has symbol in $S^{-m-n,1-m,0} \cap S^{-n,1-2m,0}$. The total order is $1-n-2m$, so the highest degree of the homogeneous terms in the symbol is $1-n-2m$. Now the local contribution to the terms with index $k=0$ in the trace expansion of this difference comes from homogeneous terms of degree $2-n-2m$ (cf. Remark 4.4), so since the terms contain only local contributions, they must vanish. This shows (5.24).

In particular, $\tilde{a}_0''(D)(D_\Pi^* D_\Pi)$ vanishes if $\tilde{a}_0''(D)(D_{\overline{\Pi}}^* D_{\overline{\Pi}})$ does so; then the eta function for D_Π is regular at 0. The eta regularity for the case $\overline{\Pi} = \Pi_+$, D equal to D_0 and selfadjoint on X with product volume element on X_c, was shown in [DW91] under the assumptions n odd and D compatible; this was extended to general n and not necessarily compatible D in Müller [M94]. It was shown for certain $\Pi(\theta)$ in [BL99, Th. 3.12]. □

In a frequently used terminology, the theorem shows that the residue of the eta function is constant on the Grassmannian of ψdo projections satisfying (5.16) and differing from $\overline{\Pi}$ by a term of order $\leq -\dim X$. We do not expect that the order can be lifted further in general.

The result on the regularity of the eta function at $s=0$ for $(-n)$-order perturbations of the product case with $\overline{\Pi} = \Pi_+$ has been obtained independently by Yue Lei [L02] at the same time as our result, by another analysis based on heat operator formulas.

The above results on the vanishing of the eta residue are concerned with situations where D equals D_0 in a selfadjoint product case. However, the fact from Corollary 5.8 that $\operatorname{Res}_{s=0} \eta(D_\Pi, s)$ is locally determined also in suitable non-product cases, should facilitate the calculation of the residue then. For example, if $D = D_0 + x_n^{n+1} A_{31}$ on X_c with a first-order tangential differential operator A_{31}, and the volume element satisfies $\partial_{x_n}^j v(x', 0) = 0$ for $1 \leq j \leq n$, then by [G02, proof of Th. 3.11], the local terms with index $k \leq 0$ in the difference between the resolvent powers are determined entirely from the interior operators D and D_0. Then when n is even, the contributions to $k=0$ vanish simply because of odd parity in ξ; this gives examples where the eta function is regular at 0 in a non-product situation.

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