COMPLEX POWERS OF RESOLVENTS OF PSEUDODIFFERENTIAL OPERATORS

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Abstract. In a work from 1995 by G. Grubb and R. Seeley, a calculus of weakly parametric pseudodifferential operators on closed manifolds was introduced and used to obtain complete asymptotic expansions of traces of resolvents and heat operators associated with the Atiyah-Patodi-Singer problem. The present paper establishes a generalization allowing not only anisotropic homogeneity in the symbols, but also including symbols of noninteger, even complex, powers of $A-\lambda$. The operators in the calculus have complete asymptotic trace expansions in the parameter (when of trace class), with polynomial and logarithmic terms.

1. Introduction

[1]For a classical (i.e., one-step polyhomogeneous) pseudodifferential operator A of positive order m on a closed manifold X, with a suitable ellipticity property, it is known that the resolvent powers $(A-\lambda)^{-N}$ composed with a classical ψdo B of order $m' \in \mathbb{R}$ (with N so large that $B(A-\lambda)^{-N}$ is trace-class) have trace expansions

$$\operatorname{Tr} B(A-\lambda)^{-N} \sim \sum_{j\in\mathbb{N}} c_j \lambda^{\frac{n+m'-j}{m}-N} + \sum_{l\in\mathbb{N}} (c'_l \log\lambda + c''_l)\lambda^{-N-l}\ ; \quad (1.1)$$

for $\lambda \to \infty$ in a sector of $\mathbb{C}$. This was shown for integer m in Grubb and Seeley [GS95, Th. 2.7] by use of a calculus of weakly polyhomogeneous symbols depending on the parameter λ. Loya introduced in [L01] a slightly different symbol calculus which allows also noninteger m. (The quoted works moreover treat parameter-dependent operators on manifolds with boundary, resp. on manifolds with singularities such as edges and corners; in the present note we shall just be concerned with nonsingular manifolds without boundary, the "interior" calculus.)

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Whereas [GS95] reduces the study to symbols (poly-)homogeneous in (ξ, μ) with $\mu = \lambda^{\frac{1}{m}}$, [L01] leaves λ as it stands, working with anisotropic homogeneity (quasi-homogeneity) in (ξ, λ); this gives a more direct check on which logarithmic terms that appear in the trace expansions. In any case, the logarithms only appear together with integer powers of λ.

Inspired by reading [L01], we found that the calculus of [GS95] can be generalized in a convenient way to allow not only anisotropic homogeneity in the symbols, but also to include symbols of noninteger, even complex, powers of $A - \lambda$; this is the subject of the present paper.

The crucial property that assures asymptotic expansions as in (1.1) is a certain expansion of the symbol in decreasing powers of λ for $\lambda \to \infty$, corresponding to a Taylor expansion in $z = 1/\lambda$ of a suitably reduced symbol, for $z \to 0$ in a sector. This was handled in [GS95] by imposing uniform symbol estimates for $z \to 0$, a point of view that is well suited to the treatment of inverses, and this is a basic point of view in the present generalization too.

A prominent application of the new calculus is the deduction of trace expansions for operators $B(A - \lambda)^{-s}$ with noninteger or even *complex* values of s:

$$\operatorname{Tr} B(A-\lambda)^{-s} \sim \sum_{j \in \mathbb{N}} c_j \lambda^{\frac{n+m'-j}{m} - s} + \sum_{l \in \mathbb{N}} (c'_l \log \lambda + c''_l) \lambda^{-s-l}. \tag{1.2}$$

Note that the logarithmic terms can here appear together with noninteger powers.

Plan of the paper: The new symbol classes are introduced in Section 2, where they are shown to contain complex powers of resolvent symbols for classical ψdo's and to have the desired Taylor expansion property. In Section 3 it is shown how kernels of the operators have diagonal expansions in the parameter with power terms and power-log terms. In Section 4 it is shown that the complex powers of resolvents belong to the calculus, and asymptotic trace formulas are deduced.

2. Parameter-dependent symbols

In the following, we use the notation

$$\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}, \quad \mathbb{N} = \{0, 1, 2, \dots\}.$$

Moreover, $\dot{\leq}$ means "$\leq$ a constant, independent of the space variable, times"; similarly $\dot{\geq}$ means "$\geq$ a constant times"; and $\doteq$ means that both $\dot{\leq}$ and $\dot{\geq}$ hold. The constants vary from case to case.

We shall use the word *sector* to denote a subset Γ of the complex plane of the form $\{re^{i\omega} \mid r>0,\, \omega\in I\}$ for some interval I strictly included in a period interval $[a, a+2\pi]$; Γ will be called a *ray*, also denoted Γ_θ, when I consists of one point $\{\theta\}$. We say that Γ is closed when it is closed as a subset of $\mathbb{C}\setminus\{0\}$. Since Γ always lies in the complement of some ray, μ^δ can be defined holomorphically for all $\mu\in\Gamma^\circ$, $\delta\in\mathbb{C}$. — In some definitions below one could take $\Gamma=\mathbb{C}\setminus\{0\}$, namely when only integer powers occur, but since we are aiming for a calculus including fractional powers, we leave this aspect out; it is not hard to include in specific situations.

Recall the usual definition of spaces of pseudodifferential symbols of order m: $S^m(\mathbb{R}^\nu\times\mathbb{R}^n)$ is the space of C^∞ functions $p(x,\xi)$ with $(x,\xi)\in\mathbb{R}^\nu\times\mathbb{R}^n$ such that

$$|\partial_x^\beta\partial_\xi^\alpha p(x,\xi)| \dot{\le} \langle\xi\rangle^{m-|\alpha|}, \text{ for all } \alpha,\beta.$$

The parameter-dependent generalization set up in [GS95] allowed showing complete asymptotic expansions of suitable families of trace-class operators in powers and logarithms of μ. [G01] introduced a generalization including powers of $|(\xi,\mu)|$ in the definition, needed in the treatment of boundary value problems. We now define an anisotropic variant. (Here the sub-σ plays a different role than the sub-δ used in [G01].)

Definition 2.1. *Let $m\in\mathbb{R}$, $\delta\in\mathbb{C}$, $\sigma\in\mathbb{R}_+$ and let Γ be a sector in $\mathbb{C}$.*

The space $S^{m,0}_\sigma(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)$ consists of the functions $p(x,\xi,\mu)\in C^\infty(\mathbb{R}^\nu\times\mathbb{R}^n\times\Gamma)$ with values in $\mathbb{C}$ or in complex $N\times N$-matrices, that are holomorphic in $\mu\in\mathring{\Gamma}$ for $|(\xi,\mu)|\ge\varepsilon$ (some $\varepsilon>0$ depending on p) and satisfy as functions of $z=\frac{1}{\mu}$, for all $j\in\mathbb{N}$,

$$\partial_z^j p(x,\xi,\tfrac{1}{z})\in S^{m+\sigma j}(\mathbb{R}^\nu\times\mathbb{R}^n),\ \text{for } \tfrac{1}{z}\in\Gamma,$$

with uniform estimates for $|z|\le 1$, $\frac{1}{z}$ in closed subsectors of Γ.

Moreover, we set

$$S^{m,\delta}_\sigma(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)=\mu^\delta S^{m,0}_\sigma(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma).$$

We often abbreviate $S^{m,\delta}_\sigma(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)$ to $S^{m,\delta}_\sigma(\Gamma)$.

When $m\in\mathbb{Z}$, $\sigma=1$ and $\delta=d\in\mathbb{Z}$, this is essentially the symbol space $S^{m,d}(\Gamma)$ defined in [GS95].

In detail, the estimates to be satisfied by functions p in $S^{m,\delta}_\sigma(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)$ are, with $z=\frac{1}{\mu}$,

$$\left|\partial_x^\beta\partial_\xi^\alpha\partial_z^j\left(z^\delta p(x,\xi,\tfrac{1}{z})\right)\right| \dot{\le} \langle\xi\rangle^{m-|\alpha|+\sigma j}, \tag{2.1}$$

uniformly for $|z| \le 1$, $\frac{1}{z}$ in closed subsectors Γ' of Γ. As usual, such function spaces are provided with the Fréchet topologies defined by the associated systems of seminorms:

$$\sup_{x\in\mathbb{R}^\nu,\, \xi\in\mathbb{R}^n,\, \frac{1}{z}\in\Gamma',\, |z|\le 1} \langle\xi\rangle^{-m+|\alpha|-\sigma j} |\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^\delta p(x,\xi,\tfrac{1}{z}))|.$$

Observe that

$$S^m(\mathbb{R}^\nu\times\mathbb{R}^n) \subset S_\sigma^{m,0}(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma) \text{ for any } \sigma>0, \text{ any } \Gamma, \tag{2.2}$$

when the symbol $p(x,\xi) \in S^m$ is considered as constant in μ. The number σ indicating the anisotropy between μ and ξ will be fixed in the calculations. For any $m \le m'$ and $k \in \mathbb{N}$ we have the inclusions

$$S_\sigma^{m,\delta}(\Gamma) \subset S_\sigma^{m',\delta}(\Gamma) \quad \text{and} \quad S_\sigma^{m,\delta}(\Gamma) \subset S_\sigma^{m,\delta+k}(\Gamma).$$

We denote

$$\bigcap_{m\in\mathbb{R}} S_\sigma^{m,\delta}(\Gamma) = S_\sigma^{-\infty,\delta}(\Gamma), \quad \bigcup_{m\in\mathbb{R}} S_\sigma^{m,\delta}(\Gamma) = S_\sigma^{\infty,\delta}(\Gamma).$$

Remark 2.2. Note that when Γ is a single ray, there is no holomorphy requirement. In fact, the requirement of holomorphy could be taken out as a side condition and ∂_z replaced by $\partial_{|z|}$ (as in [G01]); it is not necessary for the main properties of the calculus.

It can be convenient in some applications to introduce powers of $[\xi]^\sigma + \mu$ in the definition as in [G01] (where $[\xi]$ is a smooth positive version of $|\xi|$), to build in the properties of such factors shown below in Theorem 2.8. We leave this complication (which gives rise to a third upper index) out of the presentation here.

As in the case of ψdo symbols treated in [GS95], there are some straightforward rules for application of derivatives and multiplication by $z^{\delta'}$ to these symbol spaces:

Lemma 2.3. *The following mappings are continuous:*

(i) $\partial_x^\beta \partial_\xi^\alpha\colon\ S_\sigma^{m,\delta}(\Gamma) \to S_\sigma^{m-|\alpha|,\delta}(\Gamma)$,
(ii) $\partial_z^j\colon\ S_\sigma^{m,0}(\Gamma) \to S_\sigma^{m+\sigma j,0}(\Gamma)$,
(iii) $z^{\delta'}\colon\ S_\sigma^{m,\delta}(\Gamma) \xrightarrow{\sim} S_\sigma^{m,\delta-\delta'}(\Gamma)$,

for $\beta\in\mathbb{N}^\nu$, $\alpha\in\mathbb{N}^n$, $j\in\mathbb{N}$, $\delta'\in\mathbb{C}$.

(For nonzero δ, the rule for ∂_z involves the Leibniz formula.)

We also have the product rule:

Lemma 2.4. *When $p(x,\xi,\mu)$ is in $S_\sigma^{m,\delta}(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)$ and $q(x,\xi,\mu)$ is in $S_\sigma^{m',\delta'}(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)$, then the product pq is in $S_\sigma^{m+m',\delta+\delta'}(\mathbb{R}^\nu\times\mathbb{R}^n,\Gamma)$.*

Proof. Follows easily by the Leibniz rule. □

Lemma 2.5. *Let $a \in S^{m\sigma}(\mathbb{R}^\nu \times \mathbb{R}^n)$ for some $m \in \mathbb{N}$. There are estimates*

$$|\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^m a + 1)| \dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{m-j} \langle\xi\rangle^{\sigma j - |\alpha|}$$

for any choice of $\beta \in \mathbb{N}^\nu$, $\alpha \in \mathbb{N}^n$, $j \in \mathbb{N}$.

Proof. For $(\beta, \alpha, j) = 0$ we have, since $m \geq 0$,

$$|z^m a + 1| \dot{\leq} |z|^m \langle\xi\rangle^{m\sigma} + 1 \doteq (|z|\langle\xi\rangle^\sigma + 1)^m.$$

Now assume that $(\beta, \alpha, j) \neq 0$. For $j \leq m$ we get

$$\begin{aligned} |\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^m a + 1)| &\doteq |z^{m-j} \partial_x^\beta \partial_\xi^\alpha a| \dot{\leq} |z|^{m-j} \langle\xi\rangle^{m\sigma - |\alpha|} \\ &= (|z|\langle\xi\rangle^\sigma)^{m-j} \langle\xi\rangle^{\sigma j - |\alpha|} \leq (|z|\langle\xi\rangle^\sigma + 1)^{m-j} \langle\xi\rangle^{\sigma j - |\alpha|}. \end{aligned}$$

For $j > m$ we have $\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^m a + 1) = 0$. This completes the proof. □

In the following theorem and lemma we consider a square-matrix formed symbol $a(x, \xi) \in S^\sigma(\mathbb{R}^\nu \times \mathbb{R}^n)$, a closed sector Γ in $\mathbb{C}$ and an integer $m \in \mathbb{N}$, such that

$$\begin{gathered} a^m + \mu^m \text{ is invertible with an estimate} \\ |(a^m + \mu^m)^{-1}| \dot{\leq} (\langle\xi\rangle^{m\sigma} + |\mu|^m)^{-1} \end{gathered} \tag{2.3}$$

for all $(x, \xi, \mu) \in \mathbb{R}^\nu \times \mathbb{R}^n \times (\Gamma \cup \{0\})$.

For $m = 1$, when a is the principal part of a polyhomogeneous symbol $\tilde{a}$, this is the condition for *uniform parameter-ellipticity* of $p(x, \xi, \rho) = \tilde{a}(x, \xi) + \rho^\sigma$, as defined in [G96, Def. 2.1.2 2°]. It holds locally in x when the eigenvalues of a avoid the sector where $-\rho^\sigma$ runs.

Theorem 2.6. *Let Γ be a closed sector in $\mathbb{C}$, let $m \in \mathbb{N}$ and $a(x, \xi)$ be a square-matrix formed symbol in $S^\sigma(\mathbb{R}^\nu \times \mathbb{R}^n)$ satisfying* (2.3)*. Write $z := \mu^{-1}$. For each $\delta \in \mathbb{C}$ there are estimates*

$$|\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^m a(x, \xi)^m + 1)^\delta| \dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{m \operatorname{Re}\delta - j} \langle\xi\rangle^{\sigma j - |\alpha|}, \tag{2.4}$$

$$|\partial_x^\beta \partial_\xi^\alpha \partial_z^j [z^m (z^m a(x, \xi)^m + 1)^\delta]| \dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{m(\operatorname{Re}\delta + 1) - j} \langle\xi\rangle^{\sigma(j-m) - |\alpha|} \tag{2.5}$$

for all $\beta \in \mathbb{N}^\nu$, $\alpha \in \mathbb{N}^n$, $j \in \mathbb{N}$. For each $N \in \mathbb{N}$ there are estimates

$$|\partial_x^\beta \partial_\xi^\alpha \partial_z^j (a(x, \xi)^m + z^{-m})^{-N}| \dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{-j} \langle\xi\rangle^{\sigma(j - mN) - |\alpha|} \tag{2.6}$$

for all $\beta \in \mathbb{N}^\nu$, $\alpha \in \mathbb{N}^n$, $j \in \mathbb{N}$.

Proof. Set $p := z^m a^m + 1$ and $f_{(\beta,\alpha,j),t}(x, \xi, z) := (|z|\langle\xi\rangle^\sigma + 1)^{t-j} \langle\xi\rangle^{\sigma j - |\alpha|}$. Then (2.4) can be written

$$|\partial^\gamma p^\delta| \dot{\leq} f_{\gamma, m \operatorname{Re}\delta}\ , \quad \gamma \in \mathbb{N}^{\nu+n+1}.$$

Here the sign $\dot{\le}$ indicates an estimate satisfied for all $(x,\xi,z) \in \mathbb{R}^\nu \times \mathbb{R}^n \times \Gamma^{-1}$. Since

$$|p(x,\xi,z)^{-1}|^{-1} \le \rho(p(x,\xi,z)^{-1})^{-1} \quad \text{and} \quad \rho(p(x,\xi,z)) \le |p(x,\xi,z)|$$

where ρ denotes spectral radius, the spectrum of $p(x,\xi,z)$ is in the interior of the large circle with radius $R(x,\xi,z) := 2|p(x,\xi,z)|$, and in the exterior of the small circle with radius $r(x,\xi,z) := \frac{1}{2}|p(x,\xi,z)^{-1}|^{-1}$. Moreover, since $a^m + \mu^m$ is invertible for μ in a sector, the spectrum of $p - \lambda = z^m(a^m + \mu^m(1-\lambda))$ avoids $\mathbb{R}_-$. Thus we can write

$$\partial^\gamma p^\delta = \frac{i}{2\pi} \int_{\mathcal{L}} \lambda^\delta \partial^\gamma (p-\lambda)^{-1}\, d\lambda, \tag{2.7}$$

where $\mathcal{L} = \mathcal{L}(x,\xi,z)$ is the loop:

$$\begin{aligned}\mathcal{L} = \{Re^{i\theta} \mid -\pi \le \theta \le \pi\} &+ \{\rho e^{i\pi} \mid R \ge \rho \ge r\} \\ &+ \{re^{i\theta} \mid \pi \ge \theta \ge -\pi\} + \{\rho e^{-i\pi} \mid r \le \rho \le R\}.\end{aligned}$$

Since $a \in S^\sigma$, we have

$$|p| \mathrel{\dot{\le}} |z|^m \langle\xi\rangle^{m\sigma} + 1 \doteq (|z|\langle\xi\rangle^\sigma + 1)^m = f_{0,m},$$

and by (2.3) we get

$$\begin{aligned}|p^{-1}| = |z|^{-m}|(a^m + \mu^m)^{-1}| &\mathrel{\dot{\le}} |z|^{-m}(\langle\xi\rangle^{m\sigma} + |\mu|^m)^{-1} \\ &\doteq |z|^{-m}(\langle\xi\rangle^\sigma + |\mu|)^{-m} = f_{0,-m}.\end{aligned}$$

Thus $r \doteq R \doteq f_{0,m}$, from which we see that the length $|\mathcal{L}|$ of the loop $\mathcal{L}$ satisfies $|\mathcal{L}| \doteq f_{0,m}$, and that we have $|\lambda^\delta| \doteq f_{0,m}^{\operatorname{Re}\delta} = f_{0,m\operatorname{Re}\delta}$ for $\lambda \in \mathcal{L}$. We now establish an estimate

$$|(p-\lambda)^{-1}| \mathrel{\dot{\le}} f_{0,-m}$$

for λ on the four parts of the loop $\mathcal{L}$. For $|\lambda| = R$ we use the formula $(\lambda - p)^{-1} = \lambda^{-1}\sum_{k=0}^\infty (\lambda^{-1}p)^k$ to get

$$|(p-\lambda)^{-1}| \le R^{-1}\sum_{k=0}^\infty 2^{-k} \doteq R^{-1} \doteq f_{0,-m},$$

and for $|\lambda| = r$ we use $(\lambda - p)^{-1} = p^{-1}\sum_{k=0}^\infty (\lambda p^{-1})^k$ to get

$$|(p-\lambda)^{-1}| \le \tfrac{1}{2} r^{-1}\sum_{k=0}^\infty 2^{-k} \doteq r^{-1} \doteq f_{0,-m}.$$

For $\lambda = \varrho e^{\pm i\pi}$, $\varrho > 0$, we have

$$\begin{aligned}|(p-\lambda)^{-1}| &= |z|^{-m}|(a^m + \mu^m(1-\lambda))^{-1}| \\ &\mathrel{\dot{\le}} |z|^{-m}(\langle\xi\rangle^{m\sigma} + |\mu^m(1-\lambda)|)^{-1} \\ &\le |z|^{-m}(\langle\xi\rangle^{m\sigma} + |\mu|^m)^{-1} \\ &\doteq f_{0,-m}.\end{aligned}$$

When $|\gamma| = 1$, one has that $\partial^\gamma(p-\lambda)^{-1} = -(p-\lambda)^{-1}\partial^\gamma p\,(p-\lambda)^{-1}$, so in general, $\partial^\gamma(p-\lambda)^{-1}$ is a linear combination of products of the form

$$(p-\lambda)^{-1}\partial^{\gamma^1}p(p-\lambda)^{-1}\partial^{\gamma^2}p(p-\lambda)^{-1}\dots\partial^{\gamma^k}p(p-\lambda)^{-1}$$

with $k \in \mathbb{N}$, $\gamma^l \in \mathbb{N}^{\nu+n+1}$ and $\gamma^1 + \dots + \gamma^k = \gamma$. Lemma 2.5 gives that $|\partial^{\gamma^l}p| \mathrel{\dot{\leq}} f_{\gamma^l,m}$, so from

$$\begin{aligned}&|(p-\lambda)^{-1}\partial^{\gamma^1}p(p-\lambda)^{-1}\dots\partial^{\gamma^k}p(p-\lambda)^{-1}|\\ &\quad\mathrel{\dot{\leq}} f_{0,-m}f_{\gamma^1,m}f_{0,-m}\cdots f_{\gamma^k,m}f_{0,-m}\\ &\quad= f_{\gamma,-m}\end{aligned}$$

we obtain that

$$|\partial^\gamma(p-\lambda)^{-1}| \mathrel{\dot{\leq}} f_{\gamma,-m}. \tag{2.8}$$

Thus we arrive at the estimate

$$|\partial^\gamma p^\delta| \mathrel{\dot{\leq}} |\mathcal{L}|\sup_{\lambda\in\mathcal{L}}|\lambda^\delta\partial^\gamma(p-\lambda)^{-1}| \mathrel{\dot{\leq}} f_{0,m}f_{0,m\operatorname{Re}\delta}f_{\gamma,-m} = f_{\gamma,m\operatorname{Re}\delta}$$

which proves (2.4).

To prove (2.6), write $(a^m + z^{-m})^{-N} = z^{mN}p^{-N}$ and use the Leibniz formula to see that

$$\partial_x^\beta\partial_\xi^\alpha\partial_z^j\left(z^{mN}p^{-N}\right) = \sum_{j'\leq\min\{j,mN\}} c_{j'}\,z^{mN-j'}\partial_x^\beta\partial_\xi^\alpha\partial_z^{j-j'}p^{-N},$$

where

$$\begin{aligned}&|z^{mN-j'}\partial_x^\beta\partial_\xi^\alpha\partial_z^{j-j'}p^{-N}|\\ &\quad\mathrel{\dot{\leq}} |z|^{mN-j'}(|z|\langle\xi\rangle^\sigma+1)^{-mN-(j-j')}\langle\xi\rangle^{\sigma(j-j')-|\alpha|}\\ &\quad= \left(\frac{|z|\langle\xi\rangle^\sigma}{|z|\langle\xi\rangle^\sigma+1}\right)^{mN-j'}(|z|\langle\xi\rangle^\sigma+1)^{-j}\langle\xi\rangle^{\sigma(j-mN)-|\alpha|}\\ &\quad\leq (|z|\langle\xi\rangle^\sigma+1)^{-j}\langle\xi\rangle^{\sigma(j-mN)-|\alpha|}\end{aligned}$$

since $j' \leq mN$. A similar treatment of $z^m(p-\lambda)^{-1}$ gives

$$|\partial^\gamma[z^m(p-\lambda)^{-1}]| \mathrel{\dot{\leq}} (|z|\langle\xi\rangle^\sigma+1)^{-j}\langle\xi\rangle^{\sigma(j-m)-|\alpha|}. \tag{2.9}$$

Then

$$\partial^\gamma(z^m p^\delta) = \frac{i}{2\pi}\int_{\mathcal{L}}\lambda^\delta\partial^\gamma[z^m(p-\lambda)^{-1}]\,d\lambda$$

can be estimated similarly to (2.7), implying (2.5). □

The above estimates generalize those obtained in [GS95, Th. 1.17]. The present proof is different and does not need homogeneity of a in ξ (which was a prerequisite for the proof in [GS95]).

Lemma 2.7. *Let Γ be a closed sector in $\mathbb{C}$, let $a, b \in S^\sigma(\mathbb{R}^\nu \times \mathbb{R}^n)$, $m, m' \in \mathbb{N}$, and assume that $a^m + \mu^m$ and $b^{m'} + \mu^{m'}$ satisfy* (2.3)*. Then*

$$(a^m + \mu^m)^\delta (b^{m'} + \mu^{m'})^{\delta'} \in S_\sigma^{0, m\delta + m'\delta'}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma),$$

for all $\delta, \delta' \in \mathbb{C}$ with $m \operatorname{Re}\delta + m' \operatorname{Re}\delta' \leq 0$. If $\operatorname{Re}\delta \geq -1$, then

$$(a^m + \mu^m)^\delta \in S_\sigma^{\sigma m \operatorname{Re}\delta, m + m\delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma).$$

Proof. By (2.4) in Theorem 2.6, we have:

$$\begin{aligned} &|\partial_x^\beta \partial_\xi^\alpha \partial_z^j (z^m a^m + 1)^\delta \partial_x^{\beta'} \partial_\xi^{\alpha'} \partial_z^{j'} (z^{m'} b^{m'} + 1)^{\delta'}| \\ &\quad \dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{m \operatorname{Re}\delta - j} \langle\xi\rangle^{\sigma j - |\alpha|} (|z|\langle\xi\rangle^\sigma + 1)^{m' \operatorname{Re}\delta' - j'} \langle\xi\rangle^{\sigma j' - |\alpha'|} \\ &\quad = (|z|\langle\xi\rangle^\sigma + 1)^{m \operatorname{Re}\delta + m' \operatorname{Re}\delta' - (j + j')} \langle\xi\rangle^{\sigma(j+j') - |\alpha + \alpha'|} \\ &\quad \leq \langle\xi\rangle^{\sigma(j+j') - |\alpha+\alpha'|}. \end{aligned}$$

Then the claim follows when the Leibniz rule is applied to

$$\begin{aligned} &\partial_x^\beta \partial_\xi^\alpha \partial_z^j \left(\mu^{-(m\delta + m'\delta')} (a^m + \mu^m)^\delta (b^{m'} + \mu^{m'})^{\delta'} \right) \\ &\qquad = \partial_x^\beta \partial_\xi^\alpha \partial_z^j \left((z^m a^m + 1)^\delta (z^{m'} b^{m'} + 1)^{\delta'} \right). \end{aligned}$$

For the last statement, note that since $\operatorname{Re}\delta + 1 \geq 0$, one has by (2.5),

$$\begin{aligned} |\partial_x^\beta \partial_\xi^\alpha \partial_z^j [z^m (z^m a^m + 1)^\delta]| &\dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{m(\operatorname{Re}\delta + 1) - j} \langle\xi\rangle^{-\sigma m + \sigma j - |\alpha|} \\ &\dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{-j} \langle\xi\rangle^{\sigma m(\operatorname{Re}\delta + 1) - \sigma m + \sigma j - |\alpha|} \\ &\leq \langle\xi\rangle^{\sigma m \operatorname{Re}\delta + \sigma j - |\alpha|}, \end{aligned}$$

so $z^m (z^m a^m + 1)^\delta \in S_\sigma^{\sigma m \operatorname{Re}\delta, 0}$. The claim follows, since $(a^m + \mu^m)^\delta = \mu^{m + m\delta} z^m (z^m a^m + 1)^\delta$. □

We then find the following generalization of [GS95, Th. 1.17]:

Theorem 2.8. *Let Γ be a closed sector in $\mathbb{C}$ and assume that $a(x, \xi) \in S^\sigma(\mathbb{R}^\nu \times \mathbb{R}^n)$ satisfies* (2.3) *for $m = 1$. Let $\delta \in \mathbb{C}$ with $\operatorname{Re}\delta \leq 0$ and let $N \in \mathbb{N}$. Then*

$$(a + \mu)^{-N+\delta} \in S_\sigma^{0, -N+\delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \cap S_\sigma^{-\sigma N, \delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma), \tag{2.10}$$

$$(a + \mu)^{N+\delta} \in S_\sigma^{0, N+\delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) + S_\sigma^{\sigma N, \delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma). \tag{2.11}$$

If $\operatorname{Re}\delta \geq -1$, one has moreover that

$$(a + \mu)^\delta \in S_\sigma^{\sigma \operatorname{Re}\delta, 1+\delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma). \tag{2.12}$$

Proof. First we use a simple case of Lemma 2.7. With $m = 1$ and $\delta' = 0$, we find

$$(a + \mu)^\delta \in S_\sigma^{0, \delta}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma). \tag{2.13}$$

Replacing δ by $-N + \delta$ gives $(a + \mu)^{-N+\delta} \in S_\sigma^{0, -N+\delta}(\Gamma)$, showing part of (2.10). The statement in (2.12) follows directly from the last part of Lemma 2.7. (Note that for $-1 < \delta < 0$, the first upper index is negative and the second positive.)

Next, we shall show (2.11). Let μ run on a ray Γ_0 in Γ, then μ^N runs on Γ_0^N with argument θ, say. With $b = e^{i\frac{\theta}{N}}\langle\xi\rangle^\sigma$ we have

$$|b^N + \mu^N| = |e^{i\theta}\langle\xi\rangle^{\sigma N} + \mu^N| \dot{\geq} \langle\xi\rangle^{\sigma N} + |\mu|^N$$

not only in Γ_0, but in a sector $\Gamma_{00} \subset \Gamma$ corresponding to a neigbourhood of θ. Thus we can use Lemma 2.7 with $m = 1$, $\delta = N$, $m' = N$, $\delta' = -1$ to see that the product

$$p := (a+\mu)^N (b^N + \mu^N)^{-1}$$

is in $S_\sigma^{0,0}(\Gamma_{00})$. It follows that

$$(a+\mu)^N = p(e^{i\theta}\langle\xi\rangle^{\sigma N} + \mu^N) \in S_\sigma^{\sigma N,0}(\Gamma_{00}) + S_\sigma^{0,N}(\Gamma_{00}).$$

Now choose a finite covering of Γ with sectors $\Gamma_1, \dots, \Gamma_k$ such that $(a+\mu)^N$ on Γ_j equals $u_j + v_j \in S_\sigma^{\sigma N,0}(\Gamma_j) + S_\sigma^{0,N}(\Gamma_j)$, and choose a partition of unity $\varphi(\theta) = \varphi_1(\theta) + \ldots + \varphi_k(\theta)$ such that φ_j is a C^∞-function with support in the argument interval of Γ_j. Then $\varphi_j \in S_\sigma^{0,0}(\Gamma)$ when viewed as a function of (x, ξ, μ), so we get $(a+\mu)^N = u + v$ where $u = \sum_{j=1}^k \varphi_j u_j \in S_\sigma^{\sigma N,0}(\Gamma)$ and $v = \sum_{j=1}^k \varphi_j v_j \in S_\sigma^{0,N}(\Gamma)$. This shows (2.11) with $\delta = 0$. Since $(a+\mu)^{N+\delta} = (a+\mu)^N(a+\mu)^\delta$, where $(a+\mu)^\delta$ satisfies (2.13), we obtain the general form of (2.11) by use of the product rule Lemma 2.4.

To show the remaining part of (2.10), we use (2.6) in Theorem 2.6:

$$|\partial_x^\beta \partial_\xi^\alpha \partial_z^j (a + z^{-1})^{-N}| \dot{\leq} (|z|\langle\xi\rangle^\sigma + 1)^{-j}\langle\xi\rangle^{-\sigma N + \sigma j - |\alpha|} \leq \langle\xi\rangle^{-\sigma N + \sigma j - |\alpha|}$$

for $|\xi| \geq c$, which shows that

$$(a+\mu)^{-N} \in S_\sigma^{-\sigma N,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma). \tag{2.14}$$

Then $(a+\mu)^{-N+\delta} = (a+\mu)^{-N}(a+\mu)^\delta$ is in $S_\sigma^{-\sigma N,\delta}(\Gamma)$ by (2.13) and the product rule. $\square$

Remark 2.9. The statement (2.14) does not hold for noninteger N. For example, let $\sigma = 1$, $a \in S^1$ and consider $(a+\mu)^{-\frac{1}{2}}$. If this symbol were in $S_1^{-\frac{1}{2},0}(\Gamma)$, we would have $a^{\frac{1}{2}}(a+\mu)^{-\frac{1}{2}} \in S_1^{0,0}(\Gamma)$. But $a^{\frac{1}{2}}(a+\mu)^{-\frac{1}{2}} = z^{\frac{1}{2}}a^{\frac{1}{2}}(za+1)^{-\frac{1}{2}}$, where

$$\partial_z[z^{\frac{1}{2}}a^{\frac{1}{2}}(za+1)^{-\frac{1}{2}}] = z^{\frac{1}{2}}a^{\frac{1}{2}}\partial_z(za+1)^{-\frac{1}{2}} + \tfrac{1}{2}z^{-\frac{1}{2}}a^{\frac{1}{2}}(za+1)^{-\frac{1}{2}};$$

here the first term in the right hand side is bounded for $z \to 0$, but the second term blows up for $z \to 0$. Then the rules for $S_1^{0,0}(\Gamma)$ are not satisfied.

As in [GS95, Th. 1.12] we get:

Theorem 2.10. *Let $p \in S^{m,\delta}_\sigma(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$. Then the limits*

$$p_{(\delta,k)}(x,\xi) = \lim_{z\to 0} \frac{1}{k!}\partial_z^k (z^\delta p(x,\xi,\tfrac{1}{z}))$$

exist and belong to $S^{m+\sigma k}(\mathbb{R}^\nu \times \mathbb{R}^n)$, and for any $N \in \mathbb{N}$,

$$p(x,\xi,\mu) = \sum_{0\le k<N} p_{(\delta,k)}(x,\xi)\mu^{\delta-k} + r_N(x,\xi,\mu)\mu^{\delta-N},$$

where r_N is in $S^{m+\sigma N,0}_\sigma(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$.

Proof. In view of Lemma 2.3 (iii) we can take $\delta = 0$. Let Γ' be a closed subsector of Γ, and let z and $z+h$ be interior points of Γ'. Then

$$\partial_x^\beta \partial_\xi^\alpha \partial_z^j p(x,\xi,\tfrac{1}{z+h}) - \partial_x^\beta \partial_\xi^\alpha \partial_z^j p(x,\xi,\tfrac{1}{z}) = h\int_0^1 \partial_x^\beta \partial_\xi^\alpha \partial_z^{j+1} p(x,\xi,\tfrac{1}{ht+z})\,dt$$

leads to the estimate

$$\left|\partial_x^\beta \partial_\xi^\alpha \left(\partial_z^j p(x,\xi,\tfrac{1}{z+h}) - \partial_z^j p(x,\xi,\tfrac{1}{z})\right)\right| \ \dot{\le}\ |h|\langle\xi\rangle^{m-|\alpha|+\sigma(j+1)}$$

for z in the set $\Gamma'' := \{z \mid 0 < |z| \le 1,\ \frac{1}{z} \in \Gamma'\}$. Hence the map $z \mapsto \partial_z^j p(\cdot,\cdot,\frac{1}{z}) \in S^{m+\sigma(j+1)}(\mathbb{R}^\nu \times \mathbb{R}^n)$ is uniformly continuous, so it has a limit for $z \to 0$ in Γ''. Because of the uniform estimates in the seminorms of $S^{m+\sigma j}(\mathbb{R}^\nu\times\mathbb{R}^n)$, we conclude that $p_{(\delta,k)} \in S^{m+\sigma j}(\mathbb{R}^\nu\times\mathbb{R}^n)$.

Taylor's formula gives

$$\begin{aligned} r_N(x,\xi,\mu) &= \mu^N\Big(p(x,\xi,\mu) - \sum_{k<N}\mu^{-k}p_{(\delta,k)}(x,\xi)\Big)\\ &= z^{-N}\Big(p(x,\xi,\tfrac{1}{z}) - \sum_{k<N} z^k p_{(\delta,k)}(x,\xi)\Big)\\ &= \frac{1}{(N-1)!}\int_0^1 (1-t)^{N-1}\partial_z^N p(x,\xi,\tfrac{1}{tz})\,dt \end{aligned}$$

where r_N is C^∞ and holomorphic for $z \in \Gamma^\circ$ with estimates

$$\begin{aligned} |\partial_x^\beta \partial_\xi^\alpha \partial_z^j r_N(x,\xi,\mu)| &\le \frac{1}{(N-1)!}\int_0^1 (1-t)^{N-1}|\partial_x^\beta \partial_\xi^\alpha \partial_z^{N+j} p(x,\xi,\tfrac{1}{tz})|\,dt\\ &\dot{\le} \langle\xi\rangle^{m-|\alpha|+\sigma(N+j)} \end{aligned}$$

for $z \in \Gamma''$. $\square$

Thus the asymptotic expansion of $p(x,\xi,\mu)$ in decreasing powers of μ comes from a Taylor expansion of $z^\delta p(x,\xi,\frac{1}{z})$ in z at $z = 0$.

Besides such Taylor expansions, the calculus also contains asymptotic expansions of symbols in terms of decreasing order; a quite different concept. Here one can in particular define polyhomogeneous symbols as those that have expansions in homogeneous terms. The following version of homogeneity (or quasi-homogeneity) will primarily be used:

Definition 2.11. *For any* $s \in \mathbb{C}$*, a function* $p(x, \xi, \mu)$ *on* $\mathbb{R}^\nu \times \mathbb{R}^n \times \Gamma$ *is said to be* (weakly) σ-homogeneous of degree s *if*

$$p(x, t\xi, t^\sigma \mu) = t^s p(x, \xi, \mu) \quad \textit{for} \quad t, |\xi| \geq 1.$$

We denote by $S^{m,\delta}_{\sigma,\mathrm{hg}}(\Gamma)$ *the symbols in* $S^{m,\delta}_\sigma(\Gamma)$ *that are weakly* σ*-homogeneous of degree* $m + \sigma\delta$*.*

Functions for which the homogeneity property extends to $|\xi| + |\mu| \geq 1$ are called *strongly* homogeneous. Observe that the subspaces $S^{m,\delta}_{\sigma,\mathrm{hg}}(\Gamma)$ respect the mappings in Lemma 2.3.

Definition 2.12. (i) *Let* $p \in S^{\infty,\delta}_\sigma(\Gamma)$ *and* $p_j \in S^{m_j,\delta}_\sigma(\Gamma)$ *with* $m_j \searrow -\infty$*. Then we write* $p \sim \sum_{j=0}^\infty p_j$ *if* $p - \sum_{0 \leq j < N} p_j$ *is in* $S^{m_N,\delta}_\sigma(\Gamma)$ *for any* $N \in \mathbb{N}$*.*

(ii) *A symbol* p *is said to be* σ-polyhomogeneous *(of degree* $m_0 + \sigma\delta$*) if* $p \sim \sum_{j=0}^\infty p_j$ *for symbols* $p_j \in S^{m_j,\delta}_{\sigma,\mathrm{hg}}(\Gamma)$ *with* $m_j \searrow -\infty$*; the set of such symbols will be denoted* $S^{m_0,\delta}_{\sigma,\mathrm{phg}}(\Gamma)$*. When* $m_j = m_0 - j$ *for all* $j \in \mathbb{N}$*, we say that* p *is* one-step σ-polyhomogeneous*.*

There is the usual fact that for a sequence of symbols $p_j \in S^{m_j,\delta}_\sigma(\Gamma)$ with $m_j \searrow -\infty$, one can construct a symbol $p \in S^{m_0,\delta}_\sigma(\Gamma)$ such that $p \sim \sum_j p_j$ in the above sense.

The symbol spaces will most frequently be used with $\nu = n$, or with $\nu = 2n$ where the variable x is replaced by (x, y), x and $y \in \mathbb{R}^n$. They can of course also be defined over open subsets of $\mathbb{R}^n$ (or $\mathbb{R}^{2n}$), with local estimates (uniform over compact subsets).

3. Operators, kernel expansions and traces

When $p(x, \xi, \mu)$ is a symbol in one of our symbol spaces with $\nu = n$, it defines for each fixed $\mu \in \Gamma$ a pseudodifferential operator (ψdo) $P_\mu = \mathrm{OP}\big(p(x, \xi, \mu)\big)$ on $\mathbb{R}^n$ by the formula:

$$(P_\mu f)(x) = \int e^{ix\cdot\xi} p(x, \xi, \mu) \hat{f}(\xi) đ\xi \ , \quad f \in \mathcal{S}(\mathbb{R}^n);$$

here $đ\xi = (2\pi)^{-n} d\xi$. The standard rules for composition of operators give

$$\mathrm{OP}\big(p(x, \xi, \mu)\big)\, \mathrm{OP}\big(p'(x, \xi, \mu)\big) = \mathrm{OP}\big((p \circ p')(x, \xi, \mu)\big),$$

where (in the sense of oscillatory integrals)

$$(p \circ p')(x, \xi, \mu) = \int e^{iy\cdot\eta} p(x, \xi - \eta, \mu) p'(x + y, \xi, \mu)\, dy đ\eta$$

and

$$p \circ p' \sim \sum_{\alpha \in \mathbb{N}^n} \tfrac{1}{\alpha!} \partial_\xi^\alpha p \, D_x^\alpha p'. \tag{3.1}$$

($D_x = -i\partial_x$.) Recall that $S_\sigma^{m,\delta}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$ consists of those functions $p(x, \xi, \mu) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \Gamma)$ (holomorphic in $\mu \in \overset{\circ}{\Gamma}$ for $|(\xi, \mu)| \geq \varepsilon$) such that $\partial_z^j\big(z^\delta p(x, \xi, \frac{1}{z})\big)$ is in $S^{m+\delta j}(\mathbb{R}^n \times \mathbb{R}^n)$ with symbol seminorms bounded for $|z| \leq 1$, $\frac{1}{z}$ in closed subsectors of Γ, $j \in \mathbb{N}$. When $p \in S_\sigma^{m,\delta}(\Gamma)$ and $p' \in S_\sigma^{m',\delta'}(\Gamma)$, the Leibniz rule gives

$$\begin{aligned} \partial_z^j\big(z^{\delta+\delta'}(p \circ p')(x, \xi, \tfrac{1}{z})\big) &= \partial_z^j\big((z^\delta p) \circ (z^{\delta'} p')(x, \xi, \tfrac{1}{z})\big) \\ &= \sum_{k=0}^{j} \binom{j}{k} \partial_z^k\big(z^\delta p(x, \xi, \tfrac{1}{z})\big) \circ \partial_z^{j-k}\big(z^{\delta'} p'(x, \xi, \tfrac{1}{z})\big) \end{aligned}$$

from which we see that $p \circ p'$ is in $S_\sigma^{m+m',\delta+\delta'}(\Gamma)$. Thus we have the rule

$$S_\sigma^{m,\delta}(\Gamma) \circ S_\sigma^{m',\delta'}(\Gamma) \subset S_\sigma^{m+m',\delta+\delta'}(\Gamma), \tag{3.2}$$

for all $m, m' \in \mathbb{R}$, $\delta, \delta' \in \mathbb{C}$, $\sigma \in \mathbb{R}_+$.

Similarly, the rules for coordinate changes follow the usual pattern, so our μ-dependent operators can be defined in vector bundles over manifolds by the help of local trivializations.

We shall in particular use the calculus to describe the resolvents of elliptic operators, see Theorem 4.1 below.

Remark 3.1. The resolvent tempered symbol classes introduced by Loya in [L01] look different, because a Taylor expansion property related to that of Theorem 2.10 is taken as part of the definition; then one has to show that this property is preserved under compositions, inversions and other manipulations with the symbols and operators. However, Loya's class $S_{\Lambda,r}^{m,p,d}(\mathbb{R}^n)$, defined for $m, p \in \mathbb{R}$ and $d \in \mathbb{R}_+$ with $p/d \in \mathbb{Z}$, is very much like our symbol space $S_d^{m-p,\, p/d}(\Lambda)$ when $p/d \leq 0$. Loya considers only integer values of p/d, corresponding to our δ being integer.

In Proposition 3.3 below, we need the following lemma:

Lemma 3.2. *Let $f(\lambda)$ be a holomorphic function on an open sector Γ with $f(\lambda) \neq 0$ throughout, and let $c(\lambda)$ and $g(\lambda)$ be continuous functions on Γ such that $cf + g$ is holomorphic. If $c(\lambda)$ depends on the argument of λ only and $\frac{g(\lambda)}{f(\lambda)} \to 0$ for $\lambda \to \infty$ in closed subsectors of Γ, then c is constant on Γ.*

Proof. We can write $c = h + \varepsilon$ where $h := \frac{cf+g}{f}$ is holomorphic and $\varepsilon := -\frac{g}{f}$ converges to 0 as $\lambda \to \infty$. Let $\mathcal{L}$ be any closed curve in Γ and let $t > 0$. Since c depends on the argument only and h is holomorphic, we have:

$$\begin{aligned} t\int_{\mathcal{L}} c(\lambda)\,d\lambda = \int_{t\mathcal{L}} c(\lambda)\,d\lambda &= \int_{t\mathcal{L}} h(\lambda)\,d\lambda + \int_{t\mathcal{L}} \varepsilon(\lambda)\,d\lambda \\ &= \int_{t\mathcal{L}} \varepsilon(\lambda)\,d\lambda = t\int_{\mathcal{L}} \varepsilon(t\lambda)\,d\lambda \end{aligned}$$

which implies that

$$\left|\int_{\mathcal{L}} c(\lambda)\,d\lambda\right| \le \int_{\mathcal{L}} |\varepsilon(t\lambda)|\,d|\lambda| \to 0 \quad \text{for} \quad t \to \infty.$$

Thus $\int_{\mathcal{L}} c(\lambda)\,d\lambda = 0$ for all $\mathcal{L}$, so it follows that c is holomorphic; then since c is constant on rays, it must be a constant. $\square$

We shall now generalize [GS95, Th. 2.1] to the present symbols.

Proposition 3.3. *Let $p \in S^{m,0}_{\sigma,\mathrm{hg}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma)$ and assume that $p(x,\xi,\mu)$ is integrable in ξ for each μ with $|\mu| \ge 1$. Then there are functions $c, c', c_k \in C^\infty_b(\mathbb{R}^n)$ and $C_k \in C^\infty_b(\mathbb{R}^n\times\Gamma')$ (any closed $\Gamma' \subset \Gamma$), $k \in \mathbb{N}$, such that*

$$\begin{aligned} &\int_{\mathbb{R}^n} p(x,\xi,\mu)\,đ\xi \\ &\quad = (c(x) + c'(x)\log\mu)\,\mu^{\frac{m+n}{\sigma}} + \sum_{0\le k<N} c_k(x)\mu^{-k} + C_N(x,\mu)\mu^{-N}, \end{aligned}$$

for any $N \in \mathbb{N}$ and $|\mu| \ge 1$.

Here $c(x)$ and $c'(x)$ are determined from p for $|\xi| \ge 1$, with $c'(x) = 0$ if $\frac{m+n}{\sigma} \notin -\mathbb{N}$ and determined by (3.3) *and* (3.10) *below if $\frac{m+n}{\sigma} \in -\mathbb{N}$. The other coefficients depend on the full value of p.*

Proof. It suffices to show the result for large N, so we can assume that $N > -\frac{m+n}{\sigma}$.

We have the following formula from Theorem 2.10:

$$p(x,\xi,\mu) = \sum_{0\le k<N} p_k(x,\xi)\mu^{-k} + r_N(x,\xi,\mu)\mu^{-N} \tag{3.3}$$

with $p_k \in S^{m+\sigma k}(\mathbb{R}^n\times\mathbb{R}^n)$ homogeneous of degree $m+\sigma k$ in $|\xi|$ for $|\xi| \ge 1$, and with $r_N \in S^{m+\sigma N,0}_{\sigma,\mathrm{hg}}(\Gamma)$. To calculate the integral $\int p(x,\xi,\mu)\,đ\xi$ we split it into three terms

$$\int_{|\xi|\ge|\mu|^{\frac{1}{\sigma}}} p(x,\xi,\mu)\,đ\xi + \int_{|\xi|\le 1} p(x,\xi,\mu)\,đ\xi + \int_{1\le|\xi|\le|\mu|^{\frac{1}{\sigma}}} p(x,\xi,\mu)\,đ\xi. \tag{3.4}$$

The first term gives by homogeneity, when we set $\eta = |\mu|^{-\frac{1}{\sigma}}\xi$, $\omega = \mu/|\mu|$,

$$\int_{|\xi|\geq|\mu|^{\frac{1}{\sigma}}} p(x,\xi,\mu)\, đ\xi = |\mu|^{\frac{m+n}{\sigma}} \int_{|\eta|\geq 1} p(x,\eta,\omega)\, đ\eta = c^1(x,\omega)\mu^{\frac{m+n}{\sigma}}. \tag{3.5}$$

The second term gives in view of (3.3):

$$\int_{|\xi|\leq 1} p(x,\xi,\mu)\, đ\xi = \sum_{0\leq k<N} c_k^2(x)\mu^{-k} + C_N^2(x,\mu)\mu^{-N}. \tag{3.6}$$

Also for the third term in (3.4) we use (3.3). The contributions from the p_k are worked out in polar coordinates in $\mathbb{R}^n$:

$$\begin{aligned}
&\mu^{-k}\int_{1\leq|\xi|\leq|\mu|^{\frac{1}{\sigma}}} p_k(x,\xi)\, đ\xi \\
&\quad= \mu^{-k}\int_{1\leq|\xi|\leq|\mu|^{\frac{1}{\sigma}}} |\xi|^{m+\sigma k} p_k(x,\xi/|\xi|)\, đ\xi \\
&\quad= \mu^{-k}(2\pi)^{-n}\int_{S^{n-1}} p_k(x,\eta)\, dS(\eta) \int_{1\leq s\leq|\mu|^{\frac{1}{\sigma}}} s^{m+\sigma k+n-1}\, ds \qquad (3.7)\\
&\quad= \begin{cases} c_k^3(x)\mu^{-k}(|\mu|^{\frac{m+n}{\sigma}+k}-1) & \text{for } m+\sigma k+n\neq 0,\\ c'(x)\mu^{-k}\log|\mu| & \text{for } m+\sigma k+n=0,\end{cases}\\
&\quad= \begin{cases} c_k^4(x,\omega)\mu^{\frac{m+n}{\sigma}} - c_k^3(x)\mu^{-k} & \text{for } k\neq -\frac{m+n}{\sigma},\\ c'(x)\mu^{-k}(\log\mu-\log\omega) & \text{for } k=-\frac{m+n}{\sigma}.\end{cases}
\end{aligned}$$

If $\frac{m+n}{\sigma}\notin -\mathbb{N}$, the last case does not occur; then we set $c'(x)=0$. To treat the remainder r_N, let r_N^h denote the function which is homogeneous for all $\xi\neq 0$ and equals r_N for $|\xi|\geq 1$. Since $r_N\in S_\sigma^{m+\sigma N,0}(\Gamma)$,

$$|r_N^h(x,\xi,\mu)| = |\xi|^{m+\sigma N}|r_N(x,\xi/|\xi|,\mu/|\xi|^{\frac{1}{\sigma}})| \doteq\!\!\!\!\leq |\xi|^{m+\sigma N}$$

for $|\xi|\leq 1$ and $|\mu|\geq 1$, so because of the assumption $N>-\frac{m+n}{\sigma}$, the function r_N^h can be integrated into 0. With $\omega=\mu/|\mu|$, the homogeneity gives

$$\begin{aligned}
&\mu^{-N}\int_{|\xi|\leq|\mu|^{\frac{1}{\sigma}}} r_N^h(x,\xi,\mu)\, đ\xi \\
&\quad= |\mu|^{\frac{m}{\sigma}}\omega^{-N}\int_{|\xi|\leq|\mu|^{\frac{1}{\sigma}}} r_N^h(x,|\mu|^{-\frac{1}{\sigma}}\xi,\omega)\, đ\xi \\
&\quad= |\mu|^{\frac{m+n}{\sigma}}\omega^{-N}\int_{|\eta|\leq 1} r_N^h(x,\eta,\omega)\, đ\eta \\
&\quad= c_N^5(x,\omega)\mu^{\frac{m+n}{\sigma}}.
\end{aligned}$$

Setting $C^3_N(x,\mu) := \int_{|\xi|\le 1} r^h_N(x,\xi,\mu)\, đ\xi$ we then have

$$\begin{aligned}
&\mu^{-N}\int_{1\le|\xi|\le|\mu|^{\frac{1}{\sigma}}} r_N(x,\xi,\mu)\, đ\xi \\
&\quad = \mu^{-N}\int_{|\xi|\le|\mu|^{\frac{1}{\sigma}}} r^h_N(x,\xi,\mu)\, đ\xi - \mu^{-N}\int_{|\xi|\le 1} r^h_N(x,\xi,\mu)\, đ\xi \\
&\quad = c^5_N(x,\omega)\mu^{\frac{m+n}{\sigma}} - C^3_N(x,\mu)\mu^{-N}.
\end{aligned}\tag{3.8}$$

Now (3.7) and (3.8) together give:

$$\begin{aligned}
&\int_{1\le|\xi|\le|\mu|^{\frac{1}{\sigma}}} p(x,\xi,\mu)\, đ\xi \\
&\quad = \Big[-c'(x)\log\omega \;+ \sum_{0\le k<N,\, k\ne -\frac{m+n}{\sigma}} c^4_k(x,\omega) + c^5_N(x,\omega)\Big]\mu^{\frac{m+n}{\sigma}} \\
&\qquad + \; c'(x)\mu^{\frac{m+n}{\sigma}}\log\mu \;- \sum_{0\le k<N,\, k\ne -\frac{m+n}{\sigma}} c^3_k(x)\mu^{-k} - C^3_N(x,\mu)\mu^{-N} \\
&\quad = c^6_{(N)}(x,\omega)\mu^{\frac{m+n}{\sigma}} + c'(x)\mu^{\frac{m+n}{\sigma}}\log\mu \\
&\qquad - \sum_{0\le k<N} c^3_k(x)\mu^{-k} - C^3_N(x,\mu)\mu^{-N}
\end{aligned}\tag{3.9}$$

Collecting all the terms, we obtain an expansion

$$\begin{aligned}
&\int p(x,\xi,\mu)\, đ\xi \\
&\quad = (c_{(N)}(x,\omega) + c'(x)\log\mu)\mu^{\frac{m+n}{\sigma}} + \sum_{0\le k<N} c_k(x)\mu^{-k} + C_N(x,\mu)\mu^{-N},
\end{aligned}$$

where $c_{(N)}(x,\omega)$ is the sum of $c^1(x,\omega)$ from (3.5) and $c^6_{(N)}(x,\omega)$ from (3.9). Here

$$c'(x) = \begin{cases} (2\pi)^{-n}\int_{S^{n-1}} p_k(x,\eta)\, dS(\eta) & \text{if } \frac{n+m}{\sigma}\in -\mathbb{N}, \\ 0 & \text{if } \frac{n+m}{\sigma}\notin -\mathbb{N}; \end{cases}\tag{3.10}$$

also $c_{(N)}(x,\omega)$ is defined from p for $|\xi|\ge 1$. An application of Lemma 3.2 for each fixed x shows that $c_{(N)}(x,\omega)$ is independent of ω. It is also clear that it is independent of N, since another choice N' with $N' > -\frac{n+m}{\sigma}$ gives an expansion from which we infer that

$$(c_{(N)}(x) - c_{(N')}(x))\mu^{\frac{n+m}{\sigma}} \mathrel{\dot{\le}} |\mu|^{-\min\{N,N'\}} \quad \text{for} \quad |\mu|\ge 1.$$

The other coefficients will in general depend on the full value of p. $\square$

We can now show:

Theorem 3.4. *Let* $p \in S^{m,\delta}_{\sigma,\mathrm{phg}}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma)$ *with* $p \sim \sum_{j=0}^{\infty} p_j$ *for symbols* $p_j \in S^{m_j,\delta}_{\sigma,\mathrm{hg}}$*;* $m_0 = m$ *and* $m_j \searrow -\infty$*. If* $m \geq -n$*, assume furthermore that* p *and the symbols* p_j *with* $m_j \geq -n$ *are integrable in* ξ *for each* μ *with* $|\mu| \geq 1$*. Then* $\mathrm{OP}(p)$ *has a continuous kernel* $K_p(x, y, \mu)$ *with an expansion on the diagonal*

$$K_p(x, x, \mu) \sim \sum_{j=0}^{\infty} c_j(x) \mu^{\delta + \frac{m_j+n}{\sigma}} + \sum_{k=0}^{\infty} \big(c'_k(x) \log \mu + c''_k(x)\big) \mu^{\delta - k} \ , \tag{3.11}$$

for $|\mu| \to \infty$*, uniformly for* μ *in closed subsectors of* Γ*.*

The coefficient $c_j(x)$ *is determined from* $p_j(x, \xi, \mu)$ *for* $|\xi| \geq 1$*, and so is* $c'_k(x)$ *if* $k = -\frac{m_j+n}{\sigma}$ *(cf. (3.10));* $c'_k(x)$ *vanishes if* $k \notin \{\frac{m_l+n}{\sigma} \mid l \in \mathbb{N}\}$*. In this sense, these coefficients are "local". The* $c''_k(x)$ *are not in general determined by the homogeneous parts of the symbols (they are "global").*

Proof. We can write $p(x, \xi, \mu) = \mu^{\delta} p'(x, \xi, \mu)$, where $p' = \mu^{-\delta} p$ satisfies the hypotheses with δ replaced by 0, cf. Definition 2.1. Then it suffices to show the theorem for p', for this will give the expansion (3.11) of K_p by multiplication by μ^{δ}. Thus we can *assume that* $\delta = 0$ in the rest of the proof.

The hypotheses assure that all the symbols p_j and remainders $s_J = p - \sum_{0 \leq j < J} p_j$ (including $s_0 = p$) are integrable in ξ for each μ; hence the operators they define have continuous kernels

$$\begin{aligned} K_{p_j}(x, y, \mu) &= \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} p_j(x, \xi, \mu) \, đ\xi, \\ K_{s_J}(x, y, \mu) &= \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} s_J(x, \xi, \mu) \, đ\xi; \end{aligned}$$

here $K_{s_0} = K_p$. For any large N we must show that there is an expansion

$$\begin{aligned} K_p(x, x, \mu) = \sum_{0 \leq j < J} c_j(x) \mu^{\frac{m_j+n}{\sigma}} + \sum_{0 \leq k < N}^{\infty} [c'_k(x) \log \mu + c''_k(x)] \mu^{-k} \\ + O(\mu^{-N}), \end{aligned} \tag{3.12}$$

when J is taken so large that

$$\tfrac{m_J+n}{\sigma} < -N; \tag{3.13}$$

here c_j and c'_k should be as stated in the theorem. We apply Proposition 3.3 to the terms $K_{p_j}(x,x,\mu)$; this shows that they have expansions

$$K_{p_j}(x,x,\mu) = \int_{\mathbb{R}^n} p_j \, đ\xi = c_j(x)\mu^{\frac{m_j+n}{\sigma}} + c'_j(x)\mu^{\frac{m_j+n}{\sigma}} \log\mu + \sum_{k=0}^{N-1} c_{j,k}(x)\mu^{-k} + O(\mu^{-N}),$$

where $c_j(x)$ and $c'_j(x)$ are determined from p_j for $|\xi| \geq 1$ and the $c_{j,k}(x)$ need not be so. For the remainder s_J, we use Theorem 2.10:

$$s_J(x,\xi,\mu) = \sum_{0\leq k<N} s_{J,k}(x,\xi)\mu^{-k} + O(\langle\xi\rangle^{m_J+\sigma N}\mu^{-N}),$$

with $s_{J,k} \in S^{m_J+\sigma k}(\mathbb{R}^n \times \mathbb{R}^n)$. In view of (3.13), $m_J + \sigma k < -n$ for $k \leq N$, so the terms may be integrated in ξ, which gives:

$$K_{s_J}(x,x,\mu) = \sum_{0\leq k<N} c_{J,k}(x)\mu^{-k} + O(\mu^{-N}). \tag{3.14}$$

Adding the contributions, we find (3.12).

Note that there may be two contributions to each power term $c\mu^{\delta-k}$ in the cases where the $\frac{m_j+n}{\sigma}$ are integer. When (3.12) is established for all large N, all J satisfying (3.13), the full coefficient of $\mu^{\delta-k}$ will be independent of the choice of N and J, so since the c_j are determined from the p_j, also the c''_k are uniquely determined. ☐

When $P = \mathrm{OP}(p)$ is as in the above theorem and $\varphi \in C_0^\infty(\mathbb{R}^n)$, then $\varphi P \varphi$ is trace-class, and the trace equals $\int \varphi(x)^2 K_p(x,x,\mu)\,dx$, which therefore has an expansion

$$\mathrm{Tr}(\varphi P \varphi) \sim \sum_{j=0}^{\infty} c_j \mu^{\delta+\frac{m_j+n}{\sigma}} + \sum_{k=0}^{\infty} \big(c'_k \log\mu + c''_k\big)\mu^{\delta-k}\,, \tag{3.15}$$

for $|\mu| \to \infty$, uniformly for μ in closed subsectors of Γ; here the coefficients are obtained from those in (3.11) by multiplication by $\varphi(x)^2$ and integration over the support of φ. This can be used to get trace expansions of operators on a compact manifold M by use of local coordinates. (One can replace one of the factors φ by another smooth function φ_1.)

We can also easily allow operators acting on the sections of a smooth vector bundle over E; in local trivializations this gives matrices of operators, and we just have to take the fiber trace in the kernel formulas before integrating with respect to x. In conclusion:

Corollary 3.5. *Let P be a μ-dependent ψdo on a compact manifold M acting on sections of a vector bundle E such that for every coordinate patch $U \subset M$ and smooth function φ supported in U the operator $\varphi P \varphi$ in local coordinates has a symbol satisfying the assumptions of Theorem 3.4. Then P is trace-class and the trace has an expansion*

$$\operatorname{Tr}(P) \sim \sum_{j=0}^{\infty} c_j \mu^{\delta + \frac{m_j+n}{\sigma}} + \sum_{k=0}^{\infty} \big(c'_k \log \mu + c''_k\big) \mu^{\delta-k} \,, \tag{3.16}$$

for $|\mu| \to \infty$, uniformly for μ in closed subsectors of Γ.

4. Application to complex powers of resolvents

We shall apply the preceding theory to operators of the form $B(A-\lambda)^{-s}$ on a compact manifold. For fixed λ, we can use details from the construction of Seeley [S67].

First the resolvent itself is considered.

Theorem 4.1. *Let A be a one-step polyhomogeneous pseudodifferential operator of order $\sigma \in \mathbb{R}_+$, acting on the sections of an N-dimensional C^∞ vector bundle E over a compact C^∞ manifold M of dimension n. Assume that A is elliptic of order σ and that there is a nonempty open sector Γ in $\mathbb{C}$ such that the eigenvalues of the principal symbol are contained in the complementing closed sector $\Lambda = \mathbb{C} \setminus (\Gamma \cup \{0\})$.*

For any closed subsector $\Gamma' \subset \Gamma$ there is an r such that the resolvent $Q(\lambda) = (A-\lambda)^{-1}$ exists as a ψdo for $\lambda \in \Gamma'$ with $|\lambda| \geq r$. In local trivializations, the resolvent symbol $q(x,\xi,\lambda)$ is in $S^{-\sigma,0}_{\sigma,\mathrm{phg}}(\Gamma) \cap S^{0,-1}_{\sigma,\mathrm{phg}}(\Gamma)$, with principal part $q_{-\sigma} = (a_\sigma(x,\xi)-\lambda)^{-1}$ (when a_σ is modified near $\xi = 0$ to make this invertible) lying in this symbol space. Moreover, $q - q_{-\sigma}$ is in $S^{-\sigma-1,0}_{\sigma,\mathrm{phg}}(\Gamma) \cap S^{-1,-1}_{\sigma,\mathrm{phg}}(\Gamma) \cap S^{\sigma-1,-2}_{\sigma,\mathrm{phg}}(\Gamma)$

Proof. Consider the symbol a in a local trivialization; it is $N \times N$-matrix valued with entries in $S^\sigma(V \times \mathbb{R}^n)$ (V open $\subset \mathbb{R}^n$). The symbol a is a sequence $\sum_{j\geq 0} a_{\sigma-j}$ of terms homogeneous in ξ of degrees $\sigma - j$, and we construct the resolvent symbol $q(x,\xi,\lambda)$ (similarly to [S67], [GS95]) as a sequence $\sum_{l\geq 0} q_{-\sigma-l}$ of terms that are quasihomogeneous in (ξ,λ), lying in $S^{-\sigma-l,0}_{\sigma,\mathrm{hg}}(\Gamma)$. More precisely, the relation

$$\Big(\sum_{j\geq 0} a_{\sigma-j} - \lambda\Big) \circ \sum_{l\geq 0} q_{-\sigma-l} \sim 1$$

is obtained, after setting $q_{-\sigma} = (a_\sigma - \lambda)^{-1}$, by collecting the products $\neq 1$ according to homogeneity degree and equating the sums with 0;

this gives the successive formulas (cf. (3.1))

$$q_{-\sigma-l} = -q_\sigma \sum_{j<l, j+k+|\alpha|=l} \tfrac{1}{\alpha!}\partial_\xi^\alpha a_{\sigma-k} D_x^\alpha q_{-\sigma-j}, \tag{4.1}$$

for $l = 1, 2, \dots$. When this is worked out in more detail, one finds (cf. also Hörmander [H66], Nagase [N73]) that for $l \ge 1$, $q_{-\sigma-l}$ is a finite sum of terms of the form

$$g(x,\xi,\lambda) = g_1 q_{-\sigma}^{\nu_1} g_2 q_{-\sigma}^{\nu_2} \cdots q_{-\sigma}^{\nu_M} g_{M+1}, \tag{4.2}$$

where the ν_k are integers ≥ 1 and the $g_k(x,\xi)$ are ψdo symbols independent of λ and homogeneous in $|\xi| \ge 1$ of degree $r_k \in \mathbb{R}$. The index sums $\nu = \sum_{1\le k\le M} \nu_k$, $r = \sum_{1\le k\le M+1} r_k$ satisfy

$$2 \le \nu \le 2l+1, \quad -\nu\sigma + r = -\sigma - l. \tag{4.3}$$

In particular, when a_σ is "scalar," i.e. is a scalar function times the identity matrix, the factors $q_{-\sigma}$ can be commuted through the other factors to give the form

$$q_{-\sigma-l}(x,\xi,\lambda) = \sum_{1\le k\le 2l} r_{l,k}(x,\xi)(a_\sigma(x,\xi)-\lambda)^{-k-1}, \text{ when } l \ge 1; \tag{4.4}$$

here the $r_{l,k}$ are homogeneous of degree $k\sigma - l$ in $|\xi| \ge 1$.

We first observe that by Theorem 2.8,

$$q_{-\sigma}(x,\xi,\lambda) \in S_{\sigma,\mathrm{hg}}^{-\sigma,0}(\Gamma) \cap S_{\sigma,\mathrm{hg}}^{0,-1}(\Gamma).$$

For $l \ge 1$, we can in each term (4.2) use the information $q_{-\sigma} \in S_\sigma^{0,-1}$ for 0, 1 or 2 of the factors and the information $q_{-\sigma} \in S_\sigma^{-\sigma,0}$ for the rest of the factors. By the product rule (Lemma 2.4), this implies that

$$g(x,\xi,\lambda) \in S_{\sigma,\mathrm{hg}}^{-\sigma-l,0}(\Gamma) \cap S_{\sigma,\mathrm{hg}}^{-l,-1}(\Gamma) \cap S_{\sigma,\mathrm{hg}}^{\sigma-l,-2}(\Gamma) \text{ for all } l \ge 1, \tag{4.5}$$

and it follows by summation that $q_{-\sigma-l}$ is in this intersection of spaces too.

Now let q' be a symbol in $S_{\sigma,\mathrm{phg}}^{\sigma-1,-2}(\Gamma)$ such that $q' \sim \sum_{l\ge 0} q_{-\sigma-l}$ in $S_{\sigma,\mathrm{phg}}^{\sigma-1,-2}$ (it is determined modulo $S_\sigma^{-\infty,-2}$), and let $q = q_{-\sigma} + q'$; then q satisfies the assertions in the theorem.

The symbols defined in the local trivializations can be pieced together to define a parametrix $\widetilde{Q}(\lambda)$ of $A - \lambda$ on M, such that the remainders $I - (A-\lambda)\widetilde{Q}$ and $I - \widetilde{Q}(A-\lambda)$ are ψdo's with symbols in $S_\sigma^{-\infty,-2}$. The norm of this operator in $L_2(M,E)$ goes to 0 for λ going to infinity (uniformly in closed subsectors of Γ), so by a Neumann-series argument, the true resolvent $Q(\lambda) = (A-\lambda)^{-1}$ exists for large λ in such subsectors. To show that it is in our calculus and differs from $\widetilde{Q}$

by an operator of order $-\infty$, one can proceed as in [GS95, p. 502] or [G99, Th. 6.5]. $\square$

Next, we construct the complex powers $(A-\varrho)^{-s}$ by use of the resolvent. Here $(A-\varrho)^{-s}$ is defined as a Cauchy integral in the complex plane for $\operatorname{Re} s>0$, and extended to general $s\in\mathbb{C}$ by composition with positive integer powers N of $A-\varrho$:

$$(A-\varrho)^{-s}=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}(A-\varrho-\lambda)^{-1}\,d\lambda, \tag{4.6}$$

$$(A-\varrho)^{-s+N}=(A-\varrho)^{N}(A-\varrho)^{-s}=(A-\varrho)^{-s}(A-\varrho)^{N}, \tag{4.7}$$

where $\mathcal{C}$ is a suitable curve in $\mathbb{C}$ encircling the nonzero spectrum of $A-\varrho$ in the positive direction. The powers $(A-\varrho)^{-s}$ can be defined as ψdo's by Seeley's construction [S67] for *any* $\varrho\in\mathbb{C}$ (such that the operator is defined as 0 on solutions of $(A-\varrho)u=0$ if this eigenspace is nonzero). The symbols will depend holomorphically on ϱ for $\varrho\notin\operatorname{spec}A$.

In the present paper we are interested in the asymptotic behavior in ϱ for $\varrho\to\infty$ on rays; here we need to restrict the attention to rays in Γ. To allow ϱ to run on such rays, we choose the integration curve $\mathcal{C}$ as described in the following.

By a rotation we can achieve that Λ and Γ are of the form

$$\begin{aligned}\Lambda&=\{\lambda\in\mathbb{C}\setminus\{0\}\mid|\arg\lambda|\le\theta_0\},\\ \Gamma&=\{\varrho\in\mathbb{C}\setminus\{0\}\mid\arg\varrho\in\,]\theta_0,2\pi-\theta_0[\,\},\end{aligned}$$

for some $\theta_0\in[0,\pi[$. For the definition of noninteger powers λ^{-s}, we choose a cut in the complex plane along the closed negative real axis $\overline{\mathbb{R}}_-$. For any $\varepsilon>0$, let $\mathcal{C}_\varepsilon$ denote the curve

$$\begin{aligned}\{x+\varepsilon i\mid-\infty<x\le0\}&+\{\varepsilon e^{i\omega}\mid\tfrac{\pi}{2}\ge\omega\ge-\tfrac{\pi}{2}\}\\ &+\{x-\varepsilon i\mid 0\ge x>-\infty\},\end{aligned}$$

on the boundary of the set $\overline{\mathbb{R}}_-+B_\varepsilon$, $B_\varepsilon=\{\lambda\mid|\lambda|\le\varepsilon\}$.

When Γ' is a closed subsector of Γ, then Γ' is contained in the complement of

$$\Lambda_\delta=\{\lambda\in\mathbb{C}\setminus\{0\}\mid|\arg\lambda|\le\theta_0+\delta\}$$

for some $\delta>0$, and there is an $r\ge0$ such that the spectrum of A is contained in $\Lambda_\delta\cup B_r$. Then we can choose a constant $c\ge0$ such that with $\Xi:=\Lambda_\delta-c$,

$$\operatorname{spec}(A)+B_1\subset\Xi.$$

Here Ξ is an angular subset of $\mathbb{C}$ satisfying that $\{\varrho\in\Gamma'\mid|\varrho|\ge R\}$ is in the complement of Ξ for R sufficiently large.

Observe that Ξ satisfies $\Xi+\overline{\mathbb{R}}_+\subset\Xi$. Let $\varrho\in\mathbb{C}\setminus\Xi$ and $\eta\in\operatorname{spec}(A)$. Since $\eta+B_1+\overline{\mathbb{R}}_+\subset\Xi$ we have $\varrho\notin\eta+B_1+\overline{\mathbb{R}}_+$, which implies that $\eta-\varrho\notin\overline{\mathbb{R}}_-+B_1$. This proves that

$$\operatorname{spec}(A-\varrho)\subset\mathbb{C}\setminus(\overline{\mathbb{R}}_-+B_1)\quad\text{for any }\varrho\in\mathbb{C}\setminus\Xi.$$

Thus we can use $\mathcal{C}=\mathcal{C}_\varepsilon$ as integration curve for any $0<\varepsilon<1$, when $\varrho\in\mathbb{C}\setminus\Xi$. With these choices, there is room for small perturbations, assuring that the appropriate composition properties for powers can be shown.

Theorem 4.2. *Let A be as in Theorem* 4.1*. On the closed subsectors of Γ, the complex powers $(A-\varrho)^{-s}$ can be defined as ψdo's in our calculus by* (4.6) *(for large $|\varrho|$) for* $\operatorname{Re}s>0$*, and extended to general $s\in\mathbb{C}$ by* (4.7)*.*

For s with $\operatorname{Re}s>0$*, the symbol $p_s(x,\xi,\lambda)$ of $P_s=(A-\varrho)^{-s}$ in local trivializations lies in $S^{0,-s}_{\sigma,\mathrm{phg}}(\Gamma)$, with homogeneous principal part $p_{s,-s\sigma}=(a(x,\xi)-\lambda)^{-s}$ in this space, and $p_s-p_{s,-s\sigma}$ lies in $S^{-1,-s}_{\sigma,\mathrm{phg}}(\Gamma)\cap S^{\sigma-1,-s-1}_{\sigma,\mathrm{phg}}(\Gamma)$.*

Proof. For a given subsector Γ', define Ξ and choose $\mathcal{C}$ as described above, and let $\varrho\in\Gamma'\cap(\mathbb{C}\setminus\Xi)$. Since $(A-\varrho-\lambda)^{-1}$ for fixed ϱ has symbol in $S^{0,-1}_\sigma=\lambda^{-1}S^{0,0}_\sigma$, the L^2 operator norm is $O(|\lambda|^{-1})$; hence

$$(A-\varrho)^{-s}=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}(A-\varrho-\lambda)^{-1}\,d\lambda \tag{4.8}$$

defines a bounded operator in $L^2(M,E)$ for any $\operatorname{Re}s>0$.

We refer to [S67] for the verification that this operator family acts as powers of $A-\varrho$, with the appropriate composition properties.

We want to show that $(A-\varrho)^{-s}$ is defined from a symbol with the stated properties. To do this, we use Theorem 4.1 with A replaced by $A-\varrho$. Consider a coordinate patch $U\subset M$ and cut-off functions φ_1 and $\varphi_2\in C_0^\infty(U)$; then

$$\varphi_1(A-\varrho)^{-s}\varphi_2=\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}\varphi_1(A-\varrho-\lambda)^{-1}\varphi_2\,d\lambda \tag{4.9}$$

carries over to a situation in $\mathbb{R}^n$, with φ_1 and φ_2 supported in V open $\subset\mathbb{R}^n$ (we shall use the same notation there). By Theorem 4.1,

$$\varphi_1(A-\varrho-\lambda)^{-1}\varphi_2=\mathrm{OP}(\varphi_1(x)q(x,\xi,\varrho+\lambda)\varphi_2(y)),$$

for a symbol $q\in S^{-\sigma,0}_{\sigma,\mathrm{phg}}(\Gamma)\cap S^{0,-1}_{\sigma,\mathrm{phg}}(\Gamma)$ with an expansion in homogeneous terms $q\sim\sum_{j\geq0}q_{-\sigma-j}$ such that for any $J>0$, $r_J=q-\sum_{j<J}q_{-\sigma-j}\in$

$S^{-\sigma-J,0}_{\sigma,\mathrm{phg}}(\Gamma)\cap S^{\sigma-J,-2}_{\sigma,\mathrm{phg}}(\Gamma)$. Then

$$\begin{aligned}\varphi_1(A-\varrho)^{-s}\varphi_2 &= \tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}\,\mathrm{OP}(\varphi_1(x)q(x,\xi,\varrho+\lambda)\varphi_2(y))\,d\lambda\\ &= \tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}\sum_{0\le j<J}\mathrm{OP}(\varphi_1(x)q_{-\sigma-j}(x,\xi,\varrho+\lambda)\varphi_2(y))\,d\lambda\\ &\quad+\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}\,\mathrm{OP}(\varphi_1(x)r_J(x,\xi,\varrho+\lambda)\varphi_2(y))\,d\lambda.\end{aligned}$$

When considering the operators applied to smooth functions, we can exchange the integrations in ξ and λ by the Fubini theorem, so

$$\begin{aligned}&\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}\,\mathrm{OP}(\varphi_1(x)q(x,\xi,\varrho+\lambda)\varphi_2(y))\,d\lambda\\ &\qquad=\mathrm{OP}(\varphi_1(x)p_s(x,\xi,\varrho)\varphi_2(y)),\\ &\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}\,\mathrm{OP}(\varphi_1(x)q_{-\sigma-j}(x,\xi,\varrho+\lambda)\varphi_2(y))\,d\lambda\\ &\qquad=\mathrm{OP}(\varphi_1(x)p_{s,-\sigma s-j}(x,\xi,\varrho)\varphi_2(y)),\end{aligned}$$

where

$$\begin{aligned}p_s(x,\xi,\varrho)&=\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}q(x,\xi,\varrho+\lambda)\,d\lambda,\\ p_{s,-\sigma s-j}(x,\xi,\varrho)&=\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}q_{-\sigma-j}(x,\xi,\varrho+\lambda)\,d\lambda.\end{aligned}$$

The description of the symbols $p_{s,-\sigma s-j}(x,\xi,\varrho)$ is easiest and most explicit in the case where A has scalar principal symbol, so let us consider this case first. Here $q_{-\sigma-j}$ is as described in (4.4) ff. Then, as shown in [S67, (25)] by integration of the denominator powers, the symbols $p_{s,-s\sigma-j}$ are:

$$\begin{aligned}p_{s,-s\sigma}(x,\xi,\varrho)&=(a_\sigma(x,\xi)-\varrho)^{-s},\\ p_{s,-s\sigma-j}(x,\xi,\varrho)&=\sum_{1\le k\le 2j}r_{j,k}(x,\xi)C_{-s,k}(a_\sigma(x,\xi)-\varrho)^{-s-k},\ \text{for } j>0,\\ &\text{with } C_{-s,k}=(-1)^k(-s)(-s-1)\cdots(-s-k+1).\end{aligned}$$

We have from Theorem 2.8 that $(a_\sigma-\varrho)^{-s}\in S^{0,-s}_{\sigma}(\Gamma)$, and (by composition) that the negative integer powers $(a_\sigma-\varrho)^{-k}$ are in $S^{-k\sigma,0}_{\sigma,\mathrm{hg}}(\Gamma)\cap S^{(1-k)\sigma,-1}_{\sigma,\mathrm{hg}}(\Gamma)$. The coefficients $r_{j,k}$ are in $S^{k\sigma-j,0}_{\sigma,\mathrm{hg}}(\Gamma)$. Thus, by application of the product rule and summation,

$$p_{s,-s\sigma-j}\subset S^{-j,-s}_{\sigma,\mathrm{hg}}(\Gamma)\cap S^{\sigma-j,-s-1}_{\sigma,\mathrm{hg}}(\Gamma).\tag{4.10}$$

In the general case, the procedure is the same for the principal term $p_{s,-s\sigma}$; for the lower order terms we base the analysis on the fact that each $q_{-\sigma-j}$ is a finite sum of terms of the form (4.2).

The function $h(x,\xi,\varrho)=\frac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}g(x,\xi,\varrho+\lambda)\,d\lambda$ will be estimated by the method of proof of Theorem 2.6. The present ϱ corresponds to $-\mu$ there. Let $w=\frac{1}{\varrho}$ (corresponding to $-z$ there), and let $\eta=\lambda w$; then

$$q_{-\sigma}(x,\xi,\varrho+\lambda)=(a_\sigma-\varrho-\lambda)^{-1}=w(wa_\sigma-1-\eta)^{-1}=w(p-\eta)^{-1},$$

where we denote $wa_\sigma-1=p$ (then $-p=za_\sigma+1$ is as in Theorem 2.6). The integration curve may be cut down to a closed curve $\mathcal L$ around the eigenvalues of p, with the outer and inner circular parts having radii as in Theorem 2.6. Then

$$h(x,\xi,\tfrac{1}{w})=w^{s-1}\tfrac{i}{2\pi}\int_{\mathcal L}\eta^{-s}g_1[w(p-\eta)^{-1}]^{\nu_1}\dots[w(p-\eta)^{-1}]^{\nu_M}g_{M+1}\,d\eta.$$

By (2.8), we have for $\eta\in\mathcal L$:

$$\begin{aligned}|\partial_x^\beta\partial_\xi^\alpha\partial_w^k(p-\eta)^{-1}|&\dot\le(|w|\langle\xi\rangle^\sigma+1)^{-1-k}\langle\xi\rangle^{\sigma k-|\alpha|}\\&\le(|w|\langle\xi\rangle^\sigma+1)^{-1}\langle\xi\rangle^{\sigma k-|\alpha|},\end{aligned}\tag{4.11}$$

moreover, by (2.9),

$$\big|\partial_x^\beta\partial_\xi^\alpha\partial_w^k[w(p-\eta)^{-1}]\big|\le\langle\xi\rangle^{\sigma(k-1)-|\alpha|},\tag{4.12}$$

for all indices. We can now derive estimates for $h(x,\xi,\frac{1}{w})$ by using the Leibniz formula under the integral sign, applying (4.11) to 1 or 2 factors $w^{-1}q_{-\sigma}=(p-\eta)^{-1}$ and applying (4.12) to all the remaining factors $q_{-\sigma}=w(p-\eta)^{-1}$. Since

$$|\mathcal L|\dot\le|w|\langle\xi\rangle^\sigma+1\quad\text{and}\quad|(-\eta)^{-s}|\dot\le1,$$

this gives for $i=1,2$ (cf. also (4.3)):

$$\begin{aligned}&|\partial_x^\beta\partial_\xi^\alpha\partial_w^k(w^{-s+1-i}h)|\\&\quad\dot\le|\mathcal L|\sup_{\eta\in\mathcal L}\big|(-\eta)^{-s}\partial_x^\beta\partial_\xi^\alpha\partial_w^k\big(w^{-i}g_1q_{-\sigma}^{\nu_1}\dots q_{-\sigma}^{\nu_M}g_{M+1})\big)\big|\\&\quad\dot\le\langle\xi\rangle^{r-(\nu-i)\sigma+\sigma k-|\alpha|}=\langle\xi\rangle^{(i-1)\sigma-j+\sigma k-|\alpha|}.\end{aligned}$$

This shows that

$$h\in S_\sigma^{-j,-s}(\Gamma)\cap S_\sigma^{\sigma-j,-s-1}(\Gamma);$$

it is easily verified to be weakly σ-homogeneous of degree $-\sigma s-j$. Collecting the terms, we conclude that the result of (4.10) holds for $p_{s,-s\sigma-j}$ also in the general case.

Finally, there is the treatment of remainders. Let

$$s_J(x,\xi,\varrho)=\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}r_J(x,\xi,\varrho+\lambda)\,d\lambda.\tag{4.13}$$

Since $r_J \in S_\sigma^{\sigma-J,-2}(\Gamma)$,

$$\partial_x^\beta \partial_\xi^\alpha r_J(x,\xi,\varrho+\lambda) = O((\varrho+\lambda)^{-2}\langle\xi\rangle^{\sigma-J-|\alpha|})$$

for all α, β, so by integration in λ, $\partial_x^\beta \partial_\xi^\alpha s_J(x,\xi,\varrho)$ is $O(\varrho^{-s-1}\langle\xi\rangle^{\sigma-J-|\alpha|})$, which shows the basic set of estimates for having s_J lie in $S_\sigma^{\sigma-J,-s-1}(\Gamma)$. However, there is a small difficulty concerning the derivatives $\partial_w^j(\varrho^{s+1}s_J)$ with respect to $w = \frac{1}{\varrho}$. One would like to perform ∂_w^j by passing through the integration sign in (4.13) using the properties of r_J. But replacing $\frac{1}{z}$ in $r_J(x,\xi,\frac{1}{z})$ by $\varrho+\lambda = \frac{1}{w}+\lambda$ means replacing z by $w/(1+\lambda w)$, and here we have to take the following into account when we differentiate with respect to w:

$$\begin{aligned}&\partial_w \frac{w}{1+\lambda w} = \frac{1}{(1+\lambda w)^2}, \quad \partial_w^2 \frac{w}{1+\lambda w} = \frac{-2}{(1+\lambda w)^3}\lambda, \ \dots\ , \\ &\partial_w^k \frac{w}{1+\lambda w} = \frac{k!}{(1+\lambda w)^{k+1}}(-\lambda)^{k-1}, \ \dots\ .\end{aligned}$$

Then one finds that

$$|\partial_w^j r_J(x,\xi,\tfrac{1}{w}+\lambda)| \dot{\le} \sup_{1\le k\le j-1} |\partial_z^k r_J(x,\xi,\tfrac{1}{z})|_{z=w/(1+\lambda w)}|\lambda|^{j-1-k},$$

where increasing powers of $|\lambda|$ come in and may violate the integrability. So, we can apply ∂_w under the integral sign in (4.13) for a few derivatives only, depending on s.

But there is another way, taking recourse to the Taylor expansion of r_J according to Theorem 2.10:

$$\begin{aligned}r_J(x,\xi,\varrho+\lambda) = \sum_{0\le\nu<N} & r_{J,\nu}(x,\xi)(\varrho+\lambda)^{-2-\nu} \\ &+ r'_{J,N}(x,\xi,\varrho+\lambda)(\varrho+\lambda)^{-2-N},\end{aligned}$$

with $r_{J,\nu} \in S^{(1+\nu)\sigma-J}$, $r'_{J,N} \in S_\sigma^{(1+N)\sigma-J,0}(\Gamma)$. For a given order $-M$ and a positive integer N we can take $J \ge (N+1)\sigma+M$; then $(N+1)\sigma-J \le -M$. The terms in the sum over ν integrate nicely, to give a sum

$$\sum_{0\le\nu<N} c_\nu r_{J,\nu}(x,\xi)\varrho^{-s-1-\nu},$$

which becomes a simple polynomial in w after multiplication by ϱ^{s+1} (namely $\sum c_\nu r_{J,\nu} w^\nu$). This allows differentiations in w straightforwardly. For the new remainder term, the derivatives

$$\partial_w^j\Big(\varrho^{s+1}\tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}(\varrho+\lambda)^{-2-N} r'_{J,N}(x,\xi,\varrho+\lambda)\,d\lambda\Big),$$

can be studied by differentiating through the integral, as long as $j \le N + \operatorname{Re} s + 1$. All the contributions are of order $\le -M$.

So, we can adapt the choice of J both to how many derivatives ∂_w^j we want to estimate, and to how low orders the terms should have. This can be combined with the fact that the homogeneous terms $p_{s,-s\sigma-l}$ have been shown to have the right behavior:

To show that $s_{J_0} \in S_\sigma^{\sigma-J_0,-s-1}(\Gamma)$ for a given J_0, let $-M = \sigma - J_0$ and, for any given $N \in \mathbb{N}$, choose $J \geq (N+1)\sigma + M$, then

$$s_{J_0} = \sum_{J_0 \leq j < J} p_{s,-s\sigma-j} + s_J,$$

where $\partial_w^j(\varrho^{s+1}s_J) \in S^{\sigma-J_0}$, uniformly for $|w| \leq 1$, $j = 0, \dots, N$, and the terms in the sum are already known to satisfy such estimates. □

The statement in the theorem can be improved when $\operatorname{Re} s > 1$, by use of the calculus: When $s = N + s'$, $N \in \mathbb{N}$ and $\operatorname{Re} s' > 0$, then

$$p_s \in S_{\sigma,\mathrm{phg}}^{-\sigma N,-s'}(\Gamma) \cap S_{\sigma,\mathrm{phg}}^{0,-s}(\Gamma), \tag{4.14}$$

since $(A-\varrho)^{-s} = (A-\varrho)^{-N}(A-\varrho)^{-s'}$. Note also that when $\operatorname{Re} s \in]0,1]$, we can use (2.12) to see that $p_s \in S_\sigma^{-\sigma \operatorname{Re} s,-s+1}(\Gamma)$; hence for $s = s' + N$ with $\operatorname{Re} s' \in]0,1]$,

$$p_s \in S_\sigma^{-\sigma \operatorname{Re} s,-s'+1}(\Gamma), \tag{4.15}$$

assuring that the order is $-\sigma \operatorname{Re} s$ at each ϱ. (In (4.15), the indices do not in general match the homogeneity.)

The proof details based on (4.2) also work for not necessarily polyhomogeneous symbols, which could in fact be included in the treatment under suitable hypotheses; we shall not pursue this aspect here.

Finally Corollary 3.5 and Theorem 4.2 can be combined to give:

Theorem 4.3. *Let A satisfy the assumptions of Theorem 4.1, let $\operatorname{Re} s > 0$, and let B be a one-step polyhomogeneous ψdo of order $\nu \in \mathbb{R}$. For $\nu - \sigma \operatorname{Re} s < -n$, $B(A-\varrho)^{-s}$ is trace-class and the trace has an expansion*

$$\operatorname{Tr}(B(A-\varrho)^{-s}) \sim \sum_{j=0}^{\infty} c_j \varrho^{-s+\frac{\nu+n-j}{\sigma}} + \sum_{k=0}^{\infty} \big(c'_k \log \varrho + c''_k\big) \varrho^{-s-k} , \tag{4.16}$$

for $|\varrho| \to \infty$, uniformly in closed subsectors of Γ.

Proof. We have that $s = N + s'$ for some $N \in \mathbb{N}$ and s' with $\operatorname{Re} s' \in]0,1]$. Since B has symbol in $S_{\sigma,\mathrm{phg}}^{\nu,0}(\Gamma)$ (cf. (2.2)) and $(A-\varrho)^{-s}$ satisfies (4.14), we find by (3.2) that the composition of B and $(A-\varrho)^{-s}$ has symbol in $S_{\sigma,\mathrm{phg}}^{\nu-\sigma N,-s'}(\Gamma) \cap S_{\sigma,\mathrm{phg}}^{\nu,-s}(\Gamma)$. In view of (4.15), we also have that $B(A-\varrho)^{-s}$ has symbol in $S_\sigma^{\nu-\sigma \operatorname{Re} s,-s'+1}(\Gamma)$, so it is trace-class (and all terms in the symbol are integrable in ξ) since $\nu - \sigma \operatorname{Re} s < -n$.

Then Corollary 3.5 applies, both with $\{m, \delta\} = \{\nu - \sigma N, -s'\}$ and with $\{m, \delta\} = \{\nu, -s\}$, the degrees of the homogeneous symbols going down by integer steps in the symbol series. This gives (4.16), when we use the most restrictive information obtained from having $\delta = -s$. $\square$

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