POLES OF ZETA AND ETA FUNCTIONS FOR PERTURBATIONS OF THE ATIYAH-PATODI-SINGER PROBLEM

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Dedicated to Professor Norio Shimakura on the occasion of his sixtieth birthday.

Abstract. The zeta and eta functions of a differential operator of Dirac-type on a compact n-dimensional manifold, provided with a well-posed pseudodifferential boundary condition, have been shown in [G99] to be meromorphic on $\mathbb{C}$ with simple or double poles on the real axis. Extending results from [G99] we show how perturbations of the boundary condition of order $-J$ affect the poles; in particular they preserve a possible regularity of zeta at 0 and a possible simple pole of eta at 0 when $J \geq n$. This applies to perturbations of spectral boundary conditions, also when the structure is non-product and the problem is non-selfadjoint.

Let D be a first-order differential operator (e.g. a Dirac-type operator) from $C^\infty(X, E_1)$ to $C^\infty(X, E_2)$ (E_1 and E_2 Hermitian N-dimensional vector bundles over a compact n-dimensional C^∞ manifold X with boundary $\partial X = X'$), and let D_B be the L_2-realization defined by a well-posed zero-order pseudodifferential boundary condition $B(u|_{X'}) = 0$. For $\Delta_1 = {D_B}^* D_B$ and $\Delta_2 = D_B {D_B}^*$, the following expansions were shown in [G99]:

$$\text{(1)} \qquad \operatorname{Tr}(\Delta_i - \lambda)^{-m} \sim \sum_{-n \le k < 0} \tilde{a}_{i,k} (-\lambda)^{-\frac{k}{2} - m} + \sum_{k \ge 0} \big(\tilde{a}_{i,k} \log(-\lambda) + \tilde{a}'_{i,k} \big) (-\lambda)^{-\frac{k}{2} - m},$$

$$\text{(2)} \qquad \operatorname{Tr} e^{-t\Delta_i} \sim \sum_{-n \le k < 0} a_{i,k} t^{\frac{k}{2}} + \sum_{k \ge 0} \big(-a_{i,k} \log t + a'_{i,k} \big) t^{\frac{k}{2}},$$

$$\text{(3)} \qquad \Gamma(s) \operatorname{Tr} \Delta_i^{-s} \sim \sum_{-n \le k < 0} \frac{a_{i,k}}{s + \frac{k}{2}} + \frac{\nu_0(\Delta_i)}{s} + \sum_{k \ge 0} \Big(\frac{a_{i,k}}{(s + \frac{k}{2})^2} + \frac{a'_{i,k}}{s + \frac{k}{2}} \Big);$$

In (1), $m > \frac{n}{2}$ and $\lambda \to \infty$ on rays in $\mathbb{C} \setminus \mathbb{R}_+$; in (2), $t \to 0+$. (3) means that $\Gamma(s) \operatorname{Tr} \Delta_i^{-s}$, defined in a standard way for $\operatorname{Re} s > \frac{n}{2}$, extends meromorphically to $\mathbb{C}$ with the pole structure indicated in the right hand side. Here ν_0 is the dimension of the nullspace (on which Δ_i^{-s} is taken to be zero). $\operatorname{Tr} \Delta_i^{-s}$ is also known as the *zeta function* $\zeta(\Delta_i, s) = \sum_{\text{eigenv. } \lambda > 0} \lambda^{-s}$. The three expansions (1)–(3) are essentially equivalent, cf. [GS96], the k'th coefficients being interrelated by universal constants.

A fundamental example is the Atiyah-Patodi-Singer problem [APS75], where D is a Dirac operator with product structure near X' and B is taken as the orthogonal projection $\Pi_\geq$ onto the nonnegative eigenspace for an associated selfadjoint operator A on X' (the *spectral boundary condition*). For the general case without assumptions on product structure near X', an expansion up to $k<1$ was shown in [G92] with $a_{i,0}=0$, and a full expansion was shown in the joint work with Seeley [GS95].

It is important in applications to know whether the coefficient $a_{i,0}$ vanishes. Since $\Gamma(s)$ has a pole at 0, $a_{i,0}=0$ means that $\zeta(\Delta_i,s)$ is regular at 0. Then the derivative $-\zeta'(\Delta_i,0)$ is well-defined; it equals the "logarithm of the determinant" of Δ_i. (For the connection with determinants, note that $-\zeta'(\Delta_i,s)=\sum_{\text{eigenv.}\,\lambda>0}\lambda^{-s}\log\lambda$; if Δ_i is replaced by a positive matrix T, this equals $\log\det T$ for $s=0$.)

In an interesting recent paper [W99], Wojciechowski studies the regularity at 0 of the zeta and eta functions of D_B in the case where D is a selfadjoint Dirac-type operator with product structure near X', B is a pseudodifferential projection differing from the *Calderón projector* C^+ by an operator of order $-\infty$, and D_B is selfadjoint (with proof details for n odd). He does mention our paper [G99] in preprint form, but only with a vague statement that "at the moment, the problem of explicit computation of the coefficients in the expansion is open". This is not so for the particular coefficient $a_{i,0}$, since our Theorem 9.4 (showing that order $-\infty$ perturbations of the boundary condition do not change the values $a_{i,k}$), implies that $a_{i,0}=0$ in the case considered in [W99], as stated in our Corollary 9.5. This covers the result on $\zeta_{\mathcal{D}_P^2}(s)$ in [W99, Th. 0.2] (also for n even).

The purpose of the present note is to account for the consequences of [G99] and to extend the analysis to perturbations of arbitrary finite negative order, showing which of the coefficients in (1)–(3) are left unaffected. We establish similar results for eta functions, and include some improved details on the use of the polyhomogeneous calculus of [GS95].

The realization D_B of D and its adjoint $(D_B)^*$ (acting as D^* with a certain boundary condition $B^{(*)}u|_{X'}=0$) are imbedded in the larger elliptic system

$$\mathcal{D}_{\mathcal{B}}=\begin{pmatrix}0&-D_B{}^*\\ D_B&0\end{pmatrix},\ \text{with } \mathcal{R}_\mu=(\mathcal{D}_{\mathcal{B}}+\mu)^{-1}=\begin{pmatrix}\mu(\Delta_1+\mu^2)^{-1}&D_B{}^*(\Delta_2+\mu^2)^{-1}\\ -D_B(\Delta_1+\mu^2)^{-1}&\mu(\Delta_2+\mu^2)^{-1}\end{pmatrix}, \tag{5}$$

for $\mu\in\mathbb{C}\setminus i\mathbb{R}$; the resolvents $R_{i,\mu}=(\Delta_i+\mu^2)^{-1}$ can be retrieved from this.

For two choices B_1 and B_2 of B, let $B'=B_2-B_1$. Denoting $\mathcal{B}_j=\begin{pmatrix}B_j & B_j^{(*)}\end{pmatrix}$, for $j=1,2$, and $\mathcal{B}'=\mathcal{B}_2-\mathcal{B}_1$, we have that the inverses $\begin{pmatrix}\mathcal{R}_{j,\mu}&\mathcal{K}_{j,\mu}\end{pmatrix}$ of $\begin{pmatrix}\mathcal{D}+\mu\\ \mathcal{B}_j\gamma_0\end{pmatrix}$ for $\mu\in\mathbb{C}\setminus i\mathbb{R}$ satisfy

$$\begin{pmatrix}\mathcal{R}_{2,\mu}&\mathcal{K}_{2,\mu}\end{pmatrix}=\begin{pmatrix}\mathcal{R}_{1,\mu}&\mathcal{K}_{1,\mu}\end{pmatrix}\begin{pmatrix}\mathcal{D}+\mu\\ \mathcal{B}_1\gamma_0\end{pmatrix}\begin{pmatrix}\mathcal{R}_{2,\mu}&\mathcal{K}_{2,\mu}\end{pmatrix}=\begin{pmatrix}\mathcal{R}_{1,\mu}&\mathcal{K}_{1,\mu}\end{pmatrix}\begin{pmatrix}I&0\\ -\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu}&I-\mathcal{B}'\gamma_0\mathcal{K}_{2,\mu}\end{pmatrix}; \tag{6}$$

here $\gamma_0u=u|_{X'}$. It is shown in [G99, Cor. 8.3] (to which we refer for notation) that the operators have the structure $\mathcal{R}_{j,\mu}=\widetilde{Q}_{\mu,+}-K_\mu^+S'_{j,\mu}\mathcal{B}_j\gamma_0\widetilde{Q}_{\mu,+}$ and $\mathcal{K}_{j,\mu}=K_\mu^+S'_{j,\mu}\mathcal{B}_j\gamma_0$, where the $S'_{j,\mu}$ denote particular right inverses of $\mathcal{B}_jC_\mu^+$; they are weakly polyhomogeneous with symbols in $S^{0,0}$, whereas the other μ-dependent factors are strongly polyhomogeneous. Then

$$\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu}=-\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu}=-K_\mu^+S'_{1,\mu}\mathcal{B}'\gamma_0(\widetilde{Q}_{\mu,+}-K_\mu^+S'_{2,\mu}\mathcal{B}_2\gamma_0\widetilde{Q}_{\mu,+}). \tag{7}$$

Denoting $\Delta_{1,j}=D_{B_j}{}^*D_{B_j}$ and $\Delta_{2,j}=D_{B_j}D_{B_j}{}^*$, with resolvents $R_{i,j,\mu}$, we have in view of (5),

$$\begin{aligned}R_{1,2,\mu}-R_{1,1,\mu}&=\mu^{-1}\begin{pmatrix}1&0\end{pmatrix}(\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu})\begin{pmatrix}1\\0\end{pmatrix},\\ D_{B_2}R_{1,2,\mu}-D_{B_1}R_{1,1,\mu}&=-\begin{pmatrix}0&1\end{pmatrix}(\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu})\begin{pmatrix}1\\0\end{pmatrix},\end{aligned} \tag{8}$$

with similar formulas for $i = 2$. The second expression has a similar structure as in (7), and the first one has it with an extra factor μ^{-1}. We can likewise find the explicit structures of

$$
\begin{aligned}
&R^m_{1,2,\mu} - R^m_{1,1,\mu} = (R_{1,2,\mu} - R_{1,1,\mu})(R^{m-1}_{1,2,\mu} + R^{m-2}_{1,2,\mu} R_{1,1,\mu} + \dots + R^{m-1}_{1,1,\mu}),\\
&D_{B_2} R^m_{1,2,\mu} - D_{B_1} R^m_{1,1,\mu} = D_{B_2} R^m_{1,2,\mu}(R_{1,2,\mu} - R_{1,1,\mu})(R^{m-2}_{1,2,\mu} + \dots + R^{m-2}_{1,1,\mu})\\
&\qquad\qquad + (D_{B_2} R_{1,2,\mu} - D_{B_1} R_{1,1,\mu}) R^{m-1}_{1,2,\mu},
\end{aligned}
\tag{9}
$$

for higher powers m.

Theorem 1. *Let $B' = B_1 - B_2$ be of order $-J$ for some $1 \le J \le \infty$, and let $m \ge \max\{n - J, 1\}$. Then, with φ denoting a morphism from E_2 to E_1, there are expansions*

$$
\operatorname{Tr}(R^m_{i,2,\mu} - R^m_{i,1,\mu}) \sim \sum_{n-J<k<0} \tilde{c}_{i,k} \mu^{-2m-k} + \sum_{k \ge 0} (\tilde{c}_{i,k} \log \mu + \tilde{c}'_{i,k}) \mu^{-2m-k},
\tag{10}
$$

$$
\operatorname{Tr}(\varphi D_{B_2} R^m_{1,2,\mu} - \varphi D_{B_1} R^m_{1,1,\mu}) \sim \sum_{n-J<k<0} \tilde{d}_k \mu^{1-2m-k} + \sum_{k \ge 0} (\tilde{d}_k \log \mu + \tilde{d}'_k) \mu^{1-2m-k},
\tag{11}
$$

for $\mu \to \infty$ in $\mathbb{C} \setminus i\mathbb{R}$; the $\tilde{c}_{i,k}$ and $\tilde{d}_k$ vanish when $k \le J - n$.

Proof. First consider (10). We can let $i = 1$. The operator $R_{1,2,\mu} - R_{1,1,\mu}$ is of the form $\mu^{-1} \mathcal{K}_\mu \mathcal{S}_\mu \mathcal{T}_\mu$, where $\mathcal{K}_\mu$ and $\mathcal{T}_\mu$ are a strongly polyhomogeneous Poisson resp. trace operator of order 0 resp. -1, and $\mathcal{S}_\mu$ is a weakly polyhomogeneous ψdo on X' with symbol in $S^{-J,0}$, in the calculus introduced in [GS95]. If $J \ge n - 1$, the operator is trace-class and has the same trace as the ψdo on X' obtained by circular perturbation, $\mu^{-1} \mathcal{S}_\mu \mathcal{T}_\mu \mathcal{K}_\mu$. Since $\mathcal{T}_\mu \mathcal{K}_\mu$ is a strongly polyhomogeneous ψdo on X' of order -1, hence has symbol in $S^{-1,0} \cap S^{0,-1}$, the composed expression is a weakly polyhomogeneous ψdo on X' with symbol in $S^{-1-J,-1} \cap S^{-J,-2}$.

If $J < n - 1$, we need to consider a power as in (9) with $m \ge n - J$, to get a trace class operator. The operators $R_{1,j,\mu}$ are of the form $Q_{\mu,+} - \mathcal{K}_\mu \mathcal{S}'_\mu \mathcal{T}_\mu$ with $\mathcal{K}_\mu$ and $\mathcal{T}_\mu$ as above, Q_μ a strongly polyhomogeneous ψdo of degree -2 and $\mathcal{S}'_\mu$ having symbol in $S^{0,-1}$. Then by circular perturbation,

$$
\begin{aligned}
\operatorname{Tr}_X(R^m_{1,2,\mu} - R^m_{1,1,\mu}) &= \operatorname{Tr}_X(\mu^{-1} \mathcal{K}_\mu \mathcal{S}_\mu \mathcal{T}_\mu (R^{m-1}_{1,2,\mu} + \dots + R^{m-1}_{1,1,\mu}))\\
&= \operatorname{Tr}_{X'}(S_\mu), \quad S_\mu = \mu^{-1} \mathcal{S}_\mu \mathcal{T}_\mu (R^{m-1}_{1,2,\mu} + \dots + R^{m-1}_{1,1,\mu}) \mathcal{K}_\mu.
\end{aligned}
\tag{12}
$$

The various composition rules explained in [GS95] show that S_μ is a composite of weakly polyhomogeneous and strongly polyhomogeneous ψdo's on X'; its symbol lies in $S^{-m-J,-m} \cap S^{-J,-2m}$, since each factor $R_{1,j,\mu}$ results in multiplying the symbol space by $S^{-1,-1} \cap S^{0,-2}$.

Now [GS95, Th. 2.1] can be applied to the resulting ψdo on the manifold X' of dimension $n - 1$. Since the total order is $-2m - J$ and the d-index is $-2m$, the formula [GS95, (2.1)] gives an expansion (10). For the log-coefficients we use the additional information from [GS95, Th. 2.1] stating that the contribution to the coefficient $\tilde{c}_{1,k}$ of $\mu^{-2m-k} \log \mu$ when $k \ge 0$ comes entirely from the homogeneous symbol of S_μ of degree $(1 - n) - 2m - k$. Since the highest degree of homogeneity occurring in the symbol is $-2m-J$, these $\tilde{c}_{1,k}$'s vanish for $k+n-1<J$, i.e., for $k\le n-J$. This shows the statements on (10).

For (11), we have that $\varphi D_{B_2}R_{1,2,\mu}-\varphi D_{B_1}R_{1,1,\mu}$ is of the form $\mathcal{K}_\mu\mathcal{S}_\mu\mathcal{T}_\mu$, where $\mathcal{K}_\mu$, $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$ are as above. A circular perturbation gives a weakly polyhomogeneous ψdo on X', now with symbol in $S^{-1-J,0}\cap S^{-J,-1}$. For $J\ge n-1$ we can pass directly to an application of [GS95, Th. 2.1] as above. For $J<n-1$ we take a high enough m and find, very similarly to the above considerations, that circular perturbation gives a ψdo on X' with symbol in $S^{-m-J,1-m}\cap S^{-J,1-2m}$. Then an application of [GS95, Th. 2.1] gives (11). □

The cases $J=1$ and $J=\infty$ were treated in [G99, Th. 9.4 ff.]. (The considerations above on composite expressions are very similar to those in [G99, Th. 9.1]. The deduction given here in terms of resolvent powers is slightly more direct than that indicated in [G99] via μ-derivatives; the present considerations can also be used for the passage from (9.1) to (9.9)–(9.11) in [G99].)

Theorem 1 is carried over to a result on heat traces and zeta and eta functions by application of the transition rules [GS96, Cor. 2.10 and Prop. 5.1]:

Corollary 2. *Under the hypotheses of Theorem* 1, *one has:*

(13)
$$\operatorname{Tr}(e^{-t\Delta_{i,2}}-e^{-t\Delta_{i,1}})\sim\sum_{n-J<k<0}c_{i,k}t^{\frac{k}{2}}+\sum_{k\ge0}(-c_{i,k}\log t+c'_{i,k})t^{\frac{k}{2}},$$

(14)
$$\operatorname{Tr}(\varphi D_{B_2}e^{-t\Delta_{1,2}}-\varphi D_{B_1}e^{-t\Delta_{1,1}})\sim\sum_{n-J<k<0}d_kt^{\frac{k-1}{2}}+\sum_{k\ge0}(-d_k\log t+d'_k)t^{\frac{k-1}{2}},$$

(15)
$$\Gamma(s)\operatorname{Tr}((\Delta_{i,2})^{-s}-(\Delta_{i,1})^{-s})\sim$$
$$\sim\sum_{n-J<k<0}\frac{c_{i,k}}{s+\frac{k}{2}}+\frac{\nu_0(\Delta_{i,2})-\nu_0(\Delta_{i,1})}{s}+\sum_{k\ge0}\Big(\frac{c_{i,k}}{(s+\frac{k}{2})^2}+\frac{c'_{i,k}}{s+\frac{k}{2}}\Big),$$

(16)
$$\Gamma(s)\operatorname{Tr}(\varphi D_{B_2}(\Delta_{1,2})^{-s}-\varphi D_{B_1}(\Delta_{1,1})^{-s})\sim$$
$$\sim\sum_{n-J<k<0}\frac{d_k}{s+\frac{k-1}{2}}+\sum_{k\ge0}\Big(\frac{d_k}{(s+\frac{k-1}{2})^2}+\frac{d'_k}{s+\frac{k-1}{2}}\Big),$$

where the $c_{i,k}$ *and* d_k *vanish for* $k\le J-n$.

Now consider operators of Dirac-type (notation of [G99]); on a collar neighborhood U of X' they are of the form:

(17)
$$D=\sigma(\tfrac{\partial}{\partial x_n}+A+x_nP_1+P_0),$$

where x_n is a normal coordinate, A is a selfadjoint elliptic first-order differential operator in $L_2(X',E_1|_{X'})$, the P_j are differential operators of order $\le j$, and σ is a unitary morphism from $E_1|_{X'}$ to $E_2|_{X'}$. (U is identified with $X'\times[0,c]$ and the E_i are liftings of $E_i|_{X'}$ here.) The product case is the case where, moreover, the P_j are 0 on U.

The coefficients in (1)–(3) were determined from the zeta and eta expansions of D and A in [GS96, Cor. 2.7–2.8] in the product case with $B = \Pi_\geq$ (the orthogonal projection onto the nonnegative eigenspace for A); then in particular all the $\tilde{a}_{i,k}$ and $a_{i,k}$ with $k \geq 0$ vanish when n is odd; the $\tilde{a}_{i,k}$ and $a_{i,k}$ with k even ≥ 0 vanish when n is even. (The remaining coefficients are nonzero in general, cf. Gilkey-Grubb [GG98].) Combining this with Corollary 2, one finds that the same holds for perturbations $B_2 = \Pi_\geq + S$ of $B_1 = \Pi_\geq$ with S of order $-\infty$; this is formulated in [G99, Cor. 9.5]. This result includes the case $B_2 = C^+ + S'$, S' of order $-\infty$, since $C^+ - \Pi_\geq$ is of order $-\infty$ in the product case by [G99, Prop. 4.1] (C^+ is the Calderón projector for D). With the present accounting of the effect of perturbations of order $-J$, we get the following extended information:

Corollary 3. *Consider D_B in the product case with $B = \Pi_\geq + S$, S of order $-J$ (then $B - C^+$ is likewise of order $-J$). If n is odd, all $\tilde{a}_{i,k}$ and $a_{i,k}$ with $0 \leq k \leq J - n$ vanish in (1)–(3). If n is even, all the $\tilde{a}_{i,k}$ and $a_{i,k}$ with $0 \leq k \leq J - n$ and k even vanish in (1)–(3).*

We formulate some consequences for the zero'th coefficient in detail.

Corollary 4. *In (1)–(3), the coefficients $\tilde{a}_{i,0}$ and $a_{i,0}$ vanish in the following cases:*

(a) *D is a Dirac-type operator and $B - \Pi_\geq$ is a ψdo of order $\leq -n$.*
(b) *D is a Dirac-type operator in the product case, and $B - C^+$ is a ψdo of order $\leq -n$.*
(c) *D is a Dirac-type operator, $B - C^+$ is a ψdo of order $\leq -n$, and the structure near X' is so close to the product case that $\Pi_\geq - C^+$ is of order $-n$.*

Hence in these cases, the zeta function $\zeta(\Delta_i, s)$ is regular at $s = 0$.

Proof. The result in case (a) follows from Corollary 2 together with the fact that $a_{i,0}$ vanishes for Dirac-type operators with the boundary condition $\Pi_\geq u|_{X'} = 0$ ([G92]). The result in case (b) follows from Corollary 3, and the result in case (c) is an immediate consequence of the preceding ones. □

The reason that we do not claim that (a) holds with $\Pi_\geq$ replaced by C^+ is that C^+ will in general differ from $\Pi_\geq$ by an operator that is merely of order -1.

Let us also consider the *eta function*. We have from [G99, (9.9)–(9.11)]:

(18)
$$\operatorname{Tr}[\varphi D_B(\Delta_1 - \lambda)^{-m}] \sim \sum_{-n<k<0} \tilde{b}_k(-\lambda)^{-\frac{k-1}{2}-m} + \sum_{k\geq 0}\big(\tilde{b}_k \log(-\lambda) + \tilde{b}'_k\big)(-\lambda)^{-\frac{k-1}{2}-m},$$

(19)
$$\operatorname{Tr}[\varphi D_B e^{-t\Delta_1}] \sim \sum_{-n<k<0} b_k t^{\frac{k}{2}} + \sum_{k\geq 0}\big(-b_k \log t + b'_k\big)t^{\frac{k-1}{2}},$$

(20)
$$\Gamma(s)\operatorname{Tr}[\varphi D_B \Delta_1^{-s}] \sim \sum_{-n<k<0} \frac{b_k}{s + \frac{k-1}{2}} + \sum_{k\geq 0}\Big(\frac{b_k}{(s+\frac{k-1}{2})^2} + \frac{b'_k}{s+\frac{k-1}{2}}\Big),$$

with similar formulas where D_B and $D_B{}^*$ are interchanged. When $E_1 = E_2$, the *eta function* is defined by $\eta(D_B, s') = \operatorname{Tr}(D_B \Delta_1^{-\frac{s'+1}{2}})$, so in (20), s' corresponds to $s = \frac{s'+1}{2}$, and the expansion with $\varphi = I$ takes the form

(21)
$$\Gamma(\tfrac{s'+1}{2})\eta(D_B, s') \sim \sum_{-n<k<0} \frac{2b_k}{s'+k} + \sum_{k\geq 0}\Big(\frac{4b_k}{(s'+k)^2} + \frac{2b'_k}{s'+k}\Big).$$

The coefficients in (18)–(21) were determined from the zeta and eta expansions of D and A in [GS96, Cor. 4.5] in the product case with $B=\Pi_\geq$; in particular, the b_k vanish for $k\geq 0$ if n is odd, and they vanish for k even >0 if n is even. For $k=0$ and n even, b_0 is proportional to the residue of $\operatorname{Tr}_{X'}(\sigma A|A|^{-s-1})$ at $s=0$. It vanishes e.g. if $\sigma A=-A\sigma$, for in this case $\operatorname{Tr}_{X'}(\sigma A|A|^{-s-1})\equiv 0$ (see also [GS96, Cor. 2.4]).

Then Corollary 2 implies:

Corollary 5. *Consider the product case with $E_1=E_2$ and $B=\Pi_\geq+S$, S of order $-J$ (so also $B-C^+$ is of order $-J$). When n is odd, all b_k with $0\leq k\leq J-n$ vanish in* (21). *When n is even, all the b_k with $0<k\leq J-n$ and k even vanish in* (21)*; if $\sigma A=-A\sigma$, also $b_0=0$.*

Note that in the product case, D is selfadjoint if $\sigma^*=-\sigma$ and $\sigma A=-A\sigma$.

Since $\Gamma(s)$ is regular at $\frac12$, b_0 has to vanish in order for $\eta(D_B,s')$ to have a simple pole at $s'=0$. In case of a simple pole, our analysis shows that perturbations of the boundary condition by operators of order $\leq -n$ still give a simple pole. The pole vanishes under more restrictive circumstances, cf. Douglas and Wojciechowski [DW91], [GS96, Cor. 4.6], [W99].

The results in Corollary 4 extend to b_0 in view of the following theorem.

Theorem 6. *Consider the realization D_B of a Dirac-type operator D (cf.* (17)*) with $B=\Pi_\geq$. The coefficients $\tilde b_0$ and b_0 in* (18)–(20) *are the same as these coefficients in the expansions for D^0_B, where $D^0=\chi\sigma(\frac{\partial}{\partial x_n}+A)+(1-\chi)D$, with $\chi=1$ near X' and supported in $X'\times[0,c[$.*

Proof. Observe that we are here dealing with a perturbation of the interior operator, not of the boundary condition as in Theorem 1. This is advantageous, for we can then use the result of [G92, Th. 4.4], showing that the resolvent $(\Delta_1-\lambda)^{-1}$ has the form

$$(\Delta_1-\lambda)^{-1}=Q_{\lambda,+}+\chi G^0_\lambda\chi+G^{(1)}_\lambda, \tag{22}$$

where $Q_{\lambda,+}$ is the restriction to X of a parametrix Q_λ of $D^*D-\lambda$ defined on a neighboorhood of X, G^0_λ is the singular Green part of the resolvent in the product situation where D is replaced by $\sigma(\frac{\partial}{\partial x_n}+A)$ and X is replaced by $X'\times[0,\infty[$, and $G^{(1)}_\lambda$ is a singular Green operator of order -3, class 0 and *regularity* 0. Then

$$(m-1)!D(\Delta_1-\lambda)^{-m}=D\partial_\lambda^{m-1}(\Delta_1-\lambda)^{-1}=D\partial_\lambda^{m-1}Q_{\lambda,+}+D\partial_\lambda^{m-1}(\chi G^0_\lambda\chi)+D\partial_\lambda^{m-1}G^{(1)}_\lambda. \tag{23}$$

In the calculation of the trace, the first term gives no logarithms, the second term gives the expansion known for the product case, where the log-terms start with $\tilde b_0(-\lambda)^{\frac12-m}\log(-\lambda)$, and the third term gives an expansion with pure powers up to and including the term $c(-\lambda)^{\frac12-m}$ (the method of [G92] gives an $O((-\lambda)^{\frac18-m})$ after this, and the method of [GS95] gives a full expansion with log-terms for higher k). So the first log-term for $D(\Delta_1-\lambda)^{-m}$ is the same as that for $D^0((D^0_B)^*D^0_B-\lambda)^{-m}$. $\square$

One could also have shown this by arguments as in the last paragraph of the proof of [GS95, Th. 3.13], but the notation would be quite heavy.

Corollary 7. *Let $E_1 = E_2$ and assume that $\sigma A = -A\sigma$ if n is even. In the three cases* (a)–(c) *listed in Corollary 4, the coefficient b_0 in* (21) *vanishes, in other words the eta function $\eta(\Delta_i, s')$ has at most a simple pole at $s' = 0$.*

Proof. The product case with $B = \Pi_\geq$ has $b_0 = 0$, as accounted for in Corollary 5. Hence so has the non-product case with $B = \Pi_\geq$, by Theorem 6. The statements for the cases (a) and (b) then follow from Corollary 2 resp. 5, and case (c) is an immediate consequence. □

Note that we do not assume selfadjointness of D_B as in [W99].

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