PARAMETRIZED PSEUDODIFFERENTIAL OPERATORS AND GEOMETRIC INVARIANTS

GERD GRUBB
Copenhagen University Mathematics Department
Universitetsparken 5, DK-2100 Copenhagen, Denmark

Abstract. This is based on joint work with R. T. Seeley. The introduction presents the problem of parameter-dependent calculi for ψdo's and the question of trace asymptotics for Atiyah-Patodi-Singer operators. Chapter 2 establishes relations between the three operator functions: resolvent, heat operator and power operator (zeta function). Chapter 3 explains our parameter-dependent ψdo calculus with weak polyhomogeneity, showing how logarithmic terms appear in trace formulas. In Chapter 4, the APS problem is treated in the case with a product structure near the boundary, where functional calculus on the cylinder leads to precise formulas for heat trace expansions and zeta function pole structure. Finally, Chapter 5 treats the APS problem in the non-product case where the weakly polyhomogeneous ψdo calculus is used to get asymptotic trace expansions generalizing those in the product case.

1. Introduction

1.1. PARAMETER-DEPENDENT CALCULI

A typical case of an interesting parameter-dependent *pseudodifferential operator* (henceforth abbreviated to ψdo) is the resolvent $R_\lambda = (P-\lambda)^{-1}$ of a, say, strongly elliptic operator P on a compact manifold. Let the symbol of P (in a local coordinate system) be

$$p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \dots,$$

where each term $p_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ (for a positive integer m), then we write $-\lambda$ as

$$-\lambda = e^{i\theta}\mu^m, \quad \mu = |\lambda|^{1/m}, \theta \in [0, 2\pi]$$

(where i is the imaginary unit $\sqrt{-1}$), and assign to $P - \lambda$ the *principal* symbol

$$\overline{p}_m(x,\xi,\lambda) = p_m(x,\xi) + e^{i\theta}\mu^m$$

(also denoted $\overline{p}_m(x,\xi,\theta,\mu)$), and the full symbol $\overline{p} + e^{i\theta}\mu^m$ where the lower order terms are the same as those for P. The inverse of this principal symbol,

$$q_m(x,\xi,\lambda) = \overline{p}_m(x,\xi,\lambda)^{-1}$$

will then be the principal symbol of the resolvent.

Here μ can be considered as one more "cotangent variable" in addition to $\xi_1, \xi_2, \dots, \xi_n$, and $\overline{p}_m$ is homogeneous of degree m in (ξ,μ).

There is a marked difference between the case of a differential operator and that of a ψdo. In the first case, $\overline{p}_m$ is polynomial in (ξ,μ), hence homogeneous and C^∞ in $(\xi,\mu) \in \overline{\mathbf{R}}^{n+1}_+$. In the second case, the homogeneous symbol $p_m(x,\xi)$ usually has a lack of smoothness at $\xi = 0$ (it has only m bounded derivatives), so $\overline{p}_m$ will have this lack of smoothness on the whole halfline $\{(0,\mu) \mid \mu \geq 0\}$. (Alternatively, if p_m is modified in a bounded neighborhood of 0 to be C^∞, the ensuing modification of $\overline{p}_m$ takes place in an unbounded set.)

This also has an effect on the *estimates* of q_m. Here one has (with $\langle x \rangle = (|x|^2+1)^{1/2}$):

$$\begin{aligned} D^\alpha_\xi q_m &= O(\langle(\xi,\mu)\rangle^{-m-|\alpha|}), \text{ for } |\alpha| \leq m, \\ D^\alpha_\xi q_m &= O(\langle(\xi,\mu)\rangle^{-2m}\langle\xi\rangle^{m-|\alpha|}), \text{ for } |\alpha| \geq m, \end{aligned} \tag{1.1}$$

where the first line extends to all α *if and only if* p_m is polynomial in ξ. In the polynomial case one can apply the usual symbolic calculus, just in one more variable, getting simple and straightforward results, whereas in the general case the fact that only the first m estimates are standard (the so-called *regularity number* is m), gives severe trouble.

For boundary value problems there are similar phenomena. In the differential operator case, the resolvent parameter enters as another cotangent variable, on a par with the others, whereas for a *pseudodifferential boundary operator*, a resolvent parameter, when considered as a cotangent variable, gives symbolic estimates where only finitely many of them are "good". Again one assigns a regularity number to the operator, this will now be different for the different types (trace operators, Poisson operators, singular Green operators).

This phenomenon is one of the main subjects of the book Grubb [12]. It is shown there that in the application to obtain trace formulas for resolvents (and heat kernels), one get finitely many well-defined terms in an asymptotic expansion, namely as many the regularity number indicates. For resolvents in the case without boundary, there is a trick to extend the analysis to get full trace expansions with infinitely many terms, some of them logarithmic; also this is explained in [12].

More recently, we have developed a somewhat more special calculus in collaboration with Robert Seeley [14], which allows a systematic construction of full asymptotic expansions for a class of ψdo's containing the resolvents: the calculus of *weakly polyhomogeneous* operators. It is completely described for the boundaryless case (whereas the additional details needed for general pseudodifferential boundary problems only exist in a sketched form).

For differential operators with pseudodifferential boundary conditions, one can however use the weakly polyhomogeneous ψdo calculus in cases where the trace formula in question can be reduced to one for an operator *in the boundary* of the weakly polyhomogeneous kind.

The calculus was developed for, and applies in particular to, the general Atiyah-Patodi-Singer problem. We describe this in detail below.

1.2. THE ATIYAH-PATODI-SINGER PROBLEM

On a compact n-dimensional C^∞ manifold X with boundary $\partial X = X'$, consider a first-order elliptic differential operator

$$P : C^\infty(E_1) \to C^\infty(E_2)$$

between sections of vector bundles over X. E_1 and E_2 have Hermitian metrics, and X has a smooth volume element, defining Hilbert spaces structures on the sections (primarily the spaces of L_2-sections, denoted $L_2(E_i)$, and more generally the Sobolev spaces $H^s(E_i)$, $s \in \mathbf{R}$).

The restrictions of the E_i to the boundary X' are denoted E_i'. A neighborhood of X' in X has the form $X_c = X' \times [0,c]$, and there the E_i are isomorphic to the pull-backs of the E_i'. Let x_n denote the coordinate in $[0,c]$, x' the coordinate in X'. Then we assume that P is represented in X_c as

$$P = \sigma(\tfrac{\partial}{\partial x_n} + A + x_n P_1 + P_0), \tag{1.2}$$

where σ is a unitary morphism from E_1' to E_2', independent of x_n, and A is a fixed elliptic first-order differential operator on $C^\infty(E_1')$, selfadjoint with respect to the Hermitian metric in E_1' and the volume element $v(x',0)dx'$ on X' induced by the element $v(x',x_n)dx'dx_n$ on X. The P_j are smooth differential operators (in all variables) of order $\le j$; they can be taken arbitrary near X', but for larger x_n, P_1 is subject to the requirement that P be elliptic. All morphisms are assumed C^∞.

In comparison with completely general elliptic first-order operators, the assumption means (modulo homotopies) that we have restricted the attention to operators such that when the principal symbol is written near X' as $\sigma_1(x',x_n)(\mathrm{i}\xi_n I + a_1(x',x_n,\xi'))$ (with a bundle isomorphism σ_1 from E_1 to E_2 in front), then $a_1(x',0,\xi')$ is *symmetric;* cf. Grubb [13], p. 2036. The case considered by Atiyah, Patodi and Singer in [2] is the case where, furthermore, $P_1 = P_0 = 0$ in (1.2); this is often called the *product case.* Important examples are the Dirac operator and its generalizations.

We denote $u|_{X'} = \gamma_0 u$ and observe the Green's formula:

$$(Pu,w)_X - (u,P^*w)_X = -(\gamma_0 u, \sigma^*\gamma_0 w)_{X'}. \tag{1.3}$$

Since P is a first-order system, it may not be possible to formulate a well-posed boundary value problem in terms of a *differential* boundary condition (a Dirichlet condition is too much, no boundary condition is too little, and the boundary bundle structure will not in general allow putting a Dirichlet condition on some "half" of the boundary data). But using ψdo's, one can get well-posedness:

Definition 1.1 The APS boundary problem consists of finding $u \in H^1(E_1)$ for a given $f \in L_2(E_2)$, so that

$$Pu = f \text{ on } X, \quad B\gamma_0 u = 0 \text{ on } X'. \tag{1.4}$$

Here B is an orthogonal projection in $L_2(E_1')$ of the form $B = \Pi_\geq + B_0$, where $\Pi_\geq$ ($\Pi_<$, Π_R, ...) denotes the orthogonal projection onto $V_\geq$ ($V_<$, V_R, ...), the sum of eigenspaces for A with eigenvalues $\lambda \geq 0$ ($\lambda < 0$, $|\lambda| \leq R$, ...), and B_0 commutes with A and ranges in V_R for some $R \geq 0$.

The associated realization P_B is defined as the operator from $L_2(E_1)$ to $L_2(E_2)$ acting like P and with domain

$$D(P_B) = \{\, u \in H^1(E_1) \mid B\gamma_0 u = 0 \,\}; \tag{1.5}$$

it is a Fredholm operator called **the APS operator**, and **the APS index problem** consists of determining its index.

This type of boundary condition is often called a *spectral boundary condition.* The Fredholm property of P_B was shown by Seeley in [23], where it was moreover shown that the adjoint of P_B is of a related type (in view of (1.4)):

$$(P_B)^* = (P^*)_{B'}, \text{ with } B' = B^-\sigma^*, \quad B^- = I - B. \tag{1.6}$$

One of the ways to study the index of P_B is to consider the "Laplacians"

$$\Delta_1 = P_B{}^*P_B, \quad \Delta_2 = P_BP_B{}^*, \tag{1.7}$$

and search for asymptotic expansions for $t \to 0$ (with $\varepsilon > 0$):

$$\operatorname{Tr} e^{-t\Delta_i} = c_{-n,i} t^{-n/2} + \cdots + c_{-1,i} t^{-1/2} + c_{0,i} + O(t^\varepsilon),\ i = 1, 2. \tag{1.8}$$

When (1.8) holds, the index is determined by

$$\operatorname{index} P_B = \operatorname{Tr} e^{-t\Delta_1} - \operatorname{Tr} e^{-t\Delta_2} = c_{0,1} - c_{0,2}. \tag{1.9}$$

Remark 1.2 The systems $\begin{pmatrix} P \\ B\gamma_0 \end{pmatrix}$ and $\begin{pmatrix} P^* \\ B'\gamma_0 \end{pmatrix}$ are injectively elliptic (also called overdetermined elliptic or left-elliptic). The operators Δ_1 and Δ_2 are realizations of truly elliptic systems (two-sided elliptic) such as

$$\begin{pmatrix} P^*P \\ A'B\gamma_0 + B'\gamma_0 P \end{pmatrix}, \text{ resp. } \begin{pmatrix} PP^* \\ A'B'\gamma_0 + B\gamma_0 P^* \end{pmatrix}. \tag{1.10}$$

(We here use that B and B' map into complementing subspaces of $L_2(E_1')$, and we have inserted the invertible ψdo $A' = A + \Pi_0(A)$ in order to make the boundary conditions first-order. The operators are *principally* the same as in the case where $B = \Pi_\geq$, discussed in detail in [13].) Another truly elliptic system incorporating P_B and $P_B{}^*$ is discussed below in Section 5.1 (and in [14]).

Remark 1.3 If $\sigma^* = -\sigma$ and $A\sigma = -\sigma A$, then in the product case, P is formally selfadjoint. Then if furthermore $B\sigma = \sigma(I - B)$, P_B is selfadjoint. This holds in many geometrically interesting cases, see e.g. Gilkey [10].

In [2] it was shown in the product case, with $B = \Pi_\geq$, that

$$\operatorname{index} P_B = \int_X \alpha(x) - \tfrac{1}{2}\eta_A; \quad \eta_A = \eta(A, 0) + \dim \ker A; \tag{1.11}$$

where $\alpha(x)$ is a certain form defined from the symbol of P, and $\eta(A, 0)$ is the value at $s = 0$ of the eta function

$$\eta(A, s) = \operatorname{Tr}(A|A|^{-s-1}). \tag{1.12}$$

(Here A^{-s-1} is defined as 0 on the nullspace of A, and meromorphic extension is used for $\operatorname{Re} s < n$.) Formula (1.11) was extended to the non-product case in [13] as

$$\operatorname{index} P_B = \int_X \alpha(x) + \int_{X'} \beta(x') - \tfrac{1}{2}\eta_A, \tag{1.13}$$

with a boundary form $\beta(x')$ defined from the symbols of P and B at X'. The forms defined from the symbols are regarded as *local contributions*, whereas the term η_A depends on the full set-up in a global way.

Actually, [2] did not calculate the two expressions $\operatorname{Tr} e^{-t\Delta_1}$ and $\operatorname{Tr} e^{-t\Delta_2}$ separately, but only their difference. They showed for the product case that this has the same asymptotic expansion as

$$\operatorname{Tr}(e^{-t\widetilde{\Delta}_1}|_X) - \operatorname{Tr}(e^{-t\widetilde{\Delta}_2}|_X) + \operatorname{Tr}(e^{-t\Delta_1^0} - \sigma^* e^{-t\Delta_2^0}\sigma); \tag{1.14}$$

here $\widetilde{\Delta}_1 = \widetilde{P}^*\widetilde{P}$ and $\widetilde{\Delta}_2 = \widetilde{P}\widetilde{P}^*$, where $\widetilde{P}$ is a certain extension of P to bundles $\widetilde{E}_1$ and $\widetilde{E}_2$ over the double manifold $\widetilde{X}$ (cf. [2], p. 55, where the roles of E_1 and E_2 are switched on $\widetilde{X}\setminus X$); the Δ_i^0 are x_n-independent extensions of the Δ_i on X_c to the cylinder $X^0 = X' \times \mathbf{R}_+$. The first difference is well-known, and the second can be analyzed by use of functional calculus for the selfadjoint operator A; this sufficed to get the index formula in the product case.

In [13] the separate expansions (1.8) were proved with $\varepsilon = \frac{3}{8}$ in the non-product case with $B = \Pi_{\geq}$, by a combination of the general treatment of parameter-dependent ψdo boundary problems [12] with the special results from [2]. It was shown that the global term $-\frac{1}{2}\eta_A$ enters in $c_{0,i}$ for both expansions, as $-\frac{1}{4}\eta_A$ for $i = 1$, resp. $\frac{1}{4}\eta_A$ for $i = 2$.

Now the index is just one special geometric invariant connected with the APS problem. More generally, one can ask about the value of the general coefficient $c_{j-n,i}$ in (1.7), and one can ask whether there is a more detailed structure of the $O(t^\varepsilon)$ term, giving a full asymptotic expansion $\sum_{j=0}^{\infty} c_{j-n,i} t^{(j-n)/2}$ for the trace $\operatorname{Tr}\exp(-t\Delta_i)$.

These questions have been answered in two papers written in cooperation with Seeley, [14] and [15]. It is shown there that there does exist a full asymptotic expansion, which however includes also logarithmic terms $c\, t^{(j-n)/2}\log t$ for $j - n > 0$. For the product case, a precise description of the coefficients in terms of the zeta and eta functions of A is given, when B_0 ranges in the nullspace of A.

In the following we shall give an account of these results, explaining the highlights of the methods.

2. The three operator-functions

2.1. DEFINITION OF THE OPERATOR FUNCTIONS

One can associate several interesting operator-functions with an elliptic operator Q. The following have been studied extensively:

• The resolvent $(Q-\lambda)^{-1}$ and its asymptotic behaviour for $\lambda \to \infty$ on rays in $\mathbf{C}$.

• The heat operator e^{-tQ} ($t \in \mathbf{R}_+$) and its asymptotic behavior for $t \to 0+$.

• The power operator Q^{-s} and the pole structure of associated functions of $s \in \mathbf{C}$.

For the questions we address here, there are essentially equivalent formulations in terms of each of the three operator functions, and one can pass from one formulation to another by suitable transformations. Very briefly stated, the heat operator and the resolvent are related to one another by the Laplace transformation, and the heat operator and power operator are related to one another by the Mellin transformation. One can also define the heat operator and the power operator from the resolvent by suitable Cauchy integral formulas (Dunford integrals), and there is another complex integration formula involving a reciprocal sinus function going from the power function to the resolvent. (In the proofs of Theorems 2.1 and 2.3 below, we also relate the formulas to the Fourier transformation.) In the following we collect the facts on these operator functions that we need.

Much of this has been known in the literature for a long time (but not always explained as generally as here). Applications to trace asymptotics have been made earlier e.g. in Seeley [22], Duistermaat and Guillemin [8], Grubb [12], Agranovič [1], Branson and Gilkey [5]. The explanation in the following is essentially copied from [15], and is given here with full details since it may be of interest also for other purposes.

Suppose that Q is a closed operator in a Hilbert space having a resolvent $(Q-\lambda)^{-1}$ which is holomorphic in some sector $|\arg(-\lambda)| < \alpha$, with $\|(Q-\lambda)^{-1}\| = O(|\lambda|^{-1})$, and is meromorphic at 0 (in the sense that $(Q-\lambda)^{-1} - (-\lambda)^{-1}\Pi_0(Q)$ is holomorphic at 0, where $\Pi_0(Q)$ is the orthogonal projection onto the nullspace of Q). Then the power function $Z(Q,s)$ is defined for $\operatorname{Re} s > 0$ by

$$Z(Q,s) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-s}(Q-\lambda)^{-1} d\lambda, \tag{2.1}$$

where $\mathcal{C}$ is a curve

$$\begin{aligned} \mathcal{C}_{\theta,r_0} = \{\, \lambda = re^{\mathrm{i}\theta} \mid \infty > r \geq r_0 \,\} + \{\, \lambda = r_0 e^{\mathrm{i}\theta'} \mid \theta \geq \theta' \geq -\theta \,\} \\ + \{\, \lambda = re^{\mathrm{i}(2\pi-\theta)} \mid r_0 \leq r < \infty \,\}, \end{aligned} \tag{2.2}$$

with $\pi - \alpha < \theta \leq \pi$ and $r_0 > 0$ chosen so that $(Q-\lambda)^{-1}$ is holomorphic for $0 < |\lambda| \leq r_0$. If Q is invertible then $Z(Q,s) = Q^{-s}$ (further details are found e.g. in Seeley [22] or Shubin [24]); in any case, $Z(Q,s)$ is zero on the nullspace of Q, since $\int_{\mathcal{C}} \lambda^{-s-1} d\lambda = 0$. We can also write

$$Z(Q,s) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}_{\theta,0}} \lambda^{-s}(Q-\lambda)^{-1}\Pi_0^{-}(Q)\, d\lambda, \tag{2.3}$$

where $\Pi_0^-(Q) = I - \Pi_0(Q)$.

If $Z(Q,s)$ is trace class for some s, then Q has a zeta function

$$\zeta(Q,s) = \operatorname{Tr} Z(Q,s), \tag{2.4}$$

and, for appropriate operators D and values s, a "modified zeta function"

$$\zeta(D,Q,s) = \operatorname{Tr} DZ(Q,s). \tag{2.5}$$

Similarly, under appropriate conditions, we define

$$\begin{aligned} Y(Q,s) = QZ(Q^*Q, \tfrac{s+1}{2}) &= \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{-(s+1)/2} Q(Q^*Q - \lambda)^{-1} d\lambda \\ &= \tfrac{i}{2\pi} \int_{\mathcal{C}_{\theta,0}} \lambda^{-(s+1)/2} Q(Q^*Q - \lambda)^{-1} d\lambda \end{aligned} \tag{2.6}$$

(since $\Pi_0(Q^*Q) = \Pi_0(Q)$ and $Q\Pi_0(Q) = 0$, we can leave out the nullspace projection), and the eta functions

$$\eta(Q,s) = \operatorname{Tr} Y(Q,s), \quad \eta(D,Q,s) = \operatorname{Tr} D\, Y(Q,s). \tag{2.7}$$

When Q is selfadjoint,

$$\sum_{\lambda \in \operatorname{sp}(Q)\setminus\{0\}} |\lambda|^{-s} = \zeta(Q^2, \tfrac{s}{2}), \qquad \sum_{\lambda \in \operatorname{sp}(Q)\setminus\{0\}} \operatorname{sign} \lambda\, |\lambda|^{-s} = \eta(Q,s), \tag{2.8}$$

with summation over the eigenvalues, repeated according to multiplicities.

In order to move the trace inside the integral, we may represent the power function by use of a derivative of the resolvent. Note that

$$\partial_\lambda^m (Q - \lambda)^{-1} = m!(Q - \lambda)^{-m-1}. \tag{2.9}$$

If Q is a ψdo of order $r > 0$ on a compact manifold M, say, then the mth derivative of $(Q-\lambda)^{-1}$ is a ψdo of order $-(1+m)r$ and hence is trace class when $(m+1)r > \dim M$. By an integration by parts, one can replace (2.1) by

$$Z(Q,s) = \tfrac{1}{(s-1)\cdots(s-m)} \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{m-s} \partial_\lambda^m (Q - \lambda)^{-1} d\lambda, \tag{2.10}$$

whereby (2.4) can be written

$$\zeta(Q,s) = \operatorname{Tr} Z(Q,s) = \tfrac{1}{(s-1)\cdots(s-m)} \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{m-s} \operatorname{Tr} \partial_\lambda^m (Q - \lambda)^{-1} d\lambda, \tag{2.11}$$

for sufficiently large m. Similar modifications can be made when there is a factor D as in (2.5) and when eta functions as in (2.7) are studied; and the integral can be replaced by an integral over $\mathcal{C}_{\theta,0}$ when $\Pi_0^-(Q)$ is inserted in front of $d\lambda$. There are similar formulas for the symbols and kernels of the operators.

When Q is lower bounded selfadjoint, the heat operator e^{-tQ} (also called the exponential function or the semigroup generated by $-Q$) can be defined by

$$e^{-tQ} = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}'} e^{-t\lambda}(Q-\lambda)^{-1}\, d\lambda, \quad t > 0; \tag{2.12}$$

where $\mathcal{C}'$ is a curve encircling the full spectrum in the positive direction and such that $e^{-t\lambda}$ falls off for $|\lambda| \to \infty$ on the curve (e.g. one can let $\mathcal{C}'$ begin with a ray with argument $\in]0, \frac{\pi}{2}[$ and end with a ray with argument $\in]-\frac{\pi}{2}, 0[$). This is well-known from the literature, see e.g. Hille-Phillips [16], Friedman [9] or Kato [20].

The exponential function and the power function of an operator $Q \geq 0$ with resolvent as above are related to one another by the formulas:

$$\begin{aligned} Z(Q,s) &= \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tQ} \Pi_0^-(Q)\, dt, \quad \operatorname{Re} s > 0, \\ e^{-tQ}\Pi_0^-(Q) &= \tfrac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = c} t^{-s} Z(Q,s)\Gamma(s)\, ds, \quad c > 0, \end{aligned} \tag{2.13}$$

that follow e.g. from Theorem 2.3 below, with $e(t) = e^{-tQ}\Pi_0^-(Q)$, $\varphi(s) = \Gamma(s)Z(Q,s)$.

Taking $Q = S^*S$ for suitable operators S, we have accordingly (cf. (2.6)):

$$\begin{aligned} Z(S^*S,s) &= \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tS^*S} \Pi_0^-(S)\, dt, \\ e^{-tS^*S}\Pi_0^-(S) &= \tfrac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = c} t^{-s} Z(S^*S,s)\Gamma(s)\, ds, \\ Y(S,2s) = SZ(S^*S, s+\tfrac{1}{2}) &= \frac{1}{\Gamma(s+\frac{1}{2})} \int_0^\infty t^{s-\frac{1}{2}} S e^{-tS^*S}\, dt, \\ Se^{-tS^*S} &= \tfrac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = c} t^{-s} Y(S, 2s-1)\Gamma(s)\, ds. \end{aligned} \tag{2.14}$$

(Also here we can omit mention of the nullspace projection in the last two formulas.)

Again, these formulas can be composed with a suitable operator D. When the expressions are trace class (usually for $\operatorname{Re} s$ resp. c sufficiently large) one can take the trace on both sides in (2.14) (composed with D), obtaining the formulas relating zeta and eta functions to exponential function traces:

$$\begin{aligned} \zeta(D, S^*S, s) &= \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} D e^{-tS^*S} \Pi_0^-(S)\, dt, \\ \operatorname{Tr} De^{-tS^*S}\Pi_0^-(S) &= \tfrac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = c} t^{-s} \zeta(D, S^*S, s)\Gamma(s)\, ds, \\ \eta(D,S,2s) = \zeta(DS, S^*S, s+\tfrac{1}{2}) &= \tfrac{1}{\Gamma(s+\frac{1}{2})} \int_0^\infty t^{s-\frac{1}{2}} \operatorname{Tr} DSe^{-tS^*S} dt, \\ \operatorname{Tr} DSe^{-tS^*S} &= \tfrac{1}{2\pi \mathrm{i}} \int_{\operatorname{Re} s = c} t^{-s} \eta(D,S,2s-1)\Gamma(s)\, ds. \end{aligned} \tag{2.15}$$

There are similar transition formulas for the symbols and kernels of the operators.

2.2. RELATIONS BETWEEN THE RESOLVENT AND THE POWER FUNCTION

Let us first consider the passage between properties of the resolvent and properties of the power and zeta functions. In order to handle operator functions defined not only as in (2.1), but also as in (2.10), we include functions with higher order poles at 0. We denote $\{0,1,2,\dots\} = \mathbf{N}$.

Theorem 2.1 1° *Suppose that* f *is meromorphic at* 0 *with Laurent expansion*

$$f(\lambda) = \sum_{j=-k}^{\infty} h_j(-\lambda)^j,\ |\lambda| \le \rho, \tag{2.16}$$

that f *is holomorphic in the open sector* $S_{\delta_0} = \{\,\lambda \in \mathbf{C} \mid |\arg\lambda - \pi| < \delta_0\,\}$ *(for some* $\delta_0 \le \pi$*), and that* $f(\lambda) = O(|\lambda|^{-\alpha})$ *for some* $\alpha\ \in]0,1]$ *as* $\lambda \to \infty$*, uniformly in each sector* S_δ *for* $\delta < \delta_0$*. Let* $\mathcal{C}$ *be a curve* $\mathcal{C}_{\pi,r_0}$ *as in* (2.2) *(a Laurent loop, since* $\theta = \pi$*), with* $0 < r_0 < \varrho$*. Set* $f_0(\lambda) = f(\lambda) - \sum_{-k}^{-1} h_j(-\lambda)^j$*, and*

$$\zeta(s) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-s} f(\lambda)\, d\lambda, \quad \operatorname{Re} s > 1-\alpha, \tag{2.17}$$

with $\lambda^{-s} = r^{-s}e^{-\mathrm{i}s\theta}$, $r > 0$ *and* $|\theta| \le \pi$*. Then* ζ *and* f_0 *are interrelated by:*

$$\zeta(s) = \tfrac{\sin \pi s}{\pi} \int_0^\infty r^{-s} f_0(-r)\, dr, \quad 1-\alpha < \operatorname{Re} s < 1, \tag{2.18}$$

$$f_0(-\lambda) = \tfrac{1}{2\mathrm{i}} \int_{\operatorname{Re} s = \sigma} \lambda^{s-1} \tfrac{\zeta(s)}{\sin \pi s}\, ds, \quad 1-\alpha < \sigma < 1. \tag{2.19}$$

The function $\frac{\pi\zeta(s)}{\sin \pi s}$ *is meromorphic for* $\operatorname{Re} s > 1-\alpha$*, having simple poles at* $s = j+1$ *with residues* $(-1)^{j+1}\zeta(j+1) = -h_j$, $j \in \mathbf{N}$*.*

2° *Moreover, the following properties* a) *and* b) *are equivalent:*

a) f *has an asymptotic expansion as* λ *goes to infinity*

$$f(-\lambda) \sim \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} a_{j,l}\lambda^{-\alpha_j}(\log\lambda)^l, \quad 0 < \alpha_j \nearrow +\infty \tag{2.20}$$

(with $m_j \in \mathbf{N}$*), uniformly for* $-\lambda$ *in* S_δ*, for each* $\delta < \delta_0$*.*

b) $\psi(s) = \frac{\pi\zeta(s)}{\sin \pi s}$ *is meromorphic on* $\mathbf{C}$ *with the singularity structure*

$$\frac{\pi\zeta(s)}{\sin \pi s} \sim -\sum_{j=-k}^{\infty} \frac{h_j}{s-j-1} + \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} \frac{a_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}} \tag{2.21}$$

(in the sense that for large N, the left hand side minus the sums for $j \le N$ in the right hand side is holomorphic for $1-\alpha_N < \operatorname{Re} s < N+1$); and for each real C_1, C_2 and each $\delta < \delta_0$,

$$|\psi(s)| \le C(C_1, C_2, \delta) e^{-\delta |\operatorname{Im} s|}, \text{ for } |\operatorname{Im} s| \ge 1,\ C_1 \le \operatorname{Re} s \le C_2. \tag{2.22}$$

In particular, the singularities of $\psi(s)$ in $\operatorname{Re} s < 1$ are determined by the expansion (2.20) and the singular Laurent terms of $f(\lambda)$ at $\lambda = 0$, and vice versa.

3° *Let f take values in a Banach space, and be holomorphic in S_{δ_0}, and meromorphic at 0 in the sense that there is a function $\sum_{j=-k}^{-1} (-\lambda)^j H_j$ with bounded operators H_j such that $f_0(\lambda) = f(\lambda) - \sum_{-k}^{-1} (-\lambda)^j H_j$ is holomorphic for $|\lambda| < \varrho$, some $\varrho > 0$. Let $\|f(\lambda)\|$ be $O(|\lambda|^{-\alpha})$ for $\lambda \to \infty$ in subsectors S_δ with $\delta < \delta_0$. Then with $\zeta(s)$ defined by (2.17), the formulas (2.18)–(2.19) are valid.*

Proof: 1°. For $j \le -1$ and $\operatorname{Re} s > 0$, $\int_{\mathcal{C}} \lambda^{j-s} d\lambda = 0$, since the contour can be closed at ∞ in $\{|\arg \lambda| < \pi\}$. So the singular part of f, $\sum_{-k}^{-1} h_j (-\lambda)^j$, is "killed" by the integral over $\mathcal{C}$ in (2.17). For the remaining part f_0, the circular part of $\mathcal{C}$ can be reduced to the origin if $\operatorname{Re} s < 1$, reducing (2.17) to (2.18) (note that f_0 is $O(|\lambda|^{-\alpha})$ too).

The inversion (2.19) requires growth estimates for $\zeta(s)$. Replacing the integration curve by $\mathcal{C}(\delta) := \mathcal{C}_{\pi-\delta,0}$, $0 < \delta < \delta_0$, we have that

$$|\zeta(s)| = \Big| \tfrac{\mathrm{i}}{2\pi} \textstyle\int_{\mathcal{C}(\delta)} \lambda^{-s} f_0(\lambda)\, d\lambda \Big| = O(e^{(\pi-\delta)|\operatorname{Im} s|}),$$
$$1-\alpha < C_1 \le \operatorname{Re} s \le C_2 < 1. \tag{2.23}$$

For, when $\lambda = re^{\mathrm{i}(\pi-\delta)}$, we can use the estimate

$$\Big| \int_0^\infty r^{-s} e^{\mathrm{i}(\pi-\delta)(1-s)} O((1+r)^{-\alpha})\, dr \Big| \le C e^{(\pi-\delta)|\operatorname{Im} s|}, \tag{2.24}$$

and there is a similar estimate on the other half of $\mathcal{C}(\delta)$.

Now let

$$\psi(s) = \int_0^\infty r^{-s} f_0(-r)\, dr = \frac{\pi \zeta(s)}{\sin \pi s}. \tag{2.25}$$

Since $(\sin \pi s)^{-1}$ is $O(e^{-\pi|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \ge 1$, we have by (2.23) that $\psi(\sigma + \mathrm{i}\tau) = O(e^{-\delta|\tau|})$ for $1-\alpha < C_1 \le \sigma \le C_2 < 1$. Also, $\psi(\sigma + \mathrm{i}\tau)$ is the Fourier transform $\hat{F}(\tau)$ of the function $F(x) = e^{(1-\sigma)x} f_0(-e^x)$.

Since $f_0(\lambda) = O(\langle\lambda\rangle^{-\alpha})$, $F(x)$ decays exponentially as $x \to \pm\infty$, for $1-\alpha < \sigma < 1$. By Fourier inversion, $F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mathrm{i}x\tau}\psi(\sigma+\mathrm{i}\tau)d\tau$, giving (2.19), for $\lambda > 0$. It extends to $|\arg\lambda| < \delta_0$ by analytic continuation.

It is seen from (2.17) that $\zeta(s)$ is holomorphic for $\operatorname{Re} s > 1-\alpha$; and since $\zeta(j+1) = \frac{-i}{2\pi}\int_{|\lambda|=r_0}\lambda^{-j-1}f(\lambda)\,d\lambda = (-1)^j h_j$ for $j \in \mathbf{N}$, $\psi(s)$ is meromorphic for $\operatorname{Re} s > 1-\alpha$, having simple poles with residues $-h_j$.

2°. Now suppose that a) holds; then

$$f_0(-\lambda) = \sum_{j=0}^{N-1}\sum_{l=0}^{m_j} a_{j,l}\lambda^{-\alpha_j}(\log\lambda)^l - \sum_{j=-k}^{-1} h_j\lambda^j + O(|\lambda|^{-\alpha_N+\varepsilon}) \text{ for } \lambda \to \infty, \tag{2.26}$$

for $\alpha_N \geq k$, any $\varepsilon > 0$. Note that

$$\int_0^1 r^{j-s}\,dr = \frac{-1}{s-j-1} \quad \text{for } \operatorname{Re} s < j+1,$$
$$\int_1^\infty r^{\beta-s}(\log r)^l\,dr = \frac{l!}{(s-\beta-1)^{l+1}} \quad \text{for } \operatorname{Re} s > \beta+1$$

(the cases $l > 0$ follow from the case $l = 0$ by application of ∂_s^l); the right hand sides extend meromorphically to $\mathbf{C}$. Then we get from (2.25), for arbitrarily large N:

$$\begin{aligned}\psi(s) &= \int_0^1 \Big[\sum_{j=0}^{N-1} h_j r^{j-s} + r^{-s}O(r^N)\Big]dr \\ &\quad + \int_1^\infty \Big[\sum_{j=0}^{N-1}\sum_{l=0}^{m_j} a_{j,l}r^{-\alpha_j-s}(\log r)^l - \sum_{j=-k}^{-1} h_j r^{j-s} + r^{-s}O(r^{-\alpha_N+\varepsilon})\Big]dr \\ &= -\sum_{j=-k}^{N-1}\frac{h_j}{s-j-1} + \sum_{j=0}^{N-1}\sum_{l=0}^{m_j}\frac{a_{j,l}l!}{(s+\alpha_j-1)^{l+1}} + h_N(s)\end{aligned}$$

where h_N is holomorphic for $1-\alpha_N+\varepsilon < \operatorname{Re} s < N+1$, and the other terms are meromorphic on $\mathbf{C}$. This gives the singularities (2.21).

To show the decay, we use the integral in (2.23) and expand on each piece of $\mathcal{C}(\delta)$:

$$\begin{aligned}\zeta(s) = &-\tfrac{\mathrm{i}}{2\pi}\Big(\textstyle\int_0^1 + \int_1^\infty (re^{\mathrm{i}(\pi-\delta)})^{-s} f_0(re^{\mathrm{i}(\pi-\delta)})e^{\mathrm{i}(\pi-\delta)}\,dr\Big) \\ &+ \tfrac{\mathrm{i}}{2\pi}\Big(\textstyle\int_0^1 + \int_1^\infty (re^{\mathrm{i}(-\pi+\delta)})^{-s} f_0(re^{\mathrm{i}(-\pi+\delta)})e^{\mathrm{i}(-\pi+\delta)}\,dr\Big).\end{aligned} \tag{2.27}$$

The first integral from 0 to 1 is written as

$$\begin{aligned}&\tfrac{-\mathrm{i}}{2\pi}\textstyle\int_0^1 (re^{\mathrm{i}(\pi-\delta)})^{-s} f_0(re^{\mathrm{i}(\pi-\delta)})e^{\mathrm{i}(\pi-\delta)}\,dr \\ &\quad = \tfrac{-\mathrm{i}}{2\pi}\textstyle\int_0^1 \sum_{j=0}^{N-1} e^{\mathrm{i}(j+1-s)(\pi-\delta)}h_j r^{j-s}\,dr + \int_0^1 r^{-s}e^{\mathrm{i}(\pi-\delta)(1-s)}O(r^N)\,dr \\ &\qquad = \textstyle\sum_{j=0}^{N-1}\frac{-e^{\mathrm{i}(j+1-s)(\pi-\delta)}h_j}{j+1-s} + e^{\mathrm{i}(\pi-\delta)(1-s)}\int_0^1 r^{-s}O(r^N)\,dr.\end{aligned} \tag{2.28}$$

Let $|\operatorname{Im} s| \geq 1$. The sum over j extends meromorphically to $\mathbf{C}$, and its terms are $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ for $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$. The last term exists and is $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ when $\operatorname{Re} s < N+1$. Similar considerations hold for the other integral from 0 to 1. In the integrals from 1 to ∞ we expand as in (2.26), obtaining functions that are $O(e^{(\pi-\delta)|\operatorname{Im} s|})$ for $\operatorname{Re} s > 1-\alpha_N+\varepsilon$. We conclude that the estimate in (2.23) extends to $1-\alpha_N < \operatorname{Re} s < N+1$, $|\operatorname{Im} s| \geq 1$, for arbitrarily large N. Dividing by $\sin \pi s$ we find that $\psi(s)$ satisfies (2.22). This shows a) $\Longrightarrow$ b).

Conversely, assume b). Then $f_0(-\lambda)$ is given by (2.19), and we obtain the expansion (2.20) by shifting the contour of integration past the poles of $\psi(s)$. The remainder after all terms up to the singularity $s = 1-\alpha_N$ is given by the integral (2.19) but with $\sigma < 1-\alpha_N$; it is $O(|\lambda|^{-\alpha_N+\varepsilon})$ on S_δ.

3°. The proof under 1° is generalized straightforwardly to Banach spaces, with the relevant estimates valid for the norms. □

In this analysis, the poles in (2.21) may very well be considered in a general sense where we allow some of the coefficients $a_{j,l}$ to be 0; this is practical for the applications where vanishing coefficients often occur, and we shall use this point of view in the following. (So we can e.g. speak of a simple pole with residue 0 — this is usually not called a pole.)

Corollary 2.2 *When $f(\lambda)$ and $\zeta(s)$ are as in Theorem 2.1 1°–2°, then $\Gamma(s)\zeta(s)$ is meromorphic on $\mathbf{C}$ with the singularity structure*

$$\Gamma(s)\zeta(s) \sim \sum_{j=-k}^{j=-1} \frac{-\tilde{h}_j}{s-j-1} + \sum_{j=0}^{\infty}\sum_{l=0}^{m_j} \frac{\tilde{a}_{j,l}\, l!}{(s+\alpha_j-1)^{l+1}},$$
$$\tilde{h}_j = \frac{h_j}{\Gamma(-j)},\ \tilde{a}_{j,l} = \frac{a_{j,l}}{\Gamma(\alpha_j)}. \tag{2.29}$$

Thus the singularity structure (2.29) *of $\Gamma(s)\zeta(s)$ is determined from the asymptotic expansion* (2.20) *of f together with the singular part of the Laurent expansion* (2.16) *(the coefficients h_j with $-k \leq j \leq -1$), and vice versa.*

When $\delta_0 > \frac{\pi}{2}$, one has moreover, for any $\delta' < \delta_0 - \frac{\pi}{2}$, any real C_1 and C_2:

$$|\Gamma(s)\zeta(s)| \leq C'(C_1, C_2, \delta) e^{-\delta'|\operatorname{Im} s|}, \quad \textit{for } |\operatorname{Im} s| \geq 1,\ C_1 \leq \operatorname{Re} s \leq C_2. \tag{2.30}$$

Proof: Since $\pi(\sin \pi s)^{-1} = \Gamma(s)\Gamma(1-s)$, (2.29) results from (2.21) by multiplication by $\Gamma(1-s)^{-1}$, whose zeros cancel the poles $h_j/(s-j-1)$, $j \geq 0$. If $\delta - \pi/2 = \delta' > 0$, the estimate $|\zeta(s)| \leq Ce^{(\pi-\delta)|\operatorname{Im} s|}$ shown in the proof of Theorem 2.1 (and assured by (2.22)) implies (2.30), since $\Gamma(s)$ is $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \geq 1$, $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$, any $\varepsilon > 0$. (Cf. e.g. the assertion in Bourbaki [3], p. 182:

$$|\Gamma(s)| \sim \sqrt{2\pi}\, |\operatorname{Im} s|^{\operatorname{Re} s - \frac{1}{2}} e^{-\frac{\pi}{2}|\operatorname{Im} s|} \text{ for } |\operatorname{Im} s| \to \infty, \tag{2.31}$$

valid for fixed $\operatorname{Re} s$ or $\operatorname{Re} s$ in compact intervals of $\mathbf{R}$.) □

Note in particular that a case $m_j = 1$ in (2.20) corresponds to a double pole of $\Gamma(s)\zeta(s)$ at $s = 1 - \alpha_j$ (in the strict sense if $a_{j,m_j} \neq 0$).

2.3. RELATIONS BETWEEN THE POWER FUNCTION AND THE EXPONENTIAL FUNCTION

Now we shall investigate the relation between properties of exponential functions and of power and zeta functions. The general transition goes as follows:

Theorem 2.3 1° *Let $e(t)$ be a function holomorphic in a sector V_{θ_0} (for some $\theta_0 \in]0, \frac{\pi}{2}[$),*

$$V_{\theta_0} = \{t = re^{i\theta} \mid r > 0, |\theta| < \theta_0\}, \tag{2.32}$$

such that $e(t)$ decreases exponentially for $|t| \to \infty$ and is $O(|t|^a)$ for $t \to 0$ in V_δ, any $\delta < \theta_0$, for some $a \in \mathbf{R}$. Let φ be the Mellin transform of e,

$$\varphi(s) = (\mathcal{M}e)(s) := \int_0^\infty t^{s-1} e(t)\, dt, \tag{2.33}$$

for $\operatorname{Re} s > -a$. Then $\varphi(s)$ is holomorphic for $\operatorname{Re} s > -a$ and $\varphi(c + i\xi)$ is $O(e^{-\delta|\xi|})$ for $|\xi| \to \infty$, when $c > -a$ (uniformly for c in compact intervals of $] - a, \infty[$); and $e(t)$ is recovered from $\varphi(s)$ by the formula

$$e(t) = \tfrac{1}{2\pi i} \int_{\operatorname{Re} s = c} t^{-s} \varphi(s)\, ds. \tag{2.34}$$

2° *Moreover, the following properties* a) *and* b) *are equivalent:*

a) *$e(t)$ has an asymptotic expansion for $t \to 0$,*

$$e(t) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} b_{j,l} t^{\beta_j} (\log t)^l, \quad \beta_j \nearrow +\infty,\ m_j \in \mathbf{N}, \tag{2.35}$$

uniformly for $t \in V_\delta$, for each $\delta < \theta_0$.

b) *$\varphi(s)$ is meromorphic on $\mathbf{C}$ with the singularity structure*

$$\varphi(s) \sim \sum_{j=0}^{\infty} \sum_{l=0}^{m_j} \frac{(-1)^l l! b_{j,l}}{(s + \beta_j)^{l+1}}, \tag{2.36}$$

and for each real C_1, C_2 *and each* $\delta < \theta_0$,

$$|\varphi(s)| \le C(C_1, C_2, \delta) e^{-\delta |\operatorname{Im} s|}, \ |\operatorname{Im} s| \ge 1, \ C_1 \le \operatorname{Re} s \le C_2. \tag{2.37}$$

3° *Let* $f(\lambda)$ *take values in a Banach space, and be holomorphic in* $S_{\delta_0} = \{|\pi - \arg\lambda| < \delta_0\}$ *for some* $\delta_0 \in]\frac{\pi}{2}, \pi]$ *and meromorphic at* $\lambda = 0$ *(holomorphic for* $0 < |\lambda| < \varrho$*). Assume that as* $\lambda \to \infty$ *in* S_δ *(for* $\delta < \delta_0$*), some derivative* $\partial_\lambda^m f(\lambda)$ *is* $O(|\lambda|^{-1-\varepsilon})$ *for some* $\varepsilon > 0$ *(so that* $f(\lambda)$ *is* $O(|\lambda|^{m-1})$*). Let* θ_0 *and* θ *be such that* $]\theta - \theta_0, \theta + \theta_0[\subset]\pi - \delta_0, \frac{\pi}{2}[$, *let* $\mathcal{C} = \mathcal{C}_{\theta, r_0}$ *as in* (2.2) *with* $r_0 \in]0, \varrho[$, *and let*

$$e(t) = \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} e^{-t\lambda} f(\lambda)\, d\lambda, \quad \varphi(s) = \Gamma(s) \tfrac{\mathrm{i}}{2\pi} \int_{\mathcal{C}} \lambda^{-s} f(\lambda)\, d\lambda, \tag{2.38}$$

for $t \in V_{\theta_0}$ *resp.* $\operatorname{Re} s > m - \varepsilon$*. Then* $e(t)$ *is exponentially decreasing for* $t \to \infty$ *in sectors* V_δ *with* $\delta < \theta_0$*, and is* $O(|t|^{-m})$ *for* $t \to 0$*, and* $\varphi(s)$ *and* $e(t)$ *correspond to one another by* (2.33), (2.34)*. Here, when* $f(\lambda) = (Q - \lambda)^{-1}$*, then* $e(t) = e^{-tQ} \Pi_0^-(Q)$ *and* $\varphi(s) = \Gamma(s) Z(Q, s)$*.*

Proof: 1°. Note first that replacing $e(t)$ by $t^b e(t)$ replaces $\varphi(s)$ by $\varphi(s+b)$, so we can assume that $a > 0$ and then consider $c \ge 0$. The function $\varphi(s)$ is holomorphic for $\operatorname{Re} s \ge 0$ since the integrand $t^{s-1} e(t)$ is so and has an integrable majorant there.

By a change of variables $t = e^x$, we see that $\varphi_1(\xi) = \varphi(\mathrm{i}\xi)$ is the conjugate Fourier transform of $e_1(x) = e(e^x) \in L_2(\mathbf{R})$:

$$\varphi_1(\xi) = \varphi(\mathrm{i}\xi) = \int_0^\infty t^{\mathrm{i}\xi} e(t) \frac{dt}{t} = \int_{-\infty}^\infty e^{\mathrm{i}x\xi} e(e^x)\, dx = \int_{-\infty}^\infty e^{\mathrm{i}x\xi} e_1(x)\, dx,$$

so by Fourier's inversion formula,

$$e(t) = e_1(x) = \tfrac{1}{2\pi} \int_{-\infty}^\infty e^{-\mathrm{i}x\xi} \varphi_1(\xi)\, d\xi = \tfrac{1}{2\pi\mathrm{i}} \int_{\operatorname{Re} s = 0} t^{-s} \varphi(s)\, ds. \tag{2.39}$$

Similarly, $\varphi(c + \mathrm{i}\xi)$ is the conjugate Fourier transform of $e(e^x) e^{xc}$ for $c > 0$.

The hypothesis on exponential decrease of $e(t)$ in the sectors V_δ allows us to shift the path of integration in (2.33) from $t \in \mathbf{R}_+$ to $t \in e^{\mathrm{i}\delta} \mathbf{R}_+$ for $|\delta| < \theta_0$ (corresponding to a shift to $x \in \mathbf{R} + \mathrm{i}\delta$); this gives:

$$\begin{aligned} \varphi(c + \mathrm{i}\xi) &= \int_0^\infty (re^{\mathrm{i}\delta})^{c + \mathrm{i}\xi} e(re^{\mathrm{i}\delta}) \frac{dr}{r} \\ &= e^{-\delta\xi} \int_0^\infty r^{\mathrm{i}\xi} e(re^{i\delta})(re^{\mathrm{i}\delta})^c \frac{dr}{r} = e^{-\delta\xi} g(\delta, \xi, c), \end{aligned}$$

where g is bounded as a function of $\xi \in \mathbf{R}$, locally uniformly in $c \geq 0$. Taking $\delta > 0$ for $\xi > 0$ and $\delta < 0$ for $\xi < 0$, we see that $\varphi(c + i\xi)$ decreases exponentially (like $e^{-\delta|\xi|}$) for $|\xi| \to \infty$, in any vertical strip $\{s = c + i\xi \mid C_1 \leq c \leq C_2, \xi \in \mathbf{R}\}$ with $0 \leq C_1 \leq C_2$. Then we can also shift the integration path in (2.39) from $\operatorname{Re} s = 0$ to $\operatorname{Re} s = c$, $c \geq 0$. This shows 1°.

2°. Assume now in addition (2.35). Let us first write $\varphi(s)$ as

$$\varphi(s) = \int_0^1 t^{s-1} e(t)dt + \int_1^\infty t^{s-1} e(t)dt. \tag{2.40}$$

The second integral defines an entire function of s. The expansion (2.35) means that

$$e(t) = \sum_{j=0}^{N-1} \sum_{l=0}^{m_j} b_{j,l} t^{\beta_j} (\log t)^l + \varrho_N(t),$$
$$\varrho_N(t) = O(|t|^{\beta_N - \varepsilon}) \text{ for } t \to 0 \text{ in } V_\delta, \tag{2.41}$$

for $\varepsilon > 0$ and any positive integer N; we insert this in the first integral. Observe the formulas, valid for $\operatorname{Re} s > -\beta$,

$$\begin{aligned} \int_0^1 t^{s-1+\beta} (\log t)^l \, dt &= \frac{(-1)^l l!}{(s+\beta)^l}, \\ \int_0^\infty t^{s-1+\beta} (\log t)^l \, e^{-t} \, dt &= \partial_s^l \Gamma(s+\beta), \end{aligned} \tag{2.42}$$

where the cases $l > 0$ follow from the cases $l = 0$ by application of ∂_s^l. The remainder $\varrho_N(t)$ in (2.41) gives a function holomorphic for $\operatorname{Re} s > -\beta_N + \varepsilon$, and for the powers of t we use (2.42); this shows (2.36).

To show the exponential decrease of $\varphi(s)$ on general vertical strips, one can shift the contour in (2.33) and proceed much as in the proof of Theorem 2.1. Another instructive method is to insert the expansion $e^t = \sum_{\nu \geq 0} \frac{1}{\nu!} t^\nu$, that gives

$$e^t t^{\beta_j} (\log t)^l = \sum_{\nu=0}^{M-1} \tfrac{1}{\nu!} t^{\beta_j + \nu} (\log t)^l + O(t^{\beta_j + M - \varepsilon}),$$

for any $\varepsilon > 0$ and positive integer M. Then we can write

$$e(t) = e(t) e^t e^{-t} = \Big(\sum_{\beta_j + \nu < M} \sum_{l \leq m_j} b_{j,l} \tfrac{1}{\nu!} t^{\beta_j + \nu} (\log t)^l \Big) e^{-t} + \tilde{\varrho}_M(t),$$
$$\text{with } \tilde{\varrho}_M(t) = O(|t|^{M-\varepsilon}) \text{ for } t \to 0 \text{ in } V_\delta,$$

where $\tilde\varrho_M(t)$ is exponentially decreasing for $|t|\to\infty$ in V_δ since the other terms are so, and hence

$$\varphi(s)=\int_0^\infty t^{s-1}\Big(\sum_{\beta_j+\nu<M}\sum_{l\le m_j} b_{j,l}\tfrac{1}{\nu!}t^{\beta_j+\nu}(\log t)^l\Big)e^{-t}dt + \int_0^\infty t^{s-1}\tilde\varrho_M(t)\,dt. \quad (2.43)$$

The last integral defines a function that is holomorphic for $\operatorname{Re} s>-M+\varepsilon$ and exponentially decreasing (like $e^{-\delta|\operatorname{Im} s|}$) on strips $-M+\varepsilon<C_1\le \operatorname{Re} s\le C_2$, by 1°. For the contributions from the first integral we use the second formula in (2.42) together with the fact that the gamma function $\Gamma(s)$ and its derivatives are $O(e^{(-\frac{\pi}{2}+\varepsilon')|\operatorname{Im} s|})$, any $\varepsilon'>0$, for $|\operatorname{Im} s|\ge 1$, $-\infty<C_1\le\operatorname{Re} s\le C_2<\infty$, cf. e.g. [3], pp. 181–182. This gives (2.37), completing the proof of a) $\Longrightarrow$ b).

Conversely, assume b). Then $e(t)$ is given by (2.34), and we obtain the expansion (2.35) by shifting the contour of integration past the poles of $\varphi(s)$. The remainder after all terms up to and including the singularity $s=-\beta_N$ is given by an integral like (2.34) but with $c<-\beta_N$; it is $O(|t|^{\beta_N-\varepsilon})$.

3°. That $e(t)$ defined here is exponentially decreasing for $|t|\to\infty$ in V_δ, $\delta<\theta_0$, follows since $|e^{-\lambda t}|\le e^{-\gamma|t|}$ with $\gamma>0$ on the integration curve. The estimate for $t\to 0$ follows since

$$\int_{\mathcal C} e^{-\lambda t}f(\lambda)\,d\lambda=(-t)^{-m}\int_{\mathcal C}(\partial_\lambda^m e^{-\lambda t})f(\lambda)\,d\lambda=t^{-m}\int_{\mathcal C}e^{-\lambda t}\partial_\lambda^m f(\lambda)\,d\lambda$$

for $t\in V_\delta$, where $e^{-\lambda t}\partial_\lambda^m f(\lambda)$ has a fixed integrable majorant for $t\to 0$. The formula (2.33) for φ is shown by a complex change of variables, where we replace t by u/λ for each λ; when $\arg\lambda\in\,]0,\frac{\pi}{2}[$, the ray $\mathbf{R}_+$ is transformed to a ray Λ_λ with argument $-\arg\lambda\in\,]-\frac{\pi}{2},0[$, and vice versa. The integral of $u^{s-1}e^{-u}$ on such a ray is again equal to $\Gamma(s)$, as noted above. Thus (recall that $f(\lambda)$ is $O(|\lambda|^{m-1})$)

$$\int_0^\infty t^{s-1}\tfrac{i}{2\pi}\int_{\mathcal C}e^{-t\lambda}f(\lambda)\,d\lambda dt=\tfrac{i}{2\pi}\int_{\mathcal C}\int_{\Lambda_\lambda}u^{s-1}\lambda^{-s}e^{-u}f(\lambda)\,dud\lambda = \Gamma(s)\tfrac{i}{2\pi}\int_{\mathcal C}\lambda^{-s}f(\lambda)\,d\lambda. \qquad \square$$

3. Weakly polyhomogeneous symbols

3.1. POLYHOMOGENEOUS SYMBOL CLASSES

We here sketch the properties of the symbol class used to get trace expansions for the general APS problem; details are given in [14].

Consider symbols $p(x,\xi,\mu)$, where x and $\xi\in\mathbf{R}^n$, $\mu\in\Gamma$ (a sector of $\mathbf{C}\setminus\{0\}$). We shall say that:

p is **strongly homogeneous** of degree m, when

$$p(x,t\xi,t\mu)=t^m p(x,\xi,\mu) \text{ for } |\xi|^2+|\mu|^2\geq 1,\ t\geq 1,\quad (\xi,\mu)\in\mathbf{R}^n\times(\Gamma\cup\{0\}). \tag{3.1}$$

p is **weakly homogeneous** of degree m, when

$$p(x,t\xi,t\mu)=t^m p(x,\xi,\mu) \text{ for } |\xi|,t\geq 1,\ (\xi,\mu)\in\mathbf{R}^n\times\Gamma. \tag{3.2}$$

Example 3.1 Let $a(x,\xi)$ be positive and C^∞ on $\mathbf{R}^n\times\mathbf{R}^n$, and homogeneous in ξ of degree $r\in\mathbf{N}$ for $|\xi|\geq 1$. Then $a(x,\xi)+\mu^r$ and $(a(x,\xi)+\mu^r)^{-1}$ extend to:

strongly homogeneous symbols of degree r, resp. $-r$, if a is **polynomial** in ξ (it is the symbol of a differential operator);

weakly homogeneous symbols of degree r, resp. $-r$, if a is **not** polynomial in ξ (it is the symbol of a genuine ψdo).

If for example $r=n=2$, $a(x,\xi)=\xi_1^2+\xi_2^2$ enters in the first case, and $a(x,\xi)=(\xi_1^4+\xi_2^4)/(\xi_1^2+\xi_2^2)$ (for $|\xi|\geq 1$) enters in the second case.

Both cases can be shown to belong to the following symbol classes (where $(a(x,\xi)+\mu^r)^{-1}\in S^{-r,0}\cap S^{0,-r}$):

Definition 3.2 $S^{m,0}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)$ consists of the functions $p(x,\xi,\mu)$ that are holomorphic in μ for $|(\xi,\mu)|\geq\varepsilon$, $\mu\in\Gamma$, and satisfy, denoting $\frac{1}{\mu}=z$,

$$\partial_z^j p(\cdot,\cdot,\tfrac{1}{z}) \text{ is in } S^{m+j}(\mathbf{R}^n,\mathbf{R}^n) \text{ for } \tfrac{1}{z}\in\Gamma, \text{ with uniform estimates for } |z|\leq 1,\ \tfrac{1}{z}\in \text{closed subsectors of } \Gamma. \tag{3.3}$$

Moreover, we set $S^{m,d}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)=\mu^d S^{m,0}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)$.

Here $S^m(\mathbf{R}^n,\mathbf{R}^n)$ denotes the standard ψdo symbol space consisting of the functions $p(x,\xi)\in C^\infty(\mathbf{R}^n\times\mathbf{R}^n)$ such that $\partial_x^\beta\partial_\xi^\alpha p$ is $O(\langle\xi\rangle^{m-|\alpha|})$ for all $\alpha,\beta\in\mathbf{N}^n$. The rules of calculus for such symbols are well-known, see e.g. Hörmander [18], Seeley [23], Shubin [24], Hörmander [19] for various set-ups with local or global estimates in x. We call the symbols in $S^m(\mathbf{R}^n\times\mathbf{R}^n)$ *classical*, when they moreover have expansions in series of homogeneous terms (in ξ, $|\xi|\geq 1$) of degrees $m-j$, $j\in\mathbf{N}$.

When symbols $p(x,\xi)$ of order m are considered as depending on one more variable μ, they lie in $S^{m,0}$:

$$S^m(\mathbf{R}^n,\mathbf{R}^n)\subset S^{m,0}(\mathbf{R}^n,\mathbf{R}^n,\Gamma),\ \text{any } \Gamma. \tag{3.4}$$

The symbols in $S^{m,d}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)$ define ψdo's $P=\mathrm{OP}(p)$ (which depend on the parameter μ) by the usual formula:

$$\mathrm{OP}(p)f(x)=\int e^{\mathrm{i}x\cdot\xi}p(x,\xi,\mu)\hat f(\xi)\,đ\xi,\quad f\in\mathcal{S}(\mathbf{R}^n), \tag{3.5}$$

with $đ\xi=(2\pi)^{-n}d\xi$. The definition extends to more general functions and distributions f as in the nonparametrized case. When $m<-n$, $\mathrm{OP}(p)$ is an integral operator with continuous kernel $K_p(x,y,\mu)$;

$$\begin{gathered}K_p(x,y,\mu)=\int e^{\mathrm{i}(x-y)\cdot\xi}p(x,\xi,\mu)\,đ\xi,\\ \text{in particular, } K_p(x,x,\mu)=\int p(x,\xi,\mu)\,đ\xi.\end{gathered} \tag{3.6}$$

The operators have good composition rules, since $S^{m,d}\cdot S^{m',d'}\subset S^{m+m',d+d'}$, and since one can refer to the standard rules for S^m symbol classes, which must here hold uniformly in z as in (3.3). One finds for example that

$$P\in\mathrm{OP}(S^{m,d}),\ P'\in\mathrm{OP}(S^{m',d'})\implies PP'\in\mathrm{OP}(S^{m+m',d+d'}) \tag{3.7}$$

(under the usual precautions on supports or global estimates), and the resulting symbol is described by the usual formula

$$(p\circ p')(x,\xi,\mu)\sim\sum_{\alpha\in\mathbf{N}^n}\tfrac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi,\mu)(-i\partial_x)^\alpha p'(x,\xi,\mu)\text{ in }S^{m+m',d+d'}. \tag{3.8}$$

The expansion in (3.8) is an expansion in terms with decreasing m-exponents $m+m'-j$, $j\to\infty$ ($j=|\alpha|$). Such expansions enter in the theory as follows:

When $p_j\in S^{m_j,d}$ for a sequence $m_j\searrow-\infty$ (for $j\to\infty$, $j\in\mathbf{N}$), and $p\in S^{m_0,d}$, we say that $p\sim\sum_{j\in\mathbf{N}}p_j$ in $S^{m_0,d}$ if

$$p-\sum_{j<J}p_j\in S^{m_J,d}\text{ for any }J\in\mathbf{N}. \tag{3.9}$$

For any given sequence $p_j\in S^{m_j,d}$ with $m_j\searrow-\infty$, there exists a p such that (3.9) holds.

For the present special symbols there is *another* type of expansion that is of great interest:

Theorem 3.3 *When $p\in S^{m,d}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)$, then p has an expansion in terms $\mu^{d-k}p_{(d,k)}(x,\xi)$ with $p_{(d,k)}\in S^{m+k}(\mathbf{R}^n,\mathbf{R}^n)$, such that for any N,*

$$p(x,\xi,\mu)-\sum_{0\le k<N}\mu^{d-k}p_{(d,k)}(x,\xi)\in S^{m+N,d-N}(\mathbf{R}^n,\mathbf{R}^n,\Gamma). \tag{3.10}$$

In the proof one reduces to the case $d = 0$ by multiplication by μ^{-d}; then the expansion is essentially a Taylor expansion in $z = \frac{1}{\mu}$ at $z = 0$.

Note that in (3.10), the order of $p_{(d,k)}$ *increases* with increasing k, whereas the power of μ *decreases*. A very simple example is

$$(1 + |\xi|^2 + \mu^2)^{-1} = \mu^{-2}(1 - \mu^{-2}\langle\xi\rangle^2 + \mu^{-4}\langle\xi\rangle^4 - \dots).$$

Corollary 3.4 *When $p \in S^{-\infty,d}$, the kernel $K_p(x, y, \mu)$ of $\mathrm{OP}(p)$ has an expansion*

$$K_p(x, y, \mu) \sim \sum_{k\in\mathbf{N}} \mu^{d-k} K_{p,k}(x, y), \quad K_{p,k} \in C^\infty. \tag{3.11}$$

Definition 3.2 contains no homogeneity requirements, but we now define a polyhomogeneous subspace:

Definition 3.5 A symbol $p \in S^{m_0-d,d}$ is called **weakly polyhomogeneous**, when $p \sim \sum_{j\in\mathbf{N}} p_j$, with $p_j \in S^{m_j-d,d}$, $m_j \searrow -\infty$ for $j \to \infty$, $j \in \mathbf{N}$, such that the p_j are weakly homogeneous of degrees m_j (cf. (3.2)).

This will be compared with:

Definition 3.6 A function $p(x, \xi, \mu) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times (\Gamma \cup \{0\})$ is called **strongly polyhomogeneous** of degree m if there is a sequence of functions $p_j \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n \times (\Gamma \cup \{0\})$ that are strongly homogeneous of degree $m - j$ (cf. (3.1)) such that

$$\partial_x^\beta \partial_\xi^\alpha \partial_\mu^k (p - \sum_{j<J} p_j) = O(\langle(\xi, \mu)\rangle^{m-J-|\alpha|-k}), \tag{3.12}$$

for all indices, uniformly for μ in closed subsectors of $\Gamma \cup \{0\}$.

Then one has in fact:

Theorem 3.7 *When p is strongly polyhomogeneous of degree $m \in \mathbf{Z}$, then it is also weakly polyhomogeneous, with degrees $m - j$, $j \in \mathbf{N}$, and with*

$$\begin{aligned} &p \in S^{m,0} + S^{0,m} \text{ if } m \geq 0, \quad p \in S^{m,0} \cap S^{0,m} \text{ if } m \leq 0, \\ &\partial_x^\beta \partial_\xi^\alpha \partial_\mu^k p \in S^{m-|\alpha|-k,0} \cap S^{0,m-|\alpha|-k} \text{ for } |\alpha| + k \geq m, \text{ all } \beta. \end{aligned} \tag{3.13}$$

As a consequence, classical symbols of order $m \in \mathbf{Z}$ in $n + 1$ cotangent variables give strongly polyhomogeneous symbols in n cotangent variables, when one cotangent variable is replaced by μ (here $\Gamma = \mathbf{R}_+ \cup \mathbf{R}_-$).

The type of parameter-dependence entering in Theorem 3.7 was used by Agmon and by Agranovič and Vishik in resolvent studies for differential operators; for ψdo's this is the kind of parameter-dependence studied e.g. in Shubin [24] and many other works. It is a mild generalization that does not cover resolvents $(P-\lambda)^{-1}$ and parabolic operators such as $\partial/\partial t + P$ when P is truly pseudodifferential (as treated in [12]).

3.2. APPLICATIONS TO KERNEL AND TRACE EXPANSIONS

Both the expansion in Theorem 3.3 and the expansion in Definition 3.5 enter in the proof of:

Theorem 3.8 *Let p be weakly polyhomogeneous as in Definition* 3.5*, with $m_0 - d < -n$. Then* $\mathrm{OP}(p)$ *has a continuous kernel $K_p(x,y,\mu)$ with an expansion on the diagonal*

$$K_p(x,x,\mu) \sim \sum_{j=0}^{\infty} a_j(x)\mu^{m_j+n} + \sum_{k=0}^{\infty} [c_k(x)\log\mu + c_k'(x)]\mu^{d-k} \ , \tag{3.14}$$

for $|\mu| \to \infty$, uniformly for μ in closed subsectors of Γ. The coefficients $a_j(x)$ and $c_{d-m_j-n}(x)$ are determined by $p_j(x,\xi,\mu)$ for $|\xi| \ge 1$ (are "local"), while the $c_k'(x)$ are "global."

Details of proof are given in [14]. A brief explanation: One uses the general principle that "remainders contribute to c_k' terms," by Corollary 3.3. The p_j contribute with (cf. (3.6))

$$\begin{aligned} K_{p_j}(x,x,\mu) &= \int_{\mathbf{R}^n} p_j(x,\xi,\mu)\, đ\xi \\ &= \int_{|\xi|\ge|\mu|} p_j\, đ\xi + \int_{|\xi|\le 1} p_j\, đ\xi + \int_{1\le|\xi|\le|\mu|} p_j\, đ\xi = I_1 + I_2 + I_3, \end{aligned} \tag{3.15}$$

where I_1 gives part of the a_j term, I_2 gives c_k' terms, and I_3 gives the rest of a_j and c_{d-m_j-n} (if $d - m_j - n \in \mathbf{N}$) and some c_k' terms. One has of course to show that the contributions to the c_k' pile up in a controlled way.

When the operator acts on a compact boundaryless manifold, integration of $K_p(x,x,\mu)$ in x gives a similar expansion of the *trace*:

Corollary 3.9 *Let P be a μ-dependent ψdo on a compact manifold M of dimension n, with symbol satisfying the hypotheses of Theorem* 3.8 *in local coordinates. Then it is trace class, the trace satisfying*

$$\operatorname{Tr} P \sim \sum_{j=0}^{\infty} a_j\mu^{m_j+n} + \sum_{k=0}^{\infty} [c_k\log\mu + c_k']\mu^{d-k} \ , \tag{3.16}$$

for $|\mu| \to \infty$, uniformly for μ in closed subsectors of Γ. The coefficients are derived from those in (3.14) for coordinate patches by integration over M.

The result applies in particular to expressions containing a differentiated resolvent:

$$P = S\partial_\lambda^m (Q-\lambda)^{-1}, \tag{3.17}$$

where Q is a classical elliptic ψdo of positive integer order r on a compact boundaryless manifold M of dimension n_1, with principal symbol $q_r(x,\xi)$ having no eigenvalues on $\mathbf{R}_-$, S is a classical ψdo of order d, and m is chosen so that $d - r(1+m) < -n_1$. With $\mu = (-\lambda)^{1/r}$ for λ in a narrow sector Γ around $\mathbf{R}_-$, the symbol is in $S^{d-r(1+m),0} \cap S^{0,d-r(1+m)}$ and weakly polyhomogeneous. Then Theorem 3.8 and its corollary lead to an expansion of the diagonal kernel and the trace, generalizing the result of Agranovič [1] for $S = I$ (cf. [14] for details). The kernel $K(x,y,P)$ satisfies on the diagonal:

$$\begin{aligned} &K(x,x,S\partial_\lambda^m (Q-\lambda)^{-1}) \\ &\quad \sim \sum\nolimits_{j=0}^{\infty} a_j(x) \lambda^{\frac{n_1+d-j}{r}-m-1} + \sum_{k=0}^{\infty} (c_k(x)\log\lambda + c'_k(x))\lambda^{-k-m-1}, \end{aligned} \tag{3.18}$$

for $|\lambda| \to \infty$, uniformly in closed subsectors of Γ. Consequently, one has for the trace:

$$\operatorname{Tr} S\partial_\lambda^m (Q-\lambda)^{-1} \sim \sum\nolimits_{j=0}^{\infty} a_j \lambda^{\frac{n_1+d-j}{r}-m-1} + \sum_{k=0}^{\infty} (c_k \log\lambda + c'_k)\lambda^{-k-m-1}\ ; \tag{3.19}$$

where the coefficients are the integrals over M of the fiber traces of the coefficients defined in (3.18).

If S is a differential operator (in particular if $S = I$), then $c_0(x) = 0$ and the complete coefficient of λ^{-m-1} is locally determined.

If S and Q are both differential operators, we are in the well-known case where no logarithms occur, and all coefficients are locally determined (cf. [22]). This is shown by a simpler version of the above proof where the decomposition (3.15) is not needed since the symbols are smooth and homogeneous at $\xi = 0$, and it gives an expansion we may write as follows (denoting S by D):

$$\begin{aligned} K(x,x,D\partial_\lambda^m (Q-\lambda)^{-1}) &\sim \sum\nolimits_{j=0}^{\infty} b_j(x,D,Q)(-\lambda)^{\frac{n_1+d-j}{r}-m-1}, \\ \operatorname{Tr} D\partial_\lambda^m (Q-\lambda)^{-1} &\sim \sum\nolimits_{j=0}^{\infty} b_j(D,Q)(-\lambda)^{\frac{n_1+d-j}{r}-m-1}, \end{aligned} \tag{3.20}$$

for $\lambda \to \infty$ in suitable sectors. Let r be even. Then since the symbol terms homogeneous of odd degree satisfy $p(x,-\xi,\mu) = -p(x,\xi,\mu)$, their contributions to the diagonal kernel and the trace vanish (cf. (3.6)); hence

$$b_j(x,D,Q) \text{ and } b_j(D,Q) \text{ are zero for } d-j \text{ odd.} \tag{3.21}$$

For later reference we list the formula for the zeta function that follows from (3.20) by Corollary 2.2, in the case where the differential operator Q is selfadjoint ≥ 0 and of order 2. We have to take the singularity resulting from the nullspace $V_0(Q)$ (of finite dimension $\nu_0(Q)$) into account; in fact,

$$D\partial_\lambda^m (Q-\lambda)^{-1} - D\Pi_0(Q)\partial_\lambda^m(-\lambda)^{-1} \tag{3.22}$$

is holomorphic at 0. Here $\Pi_0(Q)$ is an integral operator with C^∞ kernel $K(x,y,\Pi_0(Q)) = \sum_{1\leq l\leq \nu_0} u_l(x) \otimes \overline{u}_l(y)$, where the u_l are a smooth orthonormal basis of V_0. The kernel of $D\Pi_0(Q)$ is $\sum_{1\leq l\leq\nu_0}(Du_l(x)) \otimes \overline{u}_l(y)$. Then the singularity at 0 of $K(x,x,D\partial_\lambda^m(Q-\lambda)^{-1})$, resp. $\operatorname{Tr} D\partial_\lambda^m(Q-\lambda)^{-1}$, is

$$K(x,x,D\Pi_0(Q))\partial_\lambda^m(-\lambda)^{-1}, \text{ resp. } \operatorname{Tr} D\Pi_0(Q)\partial_\lambda^m(-\lambda)^{-1}. \tag{3.23}$$

In this case (3.20) is seen to correspond, by (2.10) and Corollary 2.2, to the following pole descriptions of the diagonal kernel and trace of $\Gamma(s)DZ(Q,s)$:

$$\begin{aligned} \Gamma(s)K(x,x,DZ(Q,s)) &\sim \sum_{j=0}^\infty \frac{c_j(x,D,Q)}{s+\frac{j-n_1-d}{2}} - \frac{K(x,x,D\Pi_0(Q))}{s}, \\ \Gamma(s)\operatorname{Tr}(DZ(Q,s)) &\sim \sum_{j=0}^\infty \frac{c_j(D,Q)}{s+\frac{j-n_1-d}{2}} - \frac{\operatorname{Tr}(D\Pi_0(Q))}{s}, \end{aligned} \tag{3.24}$$

with

$$c_j(x,D,Q) = \frac{b_j(x,D,Q)}{\Gamma(m+1+\frac{j-n_1-d}{2})}, \quad c_j(D,Q) = \int_M \operatorname{tr} c_j(x,D,Q)\,dx, \\ c_j = 0 \text{ for } j-d \text{ odd.} \tag{3.25}$$

In particular, if Q is the square of a selfadjoint first-order differential operator A, and D is multiplication by $\psi(x)$, we get for the pole structure of the zeta and eta functions, taking the vanishing of coefficients into account:

$$\begin{aligned} \Gamma(s)\zeta(\psi,A^2,s) = \Gamma(s)\operatorname{Tr}(\psi Z(A^2,s)) &\sim \sum_{k=0}^\infty \frac{c_{2k}(\psi,A^2)}{s+k-\frac{n_1}{2}} - \frac{\operatorname{Tr}(\psi\Pi_0(A))}{s}, \\ \Gamma(s)\eta(\psi,A,2s-1) = \Gamma(s)\operatorname{Tr}(\psi A Z(A^2,s)) &\sim \sum_{k=0}^\infty \frac{c_{2k+1}(\psi A,A^2)}{s+k-\frac{n_1}{2}}. \end{aligned} \tag{3.26}$$

Let us also list the consequences for the heat kernel and trace, by Theorem 2.3:

$$
\begin{aligned}
K(x,x,De^{-tQ}\Pi_0^-(Q)) &\sim \sum_{j=0}^{\infty} c_j(x,D,Q)t^{\frac{j-n_1-d}{2}} - K(x,x,D\Pi_0(Q)),\\
K(x,x,De^{-tQ}) &\sim \sum_{j=0}^{\infty} c_j(x,D,Q)t^{\frac{j-n_1-d}{2}},\\
\operatorname{Tr}(De^{-tQ}) &\sim \sum_{j=0}^{\infty} c_j(D,Q)t^{\frac{j-n_1-d}{2}};
\end{aligned}
\tag{3.27}
$$

note that the effects of the nullspace projection have been cancelled out in the second and third lines. (In Theorem 2.3, a simple pole at $s=0$ for $\varphi(s)$ corresponds to a constant term for $e(t)$.)

4. The APS resolvent in the product case

4.1. GENERALITIES ON RESOLVENTS

We now return to the APS operator on a manifold with boundary, as described in Section 1.2.

One auxiliary tool is to consider an extension of P to a larger manifold without boundary. As mentioned after (1.14), one can choose a specific extension $\widetilde{P}$ to the double $\widetilde{X}$ in the product case. However, in the final formulas, the choice of extension really plays no role, since all operators are restricted back to X (more comments in [15], Remark 3.6), so we can let $\widetilde{P}$ stand for any extension satisfying the ellipticity requirements. In the general case, we just extend to a neighboring open manifold $\widetilde{X} = X \cup (X' \times]-1,0[\,)$ preserving the ellipticity hypotheses there.

We denote the extended "Laplacians" $\widetilde{\Delta}_1 = \widetilde{P}^*\widetilde{P}$ and $\widetilde{\Delta}_2 = \widetilde{P}\widetilde{P}^*$, and set

$$
Q_{i,\lambda} = (\widetilde{\Delta}_i - \lambda)^{-1}. \tag{4.1}
$$

In the product case where $\widetilde{\Delta}_i$ is selfadjoint ≥ 0 on the compact manifold $\widetilde{X}$, this is well-defined for $\lambda \in \mathbf{C}$ outside a discrete subset of $\overline{\mathbf{R}}_+$, and the zeta and eta functions as well as the heat trace for the $\widetilde{\Delta}_i$ behave as described at the end of the preceding section.

In the general case, $Q_{i,\lambda}$ is, to begin with, just defined in a parametrix sense, but it can be modified such that for sufficiently large λ

$$
(\widetilde{\Delta}_{i,+} - \lambda)Q_{i,\lambda,+} = I \text{ on } X \tag{4.2}
$$

(as explained in detail in [14], p. 508–9). Here we use the convention of defining, for an operator S on $\widetilde{X}$, the truncation S_+ to X by

$$S_+ u = r^+ S e^+ u, \tag{4.3}$$

where $e^+ u$ denotes the extension of u with $e^+ u(x', x_n) = 0$ for $x_n < 0$, and r^+ denotes restriction to $\{x_n > 0\}$. We shall also write

$$\operatorname{Tr}_+ S = \operatorname{Tr}[S_+]. \tag{4.4}$$

(The plus-subscript is often omitted when one deals with *differential* operators, since they act locally.)

The $Q_{i,\lambda}$ enter as pseudodifferential parts of the resolvents we are looking for:

$$R_{i,\lambda} = Q_{i,\lambda,+} + G_{i,\lambda}, \tag{4.5}$$

where the $G_{i,\lambda}$ are *singular Green operators* (in the notation of Boutet de Monvel [4]); s.g.o.s..

Remark 4.1 One of the well-known ways to describe the resolvent of a given boundary value problem is the following: Consider a problem

$$(P - \lambda)u = f \text{ on } X, \quad Tu = \varphi \text{ on } X', \tag{4.6}$$

where P is elliptic of order d in a bundle E over X, and T is a trace operator (from $H^d(X, E)$ to a suitable Sobolev space $H_T(X', F)$ over the boundary X'). The resolvent R_λ is the solution operator $R_\lambda : f \mapsto u$ for the problem (4.6) with $\varphi = 0$. *Assume* that $P - \lambda$, extended to a larger manifold $\widetilde{X}$, has an inverse Q_λ such that $(P - \lambda)Q_{\lambda,+} = I$ on X, where $Q_{\lambda,+}$ maps $L_2(X, E)$ into $H^d(X, E)$. *Assume* moreover that the problem (4.6) with $f = 0$ has a solution operator $K_\lambda : \varphi \mapsto u$ (such that $(P - \lambda)K_\lambda = 0$, $TK_\lambda = I$), mapping $H_T(X', F)$ into $H^d(X, E)$. Such an operator going from the boundary to the interior is called a *Poisson operator* in [4].

Then the full problem (4.6) has at most one solution for any data $\{f, \varphi\}$ in $L_2(X, E) \times H_T(X', F)$, since null-data give the null-solution. Moreover, the resolvent equals

$$R_\lambda = Q_{\lambda,+} - K_\lambda T Q_{\lambda,+}, \tag{4.7}$$

for this operator verifies $(P - \lambda)R_\lambda = I$ and $TR_\lambda = 0$ and is defined on all of $L_2(X, E)$ so it must be the unique solution operator.

In (4.7) we see the structure of the resolvent as the sum of a ψdo term and a term composed of a Poisson operator K_λ and a general type of trace operator $TQ_{\lambda,+}$; here $K_\lambda T Q_{\lambda,+}$ is an example of a singular Green operator.

Another auxiliary tool in the analysis of the inverse (4.5) is to compare it with the inverse on the cylinder $X^0 = X' \times \mathbf{R}_+$. Define

$$\begin{aligned} P^0 &= \sigma(\partial_n + A), \quad P^{0\prime} = (-\partial_n + A)\sigma^*, \text{ so} \\ P^{0\prime} P^0 &= D_n^2 + A^2, \quad P^0 P^{0\prime} = \sigma(D_n^2 + A^2)\sigma^*. \end{aligned} \tag{4.8}$$

They have a meaning on X^0, where P^0 goes from E_1^0 to E_2^0, the respective liftings of E_1' and E_2', and $P^{0\prime}$ is the formal adjoint of P^0 with respect to the product measure. They can be extended to bundles $\widetilde{E}_i^0$ over $\widetilde{X}^0 = X' \times \mathbf{R}$; the simplest choice is to take the $\widetilde{E}_i^0$ as the liftings of E_i' and extend the formulas in (4.8). We denote the extensions $\widetilde{P}^0$, $\widetilde{\Delta}_1^0 = (\widetilde{P}^0)'\widetilde{P}^0$, $\widetilde{\Delta}_2^0 = \widetilde{P}^0(\widetilde{P}^0)'$.

On the cylinder X^0 we consider the realization P_B^0 of P^0 defined by the boundary condition $B\gamma_0 u = 0$, with the Laplacians $\Delta_1^0 = P_B^{0*}P_B^0$ and $\Delta_2^0 = P_B^0 P_B^{0*}$. The resolvents are:

$$\begin{gathered} R_{i,\lambda}^0 = Q_{i,\lambda,+}^0 + G_{i,\lambda}^0, \text{ with} \\ Q_{1,\lambda}^0 = (D_n^2 + A^2 - \lambda)^{-1}, \quad Q_{2,\lambda}^0 = \sigma(D_n^2 + A^2 - \lambda)^{-1}\sigma^*, \end{gathered} \tag{4.9}$$

the $G_{i,\lambda}^0$ being singular Green operators (as in Remark 4.1).

In the product case one can show that the true resolvent $R_{i,\lambda}$ is, near X', very closely related to $R_{i,\lambda}^0$, in such a way that the singular Green contributions to the asymptotic expansions we are looking for are essentially the same. In the general case, $R_{i,\lambda}^0$ is a first order approximation in some sense, so we can take it as a point of departure for the construction of the true resolvent $R_{i,\lambda}$.

4.2. DECOMPOSITION FORMULAS IN THE PRODUCT CASE

In the product case, very precise information will be obtained for the asymptotic expansions, on the basis of exact formulas for the operators involved. Let

$$A_\lambda = (A^2 - \lambda)^{1/2}, \text{ for } \lambda \in \mathbf{C} \setminus \overline{\mathbf{R}}_+; \quad A' = A + \Pi_0(A). \tag{4.10}$$

We shall here describe the results for the case $B = \Pi_\geq$ (i.e., $B_0 = 0$) in detail. [15] moreover treats $B = \Pi_\geq + B_0$ with B_0 ranging in $V_0(A)$. In a recent manucript [6], Brüning and Lesch treat certain other boundary conditions for problems as in Remark 1.3, see Remark 4.14 below.

Using the cylindrical structure, we shall write the s.g.o. terms in (4.9) explicitly in terms of the special operator

$$(G_\lambda u)(x', x_n) = \int_0^\infty e^{-(x_n + y_n)A_\lambda} u(x', y_n)\, dy_n. \tag{4.11}$$

When G is an operator defined by $Gu = \int_0^\infty \mathcal{G}(x_n, y_n) u(x', y_n)\, dy_n$, where $\mathcal{G}$ is a function of x_n, y_n valued in operators on x'-space, we call $\mathcal{G}(x_n, y_n)$ the *normal kernel* of G, and define its *normal trace* as

$$\operatorname{tr}_n G = \int_0^\infty \mathcal{G}(x_n, x_n)\, dx_n, \tag{4.12}$$

when it exists. The normal kernel of G_λ is $e^{-(x_n+y_n)A_\lambda}$, and the normal trace is

$$\operatorname{tr}_n G_\lambda = \int_0^\infty e^{-2x_n A_\lambda}\, dx_n = (2A_\lambda)^{-1}. \tag{4.13}$$

Example 4.2 To explain how G_λ enters, consider the Dirichlet problem for $D_n^2 + A^2 - \lambda$ on X^0,

$$(D_n^2 + A^2 - \lambda)u = f, \quad \gamma_0 u = \varphi, \tag{4.14}$$

from the point of view of Remark 4.1. The Poisson operator solving (4.14) with $f = 0$ is

$$K^0_{\mathrm{Dir},\lambda}\varphi = e^{-x_n A_\lambda}\varphi, \tag{4.15}$$

and the composition $\gamma_0 Q^0_{1,\lambda,+}$ acts like

$$\begin{aligned}\gamma_0(D_n^2 + A^2 - \lambda)^{-1} e^+ f &= \gamma_0 \tfrac{1}{2A_\lambda} \int_0^\infty e^{-|x_n - y_n| A_\lambda} e^+ f(x', y_n) dy_n \\ &= \int_0^\infty \tfrac{1}{2A_\lambda} e^{-y_n A_\lambda} f(x', y_n) dy_n,\end{aligned} \tag{4.16}$$

so the singular Green operator term as in (4.7) equals the composed operator

$$\begin{aligned}G^0_{\mathrm{Dir},\lambda} f &= -K^0_{\mathrm{Dir},\lambda}\gamma_0 Q^0_{1,\lambda,+} f \\ &= \tfrac{-1}{2A_\lambda} \int_0^\infty e^{-(x_n+y_n)A_\lambda} f(x', y_n)\, dy_n = \tfrac{-1}{2A_\lambda} G_\lambda f.\end{aligned} \tag{4.17}$$

Thus the resolvent equals

$$(\Delta^0_{\mathrm{Dir}} - \lambda)^{-1} = R^0_{\mathrm{Dir},\lambda} = Q^0_{1,\lambda,+} - \tfrac{1}{2A_\lambda} G_\lambda. \tag{4.18}$$

For a Robin-type boundary condition $\gamma_0(\partial_n + S)u = 0$, where S commutes with A^2 and $A_\lambda - S$ is invertible, one finds in a similar way that the singular Green operator term in the resolvent is

$$G^0_{\mathrm{Rob},\lambda} = \tfrac{1}{2A_\lambda} \tfrac{A_\lambda + S}{A_\lambda - S} G_\lambda. \tag{4.19}$$

In particular for the Neumann condition, $G^0_{\mathrm{Neu},\lambda} = \tfrac{1}{2A_\lambda} G_\lambda$.

The actual boundary conditions mix boundary values and normal derivatives in a more complicated way; for example, Δ_1^0 has the boundary condition (cf. (1.6))

$$\Pi_{\geq}\gamma_0 u = 0, \quad \Pi_{<}\gamma_0(\partial_n + A)u = 0, \tag{4.20}$$

where we have used that $\Pi_{<} = I - \Pi_{\geq}$ and that $\sigma^*\sigma = I$. This is a Dirichlet condition on the functions of x_n valued in $V_{\geq}$ and a Robin-type condition on the functions of x_n valued in $V_{<}$, so by applying Example 4.2 to each component, we find that the singular Green term in $(\Delta_1^0 - \lambda)^{-1}$ has the form

$$G^0_{1,\lambda} = \left(\tfrac{-1}{2A_\lambda}\Pi_{\geq} + \tfrac{1}{2A_\lambda}\tfrac{A_\lambda + A}{A_\lambda - A}\Pi_{<}\right)G_\lambda \tag{4.21}$$

(which has a good sense since $-A$ is positive on $V_{<}$). Along with the corresponding formula for $G^0_{2,\lambda}$, this may be written as in [13], [15]:

$$\begin{aligned} G^0_{1,\lambda} &= G_{\mathrm{e},\lambda} + G_{\mathrm{o},\lambda} - \tfrac{\Pi_0(A)}{2\sqrt{-\lambda}}G_\lambda, \\ G^0_{2,\lambda} &= \sigma(G_{\mathrm{e},\lambda} - G_{\mathrm{o},\lambda} + \tfrac{\Pi_0(A)}{2\sqrt{-\lambda}}G_\lambda)\sigma^*, \text{ where} \\ G_{\mathrm{e},\lambda} &= \tfrac{-|A|}{2A_\lambda(|A|+A_\lambda)}G_\lambda = \left(\tfrac{-A^2}{2\lambda A_\lambda} + \tfrac{|A|}{2\lambda}\right)G_\lambda, \\ G_{\mathrm{o},\lambda} &= \tfrac{-1}{2(|A|+A_\lambda)}\tfrac{A}{|A'|}G_\lambda = \left(-\tfrac{A}{2\lambda} + \tfrac{A_\lambda}{2\lambda}\tfrac{A}{|A'|}\right)G_\lambda. \end{aligned} \tag{4.22}$$

Recall (4.10); for the last expressions it is used that $1/(|A| + A_\lambda) = (|A| - A_\lambda)/(A^2 - (A^2 - \lambda)) = |A|/\lambda - A_\lambda/\lambda$. The indices e and o refer to the evenness and oddness of the principal symbols with respect to ξ'. (The parity alternates between even and odd in the sequences of lower order symbols.) All the operators are defined and holomorphic for $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}}_+$. Moreover, $G_{\mathrm{e},\lambda}$ and $G_{\mathrm{o},\lambda}$ are holomorphic at 0 because of the factors $|A|$ and A that vanish on the nullspace.

From (4.13), (4.22) follow:

$$\operatorname{tr}_n G_{\mathrm{e},\lambda} = \tfrac{-A^2}{4\lambda A_\lambda^2} + \tfrac{|A|}{4A_\lambda \lambda}, \quad \operatorname{tr}_n G_{\mathrm{o},\lambda} = \tfrac{-A}{4\lambda A_\lambda} + \tfrac{1}{4\lambda}\tfrac{A}{|A'|}. \tag{4.23}$$

Example 4.3 The corresponding expressions for the Dirichlet and Robin-type problems considered in Example 4.2 are

$$\operatorname{tr}_n G^0_{\mathrm{Dir},\lambda} = \tfrac{-1}{4A_\lambda^2} = -\tfrac{1}{4}\tfrac{1}{A^2-\lambda}, \quad \operatorname{tr}_n G^0_{\mathrm{Rob},\lambda} = \tfrac{1}{4A_\lambda^2}\tfrac{A_\lambda + S}{A_\lambda - S}. \tag{4.24}$$

Now one can show that near X', the true s.g.o $G_{i,\lambda}$ is very similar to the cylindrical version $G^0_{i,\lambda}$.

Lemma 4.4 *(Product case.) Let* $\chi \in C_0^\infty(\mathbf{R})$ *with* $\chi(x_n) = 1$ *for* $|x_n| < \frac{c}{3}$, $\chi(x_n) = 0$ *for* $|x_n| > \frac{2c}{3}$*. Then* $G_{1,\lambda} - \chi G^0_{1,\lambda}\chi$ *is trace class in* $L_2(E_1)$ *with norm* $O(|\lambda|^{-N})$ *for* $|\lambda| \to \infty$ *with* $\arg\lambda \in [\delta, 2\pi - \delta]$*, any* $\delta > 0$*. The same is true of* $\partial_\lambda^k[G_{1,\lambda} - \chi G^0_{1,\lambda}\chi]$ *for* $k = 1, 2, \dots,$ *and of expressions* $DG_{1,\lambda} - \chi D' G^0_{1,\lambda}\chi$*, where* D *is a differential operator, constant in* x_n *near* X' *and equal to* D' *there.*

Similar estimates hold for $G_{2,\lambda} - \chi G^0_{2,\lambda}\chi$ *in* $L_2(E_2)$*, and for the operators* $(1-\chi)G^0_{i,\lambda}$ *and* $G^0_{i,\lambda} - \chi G^0_{i,\lambda}\chi$ *in* $L_2(E_i^0)$*. Here the* $G^0_{i,\lambda}$ *can be replaced by* $G_{\mathrm{e},\lambda}$ *or* $G_{\mathrm{o},\lambda}$*.*

All these functions are holomorphic in $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}}_+$*.*

The proof is given in detail in [15], using elements of [13]. It extends to show that the operators also map into H^s, any s, with $O(|\lambda|^{-N})$ estimates for any N.

We now construct the zeta functions. For this, we integrate $\lambda^{-s}R_{i,\lambda}$ along an appropriate curve $\mathcal{C}$ as in Theorem 2.1, running along the negative axis and around a small circle of radius

$$r_0 < \min\{\lambda_1(\Delta_i), \lambda_1(\widetilde{\Delta}_i), \lambda_1(A^2)\}, \tag{4.25}$$

where λ_1 denotes the smallest positive eigenvalue. ($\mathcal{C}$ could also be taken to be a curve in $\operatorname{Re}\lambda > 0$ closer to the spectra.) Then (cf. (4.3))

$$\begin{aligned} Z(\Delta_i, s) &= \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}R_{i,\lambda}\,d\lambda = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}Q_{i,\lambda,+}\,d\lambda + \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}G_{i,\lambda}\,d\lambda \\ &= Z(\widetilde{\Delta}_i, s)_+ + G_{Z,i,s}, \text{ where we have set} \\ G_{Z,i,s} &= \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}G_{i,\lambda}\,d\lambda. \end{aligned} \tag{4.26}$$

In the trace calculations in Theorem 4.6 below, we shall replace $G_{i,\lambda}$ by $G^0_{i,\lambda}$ by use of Lemma 4.4. Define the transforms

$$G_{Z,\mathrm{e},s} = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}G_{\mathrm{e},\lambda}\,d\lambda, \quad G_{Z,\mathrm{o},s} = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}G_{\mathrm{o},\lambda}\,d\lambda. \tag{4.27}$$

To describe the various G_Z, we use the function defined for $\operatorname{Re}(-t) < \operatorname{Re}s < 0$ by

$$\begin{aligned} F_t(s) &= \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}_{\pi,r_0}} \tau^{-s-1}(1-\tau)^{-t}\,d\tau \\ &= \tfrac{\mathrm{i}}{2\pi}(e^{(-s-1)\mathrm{i}\pi} - e^{(s+1)\mathrm{i}\pi})\int_0^\infty u^{-s-1}(1+u)^{-t}\,du \\ &= \tfrac{1}{\pi}\sin\pi(s+1)\,\tfrac{\Gamma(-s)\Gamma(s+t)}{\Gamma(t)} = \tfrac{\Gamma(s+t)}{\Gamma(t)\Gamma(s+1)}; \end{aligned} \tag{4.28}$$

$\mathcal{C}_{\pi,r_0}$ is taken with $r_0 \in]0,1[$, cf. (2.2). $F_t(s)$ coincides with the binomial coefficient $\binom{s+t-1}{t-1}$, also equal to $(sB(t,s))^{-1}$, where B is the beta function.

$F_t(s)$ extends meromorphically to general s and $t \in \mathbf{C}$. In particular,

$$\begin{aligned} &F_{\frac12}(s) = \tfrac{\Gamma(s+\frac12)}{\sqrt{\pi}\,\Gamma(s+1)} = \tbinom{s-\frac12}{-\frac12}, \quad F_1(s) = 1, \\ &F_0(s) = 0 \text{ if } s \neq 0, \quad F_t(0) = 1 \text{ if } \Gamma(t) \neq \infty. \end{aligned} \tag{4.29}$$

That $F_1(s) = 1$ follows directly from the first integral in (4.14), and the formula for $F_0(s)$ follows from the fact that $\frac{i}{2\pi}\int_{\mathcal{C}} \tau^{-s-1}\,d\tau = 0$ for $\operatorname{Re} s > 0$.

The formulas for the singular Green operator terms are greatly simplified when we take normal traces.

Proposition 4.5 *Define* $G_{Z,\mathrm{e},s}$ *and* $G_{Z,\mathrm{o},s}$ *by* (4.27), *cf. also* (4.13), (4.10). *Then*

$$\begin{aligned} \operatorname{tr}_n G_{Z,\mathrm{e},s} &= \tfrac14 (F_{\frac12}(s) - 1) Z(A^2, s), \\ \operatorname{tr}_n G_{Z,\mathrm{o},s} &= -\tfrac14 F_{\frac12}(s) Y(A, 2s). \end{aligned} \tag{4.30}$$

Proof: Expand the operators on X' with respect to the orthogonal eigenprojections $\{\Pi_\mu\}_{\mu \in \operatorname{sp}(A)}$ for A. Our $G_{Z,\mathrm{e},s}$ and $G_{Z,\mathrm{o},s}$ are both 0 in the zero eigenspace. Using (4.23) we find, by replacing λ by $\mu^2\tau$ for each μ,

$$\begin{aligned} \operatorname{tr}_n G_{Z,\mathrm{e},s} &= \operatorname{tr}_n \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G_{\mathrm{e},\lambda}\,d\lambda = \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}\big(\tfrac{-A^2}{4\lambda A_\lambda^2} + \tfrac{|A|}{4A_\lambda \lambda}\big)\,d\lambda \\ &= \sum_\mu \tfrac14 \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s-1}\big(-\tfrac{\mu^2}{\mu^2-\lambda} + \tfrac{|\mu|}{(\mu^2-\lambda)^{\frac12}}\big)\,d\lambda \cdot \Pi_\mu \\ &= \sum_\mu \tfrac14 |\mu|^{-2s} \tfrac{i}{2\pi}\int_{\mathcal{C}} \tau^{-s-1}\big(\tfrac{-1}{1-\tau} + \tfrac{1}{(1-\tau)^{\frac12}}\big)\,d\tau \cdot \Pi_\mu \\ &= \tfrac14(-F_1(s) + F_{\frac12}(s)) Z(A^2, s) = \tfrac14(-1 + F_{\frac12}(s)) Z(A^2, s); \end{aligned} \tag{4.31}$$

$$\begin{aligned} \operatorname{tr}_n G_{Z,\mathrm{o},s} &= \operatorname{tr}_n \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G_{\mathrm{o},\lambda}\,d\lambda = \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}\big(\tfrac{-A}{4\lambda A_\lambda} + \tfrac{1}{4\lambda}\tfrac{A}{|A'|}\big)\,d\lambda \\ &= \sum_\mu \tfrac14 \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s-1}\big(-\tfrac{\mu}{(\mu^2-\lambda)^{\frac12}} + \tfrac{\mu}{|\mu|}\big)\,d\lambda \cdot \Pi_\mu \\ &= \sum_\mu \tfrac14 \mu|\mu|^{-2s-1} \tfrac{i}{2\pi}\int_{\mathcal{C}} \tau^{-s-1}\big(\tfrac{-1}{(1-\tau)^{\frac12}} + 1\big)\,d\tau \cdot \Pi_\mu \\ &= \tfrac14(-F_{\frac12}(s) + F_0(s)) Y(A, 2s) = -\tfrac14 F_{\frac12}(s) Y(A, 2s). \quad \square \end{aligned} \tag{4.32}$$

Note that the *even part* produces a function derived from the *zeta* function of A, and the *odd part* produces a function derived from the *eta* function of A. This is the fundamental observation for the following, relating the power functions of the boundary value problem to those of A.

Now we combine this with the interior contribution, taken from the doubled manifold $\widetilde{X}$. This leads to the key result:

Theorem 4.6 *(Product case with $B_0 = 0$.) The zeta functions have the following decompositions:*

$$\Gamma(s)\zeta(\Delta_i,s) = \Gamma(s)[\zeta_+(\widetilde{\Delta}_i,s)+\tfrac14(F_{\frac12}(s)-1)\zeta(A^2,s)+(-1)^i\tfrac14 F_{\frac12}(s)\eta(A,2s)] + \tfrac1s[\operatorname{Tr}_+(\Pi_0(\widetilde{\Delta}_i)) - \nu_0(\Delta_i) + (-1)^i\tfrac14\nu_0(A)] + h_i(s), \quad (4.33)$$

where the h_i are entire. Moreover, $\Gamma(s)\zeta(\Delta_i,s)$ is $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \geq 1$, $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$, any $\varepsilon > 0$.

Here $\zeta_+(\widetilde{\Delta}_i,s) = \operatorname{Tr}_+ Z(\widetilde{\Delta}_i,s)$ (cf. (4.4)).

The basic idea in the proof goes as follows: By Lemma 4.4, the resolvent $(\Delta_i-\lambda)^{-1} = (\widetilde{\Delta}_i-\lambda)_+^{-1} + G_{i,\lambda}$ has the same asymptotic behavior for λ going to infinity as $(\widetilde{\Delta}_i-\lambda)_+^{-1} + \chi G^0_{i,\lambda}\chi$, and the last term behaves like $G^0_{i,\lambda}$. Here the contribution from $\widetilde{\Delta}_i$ is well-known; and the contributions from $G_{\mathrm{e},\lambda}$ and $G_{\mathrm{o},\lambda}$ in $G^0_{i,\lambda}$ have been dealt with in Proposition 4.5; they give the terms involving $F_{\frac12}(s)$. What remains is some adjustments due to the Laurent expansions of the resolvents at $\lambda = 0$ and the trace of $G^0_{i,\lambda}$ restricted to the nullspace of A, plus the contribution from an $O(|\lambda|^{-N})$ term; these adjustments yield the coefficient of $\frac1s$ in (4.33) and the entire function. The explanation is slightly technical because of the need to consider differentiated resolvents as in (2.11). We leave out further details; they are given in [15].

Example 4.7 For the Dirichlet realization Δ^0_{Dir} of $D_n^2 + A^2$, a calculation as in (4.31) gives, by (4.24),

$$\operatorname{tr}_n G_{Z,\mathrm{Dir},s} = \operatorname{tr}_n \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s} G^0_{\mathrm{Dir},\lambda}\,d\lambda = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} \lambda^{-s}\tfrac{-1}{4(A^2-\lambda)}\,d\lambda = -\tfrac14 Z(A^2,s).$$

Then the zeta function for the Dirichlet realization Δ_{Dir} of P^*P has the decomposition (with $h(s)$ entire):

$$\Gamma(s)\zeta(\Delta_{\mathrm{Dir}},s) = \Gamma(s)\zeta_+(\widetilde{\Delta}_1,s) - \tfrac14\Gamma(s)\zeta(A^2,s) + h(s). \quad (4.34)$$

For the Neumann case one gets this formula with $-\frac14$ replaced by $+\frac14$.

A similar analysis applies to the eta functions associated with P_B, and to functions with differential operators inserted in front. Consider e.g. the eta function $\Gamma(s)\operatorname{Tr}(\varphi P\Delta_1^{-s})$, where φ is a bundle morphism from E_2 to E_1, equal to $\varphi^0 = \varphi|_{X'}$ on $X' \times [0,c]$. (Some morphism is needed in order to allow taking the trace in $L_2(E_1)$; e.g., σ^* can be used for φ.)

Theorem 4.8 *(Product case with $B_0 = 0$.) The eta function $\Gamma(s)\eta(\varphi, P_B, 2s-1)$ has the following decomposition:*

$$\begin{aligned}\Gamma(s)\eta(\varphi, P_B, 2s-1) &\equiv \Gamma(s)\operatorname{Tr}(\varphi P \Delta_1^{-s}) \\ &= \Gamma(s)\big[\operatorname{Tr}_+(\varphi P \widetilde{\Delta}_1^{-s}) + \tfrac14 (F_{\frac12}(s-1) - 1)\eta(\varphi^0\sigma, A, 2s-1)\big] \\ &\qquad + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\varphi^0 \sigma \Pi_0(A))(s-\tfrac12)^{-1} + h_1(s), \quad (4.35)\end{aligned}$$

where $h_1(s)$ is entire. Moreover, $\Gamma(s)\eta(\varphi, P_B, 2s-1)$ is $O(e^{(-\frac{\pi}{2}+\varepsilon)|\operatorname{Im} s|})$ for $|\operatorname{Im} s| \geq 1$, $-\infty < C_1 \leq \operatorname{Re} s \leq C_2 < \infty$, any $\varepsilon > 0$.

There is a similar result for $\Gamma(s)\operatorname{Tr}(\varphi P^* \Delta_2^{-s})$, where φ is a morphism from E_1 to E_2.

4.3. PRECISE TRACE FORMULAS IN THE PRODUCT CASE

It is shown in Theorems 4.6 and 4.8 how the zeta and eta functions of the APS operator arise by simple addition of known zeta and eta functions with factors defined from $F_{\frac12}(s)$ in front.

This makes it easy to determine the pole structure! We know the pole structure of the zeta and eta functions of the operators $\widetilde{P}$, $\widetilde{\Delta}_i$ and A, and we also know the pole structure of $F_{\frac12}(s)$ from its gamma function components. The result is that we get from each decomposition a meromorphic function with poles where those functions have them; and the poles will be double when there are coincidences. Accordingly, there will be heat trace expansions with powers t^β corresponding to the simple poles $-\beta$, and powers t^β plus $t^\beta \log t$ terms corresponding to double poles $-\beta$.

We list the precise result below. An interesting aspect is that it shows a difference between the cases n even and n odd.

In the case n even, coincidences between poles give rise to double poles (hence log-terms in the heat operator formulation). At a double pole $-\beta$, the singular part consists both of a coefficient c times $(s+\beta)^{-2}$ and another coefficient c' times $(s+\beta)^{-1}$. The first coefficient c is determined from the symbols of the operators in a well-known local way, whereas the second coefficient c' is usually just globally determined.

In the case n odd, there are no coincidences, hence no double poles. But here the poles of $F_{\frac12}$ force us to evaluate the zeta and eta functions at the points midways between their well-known poles; also this gives new global coefficients. (These are the points where the poles according to (3.24) have vanishing residue (3.25), so the value can also be regarded as the second coefficient where the first one is 0.)

Now comes the detailed description:

We denote the second coefficient in the Laurent series for $\Gamma(s)\zeta(D,Q,s)$ at a pole $s_j=\frac{-j+n_1+d}{2}$ by $c'_j(D,Q)$:

$$c'_j(D,Q)=\lim_{s\to s_j}\big[\Gamma(s)\zeta(D,Q,s)-\tfrac{c_j(D,Q)}{s-s_j}\big]=\operatorname{Res}_{s=s_j}\tfrac{\Gamma(s)\zeta(D,Q,s)}{s-s_j};\tag{4.36}$$

here $\operatorname{Res}_{s=s'}$ means the residue at s'. (In case $c_j(D,Q)=0$, $c'_j(D,Q)$ is the *value* of $\Gamma(s)\zeta(D,Q,s)$ at the point.)

We also need to define some universal constants:

$$\begin{aligned}
\beta_m&=\operatorname{Res}_{s=-\frac12-m}\tfrac14F_{\frac12}(s)=\tfrac{(-1)^m}{4m!\sqrt{\pi}\,\Gamma(\frac12-m)},\\
\beta'_m&=\operatorname{Res}_{s=-\frac12-m}\tfrac14F_{\frac12}(s)(s+\tfrac12-m)^{-1},\\
\gamma_k&=\tfrac14(F_{\frac12}(\tfrac{k}{2})-1)=\tfrac14\Big[\tfrac{\Gamma(\frac{k+1}{2})}{\sqrt{\pi}\,\Gamma(1+\frac{k}{2})}-1\Big],\\
\varepsilon_m&=\operatorname{Res}_{s=-\frac12-m}\tfrac14F_{\frac12}(s)\Gamma(s)=\tfrac{(-1)^{m+1}}{4m!\sqrt{\pi}\,(m+\frac12)},\\
\delta_m&=\operatorname{Res}_{s=\frac12-m}\tfrac14F_{\frac12}(s-1)\Gamma(s)=\operatorname{Res}_{s=\frac12-m}\tfrac{1}{4\sqrt{\pi}}\Gamma(s-\tfrac12)=\tfrac{(-1)^m}{4\sqrt{\pi}\,m!};
\end{aligned}\tag{4.37}$$

here $m\in\mathbf{N}$, and the k are integers avoiding negative odd numbers. (The explicit expressions are found by use of the formula $\Gamma(s)=\pi\Gamma(1-s)/\sin\pi s$. Also β'_m can be written more explicitly, departing from the fact that $-\Gamma'(1)$ equals Euler's constant.)

From (3.24) we find, omitting vanishing coefficients,

$$\Gamma(s)\zeta_+(\widetilde{\Delta}_1,s)\sim\sum_{k=0}^{\infty}\frac{c_{2k,+}(\widetilde{\Delta}_1)}{s+k-\frac{n}{2}}-\frac{\operatorname{Tr}_+(\Pi_0(\widetilde{\Delta}_1))}{s},\tag{4.38}$$

where $c_{j,+}(\widetilde{\Delta}_1)=\int_X\operatorname{tr}c_j(x,\widetilde{\Delta}_1)\,dx$; cf. also (4.4). Since A acts on X' of dimension $n-1$, we get from (3.26):

$$\Gamma(s)\zeta(A^2,s)\sim\sum_{k=0}^{\infty}\frac{c_{2k}(A^2)}{s+k-\frac{n-1}{2}}-\frac{\nu_0(A)}{s},\tag{4.39}$$

and, for example, when ψ is a morphism in E_1,

$$\begin{aligned}
\Gamma(s)F_{\frac12}(s)\eta(\psi,A,2s)&=\frac{1}{\sqrt{\pi}\,s}\Gamma(s+\tfrac12)\zeta(\psi A,A^2,s+\tfrac12)\\
&\sim\sum_{0\le k}\frac{c_{2k+1}(\psi A,A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)(s+k+1-\frac{n}{2})}+\frac{\eta(\psi,A,0)}{s}\quad\text{if } n \text{ is odd},\\
&\sim\sum_{0\le k\ne\frac{n}{2}-1}\frac{c_{2k+1}(\psi A,A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)(s+k+1-\frac{n}{2})}\\
&\qquad+\frac{c_{n-1}(\psi A,A^2)}{\sqrt{\pi}\,s^2}+\frac{c'_{n-1}(\psi A,A^2)}{\sqrt{\pi}\,s}\qquad\text{if } n \text{ is even},
\end{aligned}\tag{4.40}$$

where $c'_{n-1}(\psi A, A^2)$ is defined as in (4.36). When $\psi = I$ then $c_{n-1}(A, A^2) = 0$ and $c'_{n-1}(A, A^2) = \sqrt{\pi}\,\eta(A, 0)$.

Insertion of these expansions in our decompositions gives:

Corollary 4.9 *The zeta function $\Gamma(s)\zeta(\Delta_i, s)$ is meromorphic on* $\mathbf{C}$, *with the following singularity structure:*

For n even:

$$\Gamma(s)\zeta(\Delta_i, s) \sim \sum_{k\geq 0} \frac{c_{2k,+}(\widetilde{\Delta}_i)}{s+k-\frac{n}{2}} - \frac{\mathrm{Tr}_+\,\Pi_0(\widetilde{\Delta}_i)}{s} + \sum_{0\leq k<\frac{n}{2}} \frac{\gamma_{n-1-2k}c_{2k}(A^2)}{s+k-\frac{n-1}{2}}$$
$$+ \sum_{k\geq\frac{n}{2}} [\frac{\beta_{k-\frac{n}{2}}c_{2k}(A^2)}{(s+k-\frac{n-1}{2})^2} + \frac{\beta_{k-\frac{n}{2}}c'_{2k}(A^2) + (\beta'_{k-\frac{n}{2}} - \frac{1}{4})c_{2k}(A^2)}{s+k-\frac{n-1}{2}}]$$
$$+ (-1)^i \tfrac{1}{4} [\sum_{0\leq k\neq\frac{n}{2}-1} \frac{c_{2k+1}(A, A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)(s+k+1-\frac{n}{2})} + \frac{\eta(A,0)+\nu_0(A)}{s}]. \quad (4.41)$$

For n odd:

$$\Gamma(s)\zeta(\Delta_i, s) \sim \sum_{k\geq 0} \frac{c_{2k,+}(\widetilde{\Delta}_i)}{s+k-\frac{n}{2}} - \frac{\mathrm{Tr}_+\,\Pi_0(\widetilde{\Delta}_i)}{s} + \sum_{k\geq 0} \frac{\gamma_{n-1-2k}c_{2k}(A^2)}{s+k-\frac{n-1}{2}}$$
$$+ \sum_{m\geq 0} \frac{\varepsilon_m \zeta(A^2, -m-\frac{1}{2})}{s+m+\frac{1}{2}}$$
$$+ (-1)^i \tfrac{1}{4} [\sum_{k\geq 0} \frac{c_{2k+1}(A, A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)(s+k+1-\frac{n}{2})} + \frac{\eta(A,0)+\nu_0(A)}{s}]. \quad (4.42)$$

[15] moreover shows the formulas where a morphism is included. The terms β'_m were missing in the Preprint version of [15].

Corollary 4.10 *The eta function $\Gamma(s)\eta(\varphi, P_B, 2s-1) = \Gamma(s)\,\mathrm{Tr}(\varphi P \Delta_1^{-s})$ is meromorphic on* $\mathbf{C}$, *with the following singularity structure:*

For n even:

$$\Gamma(s)\,\mathrm{Tr}(\varphi P \Delta_1^{-s}) \sim \sum_{k\geq 0} \frac{c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)}{s+k-\frac{n}{2}}$$
$$+ \sum_{0\leq k<\frac{n}{2}-1} \frac{\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k-\frac{n-1}{2}} + \sum_{k\geq\frac{n}{2}-1} [\frac{\beta_{k+1-\frac{n}{2}}c_{2k+1}(\varphi^0\sigma A, A^2)}{(s+k-\frac{n-1}{2})^2}$$
$$+ \frac{\beta_{k+1-\frac{n}{2}}c'_{2k+1}(\varphi^0\sigma A, A^2) + (\beta'_{k+1-\frac{n}{2}} - \frac{1}{4})c_{2k+1}(\varphi^0\sigma A, A^2)}{s+k-\frac{n-1}{2}}]$$
$$+ \frac{\mathrm{Tr}(\varphi^0\sigma\Pi_0(A))}{4\sqrt{\pi}(s-\frac{1}{2})}. \quad (4.43)$$

For n odd*:*

$$\Gamma(s)\,\mathrm{Tr}(\varphi P\Delta_1^{-s}) \sim \sum_{k\ge 0}\frac{c_{2k+1,+}(\varphi P,\widetilde{\Delta}_1)}{s+k-\frac{n}{2}} + \sum_{k\ge 0}\frac{\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A,A^2)}{s+k-\frac{n-1}{2}}$$
$$+\sum_{m\ge 0}\frac{\delta_m\eta(\varphi^0\sigma A,-2m)}{s+m-\frac12} + \frac{\mathrm{Tr}(\varphi^0\sigma\Pi_0(A))}{4\sqrt{\pi}(s-\frac12)}. \quad (4.44)$$

There are similar formulas for $\mathrm{Tr}(\varphi P^*\Delta_2^{-s})$*, with* $\varphi^0\sigma$ *replaced by* $\sigma^*\varphi^0$*.*

Let us finally list the consequences for heat traces, derived from Corollary 4.9–4.10 by use of Theorem 2.3:

Corollary 4.11 *The exponential trace* $\mathrm{Tr}(e^{-t\Delta_i})$ *has the following behavior for* $t\to 0$*.*

For n even:

$$\mathrm{Tr}(e^{-t\Delta_i}) \sim \sum_{k\ge 0} c_{2k,+}(\widetilde{\Delta}_i)\,t^{k-\frac{n}{2}} + \sum_{0\le k<\frac{n}{2}}\gamma_{n-1-2k}c_{2k}(A^2)\,t^{k-\frac{n-1}{2}}$$
$$+\sum_{k\ge\frac{n}{2}}\big[-\beta_{k-\frac{n}{2}}c_{2k}(A^2)\log t + \beta_{k-\frac{n}{2}}c'_{2k}(A^2) + (\beta'_{k-\frac{n}{2}}-\tfrac14)c_{2k}(A^2)\big]\,t^{k-\frac{n-1}{2}}$$
$$+(-1)^i\tfrac14\Big[\sum_{0\le k\ne\frac{n}{2}-1}\frac{c_{2k+1}(A,A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)}\,t^{k+1-\frac{n}{2}} + \eta(A,0)+\nu_0(A)\Big]. \quad (4.45)$$

For n odd:

$$\mathrm{Tr}(e^{-t\Delta_i}) \sim \sum_{k\ge 0} c_{2k,+}(\widetilde{\Delta}_i)\,t^{k-\frac{n}{2}}$$
$$+\sum_{k\ge 0}\gamma_{n-1-2k}c_{2k}(A^2)\,t^{k-\frac{n-1}{2}} + \sum_{m\ge 0}\varepsilon_m\zeta(A^2,-m-\tfrac12)\,t^{m+\frac12}$$
$$+(-1)^i\tfrac14\Big[\sum_{k\ge 0}\frac{c_{2k+1}(A,A^2)}{\sqrt{\pi}\,(\frac{n}{2}-k-1)}\,t^{k+1-\frac{n}{2}} + \eta(A,0)+\nu_0(A)\Big]. \quad (4.46)$$

Corollary 4.12 *The associated exponential trace* $\mathrm{Tr}(\varphi Pe^{-t\Delta_1})$ *has the following behavior for* $t\to 0$*.*

For n even:

$$\mathrm{Tr}(\varphi Pe^{-t\Delta_1}) \sim \sum_{k\ge 0} c_{2k+1,+}(\varphi P,\widetilde{\Delta}_1)\,t^{k-\frac{n}{2}}$$
$$+\sum_{0\le k<\frac{n}{2}-1}\gamma_{n-3-2k}c_{2k+1}(\varphi^0\sigma A,A^2)\,t^{k-\frac{n-1}{2}} +$$

$$+ \sum_{k\geq \frac{n}{2}-1} [-\beta_{k+1-\frac{n}{2}} c_{2k+1}(\varphi^0\sigma A, A^2)\, t^{k-\frac{n-1}{2}} \log t$$

$$+ (\beta_{k+1-\frac{n}{2}} c'_{2k+1}(\varphi^0\sigma A, A^2) + (\beta'_{k+1-\frac{n}{2}} - \tfrac{1}{4}) c_{2k+1}(\varphi^0\sigma A, A^2))\, t^{k-\frac{n-1}{2}}]$$

$$+ \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\varphi^0\sigma\Pi_0(A))\, t^{-\frac{1}{2}}. \quad (4.47)$$

For n odd*:*

$$\operatorname{Tr}(\varphi P e^{-t\Delta_1}) \sim \sum_{k\geq 0} c_{2k+1,+}(\varphi P, \widetilde{\Delta}_1)\, t^{k-\frac{n}{2}}$$

$$+ \sum_{k\geq 0} \gamma_{n-3-2k} c_{2k+1}(\varphi^0\sigma A, A^2)\, t^{k-\frac{n-1}{2}}$$

$$+ \sum_{m\geq 0} \delta_m \eta(\varphi^0\sigma A, -2m)\, t^{m-\frac{1}{2}} + \tfrac{1}{4\sqrt{\pi}} \operatorname{Tr}(\varphi^0\sigma\Pi_0(A))\, t^{-\frac{1}{2}}. \quad (4.48)$$

There are similar formulas for $\operatorname{Tr}(\varphi P^* e^{-t\Delta_2})$*, with* $\varphi^0\sigma$ *replaced by* $\sigma^*\varphi^0$*.*

The proof shows the advantage of working with the power functions, where the contributions from the boundary condition appear as simple multiplicative formulas involving the zeta and eta functions of A; this allows an exact analysis of the pole coefficients which can then be carried over to the heat expansions by Theorem 2.3. If working directly in the heat operator framework (a point of view taken up in [6]), one has to deal with convolution-type integrals.

Gilkey and Grubb [11] show that all terms, in particular the logarithmic ones, are nontrivial in general. Dowker, Apps, Kirsten and Bordag [7] find no logarithms for the Dirac operator on the ball; this is due to special symmetries and does not contradict the above since it is not a product case.

Example 4.13 For the Dirichlet problem considered in Examples 4.2, 4.3 and 4.7, formula (4.34) implies in a similar way:

$$\operatorname{Tr}(e^{-t\Delta_{\mathrm{Dir}}}) \sim \sum_{k\geq 0} c_{2k,+}(\widetilde{\Delta}_1)\, t^{k-\frac{n}{2}} - \tfrac{1}{4} \sum_{k\geq 0} c_{2k}(A^2)\, t^{k-\frac{n-1}{2}}; \quad (4.49)$$

note that all the integer and half-integer powers enter here too. There is a similar formula for the Neumann problem, with $-\frac{1}{4}$ replaced by $+\frac{1}{4}$.

Remark 4.14 In a recent study of the gluing problem for the eta-invariant, [6], Brüning and Lesch treat boundary conditions of a somewhat different nature than those considered here and in [14], [15]; moreover they depend on a parameter and the variation in this parameter is studied. We show below how those new boundary conditions can be handled in the present framework: Restrict the attention to selfadjoint operators P satisfying $\sigma^* = -\sigma$, $\sigma A = -A\sigma$ as in Remark 1.3. Let B be an orthogonal projection in $L_2(E_1')$ commuting with A^2 and satisfying

$$\begin{aligned} &\text{(i) } \sigma B = (I-B)\sigma, \\ &\text{(ii) } BAB = \alpha|A|B \text{ for some } \alpha > -1. \end{aligned} \tag{4.50}$$

([6] gives special examples of the form $B = \sigma_1\Pi_> + \sigma_2\Pi_< + B_0$ with morphisms or zero order ψdo's σ_1 and σ_2.) Because of (i), P_B is selfadjoint, and $\Delta_B = {P_B}^2$ is the realization of P^2 under the boundary condition (where $B\gamma_0$ is written $\gamma_0 B$)

$$\gamma_0 Bu = 0, \quad \gamma_0 B\sigma(\partial_n + A)u = 0. \tag{4.51}$$

For the second equation we note that when $\gamma_0 Bu = 0$, then in view of (i), $\gamma_0 B\sigma(\partial_n+A)u = \sigma\gamma_0(I-B)(\partial_n+A)(I-B)u = \sigma\gamma_0(\partial_n+(I-B)A)(I-B)u$. Here, by (i) and (ii), $(I-B)A(I-B) = -\alpha|A|(I-B)$. Thus the boundary condition may be written:

$$\gamma_0 Bu = 0, \quad \gamma_0(\partial_n - \alpha|A|)(I-B)u = 0. \tag{4.52}$$

This is a Dirichlet condition for the functions of x_n valued in $R(B)$, and a Robin-type condition as in Example 4.2 with $S = -\alpha|A|$ for the functions valued in $R(I-B)$. Then by the calculations in Example 4.2, the resolvent on X^0 is $(\Delta_B^0 - \lambda)^{-1} = Q^0_{1,\lambda,+} + G^0_{B,\lambda}$ with

$$\begin{aligned} G^0_{B,\lambda} &= \big(\tfrac{-1}{2A_\lambda}B + \tfrac{1}{2A_\lambda}\tfrac{A_\lambda-\alpha|A|}{A_\lambda+\alpha|A|}(I-B)\big)G_\lambda \\ &= \big(\tfrac{-1}{2A_\lambda} + \tfrac{A_\lambda-\alpha|A|}{2((1-\alpha^2)A^2-\lambda)} + \tfrac{1}{2(A_\lambda+\alpha|A|)}(I-2B)\big)G_\lambda. \end{aligned} \tag{4.53}$$

Now Lemma 4.4 can be extended to this case. Therefore we have as in the proof of Theorem 4.6,

$$\Gamma(s)\zeta(\Delta_B,s) = \Gamma(s)\zeta_+(\widetilde{\Delta}_1,s) + \Gamma(s)\,\mathrm{Tr}_X\,\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G^0_{B,\lambda}\,d\lambda + h(s), \tag{4.54}$$

with $h(s)$ entire; and here

$$\begin{aligned} &\mathrm{Tr}_X\,\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G^0_{B,\lambda}\,d\lambda = \mathrm{Tr}_{X'}\,\mathrm{tr}_n\,\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}G^0_{B,\lambda}\,d\lambda = \\ &\mathrm{Tr}_{X'}\,\tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}\lambda^{-s}\big(\tfrac{-1}{2A_\lambda} + \tfrac{A_\lambda-\alpha|A|}{2((1-\alpha^2)A^2-\lambda)} + \tfrac{1}{2(A_\lambda+\alpha|A|)}(I-2B)\big)\tfrac{1}{2A_\lambda}\,d\lambda. \end{aligned} \tag{4.55}$$

The term $\frac{1}{2(A_\lambda+\alpha|A|)}(I-2B)\frac{1}{2A_\lambda}$ contributes with zero, for by (i) and the fact that σ and B commute with A^2,

$$\begin{aligned}\tfrac{1}{2(A_\lambda+\alpha|A|)}(I-2B)\tfrac{1}{2A_\lambda}&=\tfrac{1}{4A_\lambda(A_\lambda+\alpha|A|)}(I-B)-\tfrac{1}{4A_\lambda(A_\lambda+\alpha|A|)}\sigma^*\sigma B\\&=\tfrac{1}{4A_\lambda(A_\lambda+\alpha|A|)}(I-B)-\sigma^*\tfrac{1}{4A_\lambda(A_\lambda+\alpha|A|)}(I-B)\sigma;\end{aligned}\tag{4.56}$$

here since the trace is invariant under circular perturbations (that we can use in a reformulation with sufficiently high λ-derivatives as in (2.10)), the contributions from these two terms will cancel each other. The remaining terms are treated as in Proposition 4.5 (we give the details for $\alpha<1$; the case $\alpha>1$ is similar and the case $\alpha=1$ is simpler):

$$\begin{aligned}&\tfrac{i}{2\pi}\int_{\mathcal{C}}\lambda^{-s}\big(\tfrac{-1}{4A_\lambda^2}+\tfrac{A_\lambda-\alpha|A|}{4A_\lambda((1-\alpha^2)A^2-\lambda)}\big)\,d\lambda\\&\quad=\sum_{\mu\in\operatorname{sp}(A)}\tfrac14\tfrac{i}{2\pi}\textstyle\int_{\mathcal{C}}\lambda^{-s}\big(\tfrac{-1}{\mu^2-\lambda}+\tfrac{1}{(1-\alpha^2)\mu^2-\lambda}-\tfrac{\alpha|\mu|}{(\mu^2-\lambda)^{\frac12}((1-\alpha^2)\mu^2-\lambda)}\big)\,d\lambda\cdot\Pi_\mu\\&\quad=\sum_{\mu}\tfrac14|\mu|^{-2s}\tfrac{i}{2\pi}\textstyle\int_{\mathcal{C}}\tau^{-s}\big(\tfrac{-1+(1-\alpha^2)^{-s}}{1-\tau}-\tfrac{\alpha}{(1-\tau)^{\frac12}(1-\alpha^2-\tau)}\big)\,d\tau\cdot\Pi_\mu\\&\quad=\tfrac14\big(-1+e^{-s\log(1-\alpha^2)}+\widetilde{F}_\alpha(s)\big)Z(A^2,s);\end{aligned}\tag{4.57}$$

with

$$\widetilde{F}_\alpha(s)=\tfrac{i}{2\pi}\int_{\mathcal{C}}\tau^{-s}\frac{-\alpha}{(1-\tau)^{\frac12}(1-\alpha^2-\tau)}\,d\tau.\tag{4.58}$$

This is a hypergeometric function whose pole structure is easily determined by use of Theorem 2.1. In fact, $\widetilde{F}_\alpha(s)$ is of the form (2.17) with $f(\tau)=-\alpha(1-\tau)^{-\frac12}(1-\alpha^2-\tau)^{-1}$. It is holomorphic on $\mathbf{C}\setminus[1,\infty[$ and has the asymptotic expansion for $-\tau\to\infty$ in closed subsectors:

$$\begin{aligned}f(-\tau)&=-\alpha\tau^{-\frac32}(1+\tfrac1\tau)^{-\frac12}(1+\tfrac{1-\alpha^2}{\tau})^{-1}\\&\sim-\alpha\tau^{-\frac32}\sum_{k\in\mathbf{N}}\tbinom{-\frac12}{k}\tau^{-k}\sum_{l\in\mathbf{N}}(\alpha^2-1)^l\tau^{-l}=\sum_{j\in\mathbf{N}}\omega_j\tau^{-\frac32-j}.\end{aligned}\tag{4.59}$$

An application of Theorem 2.1 carries the terms $\omega_j t^{-\frac32-j}$ over into simple poles at $s=-j-\frac12$ for $\frac{\pi}{\sin\pi s}\widetilde{F}_\alpha(s)$ with residues ω_j. The poles at integers $j+1$ stemming from the Taylor expansion at 0 are removed when we multiply by $\pi^{-1}\sin\pi s$. Consequently, $\widetilde{F}_\alpha(s)$ is meromorphic on $\mathbf{C}$ with simple poles at the points $-j-\frac12$, $j\in\mathbf{N}$, with residues $\pi^{-1}(-1)^{j+1}\omega_j$.

Finally,

$$\begin{aligned}\Gamma(s)\zeta(\Delta_B,s)&=\Gamma(s)\zeta_+(\widetilde{\Delta}_1,s)\\&\quad+\tfrac14\big(-1+e^{-s\log(1-\alpha^2)}+\widetilde{F}_\alpha(s)\big)\Gamma(s)\zeta(A^2,s)+h(s),\end{aligned}\tag{4.60}$$

which is meromorphic on $\mathbf{C}$ with poles at the points $(n-k)/2$, $k \in \mathbf{N}$; here the poles at the negative half-integers $-j-\frac{1}{2}$ are in general double when n is even; otherwise the poles are simple. A heat trace expansion in terms of $t^{(k-n)/2}$ and $t^{l+\frac{1}{2}} \log t$ $(k, l \in \mathbf{N})$ follows as usual by Theorem 2.3.

Note that (4.58) also implies: 1) $\widetilde{F}_\alpha(s)$ equals $\pi^{-1} \sin \pi s$ times the Mellin transform of $-\alpha(1+\tau)^{-\frac{1}{2}}(1-\alpha^2+\tau)^{-1}$ at $s-1$; cf. (2.18), (2.33).

2) $(1-\alpha^2)\widetilde{F}_\alpha(s) - \widetilde{F}_\alpha(s-1) = -\alpha F_{\frac{1}{2}}(s-1)$; cf. (4.28).

5. The general case

5.1. A GENERAL RESOLVENT CONSTRUCTION

In the non-product case the results will be more qualitative. A useful trick here is to replace the separate consideration of P_B and ${P_B}^*$ by the study of the skew-selfadjoint operator

$$\mathcal{P}_\mathcal{B} = \begin{pmatrix} 0 & -{P_B}^* \\ P_B & 0 \end{pmatrix}; \tag{5.1}$$

this is the realization of

$$\mathcal{P} = \begin{pmatrix} 0 & -P^* \\ P & 0 \end{pmatrix} \tag{5.2}$$

under the following boundary condition on $u = \{u_1, u_2\}$ (cf. (1.5)):

$$\mathcal{B}\gamma_0 u = 0, \text{ where } \mathcal{B} = \begin{pmatrix} B & B' \end{pmatrix} : \begin{matrix} L_2(E_1') \\ \times \\ L_2(E_2') \end{matrix} \to L_2(E_1'). \tag{5.3}$$

The advantage of taking P_B and ${P_B}^*$ together in this way is that $\mathcal{P}_\mathcal{B}$ is two-sided elliptic, and $\mathcal{R}_\mu = (\mathcal{P}_\mathcal{B}+\mu)^{-1}$, defined for $\mu \in \pm\Gamma_0$, $\Gamma_0 = \{\, \mu \in \mathbf{C} \setminus \{0\} \mid |\arg \mu| < \pi/2 \,\}$, satisfies

$$\mathcal{R}_\mu = (\mathcal{P}_\mathcal{B}+\mu)^{-1} = \begin{pmatrix} \mu(\Delta_1+\mu^2)^{-1} & {P_B}^*(\Delta_2+\mu^2)^{-1} \\ -P_B(\Delta_1+\mu^2)^{-1} & \mu(\Delta_2+\mu^2)^{-1} \end{pmatrix}, \tag{5.4}$$

where $(\Delta_i+\mu^2)^{-1} = R_{i,-\mu^2}$ are the resolvents we are looking for (cf. (1.7)). The diagonal terms give back the individual resolvents, and the off-diagonal terms can be used to describe eta functions instead of zeta functions.

This allows us to stay working with first-order systems (instead of passing to second order), at the cost of doubling up the size of the matrix.

We shall denote $E_1 \oplus E_2 = E$ and $E_1' \oplus E_2' = E'$.

We let $\widetilde{\mathcal{P}} = \begin{pmatrix} 0 & -\widetilde{P}^* \\ \widetilde{P} & 0 \end{pmatrix}$, where $\widetilde{P}$ is an elliptic extension of P to a bundle $\widetilde{E} = \widetilde{E}_1 \oplus \widetilde{E}_2$ over $\widetilde{X} = X \cup (X' \times \,]-1, 0[\,)$. Then $\widetilde{\mathcal{P}} + \mu$ has a parametrix Q_μ (of strongly polyhomogeneous type) for $\mu \in \pm\Gamma_0$, and as shown in detail in [14], p. 508–9, it can be modified such that for large μ in closed subsectors of $\pm\Gamma_0$,

$$(\mathcal{P} + \mu) Q_{\mu,+} = I \text{ on } X. \tag{5.5}$$

Also here, a comparison with the cylinder case (cf. (4.8)) plays a role. We denote $\begin{pmatrix} 0 & -P^{0\prime} \\ P^0 & 0 \end{pmatrix} = \mathcal{P}^0$, acting in $E^0 = E_1^0 \oplus E_2^0$. We extend $\mathcal{P}^0$ to $\widetilde{X}^0$ simply by extending the formulas (4.8) to $x_n \in \mathbf{R}$, letting $\widetilde{E}^0 = \widetilde{E}_1^0 \oplus \widetilde{E}_2^0$ be the lifting of $E' = E_1' \oplus E_2'$. Then the extended operator $\widetilde{\mathcal{P}}^0$ is skew-selfadjoint, and the resolvent is

$$\begin{gathered} Q_\mu^0 = (\widetilde{\mathcal{P}}^0 + \mu)^{-1} = \\ \begin{pmatrix} \mu (D_n^2 + A^2 + \mu^2)^{-1} & (-\partial_n + A)(D_n^2 + A^2 + \mu^2)^{-1} \sigma^* \\ -\sigma(\partial_n + A)(D_n^2 + A^2 + \mu^2)^{-1} & \mu\sigma (D_n^2 + A^2 + \mu^2)^{-1} \sigma^* \end{pmatrix}. \end{gathered} \tag{5.6}$$

In particular,

$$(\mathcal{P}^0 + \mu) Q_{\mu,+}^0 = I \text{ on } X^0. \tag{5.7}$$

Along with $\mathcal{P}_\mathcal{B}$, we study the realization $\mathcal{P}_\mathcal{B}^0$, acting like $\mathcal{P}^0$ on X^0 and with the same boundary condition (5.3) as $\mathcal{P}$. With a slight abuse of notation, we now denote

$$A_\mu = (A^2 + \mu^2)^{\frac{1}{2}}, \text{ for } \mu \in \pm\Gamma_0. \tag{5.8}$$

Lemma 5.1 *Define the ψdo from sections of E_1' to sections of $E_1' \oplus E_1'$:*

$$\mathcal{S}_{\mathcal{B},\mu} = \begin{pmatrix} B + \mu^{-1}(A_\mu + A) B^- \\ \mu^{-1}(A_\mu - A) B + B^- \end{pmatrix}; \tag{5.9}$$

and the Poisson operator from sections of E_1' to sections of E^0:

$$K_{\mathcal{B},\mu}^0 = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} e^{-x_n A_\mu} \mathcal{S}_{\mathcal{B},\mu}. \tag{5.10}$$

Then $K_{\mathcal{B},\mu}^0$ satisfies

$$\mathcal{B}\gamma_0 K_{\mathcal{B},\mu}^0 = I \textit{ on } X', \quad (\mathcal{P}^0 + \mu) K_{\mathcal{B},\mu}^0 = 0 \textit{ on } X^0. \tag{5.11}$$

The proof is a direct verification, using that B commutes with A.

In other words, $K^0_{\mathcal{B},\mu} : \psi \mapsto u$ solves the problem

$$\begin{aligned} (\mathcal{P}^0 + \mu)u &= g \text{ on } X^0, \\ \mathcal{B}\gamma_0 u &= \psi \text{ on } X', \end{aligned} \tag{5.12}$$

when $g = 0$. We note that by (5.7), the full solution operator for (5.12) is

$$\begin{pmatrix} \mathcal{R}^0_\mu & K^0_{\mathcal{B},\mu} \end{pmatrix}, \text{ where } \mathcal{R}^0_\mu = (\mathcal{P}^0_{\mathcal{B}} + \mu)^{-1} = Q^0_{\mu,+} - K^0_{\mathcal{B},\mu}\mathcal{B}\gamma_0 Q^0_{\mu,+}; \tag{5.13}$$

cf. also Remark 4.1.

Now $\mathcal{R}^0_\mu$ is principally like the true resolvent $\mathcal{R}_\mu$ at X'. However, we prefer to use a better adapted approximate resolvent, namely

$$\mathcal{R}'_\mu = Q_{\mu,+} - G_1 \text{ with } G_1 = \chi K^0_{\mathcal{B},\mu}\mathcal{B}\gamma_0 Q_{\mu,+}, \tag{5.14}$$

where $Q_{\mu,+}$ satisfies (5.5) and χ is a cut-off function as in Lemma 4.4. By (5.11), $\mathcal{R}'_\mu$ maps into the domain of $\mathcal{P}_{\mathcal{B}}$, and by (5.5), we have for large enough μ,

$$\begin{aligned} (\mathcal{P} + \mu)\mathcal{R}'_\mu &= (\mathcal{P} + \mu)Q_{\mu,+} - (\mathcal{P} + \mu)\chi K^0_{\mathcal{B},\mu}\mathcal{B}\gamma_0 Q_{\mu,+} \\ &= I - ([\mathcal{P}, \chi] + \chi(\mathcal{P} - \mathcal{P}^0))K^0_{\mathcal{B},\mu}\mathcal{B}\gamma_0 Q_{\mu,+} \\ &= I - G_2, \quad \text{with} \quad G_2 = (x_n\mathcal{P}_1 + \mathcal{P}_0)K^0_{\mathcal{B},\mu}\mathcal{B}\gamma_0 Q_{\mu,+}, \end{aligned} \tag{5.15}$$

the $\mathcal{P}_j$ denoting differential operators of order j with smooth coefficients vanishing for $x_n > \frac{2}{3}c$. (G_1 and G_2 are μ-dependent, and so are many other auxiliary operators in the following, where we do not indicate the μ-dependence explicitly.)

The exact inverse $\mathcal{R}_\mu$ of $\mathcal{P}_{\mathcal{B}} + \mu$ can then be described by

$$\mathcal{R}_\mu = \mathcal{R}'_\mu(I - G_2)^{-1} = (Q_{\mu,+} - G_1)(I - G_2)^{-1}, \tag{5.16}$$

whenever $I - G_2$ is invertible. The main point is now to show that this holds for large μ and leads to a constructive expression for $\mathcal{R}_\mu$.

For this purpose, we analyze the various factors in (5.14) and (5.15). Let us denote

$$\begin{aligned} K_0 &= e^{-x_n A_\mu}, \quad T_0 = \gamma_0 Q_{\mu,+}, \quad S_0 = \mathcal{S}_{\mathcal{B},\mu}\mathcal{B}, \\ K_1 &= \chi \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sigma \end{smallmatrix}\right) K_0, \quad K_2 = (x_n\mathcal{P}_1 + \mathcal{P}_0) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \sigma \end{smallmatrix}\right) K_0, \end{aligned} \tag{5.17}$$

here K_0 goes from $C^\infty(E'_1)$ to $C^\infty(E^0_1)$, K_1 and K_2 go from $C^\infty(E'_1 \oplus E'_1)$ to $C^\infty(E)$, T_0 goes from $C^\infty(E)$ to $C^\infty(E')$, and S_0 goes from from $C^\infty(E')$ to

$C^\infty(E_1' \oplus E_1')$. (They also define mappings beween suitable Sobolev spaces.) Then

$$G_1 = K_1 S_0 T_0, \quad G_2 = K_2 S_0 T_0. \tag{5.18}$$

In the terminology of Boutet de Monvel [4] and Grubb [12], the K_j are parameter-dependent Poisson operators and T_0 is a parameter-dependent trace operator of class 0 (trace operators of class 0 are well-defined on L_2), but their usage entered elliptic theory much earlier, cf. Seeley [21], Hörmander [17]. For the considerations of these operators, we do *not* need to introduce new and complicated symbol classes and composition rules for boundary operators, for in fact they are of the *strongly polyhomogeneous type*: When the parameter μ runs on a ray $\{\mu = \varrho e^{i\theta_0} \mid \varrho \geq 0\}$, ϱ enters like another cotangent variable on a par with $\xi_1, \dots, \xi_{n-1}$, in the sense that the standard estimates described in [4] are satisfied with $\{\xi_1, \dots, \xi_{n-1}, \varrho\}$ as the boundary cotangent variable. This is similar to the situation described in Theorem 3.7, now for boundary operators.

Let us refrain from further details (that presuppose a lengthy introdution to the calculi described in [4], [12], summarized in the appendix of [14]), but just mention a consequence we need:

Lemma 5.2 *With K_1, K_2 and T_0 defined above, and φ a morphism in E, the compositions $T_0 \varphi K_j$ are strongly polyhomogeneous ψdo's on X' of order -1. Moreover, the compositions $T_0 \varphi Q_{\mu,+} K_j$ are strongly polyhomogeneous ψdo's on X' of order -2.*

An important trick in the following is to reduce considerations of the singular Green operators G_j to considerations of ψdo's in the boundary. This is done on several levels; one is in the study of inverses that uses Lemma 5.3 below, another is in the study of traces in Section 5.2, where a cyclic permutation brings operators of the form TK into the picture.

First consider the problem of inversion of $I - G_2$. Here we shall use the elementary lemma:

Lemma 5.3 *Let $K : V \to W$ and $T : W \to V$ be linear mappings between vector spaces. Then $I - KT : W \to W$ is bijective if and only if $I - TK : V \to V$ is bijective, and*

$$(I - KT)^{-1} = I + K(I - TK)^{-1} T. \tag{5.19}$$

Proof: A straightforward verification. □

The lemma will be applied with $K = K_2$ (going from sections of $E_1' \oplus E_1'$ to sections of E) and $T = S_0 T_0$ (going the other way). This replaces the construction of the inverse of $I-KT=I-G_2$ by the construction of the inverse of $I-TK=I-S_0T_0K_2$; so that

$$(I-G_2)^{-1}=I+K_2(I-S_1)^{-1}S_0T_0 \quad \text{with} \quad S_1=S_0T_0K_2 \tag{5.20}$$

holds when $I-S_1$ is invertible. The advantage of this reduction is that S_1 is a ψdo on the boundaryless manifold X'. The factor T_0K_2 is a strongly polyhomogeneous ψdo of order -1 by Lemma 5.2, and it remains to examine the other factor in S_1 and the composition, and to apply this to construct the inverse $(I-S_1)^{-1}$.

Here we go more in details with the symbol classes introduced in Section 3.1. The following class will play a special role:

Definition 5.4 Let r be integer ≥ 0, and let $S=\mathrm{OP}(s(x,\xi,\mu))$ (or let S have the symbol s in local coordinates). S and its symbol will be called **special parameter-dependent of order** $-r$, when

$$\begin{gathered} s(x,\xi,\mu)\in S^{-r,0}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)\cap S^{0,-r}(\mathbf{R}^n,\mathbf{R}^n,\Gamma) \text{ with} \\ \partial_\mu^m s(x,\xi,\mu)\in S^{-r-m,0}(\mathbf{R}^n,\mathbf{R}^n,\Gamma)\cap S^{0,-r-m}(\mathbf{R}^n,\mathbf{R}^n,\Gamma) \end{gathered}$$

for any m, all $\partial_\mu^m s$ being weakly polyhomogeneous.

Example 5.5 To give examples, we first note that any *strongly* polyhomogeneous symbol of degree $-r$ satisfies Definition 5.4 by Theorem 3.7. But there are also important weakly polyhomogeneous examples, such as the symbol $(a(x,\xi)+\mu^r)^{-1}$ (μ in a sector Γ), where $a(x,\xi)$ is homogeneous of degree r in ξ for $|\xi|\geq 1$ and $a(x,\xi)+\mu^r$ is invertible when $\mu\in\Gamma$ (by [14], Th. 1.17).

For the operators entering in the APS problem we have:

Proposition 5.6 *The ψdo $\mathcal{S}_{\mathcal{B},\mu}$ on X', with μ running in $\pm\Gamma_0$, is special parameter-dependent of order* 0*. So are $\mathcal{B}$ and the composition $S_0=\mathcal{S}_{\mathcal{B},\mu}\mathcal{B}$.*

Proof: (Indication.) For the proof we split $\mathcal{S}_{\mathcal{B},\mu}$ in several terms:

$$\begin{aligned} \mathcal{S}_{\mathcal{B},\mu} &\equiv \begin{pmatrix} B+\mu^{-1}(A_\mu+A)B^- \\ \mu^{-1}(A_\mu-A)B+B^- \end{pmatrix} \\ &= \begin{pmatrix} \mu^{-1}(A_\mu+A)\Pi_< \\ \mu^{-1}(A_\mu-A)\Pi_\geq \end{pmatrix} + \begin{pmatrix} B \\ B^- \end{pmatrix} + \begin{pmatrix} -\mu^{-1}(A_\mu+A)B_0 \\ \mu^{-1}(A_\mu-A)B_0 \end{pmatrix}. \end{aligned} \tag{5.21}$$

The second term has a polyhomogeneous symbol in $S^0\subset S^{0,0}$ (cf. (3.4)) and is independent of μ, hence is special parameter-dependent of order 0. (This proves the statement on $\mathcal{B}$.) The third term is of order $-\infty$, and its boundedness in μ (with improved estimates for derivatives) is seen from considerations on the involved eigenspaces for eigenvalues of modulus $\le R$.

It is the first term in (5.21) that requires most of the analysis. The crucial fact used here is that

$$\begin{aligned} \mu^{-1}(A_\mu + A)\Pi_< &= \mu^{-1}(A_\mu + A)(A_\mu - A)(A_\mu + |A|)^{-1}\Pi_< \\ &= \mu(A_\mu + |A|)^{-1}\Pi_<, \\ \mu^{-1}(A_\mu - A)\Pi_\ge &= \mu^{-1}(A_\mu - A)(A_\mu + A)(A_\mu + |A|)^{-1}\Pi_\ge \\ &= \mu(A_\mu + |A|)^{-1}\Pi_\ge. \end{aligned} \tag{5.22}$$

Again $\Pi_<$ and $\Pi_\ge$ are in $S^0 \subset S^{0,0}$ and independent of μ, hence special of order 0. In view of the composition rules (cf. (3.7)), it remains to prove the statement for $\mu(A_\mu + |A|)^{-1}$. The advantage of this expression is that A_μ and $|A|$ are both "positive" (strongly elliptic), so that the inverse of $A_\mu + |A|$ can be described by a natural elliptic construction. (Details are given in [14], Proposition 3.5.) The statement on S_0 now follows from the composition rules. $\square$

These operators act on X', of dimension $n-1$ (where the space variable and cotangent variable are denoted x' and ξ'). For $s \in \mathbf{R}$ we define the space $H^{s,\mu}(\mathbf{R}^{n-1})$ as the Sobolev space with norm

$$\|u\|_{s,\mu} = \|\langle(\xi',\mu)\rangle^s \hat{u}(\xi')\|_{L_2(\mathbf{R}^{n-1})}, \tag{5.23}$$

and extend the notion to sections of a Hermitian bundle E'' over X' by use of a finite family of local coordinate systems (the space is then denoted $H^{s,\mu}(E'')$). Note that $H^{0,\mu}(E'') \simeq L_2(E'')$.

We shall need

Proposition 5.7 *Let S be a special parameter-dependent ψdo of order -1 in a bundle E'' over X', with μ running in a sector Γ. Then for $s \in \mathbf{R}$, S is continuous from $H^{s,\mu}(E'')$ to $H^{s+1,\mu}(E'')$, uniformly for μ in closed subsectors Γ' of Γ, $|\mu| \ge 1$; and its norm as an operator in $H^{s,\mu}(E'')$ satisfies*

$$\|S\|_{\mathcal{L}(H^{s,\mu}(E''))} = O(|\mu|^{-1}) \text{ for } |\mu| \to \infty,\ \mu \in \Gamma'. \tag{5.24}$$

For each Γ' there is an $r_{\Gamma'} > 0$ such that $I - S$ is invertible for $\mu \in \Gamma'$ with $|\mu| \ge r_{\Gamma'}$. The inverse equals

$$(I-S)^{-1} = I + S', \quad S' = \sum_{j=1}^{\infty} S^j, \tag{5.25}$$

where the series converges in the norm of operators in $L_2(E'')$.

Moreover, S' is a special parameter-dependent ψdo of order -1.

Proof: (Indication.) By the composition rules, S composed with an invertible ψdo with principal symbol $\langle(\xi',\mu)\rangle$ is special parameter-dependent of order 0; it is not hard to show that such an operator is continuous in $H^{s,\mu}$, uniformly as stated. This implies the asserted continuity from $H^{s,\mu}$ to $H^{s+1,\mu}$; and (5.24) follows since

$$|\mu|\|u\|_{s,\mu} \leq \text{const.}\, \|u\|_{s+1,\mu}. \tag{5.26}$$

For each sector Γ', take $r_{\Gamma'}$ so large that the operator norm of S in $L_2(E'')$ is $\leq \frac{1}{2}$ for $|\mu| \geq r_{\Gamma'}$; then (5.25) holds in operator norm.

The powers S^j are special parameter-dependent ψdo's of order $-j$, by the composition rules. Further efforts are needed to show that the sum S' is indeed a ψdo that is special parameter-dependent of order -1; see the details in [14], proof of Theorem 3.8, as explained for S_2 there. □

Now we use these facts to show:

Theorem 5.8 *The operator S_1 in* (5.20) *is a special parameter-dependent ψdo of order -1 in the bundle $E_1'' = E_1' \oplus E_1'$ over X'. Hence for each closed subsector Γ of Γ_0 (or $-\Gamma_0$) there is an $r_\Gamma > 0$ such that $I - S_1$ is invertible for $\mu \in \Gamma$ with $|\mu| \geq r_\Gamma$, with inverse*

$$(I - S_1)^{-1} = I + S_2, \quad S_2 = \sum_{j=1}^{\infty} S_1^j, \tag{5.27}$$

S_2 being a special parameter-dependent ψdo of order -1 in E_1''.

Furthermore, for such μ,

$$(I - G_2)^{-1} = I + K_2(I + S_2)S_0T_0, \tag{5.28}$$

and finally

$$\begin{aligned} \mathcal{R}_\mu &= (Q_{\mu,+} - G_1)(I - G_2)^{-1} \\ &= (Q_{\mu,+} - K_1S_0T_0)(I + K_2(I + S_2)S_0T_0) \\ &= Q_{\mu,+} - (K_1 - K_3)(I + S_2)S_0T_0, \text{ with } K_3 = Q_{\mu,+}K_2. \end{aligned} \tag{5.29}$$

Proof: In the formula (5.20) for S_1, S_0 is a special parameter-dependent ψdo of order 0 by Proposition 5.6, and T_0K_2 is a special parameter-dependent ψdo of order -1 by Lemma 5.2 and Example 5.5, so it follows from the composition rules (cf. (3.7)) that S_1 is a special parameter-dependent ψdo of order -1. Then Proposition 5.7 applies, showing the assertions for S_2.

Now the formula for $(I - G_2)^{-1}$ follows from (5.20). The first two lines in (5.29) then follow from (5.16) and (5.18). Consequently we have:

$$\begin{aligned}\mathcal{R}_\mu &= (Q_{\mu,+} - K_1 S_0 T_0)(I + K_2(I + S_2) S_0 T_0) \\ &= Q_{\mu,+} + Q_{\mu,+} K_2 (I + S_2) S_0 T_0 \\ &\quad - K_1 S_0 T_0 - K_1 S_0 T_0 K_2 (I + S_2) S_0 T_0 \\ &= Q_{\mu,+} + Q_{\mu,+} K_2 (I + S_2) S_0 T_0 \\ &\quad - K_1 S_0 T_0 - K_1 S_1 (I + S_2) S_0 T_0 \\ &= Q_{\mu,+} - (K_1 - Q_{\mu,+} K_2)(I + S_2) S_0 T_0,\end{aligned} \tag{5.30}$$

using formula (5.20) for S_1 and the fact that $I + S_1(I + S_2) = I + S_2$. This ends the proof. □

Taking the structure of the entering Poisson and trace operators into account, we have obtained:

Corollary 5.9 *For each closed subsector* Γ *of* $\pm\Gamma_0$ *one can find* $r_\Gamma > 0$ *so that the resolvent* $\mathcal{R}_\mu = (\mathcal{P}_\mathcal{B} + \mu)^{-1}$ *for* $\mu \in \Gamma$ *with* $|\mu| \geq r_\Gamma$ *is of the form*

$$\mathcal{R}_\mu = Q_{\mu,+} + KST, \tag{5.31}$$

where K *resp.* T *are a strongly polyhomogeneous Poisson resp. trace operator of degree* -1 *and* S *is a special parameter-dependent* ψ*do on* X' *of order* 0. *The detailed structure is given in* (5.29).

5.2. TRACE CALCULATIONS

Consider $\mathcal{R}_\mu = (\mathcal{P}_\mathcal{B} + \mu)^{-1}$, as described above. Since the injection of $H^s(X)$ into $L_2(X)$ is trace class for $s > n$, the terms in $\partial_\mu^m \mathcal{R}_\mu$ are trace class when $m \geq n$.

Theorem 5.10 *Let* φ *be any morphism in* $E = E_1 \oplus E_2$, *and let* $m \geq n = \dim X$. *Then*

$$\begin{aligned}\operatorname{Tr}(\varphi \partial_\mu^m (\mathcal{P}_\mathcal{B} + \mu)^{-1}) \sim a_0 \mu^{n-m-1} &+ \sum_{j=1}^{\infty} (a_j + b_j) \mu^{n-m-1-j} \\ &+ \sum_{j=0}^{\infty} (c_j \log \mu + c'_j) \mu^{-m-1-j}, \text{ as } |\mu| \to \infty,\end{aligned} \tag{5.32}$$

for μ *in closed subsectors of* $\pm\Gamma_0$. *The coefficients* a_j, b_j *and* c_j *are integrals,* $\int_X a_j(x)\,dx$, $\int_{X'} b_j(x')\,dx'$ *and* $\int_{X'} c_j(x')dx'$, *of densities locally determined by the symbols of* P *and* B, *while the* c'_j *are in general globally determined. The coefficients* c_0 *and* c'_0 *are the same as for the case where the* P_j *are zero in* (1.2) *(the product case).*

Proof: We find from (5.29):

$$\varphi\partial_\mu^m(\mathcal{P}_\mathcal{B}+\mu)^{-1} = \varphi\partial_\mu^m Q_{\mu,+} - \varphi\partial_\mu^m[K_1S_0T_0] - \varphi\partial_\mu^m[(K_1S_2 - K_3(I+S_2))S_0T_0]. \quad (5.33)$$

First, $\operatorname{Tr}(\varphi\partial_\mu^m Q_{\mu,+})$ contributes the well-known expansion $\sum_0^\infty a_j\mu^{n-m-1-j}$. For the other terms we can use the invariance of the trace under cyclic permutation of the operators, to reduce to a study of operators on X'. For the middle term we find, by the Leibniz rule:

$$\begin{aligned}\operatorname{Tr}_X(\varphi\partial_\mu^m[K_1S_0T_0]) &= \sum_{m_1+m_2+m_3=m} c_{m_1,m_2,m_3}\operatorname{Tr}_X(\varphi\partial_\mu^{m_1}K_1\,\partial_\mu^{m_2}S_0\,\partial_\mu^{m_3}T_0)\\ &= \operatorname{Tr}_{X'}\Big(\sum_{m_1+m_2+m_3=m} c_{m_1,m_2,m_3}\partial_\mu^{m_2}S_0\,\partial_\mu^{m_3}T_0\varphi\partial_\mu^{m_1}K_1\Big)\\ &= \operatorname{Tr}_{X'}\partial_\mu^m(S_0T_0\varphi K_1). \quad (5.34)\end{aligned}$$

By Lemma 5.2, $T_0\varphi K_1$ is a strongly polyhomogeneous ψdo on X' of order -1, hence special parameter-dependent by Theorem 3.7. Then since S_0 is special parameter-dependent by Proposition 5.6, it follows that $\partial_\mu^m(S_0T_0\varphi K_1)$ is a special parameter-dependent ψdo on X' of order $-m-1$.

To this we can apply our general Theorem 3.8 and its corollary, after a reduction to local trivializations by use of a partition of unity. Since the symbol has degrees $-m-1-j$, $j\ge 0$, and μ-exponent $d=-m-1$, we get an expansion in a series of locally determined terms $b_{k,1}\mu^{-m-1+(n-1)-k}$, $k\ge 0$, together with a series of terms $(c_{j,1}\log\mu + c'_{j,1})\mu^{-m-1-j}$, $j\ge 0$, with $c_{j,1}$ locally determined.

The third term is treated similarly; here the circular permutation of the terms resulting from the Leibniz rule gives a special parameter-dependent ψdo of order $-m-2$, so Corollary 3.9 gives an expansion in a series of locally determined terms $b_{k,2}\mu^{-m-2+(n-1)-k}$, $k\ge 0$, together with a series of terms $(c_{j,2}\log\mu + c'_{j,2})\mu^{-m-1-j}$, $j\ge 1$, with $c_{j,2}$ locally determined.

Taking the contributions together we get the expansion (5.32). One observes moreover that the terms $(c_0\log\mu + c'_0)\mu^{-m-1}$ in (5.32) come only from $\operatorname{Tr}(\varphi\partial_\mu^m[K_1S_0T_0])$, which leads to the last statement in the theorem. For, K_1 and S_0 are the same as for the case where the P_j and P'_j are 0. The third factor $T_0=\gamma_0Q_{\mu,+}$ uses the symbol of $(\mathcal{P}+\mu)^{-1}$ evaluated at $x_n=0$. The leading term of this is the same as for the case where P_j and P'_j are 0, and the lower order terms contribute ultimately with special parameter-dependent ψdo's of order $-m-2$ only; the first possible nonlocal and log contributions from this are the terms with μ^{-m-2} and $\mu^{-m-2}\log\mu$. $\square$

In view of (5.4), it is now easy to draw conclusions from this on asymptotic expansions for traces of λ-derivatives of $\varphi(\Delta_1-\lambda)^{-1}=\varphi(P_B{}^*P_B-\lambda)^{-1}$ and $\varphi P_B(\Delta_1-\lambda)^{-1}=\varphi P_B(P_B{}^*P_B-\lambda)^{-1}$, etc.

Corollary 5.11 *Let $\varphi_{kl}:E_l\to E_k$ be morphisms, for $k,l=1,2$.*

The traces $\operatorname{Tr}(\varphi_{11}\partial_\lambda^m(\Delta_1-\lambda)^{-1})$ and $\operatorname{Tr}(\varphi_{22}\partial_\lambda^m(\Delta_2-\lambda)^{-1})$ have asymptotic expansions (for $k=1$ resp. 2):

$$a_{0,kk}(-\lambda)^{\frac{n}{2}-m-1}+\sum_{j=1}^{\infty}(a_{j,kk}+b_{j,kk})(-\lambda)^{\frac{n-j}{2}-m-1}$$
$$+\sum_{j=0}^{\infty}(c_{j,kk}\log\lambda+c'_{j,kk})(-\lambda)^{\frac{-j}{2}-m-1};\quad (5.35)$$

and $\operatorname{Tr}(\varphi_{12}\partial_\lambda^m P_B(\Delta_1-\lambda)^{-1})$ and $\operatorname{Tr}(\varphi_{21}\partial_\lambda^m P_B{}^*(\Delta_2-\lambda)^{-1})$ have asymptotic expansions (for $\{k,l\}=\{1,2\}$ resp. $\{2,1\}$):

$$a_{0,kl}(-\lambda)^{\frac{n-1}{2}-m}+\sum_{j=1}^{\infty}(a_{j,kl}+b_{j,kl})(-\lambda)^{\frac{n-j-1}{2}-m}$$
$$+\sum_{j=0}^{\infty}(c_{j,kl}\log\lambda+c'_{j,kl})(-\lambda)^{\frac{-j-1}{2}-m};\quad (5.36)$$

with coefficients described as in Theorem 5.10.

The coefficients $c_{0,kl}$ and $c'_{0,kl}$ are the same as those for the product case.

Proof: Using (5.4), take

$$\varphi=\begin{pmatrix}\varphi_{11}&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&\varphi_{22}\end{pmatrix},\begin{pmatrix}0&\varphi_{12}\\0&0\end{pmatrix},\ \text{resp.}\ \begin{pmatrix}0&0\\\varphi_{21}&0\end{pmatrix},\quad (5.37)$$

in Theorem 5.10, and divide by μ in the first two cases. Now replace μ by $(-\lambda)^{\frac{1}{2}}$ and note that $\partial_\lambda=(2\mu)^{-1}\partial_\mu$. $\square$

These results yield asymptotic expansions of the traces of heat operators $\varphi_{11}e^{-t\Delta_1}$, $\varphi_{12}P_Be^{-t\Delta_1}$, etc., and power operators $\varphi_{11}(\Delta_1)^{-s}$, $\varphi_{12}P_B(\Delta_1)^{-s}$, etc., by use of the transition formulas in Section 2:

Theorem 5.12 *There are coefficients $\tilde a_{j,kl}$, $\tilde b_{j,kl}$, $\tilde c_{j,kl}$, $\tilde c'_{j,kl}$, related by suitable gamma factors to those in Corollary 5.11 (cf. Theorems 2.1 and 2.3) such that, with $\nu_1=\operatorname{Tr}(\varphi_{11}\Pi_0(P_B))$, $\nu_2=\operatorname{Tr}(\varphi_{22}\Pi_0(P_B{}^*))$, the zeta and* *eta functions have singularity structures described by:*

$$\begin{aligned}
&\Gamma(s)\operatorname{Tr}(\varphi_{kk}Z(\Delta_1,s)) \sim \frac{-\nu_k}{s} + \frac{\tilde a_{0,kk}}{s-\frac{n}{2}} + \sum_{j=1}^{\infty}\frac{\tilde a_{j,kk}+\tilde b_{j,kk}}{s-\frac{n-j}{2}}\\
&\qquad\qquad + \sum_{j=0}^{\infty}\Big(\frac{\tilde c_{j,kk}}{(s+\frac{j}{2})^2} + \frac{\tilde c'_{j,kk}}{s+\frac{j}{2}}\Big);\\
&\Gamma(s)\operatorname{Tr}(\varphi_{12}P_B Z(\Delta_1,s)) \ \textit{resp.}\ \Gamma(s)\operatorname{Tr}(\varphi_{21}{P_B}^* Z(\Delta_2,s))\\
&\qquad \sim \frac{\tilde a_{0,kl}}{s-\frac{n+1}{2}} + \sum_{j=1}^{\infty}\frac{\tilde a_{j,kl}+\tilde b_{j,kl}}{s-\frac{n-j+1}{2}} + \sum_{j=0}^{\infty}\Big(\frac{\tilde c_{j,kl}}{(s+\frac{j-1}{2})^2} + \frac{\tilde c'_{j,kl}}{s+\frac{j-1}{2}}\Big);
\end{aligned} \tag{5.38}$$

and the heat traces have the asymptotic behavior for $t \to 0$:

$$\begin{aligned}
&\operatorname{Tr}(\varphi_{kk}e^{-t\Delta_1}) \sim \tilde a_{0,kk}t^{-\frac{n}{2}} + \sum_{j=1}^{\infty}(\tilde a_{j,kk}+\tilde b_{j,kk})t^{\frac{j-n}{2}}\\
&\qquad\qquad + \sum_{j=0}^{\infty}(-\tilde c_{j,kk}t^{\frac{j}{2}}\log t + \tilde c'_{j,kk}t^{\frac{j}{2}}),\\
&\operatorname{Tr}(\varphi_{12}P_B e^{-t\Delta_1}) \ \textit{resp.}\ \operatorname{Tr}(\varphi_{21}{P_B}^* e^{-t\Delta_2})\\
&\sim \tilde a_{0,kl}t^{-\frac{n+1}{2}} + \sum_{j=1}^{\infty}(\tilde a_{j,kl}+\tilde b_{j,kl})t^{\frac{j-n-1}{2}} + \sum_{j=0}^{\infty}(-\tilde c_{j,kl}t^{\frac{j-1}{2}}\log t + \tilde c'_{j,kl}t^{\frac{j-1}{2}}),
\end{aligned} \tag{5.39}$$

The $\tilde c'_{j,kl}$ and ν_k are in general globally defined, while the other coefficients are local. The coefficients $\tilde c_{0,kl}$ and $\tilde c'_{0,kl}$ are the same as those for the product case.

A detailed account is given in [14]. [14] and [15] also give some information on variations of parameter-dependent situations.

Remark 5.13 Similar considerations allow the calculation of $\operatorname{Tr}(D\partial_\mu^m\mathcal{R}_\mu)$ when D is an arbitrary differential operator on X, for $m \ge n+d$, $d =$ the order of D. One finds that

$$\begin{aligned}
\operatorname{Tr}(D\partial_\mu^m\mathcal{R}_\mu) \sim a_0(D,P)\mu^{n-m+d-1} &+ \sum_{j=1}^{\infty}(a_j(D,P)+b_j(D,P_B))\mu^{n-m+d-1-j}\\
&+ \sum_{j=0}^{\infty}(c_j(D,P_B)\log\mu + c'_j(D,P_B))\mu^{-m+d-1-j}
\end{aligned} \tag{5.40}$$

(the primed coefficients global, the others local); and consequences are drawn as above for the corresponding zeta and eta functions and exponential traces.

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