NONHOMOGENEOUS NAVIER-STOKES PROBLEMS IN $\mathbf{L_p}$ SOBOLEV SPACES OVER EXTERIOR AND INTERIOR DOMAINS

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We here present our work on the solvability of completely nonhomogeneous initial-boundary value problems for the Navier-Stokes equations, in general anisotropic L_p Sobolev and Besov spaces with $p > 1$. Introducing a new twist of the method (simplifying slightly), we can now extend the results to exterior domains, for finite time intervals.

1 Introduction

In a series of papers, the author has treated the nonhomogeneous Navier-Stokes problem

$$
\begin{aligned}
\partial_t u - \Delta u + \sum_{j=1}^{n} u_j \partial_j u + \operatorname{grad} q &= f \quad \text{on } Q_{I_b} = \Omega \times I_b, \\
\operatorname{div} u &= 0 \quad \text{on } Q_{I_b}, \\
T_k\{u, q\} &= \varphi_k \quad \text{on } S_{I_b} = \Gamma \times I_b, \\
r_0 u &= u_0 \quad \text{on } \Omega;
\end{aligned}
\tag{1.1}
$$

for bounded domains $\Omega \subset \mathbf{R}^n$, $I_b =]0, b[\subset \mathbf{R}_+$, with various boundary operators T_k of Dirichlet, Neumann or intermediate type (r_0 indicates restriction to $t = 0$; further details are given below in Section 2). Strong solvability results were obtained in anisotropic L_2 Sobolev spaces in joint works with V. A. Solonnikov [11]–[14], and the results have been extended more recently to L_p Sobolev spaces [6]–[7], that we report on below (in Section 2). Besides this, we give generalizations to exterior domains (in Section 4), based on a simplified proof (in Section 3).

The main technique is to reduce the linearized problem

$$
\begin{aligned}
\partial_t u - \Delta u + \operatorname{grad} q &= f \quad \text{on } Q_{I_b}, \\
\operatorname{div} u &= 0 \quad \text{on } Q_{I_b}, \\
T_k\{u, q\} &= \varphi_k \quad \text{on } S_{I_b}, \\
r_0 u &= u_0 \quad \text{on } \Omega;
\end{aligned}
\tag{1.2}
$$

which is degenerate parabolic, to a truly parabolic pseudodifferential problem

$$\begin{aligned} \partial_t u - \Delta u + G_k u &= f_k \quad \text{on } Q_{I_b}, \\ T'_k u &= \psi_k \quad \text{on } S_{I_b}, \\ r_0 u &= u_0 \quad \text{on } \Omega; \end{aligned} \tag{1.3}$$

where the general theory of [4], [12], [9], [6] can be brought into use.

Parabolic problems of the form $\partial_t u + A(x, D_x)u = f$ (with initial and boundary conditions) are much harder when A is of pseudodifferential type than when it is a differential operator, since the singularity of the symbol of A at $\xi = 0$ has an important effect when there is an extra parameter-dependence (caused by ∂_t). While trying to extend our results to exterior domains, we were inspired by a recent collaboration with R. Seeley [10] to look for simplifications in the treatment of (1.3) such that one can take advantage of the fact that the non-differential aspects are connected with the boundary only.

We shall show below in Section 3 how an important step in the treatment of (1.3) can be broken up into three parts, treating: (i) a classical Dirichlet or Neumann heat problem, (ii) a parameter-dependent ps.d.o. problem *on the boundary* Γ, (iii) a classical Dirichlet or Neumann problem for the Laplace operator. For exterior problems, this viewpoint has the advantage that we can lean on known results for the unbounded domain, and need the technical ps.d.o. considerations only on the compact manifold Γ. It gives rather easily some extensions of the results of [7] to unbounded domains, however for bounded time intervals only.

For the unbounded time interval $\mathbf{R}_+$, the results for the Dirichlet problem in [7] do not seem readily extendible; and the new method is perhaps too rough. In fact, one may have to work in other spaces than those that we deal with here (e.g. homogeneous spaces or weighted spaces), to get really satisfactory results.

2 Results for the interior case

Consider the problems (1.1) and (1.2). Here $u(x,t)$ is the velocity vector $u = \{u_1, \dots, u_n\}$, $q(x,t)$ is the (scalar) pressure, and T_k is one of the following trace operators:

$$\begin{aligned} &T_0\{u,q\} = \gamma_0 u, \\ &T_1\{u,q\} = \chi_1 u - \gamma_0 q\,\vec{n}, \quad T_2\{u,q\} = (\chi_1 u)_\tau + \gamma_0 u_\nu \vec{n}, \\ &T_3\{u,q\} = \gamma_1 u - \gamma_0 q\,\vec{n}, \quad T_4\{u,q\} = \gamma_1 u_\tau + \gamma_0 u_\nu \vec{n}, \end{aligned} \tag{2.1}$$

where $\vec{n} = (n_1, \dots, n_n)$ is the (interior) normal at Γ, v_ν resp. v_τ denotes the normal resp. tangential component of an n-vector field v defined near Γ:

$$v_\nu = \vec{n} \cdot v, \qquad v_\tau = v - (\vec{n} \cdot v)\, \vec{n}, \tag{2.2}$$

$\gamma_k u = \partial_\nu^k u|_\Gamma$ (with $\partial_\nu = \sum_{j=1}^n n_j \partial_j$), and χ_1 is the special first order boundary operator defined via the strain tensor $S(u) = (\partial_i u_j + \partial_j u_i)_{i,j=1,\dots,n}$ as

$$\chi_1 u = \gamma_0 S(u)\, \vec{n} = \gamma_0 \big(\sum\nolimits_j (\partial_i u_j + \partial_j u_i) n_j \big)_{i=1,\dots,n}. \tag{2.3}$$

For $k = 0$ this gives the Dirichlet problem, $k = 1$ and 3 give Neumann problems, and $k = 2$ and 4 give problems with partially a Dirichlet, partially a Neumann condition. More comments on these boundary conditions in [14].

The data are assumed to satisfy

$$\begin{aligned} &\operatorname{div} u_0 = 0, \quad \text{when } k = 1 \text{ or } 3; \\ &\operatorname{div} u_0 = 0,\ \gamma_0 u_{0,\nu} = 0,\ \varphi_{k,\nu} = 0, \quad \text{when } k = 0,\ 2 \text{ or } 4. \end{aligned} \tag{2.4}$$

The problem is considered in anisotropic Bessel-potential spaces $H_p^{(s,\, s/2)}(\overline{Q}_I)^n$ and Besov spaces $B_p^{(s,\, s/2)}(\overline{Q}_I)^n$, where, as we recall, the $H_p^{(s,\, s/2)}$ spaces are generalizations of the integer case

$$H_p^{2m,m}(\overline{Q}_{I_b}) = \{\, u(x,t) \in L_p(Q_{I_b}) \mid D_x^\alpha D_t^j u \in L_p(Q_{I_b}) \text{ for } |\alpha| + 2j \le 2m \,\} \tag{2.5}$$

defined via local coordinates and restriction from

$$H_p^{(s,\, s/2)}(\mathbf{R}^n \times \mathbf{R}) = \mathrm{OP}((|\xi|^4 + \tau^2 + 1)^{-s/4}) L_p(\mathbf{R}^n \times \mathbf{R}); \tag{2.6}$$

this scale is preserved under complex interpolation. The Besov scale $B_p^{(s,\, s/2)}$ is defined slightly differently, but arises from the $H_p^{(s,\, s/2)}$ scale by suitable real interpolation. (Further details are given e.g. in [6].)

The $B_p^{(s,\, s/2)}$ spaces must be included even if one is mainly interested in solving the problem in spaces (2.6), because they are the correct boundary value spaces, as γ_j maps $H_p^{(s,\, s/2)}(\overline{Q}_{I_b})$ continuously onto the space $B_p^{(s-j-\frac{1}{p}, (s-j-\frac{1}{p})/2)}(\overline{S}_{I_b})$, for $j < s - \frac{1}{p}$. We denote by $\mathcal{B}_p^{s+2,(k)}$ the range space for T_k applied to $H_p^{(s+2,\, s/2+1)}(\overline{Q}_{I_b})^n$.

Let us first present the main results of [7] for bounded domains:

Consider systems of functions

$$\Phi_k = \{f, \varphi_k, u_0\} \in H_p^{(s,\, s/2)}(\overline{Q}_{I_b})^n \times \mathcal{B}_p^{s+2,(k)} \times B_p^{s+2-2/p}(\overline{\Omega})^n, \tag{2.7}$$

for $s > \frac{1}{p} - 1$ with $s \geq \frac{n+2}{p} - 3$. The system is said to satisfy *the compatibility condition of order* s, when

$$\begin{aligned}
&r_0 \partial_t^l \varphi_{k,\tau} = \gamma_0 u_\tau^{(l)} && \text{for } k = 0,\ 2l \leq s + 2 - \frac{3}{p}, \\
&r_0 \partial_t^l \varphi_{k,\tau} = (\chi_1 u^{(l)})_\tau && \text{for } k = 1 \text{ and } 2,\ 2l \leq s + 1 - \frac{3}{p}, \\
&r_0 \partial_t^l \varphi_{k,\tau} = \gamma_1 u_\tau^{(l)} && \text{for } k = 3 \text{ and } 4,\ 2l \leq s + 1 - \frac{3}{p}, \\
&\qquad\text{understood as} && \\
&\mathcal{I}[\partial_t^l \varphi_{k,\tau}, u_\tau^{(l)}] < \infty && \text{if } k = 0,\ 2l = s + 2 - \frac{3}{p}, \\
&\mathcal{I}[\partial_t^l \varphi_{k,\tau}, (S(u^{(l)})\vec{n})_\tau] < \infty && \text{if } k = 1 \text{ and } 2,\ 2l = s + 1 - \frac{3}{p}, \\
&\mathcal{I}[\partial_t^l \varphi_{k,\tau}, \partial_\nu u_\tau^{(l)}] < \infty && \text{if } k = 3 \text{ and } 4,\ 2l = s + 1 - \frac{3}{p};
\end{aligned} \tag{2.8}$$

here the $u^{(l)}$ are defined successively by

$$\begin{aligned}
u^{(0)} &= u_0, \\
u^{(l+1)} &= (\Delta - G_k) u^{(l)} - \kappa \sum_{m=0}^{l} \binom{l}{m} \mathcal{Q}_k(u^{(m)}, u^{(l-m)}) + r_0 \partial_t^l f_k;
\end{aligned} \tag{2.9}$$

where the G_k are certain singular Green operators stemming from the elimination of the pressure q, and

$$\mathcal{I}[\psi, v] = \int_{t\in I} \int_{x'\in\Gamma} \int_{y\in\Omega} \frac{|\psi(x',t) - v(y)|^p}{(|x'-y|^d + t)^{1+n/d}}\, dy\, d\sigma_{x'}\, dt. \tag{2.10}$$

We then define the data norm of Φ_k by

$$\mathcal{N}_{s,p,b}^{(k)}(\Phi_k) = (\|f\|^p_{H_p^{(s,\,s/2)}(\overline{Q}_{I_b})^n} + \|\varphi_k\|^p_{\mathcal{B}_p^{s+2,(k)}} + \|u_0\|^p_{B_p^{s+2-2/p}(\overline{\Omega})^n} + \mathcal{I}_{s,p,b})^{\frac{1}{p}}, \tag{2.11}$$

where $\mathcal{I}_{s,p,b} = 0$ if $s + 2 - \frac{3}{p} \notin \mathbf{N}$, and otherwise equals the possible $\mathcal{I}$ term entering in the compatibility condition. The following result on uniqueness and on the existence of solutions on large time-intervals for small enough data, and on small enough time-intervals for large data, is proved in detail in [7].

Theorem 2.1 *Let Ω be a smooth bounded open set in $\mathbf{R}^n$. Let $k = 0, 1, 2, 3$ or 4, let $s > \frac{1}{p} - 1$ with $s \geq \frac{n+2}{p} - 3$, and let $b \in \mathbf{R}_+$. Consider Φ_k as in (7), satisfying the compatibility condition of order s.*

1° *There is at most one solution $\{u, q\}$ with*

$$\{u, \operatorname{grad} q\} \in H_p^{(s+2,\, s/2+1)}(\overline{Q}_{I_b})^n \times H_p^{(s,\, s/2)}(\overline{Q}_{I_b}) \tag{2.12}$$

of the Navier-Stokes problem (1.1) for each set of data Φ_k (where q for $k = 0, 2$ or 4 is subject to the side condition $\int_\Omega q(x,t)\,dx = 0$ for almost all t).

2° *When $s \geq \frac{n+2}{p} - 3$ [$s > \frac{n+2}{p} - 3$ if $\frac{n}{2p} - \frac{3}{2} \in \mathbf{N}_+$, $p \neq 2$], there is a constant $N_{s,p,b}$ such that for data Φ_k with data norm $\mathcal{N}^{(k)}_{s,p,b}(\Phi_k) < N_{s,p,b}$ there exists a solution $\{u, q\}$ of (1.1) with (2.12), the norm depending continuously on Φ_k. When $s \geq s_0$ for some $s_0 > \frac{n+2}{p} - 3$ [$\frac{s_0}{2} - \frac{1}{p} \notin \mathbf{N}_+$ if $p \neq 2$], the norm condition for existence can be replaced by the condition $\mathcal{N}^{(k)}_{s_0,p,b}(\Phi_k) < N_{s_0,p,b}$.*

3° *When $s > \frac{n+2}{p} - 3$, one can for each $N > 0$ choose $b' \leq b$ such that there exists a solution $\{u, q\}$ of (1.1) satisfying (2.12) with b replaced by b', and with norm depending continuously on Φ_k, for any set of data Φ_k with norm $\mathcal{N}^{(k)}_{s,p,b'}(\Phi_k) < N$. For $s \geq s_0$, s_0 as above, the solution can be obtained with b' defined relative to s_0.*

The statements hold with H_p replaced by B_p throughout, even without the conditions in [...].

One concludes furthermore that $q \in H_p^{(s+1,\, s/2)}(\overline{Q}_{I_{b'}})$ when $s \geq 0$ or f is as in (2.4); in some cases q belongs to a better space, see [7], Th. 3.6.

For $k = 0$, $s = 0$, the result is consistent with Solonnikov's result [18], Th. 10.1 for $n = 3$, showing the existence of solutions in $W_p^{(2,1)}(\overline{Q}_{I_b})^n$ to the Dirichlet Navier-Stokes problem when $f \in L_p(\Omega)^n$, $\varphi = 0$, $u_0 \in \operatorname{pr}_{J_0} B_p^{2-2/p}(\overline{\Omega})^n$ and $p \geq \frac{5}{3}$.

When both f and φ_k are 0, one can get solutions with still more general initial data, e.g. in $L_n(\Omega)$, cf. [18] for the Dirichlet problem ($n = 3$), Giga-Miyakawa [3] and Giga [2] for Dirichlet and intermediate problems, and [8] for Neumann and Dirichlet problems. In [8], we use the semigroup $U(t)$ associated with $A_k = (-\Delta + G_k)_{T'_k}$ to obtain solutions e.g. in spaces $C^0(I_{b'}; H_p^r(\overline{\Omega})^n)$, when u_0 is taken in $H_p^r(\overline{\Omega})^n$, allowed for $r \geq \frac{n}{p} - 1$. [18], [3], [2] and von Wahl [20] also treat problems with $f \neq 0$, $\varphi_0 = 0$, in related spaces.

For the Dirichlet problem, we can include infinite intervals $I = \mathbf{R}_+$ in certain cases:

Theorem 2.2 *Hypotheses as in Theorem 2.1. In the Dirichlet case ($k = 0$), the existence of solutions with (2.12) for sufficiently small data extends to $b =$*

$+\infty$ *(generalizing Theorem 2.1 2°), when either 1°, 2° or 3° holds in addition to the conditions* $s > \frac{1}{p} - 1$, $s \geq \frac{n+2}{p} - 3$*:*

1° $n \leq 4$.

2° $s < \frac{3}{p}$.

3° *The data have vanishing initial values, i.e., satisfy*

$$\{f, \varphi_0, u_0\} \in H_{p\,(0)}^{(s,\,s/2)}(\overline{Q}_{\mathbf{R}_+})^n \times B_{p\,(0)}^{(s+2-\frac{1}{p},(s+2-\frac{1}{p})/2)}(\underline{F}_{\tau,\mathbf{R}_+}) \times \{0\}. \tag{2.13}$$

There is a similar generalization with H_p *replaced by* B_p*.*

The method of proof of Theorem 2.1 in [7] consists of the following four steps: 1) Reduction of the linearized problem to a truly parabolic but pseudodifferential initial-boundary value problem ([11],[14]). 2) Solution of the linear reduced parabolic problem by pseudo-differential machinery (from [9], [6]). 3) Solution of the corresponding reduced nonlinear pseudodifferential problem, by use of product estimates and iteration. 4) Conclusions for the original nonlinear problem. For Theorem 2.2 one uses moreover, that the resolvent of the linearized stationary problem is really only applied to the solenoidal space, where the spectrum for $k = 0$ is a closed subset of $\mathbf{R}_+$; this allows sharper estimates.

3 A simplified method

We shall now explain the method of proof in a version where Step 2 is simplified.

We first treat the associated resolvent problem, where ∂_t is replaced by the complex parameter $-\lambda$. To be concrete, consider (1.2) in the Neumann case $k = 1$ (which has been studied less than the Dirichlet case $k = 0$):

$$\begin{aligned} (-\Delta - \lambda)u + \operatorname{grad} q &= f \quad \text{on } \Omega, \\ \operatorname{div} u &= 0 \quad \text{on } \Omega, \\ \chi_1 u - \gamma_0 q &= \varphi \quad \text{on } \Gamma, \end{aligned} \tag{3.1}$$

with f and φ given in $H_p^s(\overline{\Omega})^n$ resp. $B_p^{s+1-\frac{1}{p}}(\Gamma)^n$; Ω bounded and smooth.

Applying $-\operatorname{div}$ to the first line in (3.1) and taking the normal component of the third line, we find:

$$\begin{aligned} -\Delta q &= -\operatorname{div} f, \\ \gamma_0 q &= 2\gamma_1 u_\nu - \varphi_\nu. \end{aligned} \tag{3.2}$$

This is a Dirichlet problem for q, so if we denote $\begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix}^{-1} = \begin{pmatrix} R_D \; K_D \end{pmatrix}$, we have

$$\begin{aligned} q &= -R_D \operatorname{div} f + K_D(2\gamma_1 u_\nu - \varphi_\nu), \text{ where} \\ R_D &: H_p^{s-1}(\overline{\Omega}) \to H_p^{s+1}(\overline{\Omega}) \text{ for } s > \frac{1}{p} - 1, \\ K_D &: B_p^{s+1-\frac{1}{p}}(\Gamma) \to H_p^{s+1}(\overline{\Omega}) \text{ for } s \in \mathbf{R}. \end{aligned} \tag{3.3}$$

Insertion of q into (3.1) gives the equations for u:

$$\begin{aligned} (-\Delta - \lambda)u + 2 \operatorname{grad} K_D \gamma_1 \operatorname{pr}_\nu u &= f + \operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D \varphi_\nu, \\ \gamma_0 \operatorname{div} u &= 0, \\ \operatorname{pr}_\tau \chi_1 u &= \varphi_\tau. \end{aligned} \tag{3.4}$$

A solution of u of (3.4) will satisfy (3.1) when q is defined from u, f and φ by (3.3) (note that $\operatorname{div} \operatorname{grad} K_D = 0$).

We write (3.4) in the short form

$$\begin{aligned} (-\Delta - \lambda)u + KTu &= f_1 \quad \text{on } \Omega, \\ T_1' u &= \psi_1 \quad \text{on } \Gamma, \end{aligned} \tag{3.5}$$

where we have set

$$\begin{aligned} &K = 2 \operatorname{grad} K_D, \quad T = \gamma_1 \operatorname{pr}_\nu, \quad T_1' = \{\operatorname{pr}_\tau \chi_1, \gamma_0 \operatorname{div}\}, \\ &f_1 = f + \operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D \varphi_\nu, \quad \psi_1 = \{\varphi_\tau, 0\}; \end{aligned} \tag{3.6}$$

here $I + \operatorname{grad} R_D \operatorname{div}$ equals the projection operator pr_J that maps $H_p^s(\overline{\Omega})^n$ onto the solenoidal space $J_p^s = \{\, u \in H_p^s(\overline{\Omega})^n \mid \operatorname{div} u = 0 \,\}$ for $s > \frac{1}{p} - 1$ (cf. [5], Example 3.14); K is a Poisson operator of order 1, and T and T_1' are trace operators of order 1.

In order to use other known properties of the Laplace operator, we now make a new reduction. Write the problem (3.5) as follows:

$$\begin{aligned} (-\Delta - \lambda)u &= f_1 - KTu \qquad &\text{on } \Omega, \\ \gamma_1 u &= \psi_1 - (T_1' - \gamma_1)u \quad &\text{on } \Gamma. \end{aligned} \tag{3.7}$$

The system $\{-\Delta - \lambda, \gamma_1\}$ is uniformly parameter-elliptic (in the sense of [9], [6]) for λ on rays with argument $\theta \in]0, 2\pi[$; and it is bijective for $\lambda \in \mathbf{C} \setminus \overline{\mathbf{R}}_+$. By a simple application of [6], the inverse is continuous for each $s > \frac{1}{p} - 1$,

$$\begin{pmatrix} -\Delta - \lambda \\ \gamma_1 \end{pmatrix}^{-1} = \begin{pmatrix} R_{N,\lambda} \; K_{N,\lambda} \end{pmatrix} : H_p^{s,\mu}(\overline{\Omega})^n \times B_p^{s+1-\frac{1}{p},\mu}(\Gamma)^n \to H_p^{s+2,\mu}(\overline{\Omega})^n, \tag{3.8}$$

uniformly for λ in sets V_ε, $\varepsilon > 0$,

$$V_\varepsilon = \{ \lambda \in \mathbf{C} \mid \arg\lambda \in [\varepsilon, 2\pi - \varepsilon], |\lambda| \geq \varepsilon \}; \tag{3.9}$$

here $\mu = |\lambda|^{\frac{1}{2}}$. (The $H_p^{s,\mu}$ and $B_p^{s,\mu}$ spaces are H_p^s and B_p^s spaces provided with norms depending on μ, as in the basic case of $H_p^{s,\mu}(\mathbf{R}^n)$, which is the space provided with the norm $\| \operatorname{OP}\big((|\xi|^2 + |\mu|^2 + 1)^{s/2}\big)u\|_p$, cf. [9]. Mapping properties like (3.8) are well-known in the literature, cf. e.g. [15], except perhaps for the extension to low values of s.) With (3.8), we can write (3.7) as:

$$u = R_{N,\lambda}(f_1 - KTu) + K_{N,\lambda}(\psi_1 - (T_1' - \gamma_1)u), \tag{3.10}$$

or, if we set

$$\Phi = R_{N,\lambda} f_1 + K_{N,\lambda}\psi_1, \quad \mathcal{K}_\lambda = \begin{pmatrix} R_{N,\lambda}K & K_{N,\lambda} \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} T \\ T_1' - \gamma_1 \end{pmatrix}, \tag{3.11}$$

as:

$$(I + \mathcal{K}_\lambda \mathcal{T})u = \Phi. \tag{3.12}$$

We observe that the operators in (3.11), when considered as depending on the parameter λ, have regularity $\frac{1}{2}$ in the sense of [4], since $R_{N,\lambda}$, $K_{N,\lambda}$ and $\mathcal{T}$ have regularity $+\infty$, and K, being of order 1, counts with regularity $\frac{1}{2}$ by [4], Prop. 2.3.14.

Now we need the elementary

Lemma 3.1 *Let $A : V \to W$ and $B : W \to V$ be linear mappings. If $I + AB : W \to W$ is bijective, then $I + BA : V \to V$ is bijective, with*

$$(I + BA)^{-1} = I - B(I + AB)^{-1}A. \tag{3.13}$$

Proof: One just has to check:

$$\begin{aligned}(I{+}BA)&(I - B(I + AB)^{-1}A)\\ &= I + BA - B(I + AB)^{-1}A - BAB(I + AB)^{-1}A\\ &= I + BA - B(I + AB)(I + AB)^{-1}A = I,\end{aligned} \tag{3.14}$$

with a similar calculation for the left composition. □

The lemma will be applied with

$$\begin{aligned} A &= \mathcal{K}_\lambda : B_p^{s+1-\frac{1}{p},\mu}(\Gamma)^{n+1} \to H_p^{s+2,\mu}(\overline{\Omega})^n, \\ B &= \mathcal{T} : H_p^{s+2,\mu}(\overline{\Omega})^n \to B_p^{s+1-\frac{1}{p},\mu}(\Gamma)^{n+1}, \quad s > \frac{1}{p} - 1, \quad \lambda \in V_\varepsilon, \end{aligned} \tag{3.15}$$

and also with the roles of A and B interchanged. The lemma shows that (21) can be uniquely solved (in these spaces) if and only if $I + \mathcal{T}\mathcal{K}_\lambda$ is invertible, in which case

$$(I + \mathcal{K}_\lambda \mathcal{T})^{-1} = I - \mathcal{K}_\lambda (I + \mathcal{T}\mathcal{K}_\lambda)^{-1} \mathcal{T}. \tag{3.16}$$

Now $I + \mathcal{T}\mathcal{K}_\lambda$ is much easier to deal with than $I + \mathcal{K}_\lambda \mathcal{T}$, since it is a parameter-dependent ps.d.o. on the boundaryless compact manifold Γ! In details, $\mathcal{T}\mathcal{K}_\lambda$ is an $(n+1)\times(n+1)$-matrix

$$\mathcal{T}\mathcal{K}_\lambda = \begin{pmatrix} TR_{N,\lambda}K & TK_{N,\lambda} \\ (T_1' - \gamma_1)R_{N,\lambda}K & (T_1' - \gamma_1)K_{N,\lambda} \end{pmatrix}, \tag{3.17}$$

where the entries are of regularity $\frac{1}{2}$, in the sense of [4].

It is *parameter-elliptic* in the sense of [4], for λ on rays in $\mathbf{C} \setminus \mathbf{R}_+$, since this is a question of bijectiveness of certain model operators at the boundary (and certain matrices), a property that can be traced all the way from (3.4) to (3.12); the parameter-ellipticity of (3.4) was shown in [14], Sect. 6. This implies that for any $\varepsilon > 0$, there is an $r(\varepsilon) > 0$ so that for any $t \in \mathbf{R}$,

$$\begin{aligned} I + \mathcal{T}\mathcal{K}_\lambda : B_p^{t,\mu}(\Gamma)^{n+1} &\xrightarrow{\sim} B_p^{t,\mu}(\Gamma)^{n+1}, \text{ uniformly for } \lambda \in W_{\varepsilon,r(\varepsilon)}, \\ W_{\varepsilon,r(\varepsilon)} &= \{\, \lambda \in \mathbf{C} \mid \arg\lambda \in [\varepsilon, 2\pi - \varepsilon], |\lambda| \geq r(\varepsilon) \,\}; \end{aligned} \tag{3.18}$$

we denote the inverse

$$(I + \mathcal{T}\mathcal{K}_\lambda)^{-1} = Q_\lambda. \tag{3.19}$$

For such λ we also have the inverse, by Lemma 3.1,

$$(I + \mathcal{K}_\lambda \mathcal{T})^{-1} = I - \mathcal{K}_\lambda Q_\lambda \mathcal{T} : H_p^{s+2,\mu}(\Omega)^n \xrightarrow{\sim} H_p^{s+2,\mu}(\Omega)^n, \tag{3.20}$$

uniformly for $\lambda \in W_{\varepsilon,r(\varepsilon)}$, when $s > \frac{1}{p} - 1$.

Altogether, we solve (3.1) by taking

$$\begin{aligned} u &= (I + \mathcal{K}_\lambda \mathcal{T})^{-1}\Phi = (I - \mathcal{K}_\lambda Q_\lambda \mathcal{T})(R_{N,\lambda} f_1 + K_{N,\lambda}\varphi_1) \\ &= (I - \mathcal{K}_\lambda Q_\lambda \mathcal{T})(R_{N,\lambda}(f + \operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D \varphi_\nu) \\ &\qquad + K_{N,\lambda}\{\varphi_\tau, 0\}), \end{aligned} \tag{3.21}$$

and defining q by (3.3). The point is here that all operators except the factor Q_λ stem from classical resolvent problems for the Laplace operator; and Q_λ is a parameter-dependent ps.d.o. on Γ.

For the other boundary conditions (the cases $k = 0, 2, 3, 4$ in (1.2)), there are similar methods; for $k = 0, 2, 4$, the roles of the Dirichlet and Neumann problems are interchanged.

Also in the original problem where $-\lambda$ is replaced by ∂_t, this approach gives some simplifications. Indeed, as described in [7], the resolvent considerations carry over to solvability of the t-dependent problem with initial data 0, formally by a Laplace transformation. Analogously to the derivation of (3.21) from (3.5) we find that the problem

$$\begin{aligned} (\partial_t - \Delta - \varrho)v + KTv &= g_1 \quad \text{on } Q_{\mathbf{R}}, \\ T_1'v &= \zeta_1 \quad \text{on } S_{\mathbf{R}}, \end{aligned} \tag{3.22}$$

with $v(x,t)$, $g_1(x,t)$ and $\zeta_1(x,t)$ supported for $t \geq 0$, and $\varrho < \inf\{\operatorname{Re}\lambda \mid \lambda \in W_{\varepsilon, r(\varepsilon)}\}$, has a solution operator described by

$$v = (I - \mathbf{KQ}\mathcal{T})(\mathbf{R_N}g_1 + \mathbf{K_N}\zeta_1); \tag{3.23}$$

with

$$\begin{aligned} \big(\mathbf{R_N}\ \mathbf{K_N}\big) &= \begin{pmatrix} \partial_t - \Delta - \varrho \\ \gamma_1 \end{pmatrix}^{-1}, \\ \mathbf{K} &= \big(\mathbf{R_N}K\ \mathbf{K_N}\big), \\ \mathbf{Q} &= (I + \mathcal{T}\mathbf{K})^{-1}. \end{aligned} \tag{3.24}$$

The latter exists since $I + \mathcal{T}\mathbf{K}$ is derived from the parameter-dependent operator $I + \mathcal{T}\mathcal{K}_{\lambda+\varrho}$ by replacing $-\lambda$ by $i\tau$ in the symbol and using a pseudo-differential definition in one more variable; here $I + \mathcal{T}\mathcal{K}_{\lambda+\varrho}$ is invertible for λ in an obtuse neighborhood of $\{\operatorname{Re}\lambda \leq 0\}$, and the calculus of [6], Theorem 3.1 1° is applicable.

The resulting estimates for the solution operator are the same as those described in detail in [7]. It is because of the constant ϱ in (3.22), that we do not obtain time-global estimates in $H_p^{(s,s/2)}$ spaces in general. See however the special considerations for the Dirichlet problem in [7].

4 Exterior problems

Consider now the case where Ω is the complement of a compact set in $\mathbf{R}^n$, still with smooth boundary Γ. We can then investigate how the method of Section

3 can be used. Applying $-\operatorname{div}$ to the first line in (3.1) and taking the normal component of the third line, we again arrive at (3.2), now an exterior Dirichlet problem for q.

Theorem 4.1 *The exterior Dirichlet problem*

$$-\Delta v = g \text{ in } \Omega, \quad \gamma_0 v = \psi \text{ on } \Gamma, \tag{4.1}$$

has a solution operator $\begin{pmatrix} R_D & K_D \end{pmatrix}$ *such that*

$$\begin{aligned} \operatorname{grad} K_D &: B_p^{s+1-\frac{1}{p}}(\Gamma) \to H_p^s(\overline{\Omega})^n \text{ for } s \in \mathbf{R}, \\ \operatorname{grad} R_D \operatorname{div} &: H_p^s(\overline{\Omega})^n \to H_p^s(\overline{\Omega})^n \text{ for } s > \frac{1}{p} - 1, \end{aligned} \tag{4.2}$$

and $\operatorname{grad} K_D$ *maps into* $\bigcap_{q>1, r\in\mathbf{R}} H_q^r(\{|x| \geq R\})$*, when* $\{|x| \geq R\} \subset \Omega$*. (More precisely,* R_D *is defined for* g *with compact support, and* $\operatorname{grad} R_D \operatorname{div}$ *is extended by continuity.)*

Here K_D *is uniquely determined by the property that* $\operatorname{grad} K_D$ *should map into functions that are* $O(|x|^{-n})$ *for* $|x| \to \infty$*. It is also uniquely determined by requiring* $\operatorname{grad} K_D$ *to map into* $\bigcap_{q>1} L_q(\{|x| \geq R\})$ *— or just into* $L_p(\{|x| \geq R\})$*, if* $p \leq n/(n-1)$*.*

Proof: The mapping K_D is constructed as follows: We want to find a solution of

$$-\Delta v = 0 \text{ in } \Omega, \quad \gamma_0 v = \psi \text{ on } \Gamma, \tag{4.3}$$

for given $\psi \in B_p^{s+1-\frac{1}{p}}(\Gamma)$, such that $\operatorname{grad} v$ is $O(|x|^{-n})$ for $|x| \to \infty$. (The derivatives will also be $O(|x|^{-n})$, and then $\operatorname{grad} u \in H_q^r(\{|x| \geq R\})$ for all r, all $q > 1$, when $\{|x| \geq R\} \subset \Omega$.) Instead of (4.3), we can study

$$-\Delta v_c = 0 \text{ in } \Omega, \quad \gamma_0 v_c = \psi - c \text{ on } \Gamma, \tag{4.4}$$

where c is a constant to be chosen freely; if v_c solves (4.4), then $v = v_c + c$ solves (4.3) (and vice versa), and they have the same gradient.

We can assume that 0 is in the complement of $\overline{\Omega}$, so that the inversion $x \mapsto x/|x|^2$ maps Ω onto $\Omega^* \setminus \{0\}$, where Ω^* is a bounded open smooth set with $0 \in \Omega^*$, and $\partial\Omega^* = \Gamma^*$ is the image of Γ. (In the following, let $n \geq 3$; there is a similar proof for $n = 2$.) Let $W(x)$ be the solution of the special Dirichlet problem

$$-\Delta W = 0 \text{ in } \Omega^*, \quad \gamma_0 W(x) = |x|^{2-n} \text{ on } \Gamma^*; \tag{4.5}$$

since $|x|^{2-n} > 0$ on Γ^*, $W(0) > 0$ by the maximum principle.

Now if a function V is harmonic in $\Omega^* \setminus \{0\}$, then the function $|x|^{2-n} V(x/|x|^2)$ is harmonic in Ω, and vice versa (the Kelvin transformation). Since ψ on Γ is carried over to $|x|^{2-n}\psi(x/|x|^2)$ on Γ^*, lying in $B_p^{s+1-\frac{1}{p}}(\Gamma^*)$, we can find a solution of (4.4) by solving

$$-\Delta V_c = 0 \text{ in } \Omega^*, \quad \gamma_0 V_c = |x|^{2-n}(\psi(x/|x|^2) - c) \text{ on } \Gamma^*. \tag{4.6}$$

The problem (4.6) has a unique solution $V_c \in H_p^{s+1}(\overline{\Omega}^*) \cap C^\infty(\Omega^*)$ for each c; and by the linearity, $V_c = V_0 - cW$, cf. (4.5). Take

$$c^* = V_0(0)/W(0); \tag{4.7}$$

then $V_{c^*}(0) = 0$. Now let

$$v_{c^*}(x) = |x|^{2-n} V_{c^*}(x/|x|^2), \tag{4.8}$$

it solves (4.4). Since $V_{c^*}(0) = 0$, $D^\alpha v_{c^*}(x)$ is $O(|x|^{1-n-|\alpha|})$ for $|x| \to \infty$, any α (seen from the Taylor expansion of V_{c^*} at 0). In particular, $\operatorname{grad} v_{c^*}$ and its derivatives are $O(|x|^{-n})$ and hence L_q integrable at ∞ for any $q > 1$. On the other hand, $V_{c^*} \in H_p^{s+1}(\overline{\Omega}^*)$ implies that v_{c^*} is in H_p^{s+1} over bounded subsets of $\overline{\Omega}$. Altogether, it is found that v_{c^*}, and hence $v = v_{c^*} + c^*$, have gradient in $H_p^s(\overline{\Omega})$ and in $\bigcap_{q>1, r\in\mathbf{R}} H_q^r(\{|x| \geq R\})$. Defining K_D as the mapping from ψ to v, we have obtained an operator with the asserted mapping properties.

To show the uniqueness, let u be a solution of (4.3) with $\operatorname{grad} u = O(|x|^{-n})$ for $|x| \to \infty$. Recall that any function $u(x)$ that is harmonic on Ω has a unique Laurent expansion

$$u(x) = \sum_{k=0}^\infty H_k^*(x) + \sum_{k=0}^\infty \frac{H_k(x)}{|x|^{n-2+2k}}, \tag{4.9}$$

where the functions $H_k^*(x)$ and $H_k(x)$ are homogeneous harmonic polynomials of degree k; cf. Brelot [1], pp. 197–202, where also sets $\{R_1 < |x| < R_2\}$ are considered. The first series in (4.9) converges uniformly on bounded sets, the second converges uniformly on sets $\{|x| \geq R\}$ contained in Ω, and the derivatives of u are represented by the termwise differentiated expressions. When u has the form (4.9),

$$\operatorname{grad} u(x) = \sum_{k=1}^\infty \operatorname{grad} H_k^*(x) + \sum_{k=0}^\infty \operatorname{grad} \frac{H_k(x)}{|x|^{n-2+2k}}, \tag{4.10}$$

where $\operatorname{grad}(H_k(x)/|x|^{n-2+2k})$ is $O(|x|^{1-n-k})$ for each $k \geq 0$. The requirement that $\operatorname{grad} u$ should be $O(|x|^{-n})$ rules out the polynomials H_k^* for $k \geq 1$ as well as the term $H_0/|x|^{n-2}$, so

$$u(x) = H_0^* + v(x), \quad v(x) = \sum_{k=1}^{\infty} \frac{H_k(x)}{|x|^{n-2+2k}}. \tag{4.11}$$

Now the Kelvin transform of v, $V(x) = |x|^{2-n} v(x/|x|^2)$ has the representation

$$V(x) = \sum_{k=1}^{\infty} H_k(x); \tag{4.12}$$

it is C^∞ on Ω^* with $V(0) = 0$. Thus V is the unique solution of (33) with $c = H_0^*$. If $\widetilde{u} = \widetilde{c} + \widetilde{v}(x)$ also solves (4.3), with $\widetilde{v} = \sum_{k=1}^{\infty} \widetilde{H}_k(x)/|x|^{n-2+2k}$, then $\widetilde{v}$ is the Kelvin transform of the unique solution $\widetilde{V}$ of (4.6) with $c = \widetilde{c}$; this also has $\widetilde{V}(0) = 0$. Now $V - \widetilde{V} = (-H_0^* + \widetilde{c})W$ (cf. (4.5)), and since $V(0) = \widetilde{V}(0) = 0$ and $W(0) \neq 0$, $\widetilde{c}$ must equal H_0^*, and $V = \widetilde{V}$. Thus u is uniquely determined.

To require $\operatorname{grad} u \in L_q(\{|x| \geq R\})$ for some $q \leq n/(n-1)$ likewise reduces u to the form (4.11), since nonzero polynomials and $|x|^{1-n} \notin L_q(\{|x| \geq R\})$; and the analysis goes as above.

Now consider R_D. To solve the problem (4.1) with $\psi = 0$ and a nonzero $g = \operatorname{div} h$, $h \in H_p^s(\overline{\Omega})^n$, we let l be a continuous linear extension operator $l : H_p^s(\overline{\Omega}) \to H^s(\mathbf{R}^n)$ such that $\operatorname{div} lf = l \operatorname{div} f$, and search for v in the form $v = r_\Omega v_1 + v_2$, where v_1 and v_2 solve

$$-\Delta v_1 = lg \text{ on } \mathbf{R}^n, \text{ resp. } \begin{cases} -\Delta v_2 = 0 \text{ in } \Omega, \\ \gamma_0 v_2 = -\gamma_0 v_1 \text{ on } \Gamma. \end{cases} \tag{4.13}$$

(r_Ω denotes restriction to Ω.) When $g = \operatorname{div} h$, we want $\operatorname{grad} v$ to depend continuously on h in $H_p^s(\overline{\Omega})^n$.

To begin with, let g have compact support. To get a convenient solution of the problem for v_1, we define R as the operator (for compactly supported f)

$$R : f \mapsto \omega_n |x|^{2-n} * f - c(f), \quad \text{where } c(f) \text{ is the constant } \frac{\int_\Gamma \gamma_0(\omega_n |x|^{2-n} * f)\, d\sigma}{\int_\Gamma 1\, d\sigma}, \tag{4.14}$$

and $\omega_n|x|^{2-n}$ is the Newton potential ($\omega_n|x|^{2-n} * f$ can be viewed as $\mathrm{OP}(|\xi|^{-2})f$). This satisfies $-\Delta Rf = f$, and

$$\int_\Gamma \gamma_0 Rf\, d\sigma = 0. \tag{4.15}$$

When $g \in H_p^{s-1}(\overline{\Omega})$, then $v_1 = Rlg \in H_{p,\mathrm{loc}}^{s+1}(\mathbf{R}^n)$ by elliptic regularity, so $\gamma_0 v_1 \in B_p^{s+1-\frac{1}{p}}(\Gamma) \subset L_p(\Gamma)$, and the expressions are well-defined.

Now insert $\gamma_0 v_1$ in the equations for v_2; then the operator K_D established above gives a solution v_2 with $\operatorname{grad} v_2 \in H_p^s(\overline{\Omega})^n$. Altogether, we take

$$R_D g \equiv v = r_\Omega v_1 + v_2 \ = r_\Omega Rlg - K_D \gamma_0 Rlg. \tag{4.16}$$

In particular, when $g = \operatorname{div} h$ is inserted, then

$$\operatorname{grad} v = r_\Omega \operatorname{grad} v_1 + \operatorname{grad} v_2 = r_\Omega \operatorname{grad} R \operatorname{div} lh - \operatorname{grad} K_D \gamma_0 R \operatorname{div} lh. \tag{4.17}$$

Here we have for $\operatorname{grad} v_1$, by Fourier transformation,

$$\operatorname{grad} v_1 = \operatorname{grad} R \operatorname{div} lh = \mathrm{OP}\big((\xi_i\xi_j/|\xi|^2)_{i,j=1,\dots,n}\big) lh. \tag{4.18}$$

The operator $\mathrm{OP}\big((\xi_i\xi_j/|\xi|^2)_{i,j=1,\dots,n}\big)$ extends to a continuous operator in $L_p(\mathbf{R}^n)^n$ by a result of Calderón and Zygmund; and it is likewise continuous in $H_p^s(\mathbf{R}^n)^n$ for all $s \in \mathbf{R}$, since

$$\mathrm{OP}\big(\langle\xi\rangle^s\big)\, \mathrm{OP}\big((\xi_i\xi_j/|\xi|^2)_{i,j=1,\dots,n}\big)\, \mathrm{OP}\big(\langle\xi\rangle^{-s}\big) = \mathrm{OP}\big((\xi_i\xi_j/|\xi|^2)_{i,j=1,\dots,n}\big). \tag{4.19}$$

Thus

$$\|r_\Omega \operatorname{grad} v_1\|_{H_p^s(\overline{\Omega})} \le \|\operatorname{grad} v_1\|_{H_p^s(\mathbf{R}^n)} \le C_1 \|lh\|_{H_p^s(\mathbf{R}^n)} \le C_2 \|h\|_{H_p^s(\overline{\Omega})}. \tag{4.20}$$

This extends to h with arbitrary support, by approximation by $\eta(x/N)h$, for $\eta \in C_0^\infty(\mathbf{R}^n)$ equal to 1 near 0 and $N \to \infty$.

For v_2, we need to know that the mapping from h (or just from $\operatorname{grad} v_1$) to $\gamma_0 v_1$ is continuous from $H_p^s(\overline{\Omega})$ to $B_p^{s+1-\frac{1}{p}}(\Gamma)$. This is shown by the help of Lemma 4.2 below. Take for Ξ a large ball containing Γ in its interior. In view of (4.15), the lemma gives that

$$\begin{aligned}\|\gamma_0 v_1\|_{B_p^{s+1-p}(\Gamma)} &\le C_3 \|v_1\|_{H_p^{s+1}(\overline{\Xi})} \le C C_3 \|\operatorname{grad} v_1\|_{H_p^s(\overline{\Xi})} \\ &\le C C_3 C_2 \|h\|_{H_p^s(\overline{\Omega})}.\end{aligned} \tag{4.21}$$

Altogether, the mapping $\operatorname{grad} R_D \operatorname{div}$, as realized in (4.17), extends by closure to a mapping with the continuity in (4.2). $\square$

In the course of the proof we used the following variant of the Poincaré inequality:

Lemma 4.2 *Let $s > \frac{1}{p} - 1$, and let Ξ be a bounded connected open subset of $\mathbf{R}^n$ with a sufficiently smooth boundary such that the injection of $H_p^{s+1}(\overline{\Xi})$ into $H_p^s(\overline{\Xi})$ is compact. Let Γ be a nonempty smooth closed hypersurface in $\overline{\Xi}$. There is a constant C such that for $u \in H_p^{s+1}(\overline{\Xi})$,*

$$\|u\|_{H_p^s(\overline{\Xi})} \le C(\|\operatorname{grad} u\|_{H_p^s(\overline{\Xi})} + |\int_\Gamma \gamma_0 u \, d\sigma|). \tag{4.22}$$

Proof: Note that $\gamma_0 u = u|_\Gamma$ is well-defined as an element of $B_p^{s+1-\frac{1}{p}}(\Gamma) \subset L_p(\Gamma)$, so that the integral has a sense. If a constant C cannot be found, there is a sequence u_k with $\|u_k\|_{H_p^s(\overline{\Xi})} = 1$ but $\|\operatorname{grad} u_k\|_{H_p^s(\overline{\Xi})} \to 0$ and $\int_\Gamma \gamma_0 u_k \, d\sigma \to 0$. Since the sequence is bounded in $H_p^{s+1}(\overline{\Xi})$, it has a subsequence u_{k_j} that is convergent in $H_p^s(\overline{\Xi})$ to a limit u_0. Since $\|\operatorname{grad} u_k\|_{H_p^s(\overline{\Xi})} \to 0$, the subsequence u_{k_j} is convergent in $H_p^{s+1}(\overline{\Xi})$, with limit u_0, and $\gamma_0 u_{k_j} \to \gamma_0 u_0$ in $B_p^{s+1-\frac{1}{p}}(\Gamma)$. Now on one hand $\|u_0\|_{H_p^s(\overline{\Xi})} = 1$, so $u_0 \neq 0$; on the other hand, $\|\operatorname{grad} u_0\|_{H_p^s(\overline{\Xi})} = 0$ so u_0 is a constant, and this constant must equal 0 since $\int_\Gamma \gamma_0 u \, d\sigma = 0$. This contradiction proves the statement. $\square$

One can also replace the integral in the right hand side of (4.22) by another supplementary term, e.g. the integral of u over some small subdomain.

The mapping K_D is established in Simader and Sohr [16] for integer $s \ge 0$ in a slightly different way, and with lower smoothness assumptions on Γ; our presentation here was inspired by conversations with B. Fuglede.

The operator $I + \operatorname{grad} R_D \operatorname{div}$ is the projection onto the solenoidal space $J_p^s = \{\, u \in H_p^s(\overline{\Omega})^n \mid \operatorname{div} u = 0 \,\}$, $s > \frac{1}{p} - 1$, generalizing the situation where Ω is bounded.

Insertion of the formula for q into (3.1) leads to (3.4) and hence (3.5) and (3.7), with the same notation as before. We find by use of Theorem 4.1 that $f_1 \in H_p^s(\overline{\Omega})^n$, when $f \in H_p^s(\overline{\Omega})^n$ and $\varphi \in B_p^{s+1-\frac{1}{p}}(\Gamma)^n$.

For exterior domains we also have (3.8), by [9], so we can write the problem in the form (3.12). Again we can apply Lemma 3.1, using that (3.15) is valid.

It is at this point that it is a great advantage that we can reduce to the inversion of $I + \mathcal{T}\mathcal{K}_\lambda$. For this is a ps.d.o. on the compact manifold Γ. It satisfies the symbol requirements for being parameter-elliptic of regularity $\frac{1}{2}$ on the rays in $\mathbf{C} \setminus \mathbf{R}_+$, hence is invertible on these rays for sufficiently large $|\lambda|$, i.e. (3.18) holds. The inverse satisfies (3.20), and we get the solution as in (3.21),

$$u = (I - \mathcal{K}_\lambda Q_\lambda \mathcal{T})(R_{N,\lambda}(f + \operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D \varphi_\nu) + K_{N,\lambda}\{\varphi_\tau, 0\}), \tag{4.23}$$

which lies in $H^{s+2,\mu}(\overline{\Omega})$. Also the considerations for the problem with $-\lambda$ replaced by ∂_t go through, and the discussion for nonzero initial values can be completed as in [7]. For the pressure q we use the formula $\operatorname{grad} q = -\operatorname{grad} R_D \operatorname{div} f + \operatorname{grad} K_D(2\gamma_1 u_\nu - \varphi_\nu)$, plus the fact that (4.2) carries over to anisotropic spaces as continuous mappings

$$\begin{aligned} \operatorname{grad} K_D &: B_p^{(s+1-\frac{1}{p},(s+1-\frac{1}{p})/2)}(\overline{S}_{\mathbf{R}}) \to H_p^{(s,(s)/2)}(\overline{Q}_{\mathbf{R}})^n, \\ \operatorname{grad} R_D \operatorname{div} &: H_p^{(s,s/2)}(\overline{Q}_{\mathbf{R}})^n \to H_p^{(s,s/2)}(\overline{Q}_{\mathbf{R}})^n, \end{aligned} \tag{4.24}$$

for $s > \frac{1}{p} - 1$. (For $\operatorname{grad} K_D$, one uses that the mapping property holds for bounded neighborhoods of Γ by [7], and that $\operatorname{grad} K_D$ maps into $\bigcap_{r\in\mathbf{R}} H_p^r(\{|x| \ge R\})^n$ when R is large enough. For $\operatorname{grad} R_D \operatorname{div}$, the property is straightforward when $s \ge 0$; to include lower s, one uses that the operator is selfadjoint (being an orthogonal projection in L_2), and that $H_p^{(s,s/2)}(\overline{\Omega})^* = H_{p'}^{(-s,-s/2)}(\overline{\Omega})^*$ for $s \in\,]\frac{1}{p} - 1, \frac{1}{p}[$, here $\frac{1}{p} + \frac{1}{p'} = 1$.)

For the other boundary conditions (the cases $k = 0, 2, 3, 4$), the above analysis can be carried through in suitably modified versions.

The application of the linear result to solve the nonlinear problem goes mechanically as in the bounded case in [7], so we arrive at the result:

Theorem 4.3 *Theorem* 2.1 *generalizes to the case of an exterior smooth domain.*

(When $p > n/(n-1)$, $\operatorname{grad} q$ is only unique up to addition of functions $c(t) \operatorname{grad} |x|^{2-n}$ for $n > 2$, resp. $c(t) \operatorname{grad} \log |x|$ for $n = 2$.)

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