Trace formulas for parameter-dependent pseudodifferential operators

Gerd Grubb [a]

[a]Copenhagen Univ. Math. Dept. Universitetsparken 5, DK-2100 Copenhagen, Denmark. E-mail grubbmath.ku.dk

Trace expansions for operator families such as the resolvent, the heat operator and the complex powers are established for elliptic problems containing pseudodifferential elements. We consider operators on closed manifolds, as well as operators on compact manifolds with boundary, where suitable boundary conditions must be added. It is found in general that one can obtain expansions, e.g. of the heat operator trace, in powers t^α and power-logarithmic terms $t^\alpha \log t$, and the stability of the coefficients under perturbations is discussed. A survey is given of the methods relying on pseudodifferential calculus that lead to these results.

1. Introduction

1.1. Trace expansions and geometric invariants

One of the ways to determine invariants of Riemannian manifolds is by showing trace expansions associated with elliptic operators on the manifold. Indeed, it has been known since the work of Minakshisundaram and Pleijel [MP49] that for a compact n-dimensional manifold X with boundary $\partial X = X'$, the zeta-function $\zeta(A_B, s) = \operatorname{Tr}[(A_B)^{-s}]$ of the Laplace–Beltrami operator $A = -\Delta$ on X with a boundary condition $Bu = 0$ at ∂X (such as the Dirichlet condition), defined for large $\operatorname{Re} s$, extends meromorphically to all of $\mathbf{C}$ with simple poles at (at most) the points $s = \frac{n-j}{2}$, $j \in \mathbf{N}$ (we write $\mathbf{N} = \{0, 1, 2, \dots\}$). The residues at these poles characterize geometric properties of the manifold, cf. e.g. McKean and Singer [MS67], and, more recently, e.g. Branson and Gilkey [BG90], Gilkey [G95], Avramidi [A90], for more general cases.

Instead of studying the powers A_B^{-s}, one may equivalently study the resolvent and its derivatives $\partial_\lambda^r (A_B - \lambda I)^{-1}$, or the heat operator e^{-tA_B}, in their dependence on λ (for $\lambda \to \infty$ in sectors of $\mathbf{C}$), resp. t (for $t \to 0+$). In fact, there are transition formulas relating A_B^{-s} and e^{-tA_B} to each other and to the resolvent (cf. e.g. Grubb and Seeley [GS96] or Grubb [G97] for details),

$$A_B^{-s} = \tfrac{1}{(s-1)\cdots(s-r)} \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^{r-s} \partial_\lambda^r (A_B - \lambda I)^{-1}\, d\lambda$$

$$= \tfrac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-tA_B}\, dt, \qquad (1)$$

$$e^{-tA_B} = t^{-r} \tfrac{i}{2\pi} \int_{\mathcal{C}'} e^{-t\lambda} \partial_\lambda^r (A_B - \lambda I)^{-1}\, d\lambda$$

$$= \tfrac{1}{2\pi i} \int_{\operatorname{Re} s = c} t^{-s} \Gamma(s) A_B^{-s}\, ds, \qquad (2)$$

which imply that there is an equivalence between a statement on the poles of $\Gamma(s) \operatorname{Tr} A_B^{-s}$, an asymptotic expansion of $\partial_\lambda^r \operatorname{Tr}(A_B - \lambda I)^{-1}$ in decreasing half-powers powers of λ, and an asymptotic expansion of $\operatorname{Tr} e^{-tA_B}$ in increasing half-powers of t; the coefficients in these expansions are closely related to the residues.

Seeley [S69], [S69′], and Greiner [G71] showed such expansions for general elliptic differential operator systems with differential boundary condition. In particular, they lead to index formulas, in view of the general observation

$$\operatorname{index} A_B = \operatorname{Tr} e^{-tA_B{}^* A_B} - \operatorname{Tr} e^{-tA_B A_B{}^*}. \qquad (3)$$

New invariants came into the picture when Seeley in [S67] initiated the study of similar questions for *pseudodifferential operators* (ψdo's) A on closed manifolds. As pointed out by Duistermaat and Guillemin [DG75], this leads to logarithmic terms in the expansions of the resolvent and heat operator traces, corresponding to double poles of $\Gamma(s) \operatorname{Tr} A^{-s}$.

Related questions for manifolds with boundary have only been taken up fairly recently. Finite expansions are known for general pseudodifferential problems from Grubb [G96] (and its first edition from 1986), and [G92] gives a finite expansion for the Atiyah–Patodi–Singer problem stopping at the index term. Full expansions for APS-problems were established by Grubb and Seeley in [GS95], [GS96], and full expansions for rather general pseudodifferential boundary problems for differential operators have been established in [G99].

For other types of manifolds, we should also mention the work of Brüning and Seeley [BS91] and of Gil [G98] concerning operators on manifolds with conical singularities, and the work of Loya [L98] treating resolvents on manifolds with singularities such as corners and edges.

Underlying these studies is always a suitable calculus of ψdo's depending on a parameter.

Plan of the paper: This is a survey paper, presenting published as well as new results. The problems in dealing with parameter-dependent ψdo's are recalled in Section 1.2. Chapter 2 is concerned with the boundaryless case. In Section 2.1, we go through a calculus that generalizes that of [GS95] in a way that allows the treatment of complex powers of resolvents [GH01]. Section 2.2 gives the application to full trace expansions with powers and logarithmic terms. Chapter 3 is concerned with manifolds with boundary. First, we show in Section 3.1 how the results for the boundaryless case can be used in the treatment of the Atiyah–Patodi–Singer problem, including a new result on the stability of the first half-integer log-term [G01″]. Next, we explain in Section 3.2 a systematic calculus for manifolds with boundary [G01′] that englobes and extends the APS methods. Finally in Section 3.3 we deal briefly with a case that is not covered by [G01′] and is needed for the study of the noncommutative residue in the Boutet de Monvel calculus [GS01].

In each of these various theories, it is the applications to full trace expansions that are in focus.

1.2. Parameter-dependent calculi

Recall that the Fourier transform

$$[\mathcal{F}f](\xi) = \hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx, \tag{4}$$

with inverse

$$[\mathcal{F}^{-1}g](x) = \int e^{+ix\cdot\xi} g(\xi)\,đ\xi$$

(where $đ\xi = (2\pi)^{-n}d\xi$), turns a differential operator $D_{x_j} = -i\partial/\partial x_j$ into the multiplication by ξ_j; more generally with multi-index notation, $D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$ is turned into multiplication by ξ^α. So a differential operator on $\mathbf{R}^n$ may be written

$$Af = \sum_{|\alpha|\le m} a_\alpha D^\alpha f = \int e^{ix\cdot\xi} \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{f}(\xi)\,đ\xi.$$

This motivates the more general definition of a pseudodifferential operator (ψdo) $P = \mathrm{OP}(p(x,\xi))$ on $\mathbf{R}^n$, by the formula:

$$Pf = \mathrm{OP}(p(x,\xi))f(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)đ\xi. \tag{5}$$

When p is independent of x, this simply means that

$$\mathrm{OP}(p(\xi))f(x) = \mathcal{F}^{-1}_{\xi\to x}[p(\xi)(\mathcal{F}f)(\xi)]. \tag{6}$$

The formula (5) is a generalization:

$$\mathrm{OP}(p(x,\xi))f(x) = \mathcal{F}^{-1}_{\xi\to x}[p(y,\xi)(\mathcal{F}f)(\xi)]|_{y=x}.$$

The function $p(x,\xi)$, called the symbol of the operator, need not be a polynomial in ξ. Instead it may be taken in the standard symbol space $S^m(\mathbf{R}^n\times\mathbf{R}^n)$, consisting of the C^∞ functions $p(x,\xi)$ such that

$$\partial_x^\beta\partial_\xi^\alpha p = O(\langle\xi\rangle^{m-|\alpha|}) \text{ for all } \alpha,\beta\in\mathbf{N}^n; \tag{7}$$

here we use the notation

$$\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}.$$

For the differential operator A, the symbol $a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ is in S^m, $m\in\mathbf{N}$, but an advantage of ψdo's is that we can take the order m negative or noninteger. For example, since $1-\Delta$ has symbol $1+|\xi|^2$, the inverse of $1-\Delta$ is the ψdo with symbol $(1+|\xi|^2)^{-1} = \langle\xi\rangle^{-2}$, and we can define the square root of $1-\Delta$ as $\mathrm{OP}(\langle\xi\rangle)$.

The rules of calculus for ψdo's with symbols in the S^m-spaces are well-known, see e.g. Hörmander [H67], Seeley [S69″], Shubin [S78], Hörmander [H85] for various set-ups with local or global estimates in x.

There is an important concept of *asymptotic series* used here: When $p \in S^{m_0}$ and there are symbols $p_j \in S^{m_j}$ for a decreasing sequence $m_j \to -\infty$ (where $j \to \infty$ in $\mathbf{N}$), we say that

$$p \sim \sum_{j\in\mathbf{N}} p_j \text{ in } S^{m_0} \text{ if}$$

$$p - \sum_{0\le j<J} p_j \in S^{m_J} \text{ for any } J \in \mathbf{N}. \tag{8}$$

This does not at all mean that the series $\sum_j p_j$ converges to p, only that for any J, the sum up to $j = J-1$ differs from p by a symbol that has the next order of magnitude $O(\langle\xi\rangle^{m_J})$ (along with the associated estimates of derivatives). This is similar to the way Taylor's formula shows how a C^∞ function is close to a polynomial of a given order, in the way that the error term is of the next order of magnitude.

For any given sequence $p_j \in S^{m_j}$ with $m_j \searrow -\infty$, one can construct a p such that (8) holds, by a generalization of the Borel construction of a C^∞ function with given Taylor coefficients.

We call the symbols in $S^m(\mathbf{R}^n \times \mathbf{R}^n)$ *classical* (or *one-step polyhomogeneous*), when they have expansions in series of terms $p_{m-j}(x,\xi)$, $j \in \mathbf{N}$, that are *homogeneous in* ξ *of degree* $m-j$ *for* $|\xi| \ge 1$, such that $p \sim \sum_{j\in\mathbf{N}} p_{m-j}$ in $S^m(\mathbf{R}^n \times \mathbf{R}^n)$.

A composition of two operators is very simple in the case of x-independent symbols:

$$\mathrm{OP}(p(\xi))\mathrm{OP}(p'(\xi)) = \mathcal{F}^{-1}p\mathcal{F}\mathcal{F}^{-1}p'\mathcal{F}$$

$$= \mathrm{OP}(p(\xi)p'(\xi)).$$

For general symbols, this holds in the first approximation:

$$\mathrm{OP}(p(x,\xi))\mathrm{OP}(p'(x,\xi)) = \mathrm{OP}((p\circ p')(x,\xi)),$$

where

$$p\circ p' \sim pp' + \sum_{|\alpha|\ge 1} \tfrac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha p\,\partial_x^\alpha p',$$

$$= \sum_{\alpha\in\mathbf{N}^n} \tfrac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha p\,\partial_x^\alpha p' \text{ in } S^{m+m'}. \tag{9}$$

The formula, sometimes called the Leibniz product, is used in the construction of approximate inverses (parametrices) for elliptic operators. If for example $a \in S^m$ with $a^{-1} \in S^{-m}$, then

$$a \circ a^{-1} = 1 - r \text{ for some } r \in S^{-1},$$

so if we take

$$a^{(-1)} \sim a^{-1} \circ \sum_{k=0}^{\infty} r^{\circ k}$$

(where $r^{\circ k} = r \circ r \circ \ldots \circ r$ with k factors), then we find that $a \circ a^{(-1)} \sim 1$. The symbol $a^{(-1)}$ is called a *parametrix symbol*; it defines an operator such that

$$\mathrm{OP}(a)\mathrm{OP}(a^{(-1)}) = 1 + R_1,$$

$$\mathrm{OP}(a^{(-1)})\mathrm{OP}(a) = 1 + R_2,$$

with ψdo's R_1 and R_2 of order $-\infty$.

In the study of resolvents one needs to carry out such parametrix constructions for symbols $a - \lambda I$ containing the extra parameter λ (and, notably, one needs a good control of how λ interacts with the other variables).

Here it is remarkably more difficult to handle pseudodifferential than differential operators.

Let for example $a(x,\xi)$ be homogeneous in ξ of order $m > 0$ for $|\xi| \ge 1$, C^∞ and *strongly elliptic:*

$$\operatorname{Re} a(x,\xi) \ge c|\xi|^m \text{ for } |\xi| \ge c_1,$$

with $c > 0$. Let $A = \mathrm{OP}(a)$, then the resolvent $Q_\lambda = (A - \lambda I)^{-1}$ is defined for $-\lambda \in \mathbf{R}_+$ with $|\lambda|$ sufficiently large. It is a ψdo with symbol

$$q(x,\xi,\lambda) = q_{-m}(x,\xi,\lambda) + q'(x,\xi,\lambda),$$

$$q_{-m} = (a(x,\xi) - \lambda)^{-1},\ q' \text{ of order } \le -m-1.$$

Here λ, or rather, $\mu = (-\lambda)^{\frac{1}{m}}$, can be considered as one more "cotangent variable" in addition to $\xi_1, \xi_2, \ldots, \xi_n$, and q_{-m} is homogeneous of degree $-m$ in (ξ,μ).

More precisely, if A is a differential operator, then m is integer and $a(x,\xi) - \lambda = a(x,\xi) + \mu^m$ is polynomial in (ξ,μ), hence homogeneous and C^∞ in $(\xi,\mu) \in \overline{\mathbf{R}}_+^{n+1}$, so $q_{-m}(x,\xi,\lambda)$ is homogeneous of degree $-m$ and C^∞ outside $(\xi,\mu) = (0,0)$. But if A is truly pseudodifferential, the strictly homogeneous version of the symbol $a(x,\xi)$ will in general have a lack of smoothness at $\xi = 0$ (its ξ-derivatives are bounded up to order $\le m$ only), so $a(x,\xi) + \mu^m$ will have this lack of smoothness on the whole halfline $\{(0,\mu) \mid \mu \ge 0\}$, and so will q_{-m}. When we modify the strictly homogeneous symbol in a bounded neighborhood of $\xi = 0$ to be C^∞, the ensuing modification of $a(x,\xi) + \mu^m$ takes place in an unbounded set.

This is also reflected in the *estimates* of q_{-m}. Here one has (for $|\xi| \ge 1$):

$$D_\xi^\alpha q_{-m} = O(\langle(\xi,\mu)\rangle^{-m-|\alpha|}), \text{ for } |\alpha| \le m,$$

$$D_\xi^\alpha q_{-m} = O(\langle(\xi,\mu)\rangle^{-2m}\langle\xi\rangle^{m-|\alpha|}), \text{ for } |\alpha| \ge m,$$

where the estimate in the first line extends to all α *if and only if* a is polynomial in ξ. In the polynomial case one can apply the usual symbolic calculus, just in one more variable, getting simple and straightforward results, whereas in the general case the fact that only the estimates up to order $\le m$ are standard (the so-called *regularity number* is only m), gives severe trouble. We give in Chapter 2 below a definition of parameter-dependent symbol classes (more restrictive than in [G96] but more general than in [GS95] and [L98]) that handles the problems with resolvents of ψdo's in an efficient way, leading to full trace expansions.

For boundary value problems there are similar phenomena. In the differential operator case, the resolvent parameter enters as another cotangent variable, on a par with the others, whereas for a *pseudodifferential boundary operator* (ψdbo), a resolvent parameter, when considered as a cotangent variable, gives symbolic estimates where only the first finitely many are "good". Again one can assign a regularity number to the operator and keep track of it in the calculus, this was an important point in [G96], leading to trace expansions with a finite number of power terms (related to the regularity number). In Chapter 3, we present several cases of more delicate (and restrictive) calculi which allow complete trace expansions with powers and logarithms.

2. Weakly polyhomogeneous pseudodifferential operators

2.1. Polyhomogeneous symbol classes

We here sketch a generalization of the symbol classes introduced in [GS95], now allowing quasi-homogeneous symbols. It was inspired from reading Loya [L98], but is adapted to treat also e.g. complex powers of resolvents. A detailed presentation is given in Grubb and Hansen [GH01].

Consider symbols $p(x,\xi,\lambda)$ that are C^∞ functions of x and $\xi \in \mathbf{R}^n$, $\lambda \in \Gamma$ (a sector of $\mathbf{C}\setminus\{0\}$). Let $\sigma \in \mathbf{R}_+$. We shall say that:

p is **strongly σ-homogeneous** of degree m, when

$$p(x,t\xi,t^\sigma\lambda) = t^m p(x,\xi,\lambda) \tag{10}$$

for $|\xi| + |\lambda|^{\frac{1}{\sigma}} \ge 1$, $t \ge 1, (\xi,\lambda) \in \mathbf{R}^n \times (\Gamma \cup \{0\})$.

p is **weakly σ-homogeneous** of degree m, when

$$p(x,t\xi,t^\sigma\lambda) = t^m p(x,\xi,\lambda) \tag{11}$$

for $|\xi| \ge 1$, $t \ge 1$, $(\xi,\lambda) \in \mathbf{R}^n \times \Gamma$.

Example 2.1. Let $a(x,\xi)$ be C^∞ on $\mathbf{R}^n \times \mathbf{R}^n$ and homogeneous in ξ of degree $r > 0$ for $|\xi| \ge 1$, taking values in a closed sector $\mathbf{C} \setminus \Gamma$. Then for $\lambda \in \Gamma$, $a(x,\xi) - \lambda$ and $(a(x,\xi) - \lambda)^{-1}$ extend to:

strongly r-homogeneous symbols of degree r, resp. $-r$, if r is integer and a is **polynomial** in ξ (it is the symbol of a differential operator);

weakly r-homogeneous symbols of degree r, resp. $-r$, if a is **not** polynomial in ξ (it is the symbol of a genuine ψdo).

For example, when $r = n = 2$, then $a(x,\xi) = \xi_1^2 + \xi_2^2$ enters in the first case, and

$$a(x,\xi) = (\xi_1^4 + \xi_2^4)/(\xi_1^2 + \xi_2^2) \text{ (for } |\xi| \ge 1)$$

enters in the second case, with $\Gamma = \mathbf{C} \setminus \overline{\mathbf{R}}_+$.

For the weakly σ-homogeneous symbols, we need extra conditions on the behavior when λ is large in comparison with ξ. Therefore we introduce the following symbol spaces:

Definition 2.2. *Let $m \in \mathbf{R}$, $\sigma \in \mathbf{R}_+$.*

$S^{m,0}_\sigma(\mathbf{R}^n\times\mathbf{R}^n,\Gamma)$ *consists of the* C^∞ *functions* $p(x,\xi,\lambda)$ *that are holomorphic in* $\lambda\in\Gamma^\circ$ *for* $|(\xi,\lambda)|\ge\varepsilon$ *(for some* $\varepsilon>0$*) and satisfy, with* $\frac{1}{\lambda}=z$,

$$\partial_z^j p(\cdot,\cdot,\tfrac{1}{z}) \text{ is in } S^{m+\sigma j}(\mathbf{R}^n\times\mathbf{R}^n) \text{ for } \tfrac{1}{z}\in\Gamma,$$

$$\text{with uniform estimates for } |z|\le 1,$$

$$\tfrac{1}{z} \text{ in closed subsectors of } \Gamma. \tag{12}$$

Moreover, we set, for any $\delta\in\mathbf{C}$,

$$S^{m,\delta}_\sigma(\mathbf{R}^n\times\mathbf{R}^n,\Gamma)=\lambda^\delta S^{m,0}_\sigma(\mathbf{R}^n\times\mathbf{R}^n,\Gamma). \tag{13}$$

The indication $(\mathbf{R}^n\times\mathbf{R}^n,\Gamma)$ is often abbreviated to (Γ).

We have:

Theorem 2.3. *When* a *is as in Example* 2.1, *then for any* $N\in\mathbf{N}$,

$$(a(x,\xi)-\lambda)^{-N}\in S^{-rN,0}_r(\Gamma)\cap S^{0,-N}_r(\Gamma).$$

$$(a(x,\xi)-\lambda)^{N}\in S^{rN,0}_r(\Gamma)+S^{0,N}_r(\Gamma). \tag{14}$$

Moreover, if $\delta\in\mathbf{C}$ *with* $\operatorname{Re}\delta\le 0$, *then*

$$(a(x,\xi)-\lambda)^{-N+\delta}\in S^{-rN,\delta}_r(\Gamma)\cap S^{0,-N+\delta}_r(\Gamma);$$

$$(a(x,\xi)-\lambda)^{N+\delta}\in S^{rN,\delta}_r(\Gamma)+S^{0,N+\delta}_r(\Gamma). \tag{15}$$

One cannot in general obtain (14) for noninteger N.

The λ-*independent* symbols fit into the calculus as follows:

Theorem 2.4. *When a symbol* $p(x,\xi)$ *of order* m *is considered as depending on one more variable* λ, *it lies in* $S^{m,0}_\sigma$:

$$S^m(\mathbf{R}^n\times\mathbf{R}^n)\subset S^{m,0}_\sigma(\mathbf{R}^n\times\mathbf{R}^n,\Gamma) \tag{16}$$

for any $\Gamma\subset\mathbf{C}$, *any* $\sigma\in\mathbf{R}_+$.

We set

$$\bigcap_{m\in\mathbf{R}} S^{m,\delta}_\sigma(\Gamma)=S^{-\infty,\delta}_\sigma(\Gamma);$$

$$\bigcup_{m\in\mathbf{R}} S^{m,\delta}_\sigma(\Gamma)=S^{\infty,\delta}_\sigma(\Gamma). \tag{17}$$

The operators have good composition rules, since clearly

$$S^{m,\delta}_\sigma(\Gamma)\cdot S^{m',\delta'}_\sigma(\Gamma)\subset S^{m+m',\delta+\delta'}_\sigma(\Gamma), \tag{18}$$

and since one can refer to the standard rules for S^m symbol classes, which must here hold uniformly in z as in (12). One finds for example that

$$P\in\operatorname{OP}(S^{m,\delta}_\sigma(\Gamma)),\ P'\in\operatorname{OP}(S^{m',\delta'}_\sigma(\Gamma))$$

$$\implies PP'\in\operatorname{OP}(S^{m+m',\delta+\delta'}_\sigma(\Gamma)) \tag{19}$$

and the resulting symbol is described by the usual formula:

$$(p\circ p')(x,\xi,\lambda)$$

$$\sim\sum_{\alpha\in\mathbf{N}^n}\tfrac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha p(x,\xi,\lambda)\,\partial_x^\alpha p'(x,\xi,\lambda) \tag{20}$$

in $S^{m+m',\delta+\delta'}_\sigma(\Gamma)$. This is an expansion in terms with decreasing m-exponents $m+m'-j$, $j\to\infty$ ($j=|\alpha|$). Here we use asymptotic series as follows:

When $p_j\in S^{m_j,\delta}_\sigma(\Gamma)$ for a decreasing sequence $m_j\to-\infty$ (for $j\to\infty$ in $\mathbf{N}$), and $p\in S^{m_0,\delta}_\sigma(\Gamma)$, we say that $p\sim\sum_j p_j$ in $S^{m_0,\delta}_\sigma(\Gamma)$ if

$$p-\sum_{j<J}p_j\in S^{m_J,\delta}_\sigma(\Gamma) \text{ for any } J\in\mathbf{N}. \tag{21}$$

For any given sequence $p_j\in S^{m_j,\delta}_\sigma(\Gamma)$ with $m_j\searrow-\infty$, one can construct a p such that (21) holds.

For the present special symbols there is *another* type of expansion that is of great interest:

Theorem 2.5. *When* $p\in S^{m,\delta}_\sigma(\mathbf{R}^n\times\mathbf{R}^n,\Gamma)$, *then the limits*

$$p_{(\delta,k)}(x,\xi)=\lim_{z\to 0}\frac{1}{k!}\partial_z^k(z^\delta p(x,\xi,\tfrac{1}{z}))$$

exist and belong to $S^{m+\sigma k}(\mathbf{R}^n\times\mathbf{R}^n)$, *and* p *has an expansion in terms* $\lambda^{\delta-k}p_{(\delta,k)}(x,\xi)$ *such that for any* N,

$$p(x,\xi,\lambda)-\sum_{0\le k<N}\lambda^{\delta-k}p_{(\delta,k)}(x,\xi) \tag{22}$$

$$\in S^{m+\sigma N,\delta-N}_\sigma(\mathbf{R}^n\times\mathbf{R}^n,\Gamma).$$

In the proof, one reduces to the case $\delta = 0$ by multiplication by $\lambda^{-\delta}$. Then the expansion is essentially a Taylor expansion in $z = \frac{1}{\lambda}$ at $z = 0$; it exists because of the uniform estimates for $z \to 0$ in the sector.

Note that in (22), the order of $p_{(\delta,k)}$ *increases* with increasing k, whereas the power of λ *decreases*. A very simple example is

$$(|\xi|^2 + \lambda)^{-1} = \lambda^{-1}\Big(1 - \frac{|\xi|^2}{\lambda} + \frac{|\xi|^4}{\lambda^2} - \dots\Big).$$

Corollary 2.6. *When $p \in S_\sigma^{-\infty,\delta}(\Gamma)$, the kernel $K(x,y,\lambda)$ of $\mathrm{OP}(p)$ has an expansion*

$$K(x,y,\lambda) \sim \sum_{k\in\mathbf{N}} \lambda^{\delta-k} K_k(x,y), \tag{23}$$

with $K_k \in C^\infty$.

Definition 2.2 contains no homogeneity requirements, but we now define a polyhomogeneous subspace:

Definition 2.7. *A symbol $p \in S_\sigma^{m_0,\delta}(\Gamma)$ is called* **(weakly) σ-polyhomogeneous** *of degree $m_0 + \sigma\delta$, when $p \sim \sum_{j\in\mathbf{N}} p_j$, with $p_j \in S_\sigma^{m_j,\delta}(\Gamma)$, $m_j \searrow -\infty$ for $j \to \infty$, $j \in \mathbf{N}$, such that the p_j are weakly σ-homogeneous of degrees $m_j + \sigma\delta$ (cf. (11)).*

If the p_j are in addition strongly σ-homogeneous (cf. (10)), and

$$\partial_x^\beta \partial_\xi^\alpha \partial_\lambda^k \Big(p - \sum_{j<J} p_j\Big) \tag{24}$$

$$= O(\langle(\xi, |\lambda|^{\frac{1}{\sigma}})\rangle^{m+\sigma\mathrm{Re}\,\delta - J - |\alpha| - \sigma k}),$$

for all indices α, β, J, then p is called **strongly σ-polyhomogeneous.**

The σ-polyhomogeneity is called classical (or one-step) when the sequence m_j is of the form $m_j = m_0 - j$, $j \in \mathbf{N}$.

Remark 2.8. One has in particular that *classical* ψdo symbols in $n+1$ cotangent variables $p(x,(\xi,\xi_{n+1}))$ give *strongly 1-polyhomogeneous* symbols when ξ_{n+1} is replaced by λ.

The type of parameter-dependence mentioned in Remark 2.8 was used by Agmon and by Agranovič and Vishik in resolvent studies for differential operators; for ψdo's this is the kind of parameter-dependence studied e.g. in Shubin [Sh78] and other works. It is a mild generalization that does not cover resolvents $(P - \lambda I)^{-1}$ and parabolic operators such as $\partial/\partial t + P$ when P is truly pseudodifferential (as treated e.g. in [G96]).

The operators behave in a standard way under coordinate transformations, so they can also be defined to act on sections of smooth vector bundles over smooth manifolds.

2.2. Applications to kernel and trace expansions

Both the expansion in Theorem 2.5 and the expansion in Definition 2.7 enter in the proof of:

Theorem 2.9. *Let $p \in S_\sigma^{m_0,\delta}(\Gamma)$ be weakly σ-polyhomogeneous as in Definition 2.7, and assume furthermore that the p_j with $m_j \geq -n$ are integrable with respect to ξ for each λ. Then $\mathrm{OP}(p)$ has a continuous kernel $K_p(x,y,\lambda)$ with an expansion on the diagonal*

$$K_p(x,x,\lambda) \sim \sum_{j=0}^{\infty} c_j(x)\lambda^{\delta + \frac{m_j+n}{\sigma}}$$

$$+ \sum_{k=0}^{\infty} [c_k'(x)\log\lambda + c_k''(x)]\lambda^{\delta-k}, \tag{25}$$

for $|\lambda| \to \infty$, uniformly for λ in closed subsectors of Γ ($|\lambda| \geq 1$). The coefficients $c_j(x)$ and $c_k'(x)$ (with $k = -\frac{m_j+n}{\sigma}$) are determined by $p_j(x,\xi,\lambda)$ for $|\xi| \geq 1$ (in particular, they are "local"), while the $c_k''(x)$ in general depend on the full symbol (are "global").

The proof, given in [GH01], is a straightforward extension of the proof of Theorem 2.1 in [GS95]. A brief explanation in local coordinates: One uses the general principle that "remainders contribute to c_k'' terms," by Corollary 2.6. The p_j contribute with

$$K_{p_j}(x,x,\lambda) = \int_{\mathbf{R}^n} p_j(x,\xi,\lambda)\, đ\xi$$

$$= \int_{|\xi|\geq|\lambda|^{\frac{1}{\sigma}}} p_j\, đ\xi + \int_{|\xi|\leq 1} p_j\, đ\xi + \int_{1\leq|\xi|\leq|\lambda|^{\frac{1}{\sigma}}} p_j\, đ\xi$$

$$= I_1 + I_2 + I_3, \tag{26}$$

where I_1 gives a power term (part of the c_j term), I_2 contributes to the c''_k terms, and I_3 gives the rest of c_j, and c'_k with $k = -\frac{m_j+n}{\sigma}$ if this is an integer ≥ 0, plus some contributions to the c''_k terms. One has of course to show that the contributions to the c''_k pile up in a controlled way.

When the operator acts on a compact boundaryless manifold, integration of $K_p(x,x,\lambda)$ (of its fiber trace if P acts in a vector bundle) in x gives a similar expansion of the *trace* of the operator:

Corollary 2.10. *Let P be a λ-dependent ψdo acting on the sections of a smooth vector bundle over a compact manifold X of dimension n, with symbol satisfying the hypotheses of Theorem* 2.9 *in local coordinates. Then it is trace-class, the trace satisfying*

$$\operatorname{Tr} P \sim \sum_{j=0}^{\infty} c_j \lambda^{\delta+\frac{m_j+n}{\sigma}} + \sum_{k=0}^{\infty} [c'_k \log \lambda + c''_k] \lambda^{\delta-k}, \tag{27}$$

for $|\lambda| \to \infty$, uniformly for λ in closed subsectors of Γ. The coefficients are derived from those in (25) *for coordinate patches by integrating the fiber traces over X.*

The result applies in particular to expressions containing a resolvent power:

$$P(\lambda) = S(A-\lambda I)^{-N}, \tag{28}$$

where A is a classical elliptic ψdo of positive order m acting in a vector bundle over a compact boundaryless manifold X of dimension n, with principal symbol $a_m(x,\xi)$ having no eigenvalues on $\mathbf{R}_-$, S is a classical ψdo of order ν acting in the bundle, and N is chosen so large that $\nu - Nm < -n$. Since

$$\partial_\lambda^r (A-\lambda I)^{-1} = r!(A-\lambda I)^{-1-r}, \tag{29}$$

this also pertains to expressions with a differentiated resolvent $S\partial_\lambda^r(A-\lambda I)^{-1}$.

By use of (14) one finds that for a narrow sector Γ around $\mathbf{R}_-$, the symbol is in $S_m^{\nu-Nm,0}(\Gamma) \cap S_m^{\nu,-N}(\Gamma)$ and weakly m-polyhomogeneous. Then Theorem 2.9 and its corollary apply. The kernel satisfies on the diagonal:

$$K(x,x,S(A-\lambda I)^{-N}) \sim \sum\nolimits_{j=0}^{\infty} c_j(x) \lambda^{\frac{n+\nu-j}{m}-N}$$

$$+ \sum_{k=0}^{\infty} (c'_k(x) \log \lambda + c''_k(x)) \lambda^{-N-k}, \tag{30}$$

for $|\lambda| \to \infty$, uniformly in closed subsectors of Γ. Consequently, one has for the trace:

$$\operatorname{Tr} S(A-\lambda I)^{-N} \sim \sum\nolimits_{j=0}^{\infty} c_j \lambda^{\frac{n+\nu-j}{m}-N}$$

$$+ \sum_{k=0}^{\infty} (c'_k \log \lambda + c''_k) \lambda^{-N-k} \tag{31}$$

(by integration of the fiber traces over X).

If S is a differential operator (in particular if $S = I$), then $c'_0(x) = 0$ and the complete coefficient of λ^{-N} is locally determined (by a generalization of the proof of this point in [GS95], Th. 2.7).

In view of (29) and the general transition formulas (1), (2), we then also get trace expansions for the following operator families:

$$\operatorname{Tr}\big(Se^{-tA}\big) \sim \sum_{j=0}^{\infty} \tilde{c}_j t^{\frac{j-n-\nu}{m}}$$

$$+ \sum_{k=0}^{\infty} (-\tilde{c}'_k \log t + \tilde{c}''_k) t^k, \tag{32}$$

$$\Gamma(s)\operatorname{Tr}\big(SA^{-s}\big) \sim \sum_{j=0}^{\infty} \frac{\tilde{c}_j}{s + \frac{j-n-\nu}{m}} - \frac{\operatorname{Tr}(S\Pi_0(A))}{s}$$

$$+ \sum_{k=0}^{\infty} \Big(\frac{\tilde{c}'_k}{(s+k)^2} + \frac{\tilde{c}''_k}{s+k} \Big). \tag{33}$$

For (32), it is required that the eigenvalues of the principal symbol of A have positive real part; (32) is considered for $t \to 0+$. In (33), the sign "$\sim$" indicates that the left hand side is meromorphic for $s \in \mathbf{C}$ with pole structure as in the right hand side; here $\Pi_0(A)$ is the orthogonal projection onto the zero eigenspace of A. The constants $\tilde{c}^{\cdot}_k$ are related to the $c^{\cdot}_k$ by universal factors.

Finally, let us mention a new type of result, made possible by the introduction of the present symbol class allowing noninteger δ:

Theorem 2.11. *Let $m \in \mathbf{R}_+$, $\nu \in \mathbf{R}$ and $s \in \mathbf{C}$. Consider $P(\lambda) = S(A-\lambda I)^{-s}$, where A is* *a classical elliptic ψdo of order m acting on the sections of a smooth vector bundle over a compact manifold X of dimension n, with principal symbol having no eigenvalues on $\mathbf{R}_-$, S is a classical ψdo of order ν, and s satisfies $\operatorname{Re} s > (n+\nu)/m$. Then for λ in a sector Γ around $\mathbf{R}_-$, the symbol of P is in $S_m^{\nu,-s}(\Gamma)$ (and in $S^{\nu-\operatorname{Re} s\, m}$ for each λ) and weakly m-polyhomogeneous. The kernel satisfies on the diagonal:*

$$K(x,x,S(A-\lambda I)^{-s}) \sim \sum\nolimits_{j=0}^{\infty} c_j(x)\lambda^{\frac{n+\nu-j}{m}-s} + \sum_{k=0}^{\infty}(c_k'(x)\log\lambda + c_k''(x))\lambda^{-s-k}, \tag{34}$$

for $|\lambda|\to\infty$, uniformly in closed subsectors of Γ. Consequently, one has for the trace:

$$\operatorname{Tr} S(A-\lambda I)^{-s} \sim \sum\nolimits_{j=0}^{\infty} c_j\lambda^{\frac{n+\nu-j}{m}-s} + \sum_{k=0}^{\infty}(c_k'\log\lambda + c_k'')\lambda^{-s-k}. \tag{35}$$

The proof uses (15); details are given in [GH01].

3. Operators on manifolds with boundary

3.1. First-order differential operators on manifolds with boundary

Resolvents and other operator families can also be studied for operators on manifolds with boundary. In the case of elliptic differential operators with differential boundary conditions, there are the fundamental studies of Seeley [S69], [S69′] and Greiner [G71] that show how trace expansions of the resolvent or heat operator can be obtained with purely power terms (no logarithms), and accordingly just simple poles in the expansion corresponding to (33).

Again, the problem is more complicated when pseudodifferential elements enter. A prominent case that has been the focus of much attention is the Atiyah–Patodi–Singer problem, that we shall briefly recall:

Let X be a compact C^∞ n-dimensional manifold with boundary X' and let D be a first-order elliptic differential operator on X;

$$D : C^\infty(X,E_1) \to C^\infty(X,E_2), \tag{36}$$

where E_1 and E_2 are N-dimensional Hermitian vector bundles over X.

The restrictions of the E_i to the boundary X' are denoted E_i'. We assume that a normal coordinate x_n has been chosen in a neighborhood X_c of the boundary X' such that the points are represented as $x = (x', x_n)$ there with $x' \in X'$, $x_n \in [0,c[$, the E_i are isomorphic to the pullbacks of the E_i' there, and there is a normal derivative ∂_{x_n}. X' is provided with the volume element $v(x',0)dx'$ induced by $v(x',x_n)dx'dx_n$ on X_c, which we view as $X_c = X' \times [0,c[$.

D can be represented on X_c as

$$D = \sigma(\tfrac{\partial}{\partial x_n} + A_1), \tag{37}$$

where σ is a homeomorphism from $E_1|_{X_c}$ to $E_2|_{X_c}$ and A_1 is a first order differential operator that acts in the x' variable at $x_n = 0$. $A_1|_{x_n=0}$ has the principal symbol $a_1^0(x',\xi')$. With the notation

$$\gamma_j u = (-i\partial_{x_n})^j u|_{x_n=0}, \tag{38}$$

we have Green's formula for D, for the sections u of E_1 and w of E_2 in the Sobolev space H^1,

$$(Du,w)_X - (u,D^*w)_X = -(\sigma\gamma_0 u, \gamma_0 w)_{X'}. \tag{39}$$

Definition 3.1. 1° *We say that D is* **"of Dirac-type"** *when σ is a unitary morphism, and*

$$A_1 = A + x_n P_1 + P_0, \tag{40}$$

where A is an elliptic first-order differential operator in $C^\infty(E_1')$ which is selfadjoint with respect to the Hermitian metric in E_1', and the P_j are differential operators of order $\le j$.

2° *The* **product case** *is the case where D is of Dirac-type and, moreover, $v(x)dx = v(x',0)dx'dx_n$ on X_c, σ is constant in x_n, and $P_1 = P_0 = 0$.*

As explained in [G92], p. 2036, unitarity of σ in (37) can be obtained by a simple homotopy near X', whereas the assumption on A_1 in 1° is an essential restriction in comparison with arbitrary first-order elliptic systems; it means that the principal symbol $a_1^0(x',\xi')$ of A_1 at $x_n = 0$ is Hermitian symmetric. P_1 and P_0 can be taken arbitrary near X', but for larger x_n, P_1 is subject to the requirement that D be elliptic.

When 1° holds, $a_1^0(x',\xi')$ equals the principal symbol $a^0(x',\xi')$ of A. Along with A one considers the orthogonal projections $\Pi_\geq, \Pi_>, \Pi_\leq, \Pi_<$ and Π_λ onto the closed spaces $V_\geq, V_>, V_\leq, V_<$ and V_λ spanned by the eigenvalues of A in $L_2(E_1')$ that are $\geq 0, > 0, \leq 0, < 0$ resp. $= \lambda$. (Since A is selfadjoint and elliptic of order 1, it has a discrete spectrum consisting of eigenvalues of finite multiplicity going to $\pm\infty$.) These projections are classical ψdo's of order 0; Π_λ is of order $-\infty$.

Atiyah, Patodi and Singer considered in [APS75] the product case. This case is also studied e.g., in [GS96], [BW93], [BL99], [W99], whereas the case where only 1° holds is studied in [G92], [GS95] and other works. Cases where not even 1° holds are studied systematically in [G99].

The problem considered in [APS75] was the boundary value problem

$$Du = f \text{ on } X, \quad \Pi_\geq \gamma_0 u = 0 \text{ on } X', \tag{41}$$

with D of Dirac-type, in the product case. This boundary condition (and slight modifications of $\Pi_\geq$ by finite dimensional eigenprojections) is often called the *spectral boundary condition.* One can also replace $\Pi_\geq$ by the exact *Calderón projector* C^+ associated with D, as e.g. in [BW93], [W99]; it differs from $\Pi_\geq$ by a ψdo in X' of order $-\infty$ in the product case ([G99], Prop. 4.1).

As shown in [G99] (with reference to [S69″]), on may replace $\Pi_\geq$ by much more general classical pseudodifferential 0-order projections B in E_1', adapted to a general elliptic first-order D, such that the problem

$$Du = f \text{ on } X, \quad B\gamma_0 u = 0 \text{ on } X', \tag{42}$$

still defines a Fredholm operator D_B (the so-called well-posed boundary conditions). Typically, B can be of the form $B = \Pi_\geq + \Pi_\geq Q \Pi_<$ with a zero-order ψdo Q, and lower-order perturbations can be added.

In all these cases (most generally in [G99]) we have obtained trace expansions of the form

$$\operatorname{Tr}(\varphi e^{-tD_B{}^* D_B}) \sim \sum_{-n\leq k<0} a_k t^{k/2} + \sum_{k=0}^{\infty} (a_k \log t + a_k') t^{k/2}, \text{ for } t \to 0, \tag{43}$$

with the associated expansions of resolvents and of power operators; here φ is a morphism in E_1. (Note that the notation differs slightly from that in (32); there is only one infinite series.)

Let us give a quick explananation of the general strategy. One can write the resolvent in the form

$$(D_B^* D_B - \lambda I)^{-1} = Q_{\lambda,+} + K_\lambda S_\lambda T_\lambda,$$

where Q_λ is a ψdo representing $(D^*D - \lambda I)^{-1}$ on an open manifold $\widetilde{X} \supset X$, $Q_{\lambda,+}$ is its truncation to X, K_λ is a Poisson operator going from X' to X and T_λ is a trace operator going from X to X' (more about such operators in Section 3.2 below), and S_λ is a ψdo on X' containing the particular information connected with the boundary condition. The resolvent derivatives $\partial_\lambda^r (D_B^* D_B - \lambda I)^{-1}$ have a similar form (cf. (29)). Then (for r so large that the operator is trace-class) one finds by circular permutation

$$\begin{aligned}
&\operatorname{Tr}_X(\varphi \partial_\lambda^r (D_B^* D_B - \lambda I)^{-1}) \\
&\quad = \operatorname{Tr}_X(\varphi \partial_\lambda^r Q_{\lambda,+}) + \operatorname{Tr}_X(\varphi \partial_\lambda^r (K_\lambda S_\lambda T_\lambda)) \\
&\quad = \operatorname{Tr}_X(\varphi \partial_\lambda^r Q_{\lambda,+}) + \operatorname{Tr}_{X'} \partial_\lambda^r (T_\lambda \varphi K_\lambda S_\lambda),
\end{aligned} \tag{44}$$

where the difficult term is turned into the trace of a ψdo on the boundary. The trace of $\varphi \partial_\lambda^r Q_{\lambda,+}$ is well-known (has an expansion like in (31) without the log-terms), and the operator on X' is treated by use of our weakly parametric calculus for ψdo's, giving an expansion with powers and logs. Taken together, they lead to (43).

An important question in this connection is what the value of the coefficients are, and how they reflect the geometry of the problem. In [GS96], the coefficients in the product case with spectral boundary condition were described in terms of the zeta and eta functions of A. In the non-product case in a natural setting with spectral boundary condition, the first three terms in (43) for $n \geq 3$ were determined by Dowker, Gilkey and Kirsten in [DGK99] (moreover, the latter two authors have recently determined the next term in the case $n \geq 4$).

A basic information is of course whether some of the coefficients in (43) *vanish* under suitable circumstances. Here are some results to that effect:

It is known, first from [G92] for spectral boundary conditions in the case of Definition 3.1 1°, and then for increasingly general cases (including the condition with $B = C^+$ in the product case [G99], for odd n [W99], see moreover [G01]), that a_0 vanishes when $\varphi = I$. The coefficient a_0' "behind" a_0 is of great interest; it is global and contains the eta invariant of A.

For the product case with a spectral boundary condition, much more was shown in [GS96]: When n is odd, *all* the log-coefficients a_k with $k > 0$ vanish. When n is even, the log-coefficients with k even > 0 vanish (so there are only higher order log-terms of the form $c\,t^{l+\frac{1}{2}}\log t$, $l \in \mathbf{N}$).

The question of whether the remaining log-coefficients are generically nonzero was answered in the affirmative by Gilkey and Grubb in [GG98]. We have recently (in [G01″]) considered perturbations of the product case, in order to find out to what extent the logarithmic terms preserve their properties. The first half-integer log-term $a_1 t^{1/2}\log t$ in (43) has in the product case, when $\varphi(x', x_n) = \varphi^0(x')$ on X_c, the coefficient

$$a_1 = -\pi^{-1} c_1(\varphi^0, A^2) \tag{45}$$

according to [GS96], where c_1 is the coefficient of $t^{1/2}$ in the heat trace expansion for A^2 on X':

$$\operatorname{Tr}(\varphi^0 e^{-tA^2}) \sim \sum_{k=1-n}^{\infty} c_k(\varphi^0, A^2) t^{k/2} \tag{46}$$

($c_k(\varphi^0, A^2) = 0$ for $k - n + 1$ odd). When n is odd, $c_1(\varphi^0, A^2) = 0$, and when n is even, it is generically nonzero (cf. [GG98]). In [G01″] we show the following new result:

Theorem 3.2. *Let D^0 be as in Definition 3.1 2° and let D be a perturbation where A is replaced by $A + x_n P_1$ near X', P_1 a first-order tangential differential operator. Let $B^0 = \Pi_\geq$ and $B = \Pi_\geq + S$ with S a classical ψdo of order $\leq -n-1$. Let $\varphi(x', x_n) = \varphi^0(x')$ for $x_n \in [0, c[$. Then the coefficient a_1 in (43) is the same for $D^0_{B^0}$ and D_B. When n is odd, it is zero; when n is even, it is as in (45), (46), generically nonzero.*

For perturbations $A + x_n P_1 + P_0$ with $P_0 \neq 0$ there does not seem to be a similar stability.

Let us finally mention that when only the boundary condition is perturbed, there is the following phenomenon, formulated explicitly in [G01] on the basis of results from [G99]:

Theorem 3.3. *When B is replaced by $B' = B + S$ with S of order $-J \leq -1$, then the coefficients a_k in (43) are the same for D_B and $D_{B'}$ for $k \leq J - n$.*

There are similar results on trace expansions for $D_B e^{-tD_B{}^* D_B}$, corresponding to the eta function $\operatorname{Tr}(D_B(D_B{}^* D_B)^{-s})$.

3.2. Pseudodifferential boundary operators

To address the same questions for more general boundary value problems connected with pseudodifferential operators, we begin by recalling some elements of the Boutet de Monvel calculus [BM71] of boundary problems.

First let us describe the *model case*, concerned with operators on the half-axis $\mathbf{R}_+$. The variable is denoted x_n. We denote by $r^\pm$ the restriction operators from distributions on $\mathbf{R}$ to distributions on $\mathbf{R}_\pm$, and we denote by e^+ the "extension by zero" operator (applied to functions on $\mathbf{R}_+$):

$$e^+ f(x_n) = f(x_n) \text{ if } x_n > 0,$$

$$e^+ f(x_n) = 0 \text{ if } x_n < 0; \tag{47}$$

with a corresponding definition of e^-. Out of the Schwartz space $\mathcal{S}(\mathbf{R})$ of rapidly decreasing C^∞ functions we construct the spaces

$$\mathcal{S}_\pm = \mathcal{S}(\overline{\mathbf{R}}_\pm) = r^\pm \mathcal{S}(\mathbf{R}),$$

$$\dot{\mathcal{S}} = e^+\mathcal{S}_+ \dotplus e^-\mathcal{S}_- \dotplus \mathbf{C}[\delta']; \tag{48}$$

here $\mathbf{C}[\delta']$ indicates the "polynomials" in δ', i.e., linear combinations of derivatives of δ, $\sum_{0\leq j\leq k} \alpha_j \partial_j \delta$. The Fourier transformed spaces are denoted

$$\mathcal{H}^+ = \mathcal{F} e^+ \mathcal{S}_+,$$

$$\mathcal{H}^- = \mathcal{F}(e^-\mathcal{S}_- \dotplus \mathbf{C}[\delta']) = \mathcal{H}^-_{-1} \dotplus \mathbf{C}[\xi_n], \tag{49}$$

$$\mathcal{H} = \mathcal{F}\dot{\mathcal{S}} = \mathcal{H}^+ \dotplus \mathcal{H}^-.$$

A constant-coefficient ψdo $P = \mathrm{OP}_n(p(\xi_n))$ of integer order m on $\mathbf{R}$ is said to satisfy *the transmission condition* when $p(\xi_n) \in \mathcal{H}$. Another description is that p has an asymptotic series development $p(\xi_n) \sim \sum_{m \geq j > -\infty} s_{m-j}\xi_n^j$, in the sense that

$$\partial_{\xi_n}^k \Big[\xi_n^l p(\xi_n) - \sum_{m \geq j > m-N} s_{m-j}\xi_n^{j+l}\Big]$$
$$\text{is } O(\langle \xi_n \rangle^{m+l-k-N})$$

for all indices $k, l, N \in \mathbf{N}$. A third description is that $\tau^m p(\tau^{-1})$ extends from $\tau > 0$ and $\tau < 0$ to be C^∞ at $\tau = 0$. Cf. e.g. [G96], Sect. 2.2.

When p is of this kind, the *truncated ψdo P_+*, defined on functions on $\mathbf{R}_+$ by

$$P_+ u = r^+ P e^+ u, \tag{50}$$

maps $\mathcal{S}_+$ into $\mathcal{S}_+$ (no singularities arise at $x_n = 0$).

Besides the standard trace operators (38), we introduce *trace operators of class* 0; for functions u on $\mathbf{R}_+$ they are mappings

$$T = \mathrm{OPT}_n(\tilde t) : u(x_n) \mapsto \int_0^\infty \tilde t(x_n) u(x_n)\, dx_n \tag{51}$$

with $\tilde t(x_n) \in \mathcal{S}_+$; more generally, a trace operator *of class* $r \geq 0$ is an operator of the form

$$T_1 u = \sum_{0 \leq j \leq r-1} s_j \gamma_j u + Tu \tag{52}$$

with T as in (51). Such operators map functions on $\mathbf{R}_+$ into numbers. There is another (dual) type of mappings going the other way; the *Poisson operators* (in some texts called potential operators)

$$K = \mathrm{OPK}_n(\tilde k) : v \mapsto \tilde k(x_n) v \tag{53}$$

with $\tilde k(x_n) \in \mathcal{S}_+$. Finally, there is an operator type from functions on $\mathbf{R}_+$ to functions on $\mathbf{R}_+$, called a *singular Green operator* (s.g.o.). Denote

$$\mathcal{S}_{++} = \mathcal{S}(\overline{\mathbf{R}}^2_{++}) = r^+_{x_n} r^+_{y_n} \mathcal{S}(\mathbf{R}^2), \tag{54}$$

$\mathbf{R}^2_{++} = \mathbf{R}_+ \times \mathbf{R}_+$. A singular Green operator *of class* 0 is an integral operator with kernel $\tilde g(x_n, y_n) \in \mathcal{S}_{++}$,

$$Gu = \mathrm{OPG}_n(\tilde g) u = \int_0^\infty \tilde g(x_n, y_n) u(y_n)\, dy_n. \tag{55}$$

A general s.g.o. G_1 of class $r \geq 0$ is of the form

$$G_1 u = \sum_{0 \leq j \leq r-1} K_j \gamma_j u + Gu, \tag{56}$$

where the K_j are Poisson operators and G is a singular Green operator of class 0 as defined in (55).

These are the basic ingredients in the calculus. In more generality, one allows x_n-dependence in the symbol p of P, and one defines operators relative to $\overline{\mathbf{R}}^n_+$ by letting the operators act in the tangential variables $x' = (x_1, \dots, x_{n-1})$ as pseudodifferential operators and in the normal variable x_n as described above. Then the functions p, $\tilde t$, $\tilde k$, $\tilde g$ are taken to depend moreover on $(x', \xi') \in \mathbf{R}^{2(n-1)}$. For our purposes, we furthermore let the functions depend on a parameter μ. Denoting the ψdo definition with respect to x' by OP' (cf. (5)), we then get operators defined relative to $\overline{\mathbf{R}}^n_+$:

Truncated ψdo's:

$$\mathrm{OP}\,(p(x,\xi,\mu))_+ = \mathrm{OP}'\mathrm{OP}_n(p(x,\xi,\mu))_+,$$

trace operators of class 0:

$$\mathrm{OPT}(\tilde t(x', x_n, \xi', \mu)) = \mathrm{OP}'\mathrm{OPT}_n(\tilde t(x', x_n, \xi', \mu)),$$

Poisson operators:

$$\mathrm{OPK}(\tilde k(x', x_n, \xi', \mu)) = \mathrm{OP}'\mathrm{OPK}_n(\tilde k(x', x_n, \xi', \mu)),$$

s.g.o.s of class 0:

$$\mathrm{OPG}(\tilde g(x', x_n, y_n, \xi', \mu)) = \mathrm{OP}'\mathrm{OPG}_n(\tilde g(x', x_n, y_n, \xi', \mu)). \tag{57}$$

The definition of trace operators and s.g.o.s is extended to class > 0 as in (52), (56).

Example 3.4. As a simple and important example defining a Poisson operator, we mention $\tilde k(x_n, \xi', \mu) = e^{-|(\xi',\mu)| x_n}$ with $\mu \in \mathbf{R}_+$; here $\mathrm{OPK}(e^{-|(\xi',\mu)|x_n})$ is the solution operator for the Dirichlet problem

$$(\mu^2 - \Delta) u(x) = 0 \text{ on } \mathbf{R}^n_+, \quad u(x', 0) = v(x'). \tag{58}$$

A crucial question for a parameter-dependent calculus is now *how to prescribe the dependence* on all symbolic variables x', ξ', x_n, y_n, μ taken together, in order to get operators with the properties we want. We shall here only deal with 1-polyhomogeneity (the case $\sigma = 1$ in Definition 2.2), where μ enters in homogeneity considerations with the same power as ξ'.

The calculus may be explained in another way (the "complex formulation") where the Fourier transform in x_n plays a great role. This involves some complicated spaces of holomorphic funtions of ξ_n, defined in terms of $\mathcal{H}$ and $\mathcal{H}^{\pm}$ (49), that we want to keep out of the presentation here. On the other hand, in the complex explanation, ξ_n enters in the homogeneities of *symbols* in the same way as the other variables ξ', μ, whereas in the present "real formulation", x_n and y_n enter in the functions $\tilde{t}$, $\tilde{k}$, $\tilde{g}$ (called the *symbol-kernels*) in a quasi-homogeneous way oppposite to that of (ξ', μ). (For example, the *symbol* $k = \mathcal{F}_{x_n \to \xi_n} e^+ \tilde{k}$ of the operator described in Example 3.4 is $k(\xi, \mu) = (|(\xi', \mu)| + i\xi_n)^{-1}$, which is homogeneous of degree -1, whereas the *symbol-kernel* $\tilde{k}(x_n, \xi', \mu) = e^{-|(\xi', \mu)| x_n}$ has a certain quasi-homogeneity.)

We shall now let the symbol spaces defined in Definition 2.2 be based on $x' \in \mathbf{R}^{n-1}$ and take values in a Banach space B, consisting e.g. of functions of $x_n \in \mathbf{R}_+$. We let $\sigma = 1$ and leave it out of the notation, we denote the parameter by μ, and we replace δ by an integer d. Now we need to take powers of $|(\xi', \mu)|$ into the definition. To smooth out the behavior near $(\xi', \mu) = 0$, it is convenient to replace $|(\xi', \mu)|$ by $[(\xi', \mu)]$; here $[x]$ denotes a C^∞ function of $x \in \mathbf{R}^N$ satisfying:

$$[x] = |x| \text{ for } |x| \geq 1,\ [x] \in [\tfrac{1}{2}, 1] \text{ for } |x| \leq 1. \quad (59)$$

Note that $[(\xi', \frac{1}{z})] = |(\xi', \frac{1}{z})|$ for $|z| \leq 1$, and that

$$|(\xi', \tfrac{1}{z})| = |(\xi', \mu)| = |\mu|\, |(\xi'/\mu, 1)|$$
$$= |\mu|\, \langle \xi'/\mu \rangle = |z|^{-1} \langle z\xi' \rangle, \text{ when } \mu = \tfrac{1}{z}. \quad (60)$$

$[(\xi', \mu)]$ is also written $[\xi', \mu]$; it will in the following be denoted κ (as in [G96]), so from now on,

$$\kappa = [\xi', \tfrac{1}{z}] = [\xi', \mu], \text{ with } \mu = \tfrac{1}{z}. \quad (61)$$

Definition 3.5. *Let $m \in \mathbf{R}$, d and $s \in \mathbf{Z}$. Then $S^{m,0,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, B)$ consists of the C^∞ functions $p(x', \xi', \mu)$ valued in B which satisfy, with $\frac{1}{\mu} = z$,*

$$\partial^j_{|z|} p(\cdot, \cdot, \tfrac{1}{z}) \in S^{m+j}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, B) \text{ for } \tfrac{1}{z} \in \Gamma,$$
$$\text{with uniform estimates for } |z| \leq 1,$$
$$\tfrac{1}{z} \text{ in closed subsectors of } \Gamma. \quad (62)$$

Moreover, we define

$$S^{m,d,s}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, B)$$
$$= \mu^d [(\xi', \mu)]^s S^{m,0,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, B). \quad (63)$$

The indication $(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, B)$ will often be abbreviated to (Γ, B). Keeping the identification of μ with $\frac{1}{z}$ in mind, we shall also say that $p(x', \xi', \frac{1}{z})$ lies in $S^{m,d,s}(\Gamma, B)$.

We have left the requirement of being holomorphic in $\mu \in \Gamma^\circ$ out of the definition since $\kappa = [\xi', \mu]$ is not so (one could instead work with a variant of κ that is holomorphic on suitable sectors, as in [G96], (A.2″)–(A.2‴)). The symbol is assumed C^∞ in $\mu \in \Gamma$ considered as a subset of $\mathbf{R}^2$. Moreover, we write $\partial_{|z|}$ (instead of ∂_z), since it is a control of the radial derivative that is needed (uniformly when the argument of z runs in a compact interval), and we shall in the following let $|z|$ enter as a real parameter.

To define symbol-kernels for the boundary operators, we use the cases $B = L_\infty(\mathbf{R}_+)$ and $B = L_\infty(\overline{\mathbf{R}}^2_{++})$, with variables x_n or u_n, resp. (x_n, y_n) or (u_n, v_n); these variables will then be mentioned in the detailed description of the function.

Definition 3.6. *Let $m \in \mathbf{R}$, d and $s \in \mathbf{Z}$.*

1° *The space $\mathcal{S}^{m,d,s}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, \mathcal{S}_+)$ (briefly denoted $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_+)$) consists of the functions $\tilde{f}(x', x_n, \xi', \mu)$ in $C^\infty(\mathbf{R}^{n-1} \times \overline{\mathbf{R}}_+ \times \mathbf{R}^{n-1} \times \Gamma)$ satisfying, for all $l, l' \in \mathbf{N}$,*

$$\langle z\xi' \rangle^{l-l'} u_n^l \partial^{l'}_{u_n} \tilde{f}(x', |z| u_n, \xi', \tfrac{1}{z})$$
$$\in S^{m,d,s+1}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, L_{\infty, u_n}(\mathbf{R}_+)) \quad (64)$$

(equivalently, $u_n^l \partial_{u_n}^{l'} \tilde f(x', |z|u_n, \xi', \frac{1}{z})$ *belongs to* $\in S^{m,d+l-l',s+1-l+l'}(\Gamma, L_{\infty,u_n}(\mathbf{R}_+)))$.

2° *The space* $\mathcal{S}^{m,d,s}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, \mathcal{S}_{++})$ *(briefly denoted* $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_{++})$*) consists of the functions* $\tilde f(x', x_n, y_n, \xi', \mu)$ *in* $C^\infty(\mathbf{R}^{n-1} \times \overline{\mathbf{R}}^2_{++} \times \mathbf{R}^{n-1} \times \Gamma)$ *satisfying, for all* $l, l', k, k' \in \mathbf{N}$,

$$\langle z\xi'\rangle^{l-l'+k-k'} u_n^l \partial_{u_n}^{l'} v_n^k \partial_{v_n}^{k'} \tilde f(x', |z|u_n, |z|v_n, \xi', \tfrac{1}{z})$$

$$\in S^{m,d,s+2}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}, \Gamma, L_{\infty,u_n,v_n}(\mathbf{R}^2_{++})). \quad (65)$$

In detail, the statement in (64) means that for all j,

$$\|\partial_{|z|}^j (z^d \kappa^{-s-1} \langle z\xi'\rangle^{l-l'} u_n^l \partial_{u_n}^{l'} \tilde f(x', |z|u_n, \xi', \tfrac{1}{z}))\|_{L_\infty}$$

$$\dot\leq \langle \xi'\rangle^{m+j}, \quad (66)$$

with similar estimates for derivatives $\partial_{\xi'}^\alpha \partial_{x'}^\beta$ with m replaced by $m - |\alpha|$. There is a related explanation of (65). We here use $\dot\leq$ to indicate "$\leq$ a constant times"; also $\dot\geq$ will be used, and $\doteq$ is used when both $\dot\leq$ and $\dot\geq$ hold.

The third index s is included to keep track of factors κ in a manageable way. When $s = 0$, we may lave it out of the notation. For the trace formulas later on, it is important to know that we always have inclusions (in view of (14)):

$$S^{m,d,s} \subset S^{m+s,d,0} \cap S^{m,d+s,0} \text{ if } s \leq 0,$$

$$S^{m,d,s} \subset S^{m+s,d,0} + S^{m,d+s,0} \text{ if } s \geq 0, \quad (67)$$

for S- as well as for $\mathcal{S}$-spaces.

In applications, the symbols will often be assumed to be holomorphic in μ for $\mu \in \Gamma^\circ$ with $|(\xi', \mu)| \geq \varepsilon$ (some $\varepsilon > 0$); such symbols are called holomorphic in μ, and this property is preserved in compositions.

Definition 3.7.

1° *The functions in* $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_+)$ *are the* **Poisson symbol-kernels** *and* **trace symbol-kernels of class 0**, *of degree* $m + d + s$, *in the parametrized calculus.*

2° *The functions in* $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_{++})$ *are the* **singular Green symbol-kernels of class** 0 *and degree* $m + d + s$ *in the parametrized calculus.*

One also defines subspaces of these symbol-kernel spaces, consisting of the functions that are series of terms with decreasing m-index and quasi-homogeneity in (x_n, ξ', μ) or (x_n, y_n, ξ', μ) (the Fourier transformed terms will be truly homogeneous in (ξ_n, ξ', μ) resp. $(\xi_n, \eta_n, \xi', \mu)$ for $|\xi'| \geq c > 0$). These symbol-kernels are called (weakly) polyhomogeneous.

Remark 3.8. By Fourier transformation in x_n of the functions extended by 0 for $x_n < 0$, $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_+)$ is carried over into the space $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{H}^+)$ of **Poisson symbols** of degree $m + s + d$, and co-Fourier transformation (formula (4) with minus replaced by plus) gives the space $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{H}^-_{-1})$ of **trace symbols of class** 0 and degree $m + s + d$.

By Fourier transformation in x_n and co-Fourier transformation in y_n of the functions in $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_{++})$ extended by 0 for $x_n < 0, y_n < 0$, we get the space $\mathcal{S}^{m,d,s}(\Gamma, \mathcal{H}^+ \hat\otimes \mathcal{H}^-_{-1})$ of **singular Green symbols of class** 0 and degree $m+d+s$.

The explanation for the $+1$ resp. $+2$ in the s-index in (64) resp. (65) is that with this choice, $m + d + s$ is consistent with the top degree of homogeneity in the Fourier transformed situation for polyhomogeneous symbols, where the scalings in x_n and y_n lead to shifts in the indices. Further details in [G01′].

Remark 3.9. To motivate the scaling in x_n in the above definition of parameter-dependent boundary symbols, consider the symbol-kernel in the basic Example 3.4, $\tilde k(x_n, \xi', \mu) = e^{-|(\xi',\mu)|x_n}$. Setting $\mu = \frac{1}{z}$, we find for $\tilde k(x_n, \xi', \frac{1}{z}) = e^{-\frac{1}{z}\langle z\xi'\rangle x_n}$ (cf. (60)), by use of the formula (with $c_j = j^j e^{-j}$)

$$\sup_{x_n \geq 0} x_n^j e^{-ax_n} = c_j a^{-j} \text{ for } a > 0, \quad (68)$$

that $\sup_{x_n \geq 0} |\partial_z^j \tilde k(x_n, \xi', \frac{1}{z})|$ behaves like z^{-j} for each ξ'. This does not comply with the uniform estimates in $z \leq 1$ required in (62) (with $B = L_\infty(\mathbf{R}_+)$). But if we replace x_n by zu_n, we find using (68) that the resulting function $\tilde k_1(u_n, \xi', z) = \tilde k(zu_n, \xi', \frac{1}{z}) = e^{-\langle z\xi'\rangle u_n}$ has $\sup_{u_n \geq 0} |\partial_z^j e^{-u_n\langle z\xi'\rangle}| \mathrel{\dot\leq} \langle \xi'\rangle^j$, $j \in \mathbf{N}$, which fits well with (62). (It is easy to check a few steps by hand calculation; a systematic proof is given in [G01′], Th. 3.2.)

On the other hand, because of the scaling, the example $e^{-[\xi']x_n}$ (constant in μ) *does not* fit into the calculus. This is an unpleasant difficulty in the applications: μ-independent ψdo's on the boundary $\mathbf{R}^{n-1}$ are included, but μ-independent Poisson, trace and singular Green operators are in general not so; Theorem 2.4 does not extend to these new symbol families.

We also remark that the generalization of Theorem 2.5 to the new s.g.o.s is not really convenient.

It is easy to show that composition of the operators in the list (57) leads to other operators of these types. The basic step is the composition with respect to the variable x_n, denoted $\circ_n$. Here we have for example:

Proposition 3.10. *Let*

$$\tilde g(x',x_n,y_n,\xi',\mu)\in\mathcal S^{m,d,s}(\Gamma,\mathcal S_{++}),\tag{69}$$

$$\tilde t(x',x_n,\xi',\mu),\tilde k(x',x_n,\xi',\mu)\in\mathcal S^{m,d,s}(\Gamma,\mathcal S_{+}),$$

and let $\tilde g'$, $\tilde t'$ and $\tilde k'$ be given similarly with m, d, s replaced by m', d', and s'. Define

$$m''=m+m',\ d''=d+d',\ s''=s+s'.\tag{70}$$

Then (suppressing the variables (x',ξ') in the formulas)

1° $$\tilde k\circ_n\tilde t'=\tilde k(x_n,\mu)\tilde t'(y_n,\mu)\in\mathcal S^{m'',d'',s''}(\Gamma,\mathcal S_{++}),$$

2° $$\tilde t\circ_n\tilde k'=\int_0^\infty\tilde t(x_n,\mu)\tilde k'(x_n,\mu)\,dx_n\in\mathcal S^{m'',d'',s''+1}(\Gamma,\mathbf C),$$

3° $$\tilde t\circ_n\tilde g'=\int_0^\infty\tilde t(x_n,\mu)\tilde g'(x_n,y_n,\mu)\,dx_n\in\mathcal S^{m'',d'',s''+1}(\Gamma,\mathcal S_{+}),$$

4° $$\tilde g\circ_n\tilde k'=\int_0^\infty\tilde g(x_n,y_n,\mu)\tilde k'(y_n,\mu)\,dy_n\in\mathcal S^{m'',d'',s''+1}(\Gamma,\mathcal S_{+}),$$

5° $$\tilde g\circ_n\tilde g'=\int_0^\infty\tilde g(x_n,z_n,\mu)\tilde g'(z_n,y_n,\mu)\,dz_n\in\mathcal S^{m'',d'',s''+1}(\Gamma,\mathcal S_{++}).$$

Here 1° is a simple product result. For 2°, we use that, according to (66),

$$|z^d\kappa^{-s-1}\tilde t|\mathrel{\dot\le}\langle\xi'\rangle^m,$$

$$|z^{d'}\kappa^{-s'-1}(1+\langle z\xi'\rangle^2u_n^2)\tilde k'|\mathrel{\dot\le}\langle\xi'\rangle^{m'},$$

with $x_n=|z|u_n$. Then we have on each ray $\mu=re^{i\theta}$, for $|\mu|\ge1$:

$$\Big|z^{d+d'}\kappa^{-s-s'-1}\int_0^\infty\tilde t(x_n)\tilde k'(x_n)\,dx_n\Big|$$

$$\mathrel{\dot\le}\langle\xi'\rangle^m\Big|\int_0^\infty z^{d'}\kappa^{-s'}\tilde k'(|z|u_n)\,z\,du_n\Big|$$

$$\mathrel{\dot\le}\langle\xi'\rangle^{m+m'}|z\kappa|\int_0^\infty(1+\langle z\xi'\rangle^2u_n^2)^{-1}du_n$$

$$\mathrel{\dot\le}\langle\xi'\rangle^{m+m'},$$

since $|z\kappa|=\langle z\xi'\rangle$ (cf. (60), (61)). A similar pattern is found for all the derivatives in x', ξ' and z (using the Leibniz rule inside the integral); this shows 2°. The other results are obtained in essentially the same way.

Full composition rules in all variables are as in (20) for the tangential variables and as in Proposition 3.10 for the normal variable.

There is a special mapping that is important for trace calculations, namely the application of the *normal trace* tr_n

$$(\mathrm{tr}_n\tilde g)(x',\xi',\mu)=\int_0^\infty\tilde g(x',x_n,x_n,\xi',\mu)\,dx_n\tag{71}$$

to a singular Green symbol-kernel $\tilde g$ of class 0; this gives the trace of the operator in the x_n-variable defined by $\tilde g$. It is a ψdo symbol belonging to our calculus:

Theorem 3.11. *Let $\tilde g$ be a singular Green symbol-kernel on $\overline{\mathbf R}^n_+$:*

$$\tilde g(x',x_n,y_n,\xi',\mu)\in\mathcal S^{m,d,s-1}(\Gamma,\mathcal S_{++}).\tag{72}$$

Then the normal trace of $\tilde g$ is a ψdo symbol on $\mathbf{R}^{n-1}$ satisfying

$$\operatorname{tr}_n \tilde g(x',\xi',\mu) \in S^{m,d,s}(\Gamma,\mathbf{C}). \tag{73}$$

We refer to [G01′] for the detailed proof. When $G = \mathrm{OPG}(\tilde g)$, the ψdo with symbol $\operatorname{tr}_n \tilde g$ will be denoted $\operatorname{tr}_n G$.

There are further composition rules, involving also a parameter-dependent pseudodifferential operator on $\mathbf{R}^n$, truncated to $\mathbf{R}^n_+$ and satisfying a suitable transmission condition at $x_n = 0$ taking the parameter into account. The basic property needed here is that when Q is a μ-dependent ψdo on $\mathbf{R}^n$, the trace operator T and the Poisson operator K defined by

$$Tu = \gamma_0 Q_+ u, \quad Kv = r^+ Q(v(x') \otimes \delta(x_n)), \tag{74}$$

should have symbols of the type introduced in Definition 3.6. This restricts considerably the generality of the ψdo's that are allowed. Resolvents of differential operators, and strongly polyhomogeneous ψdo's (with symbols stemming from ψdo's in one more cotangent variable which is replaced by μ) enter, but ψdo's that are constant in μ will in general not be included, since they produce μ-independent Poisson and trace operators by (74), cf. Remark 3.9.

The precise definition of the appropriate transmission condition, and the proof of the full composition rules, make heavy use of the complex formulation of the theory and will be omitted here. It is found that when $\mathcal{A}$ and $\mathcal{A}'$ are systems of operators belonging to the calculus, so is their composition $\mathcal{A}''$,

$$\mathcal{A}\mathcal{A}' = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} \begin{pmatrix} P'_+ + G' & K' \\ T' & S' \end{pmatrix} = \begin{pmatrix} P''_+ + G'' & K'' \\ T'' & S'' \end{pmatrix} = \mathcal{A}'', \tag{75}$$

with resulting symbol classes where the indices are added and modified like in Proposition 3.10.

The operators behave in a standard way under coordinate transformations, so they can be defined to act on the sections of smooth vector bundles over smooth compact manifolds X with boundary X'. (Suitable noncompact manifolds may be included too, cf. e.g. [G96].)

Now let us consider trace expansions. When $\mathcal{A}$ is a μ-dependent system as in (75), of order $m < -n$ and class 0, going from $C^\infty(X,E)\times C^\infty(X',F)$ to itself (where E and F are vector bundles over X resp. X'), then

$$\operatorname{Tr}\mathcal{A} = \operatorname{Tr}_X(P_+ + G) + \operatorname{Tr}_{X'} S.$$

For $\operatorname{Tr}_X P_+$ and $\operatorname{Tr}_{X'} S$ we have results as in Theorem 2.9 and Corollary 2.10, based on the theory for boundaryless manifolds. But $\operatorname{Tr}_X G$ demands new results. It is here that Theorem 3.11 is extremely useful, because it allows us to write, in each local coordinate patch at the boundary,

$$\operatorname{Tr}_X G = \operatorname{Tr}_{X'}(\operatorname{tr}_n G).$$

Here $\operatorname{tr}_n G$ is a ψdo on the boundary, which is itself a boundaryless manifold, and the symbol of $\operatorname{tr}_n G$ belongs to the weakly parametric calculus of [GS95], so Theorem 2.9 and Corollary 2.10 can be applied in dimension $n-1$ to give a trace expansion for this term. One finds for example:

Theorem 3.12. *Let G be a singular Green operator of class 0 on X, with polyhomogeneous symbol-kernel lying in $\mathcal{S}^{m,d,-1}(\mathbf{R}^{n-1}\times\mathbf{R}^{n-1},\Gamma,\mathcal{S}_{++})$ in local coordinates (whereby $\operatorname{tr}_n G$ has symbol in $S^{m,d,0}(\mathbf{R}^{n-1},\Gamma,\mathbf{C})$ in local coordinates on X'), holomorphic in μ. Assume moreover that the homogeneous terms in the symbol of $\operatorname{tr}_n G$ lying in $S^{m-j,d,0}$ with $m-j \ge -n$ are integrable in ξ'. Then G is trace-class, and the trace of G has an asymptotic expansion*

$$\operatorname{Tr} G \sim \sum_{j=0}^{\infty} c_j \mu^{m+d+n-1-j} + \sum_{k=0}^{\infty} (c'_k \log\mu + c''_k)\mu^{d-k} \tag{76}$$

for $\mu\to\infty$ in closed subsectors of Γ. The coefficients c_j and c'_k are determined from the homogeneous terms in the symbol.

The use of tr_n here replaces the circular permutation used in a special case in (44).

This calculus covers all cases connected with Dirac operators, and also much more general cases, where the singular Green operators are not finite sums of products of particular Poisson and trace operators. The calculus is used in [G01″].

It should be noted that the "interior" ψdo's P allowed in this calculus are not nearly as general as for example those considered in [G96] (since most μ-independent operators do not fit in). This does not exclude that an expansion like (76) can hold for other operators, since what we really need in order to show this is that we get a ψdo in the weakly parametric calculus of [GS95] or [GH01] *after performing the reduction to* X' (the application of tr_n).

In the last section we shall consider a case with somewhat different ingredients: There is a μ-independent interior ψdo involved, but the parameter-dependence occurs in a factor which is a very nice operator having a rational symbol with full control over the poles: the resolvent of a Dirichlet Laplacian. In this case it is possible to make a full analysis of trace formulas by residue calculus and use of special functions.

3.3. The non-commutative residue

Wodzicki introduced in [W84] a trace functional (a linear functional vanishing on commutators) for the algebra of classical ψdo's S on a compact manifold, called the *non-commutative residue* $\mathrm{res}(S)$, and identified it with the residue at zero in the trace expansion of SA^{-s}:

$$\mathrm{res}(S) = \mathrm{ord}\, A \cdot \mathrm{Res}_{s=0}\mathrm{Tr}(SA^{-s}), \tag{77}$$

here A is an auxiliary elliptic operator. In the corresponding formulation for the iterated resolvent $S(A-\lambda I)^{-N}$, $\mathrm{res}(S)$ is the first log-coefficient c_0' in the trace expansion (31) (up to a universal factor). Wodzicki also describes $\mathrm{res}(S)$ by an integral formula in terms of the symbol of S. See also Guillemin [Gu85].

For operators $P_+ + G$ in the Boutet de Monvel calculus, a similar trace functional was introduced by Fedosov, Golse, Leichtnam and Schrohe [FGLS96], defined by an integral formula in terms of the symbols of P and G. However, this was not shown to be a residue as in (77), nor a log-coefficient in a resolvent trace, simply because such trace formulas were not sufficiently developed for the Boutet de Monvel calculus.

In a joint work with Schrohe [GS01] we have attacked this problem. It is interesting that in this case it is a log-term one is after, whereas the log-terms may seem to be something of a nuisance in the previous cases.

We consider the product

$$(P_+ + G)(P_{1,D} - \lambda I)^{-N},$$

where $P_{1,D}$ stands for the Dirichlet realization of a strongly elliptic second-order differential operator P_1 with scalar principal symbol near X'. The calculus of [G01′] is not helpful, since it does not allow general interior parameter-independent factors. We use instead that $Q_\lambda = (P_1 - \lambda I)^{-1}$ has a symbol at X' consisting of rational functions of ξ_n with a pole $i\kappa^+(x',\xi',\mu)$ with positive imaginary part and a pole $-i\kappa^-(x',\xi',\mu)$ with negative imaginary part, where κ^+ and κ^- behave nicely as functions of (ξ',μ), $\mu = (-\lambda)^{\frac{1}{2}}$.

This is coupled with Laguerre expansions of the symbols of P and G, which break the composed symbols up in small pieces that can be treated by residue calculus and summed up afterwards. The Laguerre expansions used here are expansions in the following orthogonal basis of $\mathcal{H}^+$ (cf. (49)):

$$\hat\varphi_k'(\xi_n,\sigma) = \frac{(\sigma - i\xi_n)^k}{(\sigma + i\xi_n)^{k+1}}, \quad k \in \mathbf{N}, \quad \sigma = \langle \xi' \rangle,$$

and the corresponding orthogonal basis of $e^+\mathcal{S}_+$:

$$\varphi_k'(x_n,\sigma) = H(x_n)(\sigma - \partial_{x_n})^k (x_n^k e^{-\sigma x_n})/k!,$$

where $H(x_n) = 1_{\mathbf{R}_+}$.

(One might think that the composition of a μ-independent operator with a μ-dependent operator could in general be handled by expansion of the former symbol in Laguerre functions $\hat\varphi_k'(\xi_n,\langle\xi'\rangle)$ and the latter symbol in Laguerre functions $\hat\varphi_k'(\xi_n,\langle(\xi',\mu)\rangle)$, but this gives severe problems with the *convergence* of the resulting series.)

We show in [GS01] that the normal trace of each piece belongs to the calculus of [GS95], hence defines an operator with a trace expansion as in Corollary 2.10, and *moreover*, that these trace expansions sum up to a similar trace expansion of the full operator. Keeping a special check of the first log-coefficient c_0', we show that it does indeed satisfy the integral formula of [FGLS96] for $\operatorname{res}(P_+ + G)$.

4. Concluding remarks

After having recalled the definition and basic properties of pseudodifferential operators (ψdo's) on $\mathbf{R}^n$, we explained the special problems connected with parameter-dependent ψdo's such as the resolvent, and we introduced a symbol calculus for λ-dependent operators that allows showing complete trace expansions (for operators on compact manifolds) in powers λ^α and logarithmic terms $\lambda^\alpha \log \lambda$. Heat trace expansions are likewise obtained, as well as the pole structure of zeta functions. An improved calculus allows treatment of complex powers of resolvents too.

We have moreover explained a similar theory for operators on manifolds with boundary, in particular Dirac operators with spectral boundary conditions. New results on the vanishing or the stability of the logarithmic terms were presented, and we ended up by showing how the noncommutative residue of a pseudodifferential boundary operator enters as a log-coefficient.

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