LOGARITHMIC TERMS IN ASYMPTOTIC EXPANSIONS OF HEAT OPERATOR TRACES

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Abstract. Let P be an elliptic selfadjoint positive classical pseudodifferential operator of order d on a compact m-dimensional manifold without boundary. The heat trace of P has an asymptotic expansion in $t^{(l-m)/d}$ and $t^k \log t$ for $l = 0, 1, 2, \ldots$ and $k = 1, 2, \ldots$. We show that the coefficients of all terms in this expansion are nontrivial for a dense set of P. We show that the coefficient of the $t^{(l-m)/d}$ term is not locally computable when $(l-m)/d$ is a positive integer; the remaining coefficients are known to be locally computable. — Let P_B be an operator of Dirac type on a compact n-dimensional manifold with smooth boundary such that the structures are product near the boundary; here a spectral boundary condition is imposed. Let $\Delta_1 = P_B{}^* P_B$ and $\Delta_2 = P_B P_B{}^*$. If n is even, the heat trace of Δ_i has an asymptotic expansion in $t^{(l-n)/d}$ and $t^{k+1/2} \log t$ for $l = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$; if n is odd, there is an expansion without the $t^{k+1/2} \log t$ terms. We show that all coefficients (all but one if n is odd) are nontrivial for a dense set of operators.

1. Pseudodifferential operators on manifolds without boundary.

Let M be a compact boundaryless m-dimensional C^∞ manifold provided with a smooth volume element, let E be a smooth Hermitian vector bundle over M, let d be a positive integer, and let P be a classical pseudodifferential operator (ψdo) in E of order d which is elliptic, selfadjoint and positive (> 0); such a P will be said to be *admissible*. We refer to Seeley [15, 17], Greiner [10], Duistermaat and Guillemin [7], Grubb [12], Agranovič [1], and Grubb and Seeley [13] for proofs of the following analytic results.

Let e^{-tP} be the solution operator $e^{-tP} : f \mapsto u$ for the heat equation $\partial_t u + Pu = 0$ with initial value $u|_{t=0} = f$. This operator is trace class for each $t > 0$, and as $t \downarrow 0$ there is an asymptotic expansion of the form:

$$h(P,t) := \operatorname{Tr} e^{-tP} \sim \textstyle\sum_{l=0}^\infty a_l(P) t^{(l-m)/d} + \sum_{k=1}^\infty b_k(P) t^k \log t. \tag{1.1}$$

For $\operatorname{Re}(s) >> 0$, let $\zeta(P,s) := \operatorname{Tr} P^{-s}$; this has a meromorphic extension to $\mathbb{C}$ with isolated simple poles. The Mellin transform yields the relationship

$$\Gamma(s)\zeta(P,s) = \textstyle\int_0^\infty t^{s-1} h(P,t) dt. \tag{1.2}$$

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Since $h(P,t)$ decays exponentially as $t \to \infty$, one can use equations (1.1) and (1.2) to see that $\Gamma(s)\zeta(P,s)$ has a meromorphic extension to $\mathbb{C}$ with poles at the points $(m-l)/d$, $l = 0,1,2,\ldots$. Let $\mathbb{N} := \{1,2,...\}$. The poles at the points $s = (m-l)/d \notin -\mathbb{N}$ are (at most) simple, and the poles at the points $s \in -\mathbb{N}$ are (at most) double. (The concept of poles is used in a general sense where residues and other Laurent coefficients can be zero.) There is the following straightforward relationship between the heat trace coefficients and the coefficients of the Laurent expansions at these points:

$$\begin{aligned} a_l(P) &= \mathrm{Res}_{s=(m-l)/d}\Gamma(s)\zeta(P,s), \text{ and} \\ b_k(P) &= -\mathrm{Res}_{s=-k}(s+k)\Gamma(s)\zeta(P,s). \end{aligned} \tag{1.3}$$

The asymptotic expansion of $h(P,t)$ determines the pole structure of $\Gamma(s)\zeta(P,s)$ and conversely, the pole structure of $\Gamma(s)\zeta(P,s)$ determines the asymptotic expansion of $h(P,t)$.

If P is a differential operator, then $b_k(P) = 0$ for all k, and $a_l(P) = 0$ when l is odd (in this case the order d of P is necessarily even). There is a similar expansion, given in equation (2.1) below, when the differential operator P is considered on a compact manifold with boundary and is provided with a local elliptic boundary condition.

If P is merely assumed ≥ 0, P^{-s} is defined to be zero on the nullspace $V_0(P)$, and the transition between the heat trace expansion (1.1) and the pole structure (1.3) continues to hold when the residue at 0 is modified by subtraction of $\dim(V_0(P))$.

We say that a property holds generically for the values of a parameter in $\mathbb{R}^\nu$ (or $(\overline{\mathbb{R}}_+)^\nu$ or another complete metric vector space) close to x_0 if it holds for the points in some small ball about x_0 minus a set of Baire category I (recall that the sets of Baire category I are countable unions of nowhere dense sets). We denote the imaginary unit ($\sqrt{-1}$) by i.

1.4 Theorem. *Let M be a compact boundaryless C^∞ manifold, E a C^∞ vector bundle over M and d a positive integer. Let P be any elliptic, selfadjoint positive classical pseudodifferential operator of order d in E. There exists a selfadjoint classical pseudodifferential operator Q of order $d-1$ in E commuting with P such that for generic small values of a and b, $a_l(P+aQ+b) \neq 0$ for all $l \geq 0$ and $b_k(P+aQ+b) \neq 0$ for all $k \geq 1$.*

1.5 Remark. Let m and k be odd and let $d=1$. By considering the square root of an operator of Laplace type, Cognola et al. [6] construct operators where b_k is non-trivial.

Proof. Let $P_1 := P^{1/d}$. For real parameters $\vec{\varepsilon} = (\varepsilon_1, ..., \varepsilon_{d-1})$ and ϱ, define:

$$P_2(\vec{\varepsilon},\varrho) := P + \varepsilon_1 P_1^{d-1} + ... + \varepsilon_{d-1}P_1 + \varrho.$$

By Seeley [15], $P_2(\vec{\varepsilon},\varrho)$ is an admissible d'th order ψdo for small values of $\vec{\varepsilon}$ and ϱ. Let $1 \leq i \leq d-1$. Then

$$\begin{aligned} &\partial_{\varepsilon_i} \operatorname{Tr} e^{-tP_2(\vec{\varepsilon},0)} = -t \operatorname{Tr}\{P_1^{d-i} e^{-tP_2(\vec{\varepsilon},0)}\}, \text{ and hence} \\ &\partial_{\varepsilon_i}\Gamma(s)\zeta(P_2(\vec{\varepsilon},0),s)|_{\vec{\varepsilon}=0} = -\Gamma(s+1)\zeta(P,s+i/d). \end{aligned}$$

Note that $a_0(P) > 0$ (it is an integral of the principal symbol, see for example [15]). Thus the residue of $\Gamma(s)\zeta(P,s)$ at $s = m/d$ is nonzero. Since $\Gamma(m/d)$ is regular, $\zeta(P,s)$ has a non-trivial simple pole when $s = m/d$. Thus $\zeta(P, s+i/d)$ has a simple pole with non-trivial residue at $s(i) := (m-i)/d$. Since $s(i) > -1$, $\Gamma(s(i)+1)$ is regular so $\partial_{\varepsilon_i}\Gamma(s)\zeta(P_2(\vec{\varepsilon},0),s)$ has a non-trivial simple pole at $s(i)$ when $\vec{\varepsilon} = 0$. The variation of the residue is the residue of the variation in this instance. Thus

$$\partial_{\varepsilon_i} \operatorname{Res}_{s=s(i)} \Gamma(s)\zeta(P_2(\vec{\varepsilon},0),s) = \operatorname{Res}_{s=s(i)} \partial_{\varepsilon_i}\Gamma(s)\zeta(P_2(\vec{\varepsilon},0),s) \neq 0$$

and $\partial_{\varepsilon_i} a_i(P_2(\vec{\varepsilon},0)) \neq 0$ at $\vec{\varepsilon} = 0$. Thus we may choose $\vec{\varepsilon}$ so that $a_i(P_2(\vec{\varepsilon},0)) \neq 0$ for $1 \leq i \leq d-1$; $a_0(P_2(\vec{\varepsilon},0)) = a_0(P)$ is always nonzero. Since

$$\begin{aligned} &h(P_2(\vec{\varepsilon},\varrho),t) = h(P_2(\vec{\varepsilon},0),t)e^{-t\varrho}, \\ &a_l(P_2(\vec{\varepsilon},\varrho)) = \textstyle\sum_{0\leq j\leq l/d}(-\varrho)^j a_{l-dj}(P_2(\vec{\varepsilon},0))/j!. \end{aligned} \tag{1.6}$$

Choose j so that $l - dj = i$ with $0 \leq i < d$. Then $a_{l-dj}(P_2(\vec{\varepsilon},0)) \neq 0$ so $a_l(P_2(\vec{\varepsilon},\varrho))$ is a non-trivial polynomial in ϱ and is nonzero for generic ϱ. This shows that there exists an admissible ψdo P_2 which has the same leading symbol as P and which commutes with P so that $a_l(P_2) \neq 0$ for $l \geq 0$.

We now study the invariants b_k. Let $P_3(\tau_1,\tau_0) := P_1^2 + \tau_1 P_1 + \tau_0$; P_3 is an admissible second order ψdo for small values of τ_0 and τ_1 [15]. The argument given above shows that τ_0 and τ_1 can be chosen so $a_l(P_3(\tau_1,\tau_0)) \neq 0$ for all $l \geq 0$. Let $P_4 = \sqrt{P_3}$; it is an admissible first order ψdo [15]. Since $a_{m+1}(P_3) \neq 0$ and since Γ is regular at $s = -1/2$, $\zeta(P_3,s)$ has a non-trivial simple pole at $s = -1/2$. Thus at $s = -1$, $\zeta(P_4,s) = \zeta(P_3,s/2)$ has a non-trivial simple pole and $\Gamma(s)\zeta(P_4,s)$ has a double pole so $b_1(P_4) \neq 0$. Let $P_5(\tau_2) := P_4 + \tau_2$; P_5 is an admissible first order ψdo for τ_2 small. Then $h(P_5(\tau_2),t) = h(P_4,t)e^{-t\tau_2}$ so

$$b_k(P_5(\tau_2)) = \textstyle\sum_{0\leq j<k}(-\tau_2)^j b_{k-j}(P_4)/j!.$$

This is a non-trivial polynomial in τ_2 so we can choose τ_2 so that $b_k(P_5(\tau_2)) \neq 0$ for $k \geq 1$; this implies that $\Gamma(s)\zeta(P_5(\tau_2),s)$ has a double pole at $s \in -\mathbb{N}$. Let $P_6 = P_5^d$; it is an admissible ψdo of order d. Then $\Gamma(s)\zeta(P_6,s) = \Gamma(s)\zeta(P_5(\tau_2),ds)$ has a double pole at $s \in -\mathbb{N}$ so $b_k(P_6) \neq 0$ for $k \geq 1$. This shows that there exists an admissible ψdo P_6 which has the same leading symbol as P and which commutes with P so that $b_k(P_6) \neq 0$ for $k \geq 1$.

For $0 \leq \tau_3 \leq 1$, let $P_7(\tau_3) = \tau_3 P_2 + (1-\tau_3)P_6$; it is an admissible ψdo of order d. The invariants a_l for $0 \leq l < d$ and b_1 are non-trivial polynomials in τ_3 so we can choose τ_3 so $a_l(P_7(\tau_3)) \neq 0$ for $0 \leq l < d$ and so $b_1(P_7(\tau_3)) \neq 0$. Let $Q = P_7(\tau_3) - P$; Q is a selfadjoint ψdo of order $d-1$ which commutes with P. Let $P(a,b) = P + aQ + b$; this is an admissible ψdo of order d for small values of (a,b). Then $a_l(P(a,0))$ for $0 \leq l < d$ and $b_1(P(a,0))$ are non-trivial polynomials in a; hence they are nonzero for generic values of a and we restrict to such values of a henceforth. Since $h(P(a,b),t) = h(P(a,0),t)e^{-tb}$, $a_l(P(a,b))$ for $l \geq 0$ and $b_k(P(a,b))$ for $k \geq 1$ are non-trivial polynomials in b; hence they are non-trivial for generic values of b. $\square$

Fix the order d, the dimension m of M and the rank r of E. Choose a local coordinate system on M and a local frame for E. A *local formula* $\mathcal{A}(P)(x)$ is simply a smooth function of the values at x of a finite number of derivatives of a finite number of terms (up to a fixed number n_0) in the asymptotic expansion of the total symbol of P such that $\mathcal{A}(P)(x)$ is defined for all admissible P; this formula is said to be *invariant* if the value is independent of the particular local coordinate system and frame which is chosen. A scalar valued function $a(P)$ is said to be *locally computable* if there is an invariant local formula so that $a(P) = \int_M \mathcal{A}(P)(x)$. When P is an admissible pseudodifferential operator, the invariants $a_l(P)$ for $(l-m)/d \notin \mathbb{N}$ are locally computable and the invariants $b_k(P)$ for $k \in \mathbb{N}$ are locally computable, by formulas based on the rules for composition and inversion of ψdos (Seeley [15]).

1.7 Theorem. *If $(l-m)/d = k \in \mathbb{N}$, then $a_l(P)$ is not locally computable.*

Proof. Suppose the contrary; let $\mathcal{A}_l$ be the corresponding local formula for fixed (m, d, r, n_0). Let g be a Riemannian metric on $M := S^m$. Suppose first $m > 1$. Let $\Delta(g) := (\Delta_0(g)^2 + |R(g)|^2)^{1/4} \otimes I_r$ acting on a trivial bundle of fiber dimension r where $\Delta_0(g)$ is the scalar Laplacian and where $|R(g)|^2$ is the norm of the total curvature tensor. Then $\Delta(g)$ is a natural first order elliptic selfadjoint classical ψdo with $\Delta(c^{-2}g) = c\Delta(g)$. Since S^m does not admit a flat metric, $|R(g)|^2$ does not vanish identically so $\Delta(g)$ is positive and hence admissible. If $m = 1$, let $\Delta(g)$ be $\Delta_0(g)^{1/2}$ with coefficients in r copies of the Möbius bundle; again $\Delta(g)$ is admissible and $\Delta(c^{-2}g) = c\Delta(g)$. The operator

$$P(g, \vec{\tau}) := \left\{(\Delta(g)^2 + \tau_1\Delta(g) + \tau_0)^{1/2} + \tau_2\right\}^d$$

is admissible when the components of $\vec{\tau}$ are nonnegative. Furthermore, the argument used to prove Theorem 1.4 shows that $b_k(P(g, \vec{\tau}))$ is nonzero for generic small $\vec{\tau}$ with nonnegative components. For $c > 0$, let $g(c) := c^{-2}g$, $\tau_1(c) := c\tau_1$, $\tau_0(c) := c^2\tau_0$, and $\tau_2(c) := c\tau_2$. Then $P(g(c), \vec{\tau}(c)) = c^d P(g, \vec{\tau})$. We will show further below that there exists an asymptotic expansion as $c \downarrow 0$ of the form:

$$(1.8) \qquad \mathcal{A}_l(P(g(c), \vec{\tau}(c))) = \textstyle\sum_{0 \le n \le N} c^n \mathcal{A}_{l,n}(g, \vec{\tau}) + O(c^{N+1}), \text{ for any } N.$$

Since $dvol(g(c)) = c^{-m} dvol(g)$, we integrate equation (1.8) to see that

$$(1.9) \qquad a_l(P(g(c), \vec{\tau}(c))) = \textstyle\sum_{0 \le n \le N} c^{n-m} a_{l,n}(g, \vec{\tau}) + O(c^{N+1-m}).$$

On the other hand, since $P(g(c), \vec{\tau}(c)) = c^d P(g, \vec{\tau})$, we may equate asymptotic expansions of $h(c^d P, t)$ and $h(P, c^d t)$ and compare the coefficients of t^k and $t^k \log t$ to see that $b_k(cP) = c^k b_k(P)$ and that

$$(1.10) \qquad a_l(P(g(c), \vec{\tau}(c))) = c^k\{a_l(P(g, \vec{\tau})) + d \log c\, b_k(P(g, \vec{\tau}))\}.$$

Since $b_k(P(g, \vec{\tau}))$ is nonzero for generic small values of $\vec{\tau}$, the expansion in equation (1.9) is inconsistent with the expansion in equation (1.10). This contradiction implies that a_l is not locally computable.

To establish equation (1.8) we generalize an argument given in Gilkey [8]. Fix $x_0 \in M$ and choose a system of local coordinates X on M centered at x_0. Introduce formal variables $g_{ij}(X, g) := g(\partial_i^X, \partial_j^X)$ and $g_{ij/\alpha}(X, g) := \partial_x^\alpha g_{ij}(X, g)$. Then

$\mathcal{A}_l(P(g,\vec{\tau}))$ is an invariantly defined smooth function of the variables $g_{ij/\alpha}$ and $\vec{\tau}$ whose value is independent of the particular coordinate system X which is chosen. This function is defined for g_{ij} positive definite and $\tau_i \geq 0$; there is no restriction on the $g_{ij/\alpha}(X,g)$ variables for $|\alpha| > 0$. We now see that the restriction $P > 0$ was inessential; a local formula can not detect the globally defined kernel and hence we can work with any natural selfadjoint nonnegative operator $P(g)$. Let $X_c = c^{-1}X$ be a new coordinate system on M centered at x_0. Then (see [8] for details):

$$
\begin{aligned}
&g_{ij/\alpha}(X_c, c^{-2}g)(x_0) = c^{|\alpha|} g_{ij}(X,g)(x_0), \text{ so} \\
&\mathcal{A}_l(P(g(c),\vec{\tau}(c))) = \mathcal{A}_l(c^{|\alpha|} g_{ij/\alpha}(X,g)(x_0), \vec{\tau}(c))
\end{aligned}
$$

is a smooth function of c at $c = 0$. We expand this function in a Taylor series about $c = 0$ to derive the expansion given in equation (1.8); it is then immediate that the individual terms in this expansion are invariant separately. $\square$

Theorem 1.4 shows that the set of admissible ψdos for which all the invariants $a_l(P)$ and $b_k(P)$ do not vanish is a dense set (in a suitable topology). We shall now show that the set of admissible partial differential operators for which the invariants $a_l(P)$ do not vanish for all even l is dense in the set of admissible partial differential operators. Here we cannot in general choose the perturbation to commute with P.

1.11 Theorem. *Let M be a compact boundaryless C^∞ manifold, E a C^∞ vector bundle over M and d a positive integer. Let P be any elliptic, selfadjoint positive differential operator of order $2d$ in E. There exists a selfadjoint differential operator Q of order $2d-2$ on M such that for generic small values of a, $a_l(P+aQ) \neq 0$ for l even and ≥ 0.*

Proof. First we recall the explicit combinatorial formulas for the invariants $a_{2j}(P)$ derivable from Seeley [15] (further details can be found in [9] or [11]). Let $p_d + ... + p_0$ be the total symbol of the differential operator P. For $\lambda \in \mathbb{C} \setminus [0,\infty[$, set

$$
\begin{aligned}
&q_{-d} := (p_d - \lambda)^{-1} \text{ and inductively set} \\
&q_{-d-l}(x,\xi,\lambda) := -q_{-d} \textstyle\sum_{|\alpha|+d+j-k=l,\, j<l} (-\mathrm{i})^{|\alpha|} \partial_\xi^\alpha p_k \partial_x^\alpha q_{-d-j}/\alpha!.
\end{aligned}
$$

Let $k_m := \mathrm{i}(2\pi)^{-m-1}$ and let $\mathcal{C}$ be a suitably chosen contour in $\mathbb{C}$ about the positive real axis. Then

$$
a_l(P) = k_m \textstyle\int_{T^*M} \int_{\mathcal{C}} e^{-\lambda} \operatorname{Tr} q_{-d-l}(x,\xi,\lambda)\, d\lambda d\xi dx.
$$

Use a partition of unity to construct an operator Δ_0 in E with leading symbol given by a Riemannian metric on M. Let $P_1(\vec{\varepsilon},\varrho) := P + \varepsilon_1 \Delta_0^{d-1} + ... + \varepsilon_{d-1}\Delta_0 + \varrho$. Then

$$
\partial_{\varepsilon_j} a_{2j}(P_1(\vec{\varepsilon},0))|_{\vec{\varepsilon}=0} = -C_{m,2j} \textstyle\int_{T^*M} |\xi|^{2(d-j)} \int_{\mathcal{C}} e^{-\lambda} \operatorname{Tr} q_{-d}(x,\xi,\lambda)^2\, d\lambda d\xi dx \neq 0.
$$

Thus we may choose $\vec{\varepsilon}$ so that $a_{2j}(P_1(\vec{\varepsilon},0)) \neq 0$ for $0 < j < d$; $a_0(P_1(\vec{\varepsilon},0))$ is always nonzero. Since $h(P(\vec{\varepsilon},\varrho),t) = h(P_1(\vec{\varepsilon},0),t)e^{-t\varrho}$, there exists $(\vec{\varepsilon},\varrho)$ so that $a_l(P_1(\vec{\varepsilon},\varrho)) \neq 0$ for l even and ≥ 0. We set $Q := P - P_1(\vec{\varepsilon},\varrho)$. Then $a_l(P+aQ)$ is a non-trivial polynomial in a and hence is nonzero for generic a. $\square$

We say that a second order differential operator D is of *Laplace* type if the leading symbol of D is scalar and is given by a Riemannian metric; $D = -\sum_{ij} g^{ij}\partial_i\partial_j +$ lower order terms. We say that a first order differential operator A is of *Dirac type* if A^2 is of Laplace type. Let $\mathrm{Clif}^c(\mathbb{R}^m)$ denote the complex Clifford algebra. If e_i is the usual orthonormal basis for $\mathbb{R}^m$, this is the universal complex unital algebra generated by the e_i subject to the Clifford commutation relations

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

The algebra $\mathrm{Clif}^c(\mathbb{R}^{2k})$ has a unique complex irreducible representation S of dimension 2^k; the algebra $\mathrm{Clif}^c(\mathbb{R}^{2k+1})$ has two inequivalent complex irreducible representations S_i of dimension 2^k. Every complex representation of these algebras can be expressed uniquely in terms of S or in terms of S_1 and S_2, see Atiyah, Bott, and Shapiro [2] for details. Let M be a compact connected boundaryless C^∞ manifold. Let $\mathcal{D}(M)$ be the space of selfadjoint operators of Dirac type on M; this is a complete metric space in a suitable topology. The leading symbol of an operator $A \in \mathcal{D}(M)$ defines a $\mathrm{Clif}^c(M)$ module structure on the fibers of the vector bundle on which A acts. Let m be odd. If M is orientable, let $\mathcal{D}(M, r_1, r_2)$ be the space of operators giving rise to a module structure isomorphic to $r_1 S_1 + r_2 S_2$. If M is not orientable, locally the structure is always of the form $r(S_1 + S_2)$ and we denote this space by $\mathcal{D}(M, r, r)$. If m is even, let $\mathcal{D}(M, r)$ be the space of operators giving rise to the module structure rS. If m is odd, $\mathcal{D}(M)$ is the disjoint union of the $\mathcal{D}(M, r_1, r_2)$ while if m is even, $\mathcal{D}(M)$ is the disjoint union of the $\mathcal{D}(M, r)$. $\mathcal{D}(M)$ is a Fréchet space, e.g. with the seminorms defining the C^∞ spaces of coefficients in a finite system of local coordinate patches (also global seminorms could be defined).

We shall need the following technical result.

1.12 Lemma. *Let M be a compact boundaryless C^∞ manifold, E a C^∞ vector bundle over M, D an operator of Laplace type in E, and $\psi_i \in C^\infty(\mathrm{End}(E))$. Let $D(\varepsilon) := D + \varepsilon\psi_1 + \varepsilon^2\psi_2$. Expand $a_{2l}(D(\varepsilon)) = \sum_{0\le i\le 2l} a_{2l,i}(D, \psi_1, \psi_2)\varepsilon^i$ as a polynomial in ε. Then*

$$a_{2l,2l}(D, \psi_1, \psi_2) = (4\pi)^{-m/2}(-1)^l/l! \int_M \mathrm{Tr}(\psi_2^l).$$

Proof. Let $D_1 = -(g^{ij}\partial_i\partial_j + A^k\partial_k + B)$ be an operator of Laplace type where A^k and B are endomorphisms of E. We define:

$$\mathrm{ord}(\partial_x^\alpha g^{ij}) := |\alpha|, \;\; \mathrm{ord}(\partial_x^\beta A^k) := |\beta| + 1, \;\text{and}\; \mathrm{ord}(\partial_x^\gamma B) := |\gamma| + 2.$$

The combinatorial formula given in the proof of Theorem 1.11 shows $a_{2l}(D_1)$ is the trace of a non-commutative polynomial in the variables $\partial_x^\alpha g_{ij}$ (for $|\alpha| > 0$), $\partial_x^\beta A^k$, and $\partial_x^\gamma B$ which is homogeneous of order $2l$ with coefficients which are smooth functions of the g_{ij} variables. See [9, Lemma 1.8.3] for further details. The coefficient of ε^{2l} in $a_{2l}(D(\varepsilon))$ must therefore be of the form $c(m,l)\int_M \mathrm{Tr}(\psi_2^l)$; ψ_1 does not enter. We can evaluate this constant by taking $\psi_1 = 0$ and $\psi_2 = I$. Then $h(D + \varepsilon^2, t) = h(D, t)e^{-\varepsilon^2 t}$, so $a_{2l}(D + \varepsilon^2) = (-1)^l\varepsilon^{2l}a_0(D)/l!$ plus lower order terms in ε. We use the identity $a_0(D) = (4\pi)^{-m/2}vol(M)\dim(E)$ to complete the proof. $\square$

We now study the invariants $a_l(A^2)$ for operators A of Dirac type.

1.13 Theorem.

(1) *Let M be a compact connected boundaryless C^∞ manifold of dimension $m > 1$, and let $A \in \mathcal{D}(M)$. Then $a_{2l}(A^2) \neq 0$ holds generically for operators close to A in $\mathcal{D}(M)$.*

(2) *If $A \in \mathcal{D}(S^1, r_1, r_2)$ with $r_1 r_2 = 0$, then $a_{2l}(A^2) = 0$ for all $l > 0$.*

(3) *If $r_1 r_2 \neq 0$ and $A \in \mathcal{D}(S^1, r_1, r_2)$, then $a_{2l}(A^2) \neq 0$ holds generically for operators close to A in $\mathcal{D}(S^1, r_1, r_2)$.*

Proof. The invariants a_{2l} are given by local formulas so they are continuous on $\mathcal{D}$. Consequently, to prove assertions (1) and (3), it suffices to show for each l that $a_{2l}(A^2)$ does not vanish on a dense set. The proof of (1) essentially follows from work of Branson and Gilkey [4]. We outline the proof since there is one technical point that needs amplification which was omitted in [4]. Let $A \in \mathcal{D}(M)$. Let $A(\varepsilon) := A + \varepsilon$. We compute:

$$\begin{aligned}
&\textstyle\sum_i \partial_\varepsilon^2 a_i(A(\varepsilon)^2) t^{(i-m)/2} \sim \partial_\varepsilon^2 \operatorname{Tr}(e^{-tA(\varepsilon)^2})\\
=&\partial_\varepsilon \operatorname{Tr}(-2tA(\varepsilon)e^{-tA(\varepsilon)^2}) = \operatorname{Tr}((-2t + 4t^2 A(\varepsilon)^2)e^{-tA(\varepsilon)^2})\\
=&2t(-1-2t\partial_t)\operatorname{Tr}(e^{-tA(\varepsilon)^2}) \sim \textstyle\sum_j 2(-1+m-j)a_j(A(\varepsilon)^2)t^{(j-m+2)/2}.
\end{aligned}$$

We compare coefficients of t in the two asymptotic expansions and set $i = 2l$ and $j = 2l - 2$ to see:

$$\partial_\varepsilon^2 a_{2l}(A(\varepsilon)^2) = 2(1 + m - 2l)a_{2l-2}(A(\varepsilon)^2). \tag{1.14}$$

Suppose that m is even or that $2l < m$. Then $m + 1 - 2l \neq 0$, and equation (1.14) can be applied recursively to construct a non-zero constant $c(m, l)$ so that

$$\partial_\varepsilon^{2l} a_{2l}((A + \varepsilon)^2)|_{\varepsilon=0} = c(m, l)a_0(A^2) \neq 0.$$

This shows that a_{2l} is nonzero on a dense set. It remains to consider the cases where m is odd and $2l > m$. Again, we can find $c(m, k) \neq 0$ so that

$$\partial_\varepsilon^{2k} a_{m+1+2k}(A(\varepsilon)^2) = c(m, k)a_{m+1}(A(\varepsilon)^2).$$

Thus it suffices to prove that $a_{m+1}(A(\varepsilon)^2)$ is nonzero on a dense set. If $f \in C^\infty(M)$, there is an expansion

$$\operatorname{Tr}(fAe^{-tA^2}) \sim \textstyle\sum_{l=0}^\infty a_l(f, A, A^2)t^{(l-m-1)/2}.$$

Let $A(\varrho) := A + \varrho f$. We compute

$$\begin{aligned}
&\textstyle\sum_{i=0}^\infty \partial_\varrho a_i((A + \varrho f)^2)|_{\varrho=0} t^{(i-m)/2} \sim \partial_\varrho \operatorname{Tr}(e^{-t(A+\varrho f)^2})|_{\varrho=0}\\
=& -2t \operatorname{Tr}(fAe^{-tA^2}) \sim -2\textstyle\sum_{j=0}^\infty a_j(f, A, A^2)t^{(j-m+1)/2}.
\end{aligned}$$

We compare coefficients of t in the two asymptotic expansions and set $i = m + 1$ and $j = m$ to see

$$\partial_\varrho a_{m+1}((A + \varrho f)^2)|_{\varrho=0} = -2a_m(f, A, A^2).$$

The invariants $a_l(f,A,A^2)$ are locally computable;

$$a_l(f,A,A^2)=\int_M f(x)\mathcal{A}_l(A,A^2)(x).$$

Thus to show that $a_{m+1}(A^2)$ is generically non-zero, it suffices to show that the local formula $\mathcal{A}_m(A,A^2)(x)$ does not vanish identically for a dense set of operators A. Relative to a system of local coordinates and in a local frame for E, we may express the operator as $A=\sum_i\gamma_i\partial_i+b$. Fix $x_0\in M$ and normalize the choice of coordinates so that $g_{ij}(x_0)=\delta_{ij}$. Fix (m,r_1,r_2). We can normalize the local frame on the vector bundle in question so that the γ_i have a standard form at x_0. Then $\mathcal{A}_m(A,A^2)(x_0)$ is a polynomial in the matrix components of b and its derivatives and in the matrix components of the derivatives of the γ_i which is universally defined. Thus we need only show that this polynomial is non-trivial; the topology of the underlying manifold M plays no role. For $m>3$ odd, the product argument described in [4, page 81] preserves the structure constants (r_1,r_2) and reduces this to the case $m=3$. The case $m=3$ follows from [4, Theorem 4.1 (d)]. This completes the proof of assertion (1). We note that the argument given in [4] did not take into account the need to specify the structure constants (r_1,r_2) and was incomplete at this point.

Suppose that $m=1$. Parametrize the circle by arc length to write $A=\gamma\partial_x+b$ where $\gamma^2=-I$. If $r_1=0$ or if $r_2=0$, then γ is scalar so $A=\pm\mathrm{i}\partial_x+b$. Choose a local primitive B for b. Then $A=\pm\mathrm{i}e^{\pm\mathrm{i}B}\partial_x e^{\mp\mathrm{i}B}$ so A is locally gauge equivalent to $\pm\mathrm{i}\partial_x$ and all the higher order local invariants of A vanish. This proves assertion (2). If $r_1r_2\neq0$, we can choose $\hat\gamma$ selfadjoint so that $\hat\gamma\gamma+\gamma\hat\gamma=0$ and so that $\mathrm{Tr}(\hat\gamma^2)\neq0$. Set $A(\varepsilon):=A+\varepsilon\hat\gamma$. Then we have $A(\varepsilon)^2=A^2+\varepsilon\psi+\varepsilon^2\hat\gamma^2$ where $\psi=b\hat\gamma+\hat\gamma b$ is an operator of order zero. By Lemma 1.12, the coefficient of ε^{2l} in $a_{2l}(A(\varepsilon)^2)$ is non-trivial and assertion (3) follows. $\square$

Let D be a self-adjoint positive operator of Laplace type and let $u\in\mathbb{C}$. Let $L_{u,j}(D)$ for $j\geq-1$ be the j^{th} coefficient in the Laurent expansion of $\Gamma(s)\zeta(D,s)$ about $s=u$; $L_{u,-1}(D)=a_{2n}(D)$ if $u=(m-2n)/2$ for some n and $L_{u,-1}(D)=0$ otherwise. If m is even, let $\mathcal{D}(M,r_1,r_2)=\mathcal{D}(M,r_1)$. For $A\in\mathcal{D}(M,r_1,r_2)$ with $\ker(A)=0$, we consider the invariants $L_{u,j}(A^2)$. For generic values of ε, $A+\varepsilon$ is invertible; we restrict to such values of ε henceforth. Let $\tau>0$, let $\varrho\in\mathbb{R}\setminus\{0\}$, and let μ be the multiplicity of the lowest eigenvalue λ of A^2. We have

(1.15) $\partial_\varepsilon^{2k}\Gamma(s)\zeta((A+\varepsilon)^2,s)=2s(2s+1)...(2s+2k-1)\Gamma(s)\zeta(A(\varepsilon)^2,s+k)$,

(1.16) $\partial_\tau^k\Gamma(s)\zeta(A^2+\tau,s)=(-1)^ks(s+1)...(s+k-1)\Gamma(s)\zeta(A^2+\tau,s+k)$,

(1.17) $\lim\limits_{k\to\infty}\lambda^{s+k}\zeta(A^2,s+k)=\mu$.

Note that $\zeta((\varrho A)^2,s)=|\varrho|^{-2s}\zeta(A^2,s)$. We expand $|\varrho|^{-2s}$ and $\zeta(A^2,s)$ in Laurent series separately, multiply the two series together, and collect terms to see that

(1.18) $$L_{u,j}((\varrho A)^2)=|\varrho|^{-2u}\sum_{-1\leq k\leq j}L_{u,k}(A^2)(-2)^{j-k}(\log|\varrho|)^{j-k}/(j-k)!$$

1.19 Lemma. *Let (u,m,r_1,r_2) be given. There exists $A \in \mathcal{D}(S^m,r_1,r_2)$ so that $L_{u,0}(A) \neq 0$.*

Proof. We shall assume $r_1 = 1$ and $r_2 = 0$; taking direct sums and replacing A by $-A$ defines operators with arbitrary structure constants and reduces the proof of the lemma to this special case. Let $A_1 \in \mathcal{D}(S^m,1,0)$ be the Dirac operator defined by the spin structure on S^m. Suppose that $2u$ is not a negative odd integer, and consider a $k \in \mathbb{N}$. Since $2u(2u+1)...(2u+2k-1)\Gamma(u) \neq 0$, we can use equations (1.15) and (1.17) to see that for sufficiently large k, $\partial_\varepsilon^{2k} L_{u,-1}((A_1+\varepsilon)^2) = 0$ and $\partial_\varepsilon^{2k} L_{u,0}((A_1+\varepsilon)^2) \neq 0$. This shows that $L_{u,0}((A_1+\varepsilon)^2) \neq 0$ for generic values of ε.

For the remainder of the proof, we shall assume $2u$ is a negative odd integer. Suppose that $m = 1$. Let $\zeta(s) := \sum_{n>0} n^{-s}$ be the Riemann zeta function. The functional equation $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$ shows that $\zeta(u) \neq 0$. The eigenvalues of the Dirac operator $A := -\mathrm{i}\partial_\theta$ on the Möbius bundle over the circle are $\{n+1/2\}$ for $n \in \mathbb{Z}$. Since $\zeta(s,A^2) = 2^{2s+1}(1-2^{-2s})\zeta(2s)$, $\Gamma(u)\zeta(u,A^2) \neq 0$ so $L_{u,0}(A^2) \neq 0$.

Suppose $m > 1$ is odd. Choose $l > 0$ so that $u = (m-2l)/2$. By Theorem 1.13, there exists $A_2 \in \mathcal{D}(S^m,1,0)$ close to the Dirac operator on S^m so that $a_{2l}(A_2^2) \neq 0$. We set $j = 0$ in Equation (1.18) to see $L_{u,0}((\varrho A_2)^2) \neq 0$ for generic values of ϱ.

Suppose that m is even. The spin bundle on S^m decomposes into the half spin bundles $\mathcal{S}^\pm$. Let $\gamma_0 = \pm 1$ on $\mathcal{S}^\pm$; γ_0 anti-commutes with the Dirac operator A_1. Let $A_2(\tau) := A_1 + \gamma_0\tau^{1/2}$. Since $(-u)(-u-1)...(-u-k+1)\Gamma(u) \neq 0$, and since $A_2(\tau)^2 = A_1^2 + \tau$, equations (1.16) and (1.17) show that $\partial_\tau^k L_{u,0}(A_2(\tau)^2) \neq 0$ for large k. $\square$

As recalled earlier, the invariants $L_{u,-1}$ are locally computable. On the other hand:

1.20 Theorem. *The invariants $L_{u,j}$ are not locally computable for $j \geq 0$.*

Proof. We fix (m,r_1,r_2). Suppose that $L_{u,j}$ is given by a local formula $\mathcal{L}_{u,j}$. Let $\varrho \in \mathbb{R} \setminus \{0\}$. Let X be a system of local coordinates centered at $x_0 \in M$. Let $A = \sum_i \gamma_i\partial_i + b$. Let $\gamma_{i/\alpha} := \partial_x^\alpha \gamma_i$ and $b_{/\beta} := \partial_x^\beta b$. Then $\mathcal{L}_{u,j}(A)$ is an invariantly defined smooth function of the variables $\gamma_{i/\alpha}$ and $b_{/\beta}$ whose value is independent of the particular coordinate system which is chosen. This function is defined for γ_i satisfying the Clifford commutation relations; there are no restrictions on the other variables. Let $\varrho \in \mathbb{R} \setminus \{0\}$ and let $X_\varrho = \varrho^{-1}X$. Then

$$\begin{aligned}
&\gamma_{i/\alpha}(X_\varrho,\varrho A)(x_0) = \varrho^{|\alpha|}\gamma_{i/\alpha}(X,A)(x_0),\\
&b_{/\beta}(X_\varrho,\varrho A)(x_0) = \varrho^{1+|\beta|}b_{/\beta}(X,A)(x_0), \text{ and}\\
&\mathcal{L}_{u,j}((\varrho A)^2)(x_0) = \mathcal{L}_{u,j}(\varrho^{|\alpha|}\gamma_{i/\alpha}(X,A),\varrho^{1+|\beta|}b_{/\beta}(X,A))(x_0).
\end{aligned}$$

Thus $\mathcal{L}_{u,j}((\varrho A)^2)$ is smooth at $\varrho = 0$. We expand this function in a Taylor series about $\varrho = 0$ to show

$$\mathcal{L}_{u,j}((\varrho A)^2) = \textstyle\sum_{0 \leq n \leq N} \mathcal{L}_{u,j,2n}(A^2)\varrho^{2n} + O(\varrho^{2N+2}), \text{ for any } N;$$

only even powers of ϱ appear since $\mathcal{L}_{u,j}((\varrho A)^2)$ is an even function of ϱ. We integrate this expression with respect to the metric defined by the leading symbol of A to see

$$L_{u,j}((\varrho A)^2) = \sum_{0\le n\le N} L_{u,j,2n}(A^2)|\varrho|^{2n-m} + O(\varrho^{2N+2-m}). \tag{1.21}$$

Use Lemma 1.19 to choose $A \in \mathcal{D}(S^m, r_1, r_2)$ so that $L_{u,0}(A) \ne 0$. If $j > 0$, the presence of $(\log|\varrho|)^j L_{u,0}$ in equation (1.18) contradicts equation (1.21). If $j = 0$ and if $u \ne (m-2n)/2$ for $n \ge 0$, then $L_{u,-1}(A^2) = 0$. Thus equation (1.18) implies $L_{u,0}((\varrho A)^2) = |\varrho|^{-2u} L_{u,0}(A^2)$; this contradicts equation (1.21) since the power of ϱ is not of the correct form.

Suppose that $j = 0$ and that $u = (m-2n)/2$ for some n. If $m > 1$, use Theorem 1.13 to choose A so that $L_{u,-1}(A^2) \ne 0$. The presence of $(\log|\varrho|)L_{u,-1}(A^2)$ in equation (1.18) contradicts equation (1.21). If $m = 1$, then $u = (1-2n)/2$. If $n = 0$, $L_{u,-1}(A) = a_0(A^2) \ne 0$ and the same argument shows $L_{u,-1}$ is not locally computable. Suppose $n \ge 1$. Choose $A \in \mathcal{D}(S^1, 1, 0)$ so that $L_{u,0}(A) \ne 0$; we take the direct sum of copies of A and of $-A$ to treat the general case. Let $A(\varrho) = \varrho A$ for $\varrho \ne 0$. Since $L_{u,-1}(A(\varrho)^2) = 0$, we have $\mathcal{L}_{u,0}(A(\varrho)^2) = \varrho^{2n}\mathcal{L}_{u,0}(A^2)$. The operator $A(\varrho)$ is locally gauge equivalent to the operator A; consequently $\mathcal{L}_{u,0}(A(\varrho)^2) = \mathcal{L}_{u,0}(A^2)$. Since $n \ne 0$, $\mathcal{L}_{u,0}(A) = 0$ so $L_{u,0}(A) = 0$ which is false. $\square$

2. Operators of Dirac type with spectral boundary conditions.

Let X be a compact connected n-dimensional C^∞ manifold with smooth boundary $M = \partial X$ (of dimension $m = (n-1)$). Let D be a realization of a second order strongly elliptic differential operator with a *local* boundary condition. Then equation (1.1) generalizes to become

$$h(D,t) := \operatorname{Tr} e^{-tD} \sim \sum_{l=0}^\infty a_l(D) t^{(l-n)/2}. \tag{2.1}$$

For example, if we let D act like $-\partial_\theta^2 + c$ on the interval $[0,\pi]$ with Dirichlet boundary condition, then $h(D,t) = (\sqrt{\pi}\, t^{-1/2} - 1/2)e^{-ct} + O(t^k)$ for any k; this provides an example where all the coefficients a_l in equation (2.1) are nonzero.

If a non-local boundary condition is imposed (as in Atiyah, Patodi, and Singer [3]), then there is an asymptotic expansion which can furthermore contain logarithmic terms. Let us recall the setting of [3], [13]. Choose a collared neighborhood $X_c := M \times [0, c[$ of M in X for some $c > 0$. Let x_n denote the coordinate in $[0, c[$ (it is considered as the normal coordinate). Let X have a smooth volume element ν_X and suppose there is a volume element ν_M on M so that $\nu_X = \nu_M\, dx_n$ on X_c. Let E_i be Hermitian C^∞ vector bundles over X and let

$$P : C^\infty(E_1) \to C^\infty(E_2)$$

be a first-order elliptic differential operator from E_1 to E_2. Let E_i' denote the restriction of the bundles E_i to the boundary M. On X_c, the E_i are isomorphic to the pull-backs of the E_i'. Let ∂_n denote the normal derivative. We assume on X_c that $P = \sigma(\partial_n + A)$ where σ is a unitary morphism from E_1' to E_2', independent of x_n, and where A is a fixed elliptic first order differential operator on $C^\infty(E_1')$ which is selfadjoint in $L_2(E_1')$, defined with respect to the Hermitian metric in E_1'. In this setting, we shall say that the structures are *product near the boundary.*

The *APS operator* P_B is defined as the operator from $L_2(E_1)$ to $L_2(E_2)$ acting like P and with domain defined by a nonlocal (so-called *spectral*) boundary condition:

$$D(P_B) = \{ u \in H^1(E_1) \text{ (Sobolev space)} \mid B(u|_M) = 0 \};$$

here B is an orthogonal projection in $L_2(E_1')$ of the form $B = \Pi_{\geq} + B_0$, where $\Pi_{\geq}$ is the orthogonal projection onto the sum of eigenspaces for A with eigenvalues $\lambda \geq 0$, and B_0 commutes with A and ranges in $V_0(A)$. (More general boundary conditions are considered in Grubb and Seeley [14] and in Brüning and Lesch [5].) By [16], P_B is a Fredholm operator.

Now consider the associated second order operators

$$\Delta_1 := {P_B}^* P_B \text{ and } \Delta_2 := P_B {P_B}^*.$$

The following analogues of the expansion (2.1) for the heat traces of these operators $h(\Delta_i, t) := \operatorname{Tr} e^{-t\Delta_i}$ were established in [13]. If $n = \dim(X)$ is even, then

$$h(\Delta_i, t) \sim \textstyle\sum_{l=0}^{\infty} a_l(\Delta_i) t^{(l-n)/2} + \sum_{k=0}^{\infty} b_k(\Delta_i) t^{k+1/2} \log t, \tag{2.2}$$

with coefficients satisfying, for suitable universal constants $\beta(k,n)$ and $\gamma(k,n) \neq 0$:

$$\begin{aligned} b_k(\Delta_i) &= \beta(k,n) a_{2k+n}(A^2), \\ a_{2k}(\Delta_i) &= a_{2k,+}(\widetilde{\Delta}_i) + f_k(A), \\ a_{2k+1}(\Delta_i) &= \gamma(k,n) a_{2k}(A^2) \text{ for } k < n/2, \\ a_{2k+1}(\Delta_i) &= f_k'(A) \text{ for } k \geq n/2. \end{aligned} \tag{2.3}$$

Here $a_{2k,+}(\widetilde{\Delta}_i) = \int_X \mathcal{A}_{2k}(\widetilde{\Delta}_i)(x)$ where the $\mathcal{A}_{2k}(\widetilde{\Delta}_i)(x)$ are the local formulas defining the coefficients in the heat trace expansions for $\widetilde{\Delta}_1 = \widetilde{P}^* \widetilde{P}$ resp. $\widetilde{\Delta}_2 = \widetilde{P}\widetilde{P}^*$, with $\widetilde{P}$ denoting the extension of P to the double $\widetilde{X}$ described in [3]. The $f_k(A)$ are locally computable functions of A when $2k \neq n$, and the $f_k'(A)$ are, by Theorem 1.20, not locally computable.

If n is odd, the $\log t$ terms do not appear and the expansion has a form similar to that given in equation (2.1):

$$h(\Delta_i, t) \sim \textstyle\sum_{l=0}^{\infty} a_l(\Delta_i) t^{(l-n)/2}, \tag{2.4}$$

with

$$\begin{aligned} a_{2k}(\Delta_i) &= a_{2k,+}(\widetilde{\Delta}_i) + g_k'(A), \\ a_{2k+1}(\Delta_i) &= \gamma(k,n) a_{2k}(A^2) \text{ for } 2k+1 \neq n, \\ a_n(\Delta_i) &= g''(A). \end{aligned} \tag{2.5}$$

where the $g_k'(A)$ are 0 for $k < n/2$ and are, by Theorem 1.20, not locally computable for $k > n/2$.

Let $\mathcal{P}(X)$ be the space of all operators of Dirac type over X such that the structures are product near the boundary. Then the tangential operator A is of Dirac type on M. If n is even, let $\mathcal{P}(X, r_1, r_2)$ be the subset of operators such that $A \in \mathcal{D}(M, r_1, r_2)$, with structure constants r_i independent of the particular boundary component considered. In the following theorem, we show that the invariants of the expansions (2.2) and (2.4) are non-trivial.

2.6 Theorem. *Consider P_B with P of Dirac type.*

(1) *Let $n = 2$. If $r_1 r_2 = 0$, then $b_k(\Delta_i) = 0$ for all k if $P \in \mathcal{P}(X, r_1, r_2)$.*

(2) *Let $n = 2$. If $r_1 r_2 \neq 0$, then $a_1(\Delta_i) \neq 0$ and $b_k(\Delta_i) \neq 0$ for $k \geq 0$ holds generically for operator close to P in $\mathcal{P}(X, r_1, r_2)$.*

(3) *Let $n \geq 4$ be even. Then $a_l(\Delta_i) \neq 0$ for l odd $< n$ and $b_k(\Delta_i) \neq 0$ for $k \geq 0$ holds generically for operators close to P in $\mathcal{P}(X)$.*

(4) *Let n be odd. Then $a_l(\Delta_i) \neq 0$ for l odd $\neq n$ holds generically for operators close to P in $\mathcal{P}(X)$.*

(5) *Let n be even, let $r_1 r_2 \neq 0$ and let $P \in \mathcal{P}(X, r_1, r_2)$. Then $a_l(\Delta_i) \neq 0$ for even l holds generically for operators close to P in $\mathcal{P}(X, r_1, r_2)$.*

(6) *Let n be odd and let $P \in \mathcal{P}(X, r)$. Then $a_l(\Delta_i) \neq 0$ for even l holds generically for operators close to P in $\mathcal{P}(X)$.*

Proof. The first 4 assertions follow immediately from Theorem 1.13 in view of the formulas (2.3), (2.5) for the coefficients in question.

When l is even, (2.3) and (2.5) show that the invariants a_l depend on the behavior of P in the interior; we exploit this fact in the proof. Let φ be a smooth function on X which vanishes near the boundary and which has support in a small coordinate neighborhood $\mathcal{O}$ on X. On $\mathcal{O}$, we write $P = \sum_i \sigma_i e_i + b$ where e_i is a local orthonormal frame for the tangent bundle of X. We use σ_1 to identify E_1 and E_2 over $\mathcal{O}$ and therefore assume without loss of generality that $\sigma_1 = I$. The condition that $P^* P$ has leading symbol given by the metric tensor then yields that the γ_i are skew-adjoint and satisfy the Clifford commutation conditions $\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}$ for $2 \leq i \leq n$. Under the assumptions of the theorem, we can find γ_0 selfadjoint with $\gamma_0^2 = I$ so that $\gamma_0 \gamma_i + \gamma_i \gamma_0 = 0$ for $2 \leq i \leq n$. We let $P(\varepsilon) := P + \varepsilon \varphi \gamma_0$. Then the commutation relations involved imply there exists an operator ψ of order zero so that $\Delta_i(\varepsilon) = \Delta_i(0) + \varepsilon \psi + \varepsilon^2 \varphi^2$.

Consider the coefficients $a_{2j}(\widetilde{\Delta}_i(\varepsilon))$ in the heat trace for the associated Laplacians $\widetilde{\Delta}_i$ on the doubled manifold $\widetilde{X}$. Here $\widetilde{\Delta}(\varepsilon) = \widetilde{\Delta}(0) + \varepsilon \widetilde{\psi} + \varepsilon^2 \widetilde{\varphi}^2$. By Lemma 1.12, $a_{2j}(\widetilde{\Delta}_i(\varepsilon))$ is a non-trivial polynomial in ε. The same holds for the invariant $a_{2j,+}(\Delta_i(\varepsilon)) = \frac{1}{2} a_{2j}(\widetilde{\Delta}_i(\varepsilon))$. Since $f_k(A)$ in (2.3) and $g_k'(A)$ in (2.5) depend only on the behavior of P near the boundary, and φ has support in the interior of X,

$$a_{2j}(\Delta_i(\varepsilon)) - a_{2j}(\Delta_i(0)) = a_{2j,+}(\widetilde{\Delta}_i(\varepsilon)) - a_{2j,+}(\widetilde{\Delta}_i(0))$$

is a non-trivial polynomial in ε. Thus $a_{2j}(\Delta_i(\varepsilon))$ is nonzero for all j for generic values of ε near 0 and the theorem follows. $\square$

For the odd dimensional case we conclude, since a union of two sets of Baire category I is of Baire category I:

2.7 Corollary. *Let n be odd and consider P_B as above. Then all coefficients $a_l(\Delta_i)$ except possibly $a_n(\Delta_i)$ are nonzero generically for operators close to P in $\mathcal{P}(X)$.*

In the even dimensional case, we can include all the remaining coefficients as follows:

2.8 Theorem. *Let n be even and consider P as above. If $n = 2$, assume $r_1 r_2 \neq 0$. Then all coefficients are nonzero for operators in a dense subset of a neighborhood of P in $\mathcal{P}(X, r_1, r_2)$.*

Proof. We already have that the coefficients b_k and a_l with $l \leq n$ or l even are nonzero generically for P_1 near P. We shall show that there is a P_2 close to P_1 such that also the a_l with l odd $> n$ are nonzero.

Let $P_1(\tau) = e^\tau P_1$. The corresponding Laplacian is $\Delta_{1,i}(\tau) = e^{2\tau}\Delta_{1,i}(0)$; the spectral boundary condition is unchanged. Thus $h(\Delta_{1,i}(\tau), t) = h(\Delta_{1,i}(0), e^{2\tau}t)$. Let $2k+1 = l-n$. We compare coefficients in the asymptotic expansion to see that

$$a_l(P_1(\tau)) = e^{\tau(l-n)}\{a_l(P_1(0)) + 2\tau b_k(P_1(0))\}.$$

Since b_k is nonzero, a_l is nonzero for τ in a dense set. $\square$

We have not investigated whether the a_l with l odd $> n$ are continuous on $\mathcal{P}(X, r_1, r_2)$ and can therefore not conclude they are generically nonzero.

Let d_X and δ_X be the derivative and the coderivative on X. Then $d_X + \delta_X$ belongs to $\mathcal{P}(X, r, r)$ if n is even and $d_X + \delta_X \in \mathcal{P}(X)$ if n is odd so these theorems provide non-trivial examples in all dimensions.

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