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REMARKS ON NONLOCAL TRACE EXPANSION COEFFICIENTS

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Dedicated to Krzysztof P. Wojciechowski on his 50th birthday

In a recent work, Paycha and Scott establish formulas for all the Laurent coefficients of $\mathrm{Tr}(AP^{-s})$ at the possible poles. In particular, they show a formula for the zero'th coefficient at $s = 0$, in terms of two functions generalizing, respectively, the Kontsevich-Vishik canonical trace density, and the Wodzicki-Guillemin noncommutative residue density of an associated operator. The purpose of this note is to provide a proof of that formula relying entirely on resolvent techniques (for the sake of possible generalizations to situations where powers are not an easy tool). — We also give some corrections to transition formulas used in our earlier works.

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1. Introduction

In an interesting new work [PS], Sylvie Paycha and Simon Scott have obtained formulas for all the coefficients in Laurent expansions of zeta functions $\zeta(A, P, s) = \mathrm{Tr}(AP^{-s})$ around the poles, in terms of combinations of finite part integrals and residue type integrals, of associated logarithmic symbols. We consider classical pseudodifferential operators (ψdo's) A and P of order $\sigma \in \mathbb{R}$ resp. $m \in \mathbb{R}_+$ acting in a Hermitian vector bundle E over a closed n-dimensional manifold X, P being elliptic with principal symbol eigenvalues in $\mathbb{C} \setminus \mathbb{R}_-$. The basic formula is the following formula for $C_0(A, P)$, where $C_0(A, P) - \mathrm{Tr}(A\Pi_0(P))$ is the regular value of $\zeta(A, P, s)$ at $s = 0$:

$$C_0(A, P) = \int_X \big(\mathrm{TR}_x(A) - \tfrac{1}{m}\,\mathrm{res}_{x,0}(A \log P)\big)\, dx. \tag{1}$$

The integrand is defined in a local coordinate system by:

$$\mathrm{TR}_x(A) = \fint \operatorname{tr} a(x,\xi)\, đ\xi,\ \operatorname{res}_{x,0}(A\log P) = \int_{|\xi|=1} \operatorname{tr} r_{-n,0}(x,\xi) đS(\xi), \quad (2)$$

where $\fint a(x,\xi)\, đ\xi$ is a finite part integral, and r is the symbol of $R = A\log P$, of the form

$$r(x,\xi) \sim \sum_{j\geq 0,\, l=0,1} r_{\sigma-j,l}(x,\xi)(\log[\xi])^l\,;$$

$r_{\sigma-j,l}$ homogeneous in ξ of degree $\sigma - j$ for $|\xi| \geq 1$, $[\xi]$ positive equal to $|\xi|$ for $|\xi| \geq 1$. (Here $r_{-n,0}$ is set equal to 0 when $\sigma - j$ does not hit $-n$.) Moreover, the expression $\big(\mathrm{TR}_x(A) - \operatorname{res}_{x,0}(A\log P)\big)\, dx$ has an invariant meaning as a density on X, although its two terms individually do not so in general. (In these formulas we use the conventions $đ\xi = (2\pi)^{-n} d\xi$, $đS(\xi) = (2\pi)^{-n} dS(\xi)$, where $dS(\xi)$ indicates the usual surface measure on the unit sphere. tr indicates fiber trace.)

Formula (1) generalizes the formula

$$C_0(A,P) = \mathrm{TR}\, A$$

(the canonical trace), which holds in particular cases, cf. Kontesevich and Vishik [KV], Lesch [L], Grubb [G4]. The general formula (1) is shown in [PS] by use of holomorphic families of ψdo's (depending holomorphically on their complex order z); in particular complex powers of P. The purpose of this note is to derive it by methods relying on the knowledge of the resolvent $(P-\lambda)^{-1}$. This is meant to facilitate generalizations to manifolds with boundary, where powers of operators are not an easy tool.

— We take the opportunity here to correct, in the appendix, some inaccuracies in earlier papers, mainly concerning the relations between expansion coefficients in resolvent traces and zeta functions.

2. Preliminaries

Recall the expansion formulas for the resolvent kernel and trace, worked out in local coordinates in Grubb and Seeley [GS1], when $N > \frac{\sigma+n}{m}$:

$$K\big(A(P-\lambda)^{-N}, x, x\big) \sim \sum_{j\in\mathbb{N}} \tilde{c}_j^{(N)}(x)(-\lambda)^{\frac{\sigma+n-j}{m}-N} + \sum_{k\in\mathbb{N}} \big(\tilde{c}_k^{(N)\prime}(x)\log(-\lambda) + \tilde{c}_k^{(N)\prime\prime}(x)\big)(-\lambda)^{-k-N},$$

$$\operatorname{Tr}\big(A(P-\lambda)^{-N}\big) \sim \sum_{j\in\mathbb{N}} \tilde{c}_j^{(N)}(-\lambda)^{\frac{\sigma+n-j}{m}-N} + \sum_{k\in\mathbb{N}} \big(\tilde{c}_k^{(N)\prime}\log(-\lambda) + \tilde{c}_k^{(N)\prime\prime}\big)(-\lambda)^{-k-N}, \quad (3)$$

for $\lambda \to \infty$ on rays in a sector around $\mathbb{R}_-$. The second formula is deduced from the first one by integrating the fiber trace in x. We denote $\{0,1,2,\dots\} = \mathbb{N}$. (More precisely, [GS1] covers the cases where m is integer; the noninteger cases are included in Loya [Lo], Grubb and Hansen [GH].)

The $\tilde{c}_k^{(N)\prime}(x)$ and $\tilde{c}_k^{(N)\prime}$ vanish when

$$\sigma + n + mk \notin \mathbb{N};$$

this holds for all k when m is integer and σ is noninteger. When

$$\sigma + n + mk = j \in \mathbb{N},$$

$\tilde{c}_j^{(N)}(x)$ and $\tilde{c}_k^{(N)\prime\prime}(x)$ are both coefficients of the power $(-\lambda)^{-k-N}$; their individual values depend on the localization used (as worked out in detail in [G4]), and it is only the sum $\tilde{c}_j^{(N)} + \tilde{c}_k^{(N)\prime\prime}$ that has an invariant meaning.

The coefficients depend on N; when $N = 1$, we drop the upper index (N). We are particularly interested in the coefficient of $(-\lambda)^{-N}$, for which we shall use the notation

$$\widetilde{C}_0^{(N)}(A,P) = \tilde{c}_{\sigma+n}^{(N)} + \tilde{c}_0^{(N)\prime\prime}; \quad (4)$$

here we have for convenience set

$$\tilde{c}_{\sigma+n}^{(N)} = 0 \text{ when } \sigma + n \notin \mathbb{N}. \quad (5)$$

Recall that there is, equivalently to the expansion in (3), an expansion formula for complex powers:

$$\Gamma(s)\operatorname{Tr}(AP^{-s}) \sim \sum_{j\in\mathbb{N}} \frac{c_j}{s + \frac{j-\sigma-n}{m}} - \frac{\operatorname{Tr}(A\Pi_0(P))}{s} + \sum_{k\in\mathbb{N}} \Big(\frac{c_k'}{(s+k)^2} + \frac{c_k''}{s+k}\Big). \quad (6)$$

This means that $\Gamma(s)\operatorname{Tr}(AP^{-s})$, defined as a holomorphic function for $\operatorname{Re} s > \frac{\sigma+n}{m}$, extends meromorphically to $\mathbb{C}$ with the pole structure indicated in the right hand side. Here

$$\Pi_0(P) = \tfrac{i}{2\pi}\int_{|\lambda|=\varepsilon} (P-\lambda)^{-1}\, d\lambda$$

is the projection onto the generalized nullspace of P (on which P^{-s} is taken to be zero). Again, the c'_k vanish when $\sigma + n + mk \notin \mathbb{N}$. We denote

$$C_0(A, P) = c_{\sigma+n} + c''_0, \tag{7}$$

the basic coefficient (setting $c_{\sigma+n}$ equal to 0 when $\sigma + n \notin \mathbb{N}$).

The transition between (3) and (6) is accounted for e.g. in Grubb and Seeley [GS2], Prop. 2.9, (3.21). The coefficient sets in (3) and (6) are derivable from one another. The coefficients $\tilde{c}_j^{(N)}$ and c_j, resp. $\tilde{c}_k^{(N)\prime}$ and c'_k, are proportional by universal nonzero constants. This holds also for $\tilde{c}_k^{(N)\prime\prime}$ and c''_k, when the c'_k vanish. In general, there are linear formulas for the transitions between $\{\tilde{c}_k^{(N)\prime}, \tilde{c}_k^{(N)\prime\prime}\}$ and $\{c'_k, c''_k\}$. (For $N = 1$, [GS2], Cor. 2.10, would imply that $\tilde{c}''_k$ and c''_k are proportional in general, but in fact, the formulas for the $\tilde{a}_{j,l}$ given there are only correct for $l = m_j$, whereas for $l < m_j$ there is an effect from derivatives of the gamma function that was overlooked.) One has in particular that $\tilde{c}_0^{(N)\prime} = c'_0$ for all N.

Division of (6) by $\Gamma(s)$ gives the pole structure of $\zeta(A, P, s)$:

$$\zeta(A, P, s) = \operatorname{Tr}(AP^{-s}) \sim \sum_{j \in \mathbb{N}} \frac{c'''_j}{s + \frac{j-\sigma-n}{m}}, \tag{8}$$

where c'''_j is proportional to c_j if $\frac{j-\sigma-n}{m} \notin \mathbb{N}$, and c'''_j is proportional to c'_k if $\frac{j-\sigma-n}{m} = k \in \mathbb{N}$.

One can study the Laurent series expansions of $\zeta(A, P, s)$ at the poles by use of (3). We now restrict the attention to the possible pole at $s = 0$.

Write the Laurent expansion at 0 as follows:

$$\begin{aligned} \zeta(A, P, s) = C_{-1}(A, P)s^{-1} &+ (C_0(A, P) - \operatorname{Tr}(A\Pi_0(P)))s^0 \\ &+ \sum_{l \geq 1} C_l(A, P)s^l. \end{aligned} \tag{9}$$

It is known from Wodzicki [W], Guillemin [Gu], that

$$C_{-1}(A, P) = c'_0 = \tfrac{1}{m} \operatorname{res}(A), \text{ independently of } P; \tag{10}$$

it vanishes if $\sigma + n \notin \mathbb{N}$ or the symbols have certain parity properties. From [KV], [L], [G4] we have that

$$C_0(A, P) = \operatorname{TR} A$$

when $\sigma + n \notin \mathbb{N}$, and in certain parity cases (given in [KV] for n odd, [G4] for n even, more details at the end of Section 4). Also $C_1(A, P)$ is of interest, since the zeta determinant of P satisfies

$$\log \det P = -C_1(I, P) = C_0(\log P, P) \tag{11}$$

(cf. Okikiolu [O], [G4]); here it is useful to know that expansions like (3) but with higher powers of $\log(-\lambda)$ hold if A is log-polyhomogeneous, cf. [L] and [G4].

We have for general $N \geq 1$:

Lemma 2.1. *When* $N > \frac{\sigma+n}{m}$ *(so that* $A(P-\lambda)^{-N}$ *is trace-class), then*

$$\begin{aligned} \tilde{c}_0^{(N)\prime} &= c_0' = \tfrac{1}{m}\operatorname{res} A, \\ \widetilde{C}_0^{(N)}(A,P) &= C_0(A,P) - \alpha_N c_0', \text{ where} \\ \alpha_N &= 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{N-1}. \end{aligned} \tag{12}$$

Proof. Denote $N-1 = M$, then

$$(P-\lambda)^{-N} = (P-\lambda)^{-M-1} = \tfrac{1}{M!}\partial_\lambda^M (P-\lambda)^{-1}. \tag{13}$$

The transition from (3) to information on $\zeta(A,P,s)$ is obtained by use of the formula

$$AP^{-s} = \tfrac{M!}{(s-M)\dots(s-1)}\tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{M-s}\tfrac{1}{M!}\partial_\lambda^M A(P-\lambda)^{-1}(I-\Pi_0(P))\,d\lambda, \tag{14}$$

where $\mathcal{C}$ is a curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ around the nonzero spectrum of P. Here we can take traces on both sides and apply [GS2], Prop. 2.9, to

$$f(\lambda) = \operatorname{Tr}\big(A\tfrac{1}{M!}\partial_\lambda^M (P-\lambda)^{-1}(I-\Pi_0(P))\big), \tag{15}$$

defining

$$\varrho(s) = \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s} f(\lambda)\,d\lambda, \tag{16}$$

for $\operatorname{Re} s$ large, and extending meromorphically. Then

$$\zeta(A,P,s) = \tfrac{M!}{(s-M)\dots(s-1)}\varrho(s-M). \tag{17}$$

(Note that $\zeta(A,P,s) = \varrho(s)$ if $N=1$.) The expansion coefficients of $f(\lambda)$ in powers and log-powers are universally proportional to the pole coefficients of

$$\psi(s) = \frac{\pi}{\sin(\pi s)}\varrho(s)$$

at simple and double poles, for each index, as accounted for in [GS2], Prop. 2.9.

When we apply this to $\zeta(A,P,s)$, we must take the factors

$$g_M(s) = \tfrac{M!}{(s-M)\dots(s-1)}$$

and $\frac{1}{\pi}\sin(\pi(s-M))$ into account. We have

$$\zeta(A,P,s) = g_M(s)\tfrac{1}{\pi}\sin(\pi(s-M))\psi(s-M). \tag{18}$$

By [GS2], Prop. 2.9, a pair of terms $a(-\lambda)^{-M-1}\log(-\lambda) + b(-\lambda)^{-M-1}$ in the expansion of $f(\lambda)$ carries over to the pair of terms $\frac{a}{(s+M)^2} + \frac{b}{s+M}$ in the pole structure of $\psi(s)$, whereby

$$\psi(s-M) = \frac{a}{s^2} + \frac{b}{s} + O(1), \text{ for } s \to 0. \tag{19}$$

Now it is easily checked that

$$\begin{aligned} \tfrac{1}{\pi}\sin(\pi(s-M)) &= (-1)^M(s + cs^3 + O(s^5)), \\ g_M(s) &= (-1)^M(1 + (1 + \tfrac{1}{2} + \dots + \tfrac{1}{M})s + O(s^2)), \end{aligned} \tag{20}$$

for $s \to 0$. Then, with α_N defined in (12),

$$\begin{aligned} &\zeta(A,P,s) \\ &= (s + cs^3 + O(s^5))(1 + \alpha_{M+1}s + O(s^2))\Big(\frac{a}{s^2} + \frac{b}{s} + O(1)\Big) \\ &= \frac{a}{s} + (b + \alpha_{M+1}a) + O(s), \end{aligned} \tag{21}$$

for $s \to 0$. For $f(\lambda)$ in (16) we have this situation with

$$a = \tilde c_0^{(N)\prime} \quad \text{and} \quad b = \widetilde C_0^{(N)}(A,P) - \operatorname{Tr}(A\Pi_0(P)),$$

so (21) holds with these values. In view of (9), (10), this shows (12). □

If one writes $f(\lambda)$ in the lemma as a sum $f_1(\lambda) + f_2(\lambda)$, where f_1 has the sum over j in (3), respectively f_2 has the sum over k in (3), as asymptotic expansions, the lemma can be applied to f_1 and f_2 separately, relating their coefficients to those of the poles of the corresponding functions of s.

Remark 2.1. Formula (12) gives a correction to our earlier papers [G1–5] and Grubb and Schrohe [GSc1–2], where it was taken for granted that $C_0(A,P)$ would equal $\tilde c_{\sigma+n}^{(N)} + \tilde c_0^{(N)\prime\prime}$ for any N. Fortunately, the correction has no consequence for the results in those papers, which were either concerned with the value of $C_0(A,P)$ when $c_0' = 0$, or its value *modulo local terms* (c_0' is local), or values of combined expressions where c_0'-contributions cancel out. More on corrections in the appendix.

In [GS2], Cor. 2.10 was not used in the argumentation, which was based directly on Prop. 2.9 and the primary knowledge of zeta expansions.

We shall now analyze $C_0(A,P)$ further, showing (1) by resolvent considerations. Our proof is based on an explicit calculation of one simple special case, together with the use of the trace defect formula

$$C_0(A,P) - C_0(A,P') = -\tfrac{1}{m}\operatorname{res}(A(\log P - \log P')). \tag{22}$$

This formula is well-known from considerations of complex powers of P ([O], [KV], Melrose and Nistor [MN]), but can also be derived directly from resolvent considerations [G6].

3. The trace defect formula for general orders

In [G6], the arguments for (22) are given in detail in cases where $m > \sigma + n$, whereas more general cases are briefly explained by reference to Remark 3.12 there. For completeness, we give the explanation in detail here. This is a minor technical point that may be skipped in a first reading. Denote as in [G6]

$$S_\lambda = A((P-\lambda)^{-1} - (P'-\lambda)^{-1}), \tag{23}$$

where P and P' are of order $m > 0$; then (cf. (13))

$$A((P-\lambda)^{-N} - (P'-\lambda)^{-N}) = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} S_\lambda \equiv S_\lambda^{(N)}. \tag{24}$$

S_λ and $S_\lambda^{(N)}$ have symbols $s(x,\xi,\lambda)$ respectively

$$s^{(N)}(x,\xi,\lambda) = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} s(x,\xi,\lambda)$$

in local trivializations.

The difference of the two expansions (3) with P resp. P' inserted satisfies

$$\begin{aligned} \operatorname{Tr}\big(S_\lambda^{(N)}\big) &\sim \sum_{j\in\mathbb{N}} \tilde{s}_j^{(N)} (-\lambda)^{\frac{\sigma+n-j}{m}-N} \\ &+ \tilde{s}_0^{(N)\prime\prime}(-\lambda)^{-N} + \sum_{k\geq 1}\big(\tilde{s}_k^{(N)\prime}\log(-\lambda) + \tilde{s}_k^{(N)\prime\prime}\big)(-\lambda)^{-k-N}; \end{aligned} \tag{25}$$

in view of Lemma 2.1, the contributions from $\operatorname{res} A$ cancel out and the coefficient of $(-\lambda)^{-N}$ equals $C_0(A,P) - C_0(A,P')$.

The symbol $s(x,\xi,\lambda)$ is analyzed in [G6], Prop. 2.1. For $s^{(N)}$, we conclude that the homogeneous terms have at least $N+1$ factors of the form $(p_m - \lambda)^{-1}$ or $(p'_m - \lambda)^{-1}$, hence the strictly homogeneous version of the symbol of order $\sigma - Nm - j$ satisfies

$$|s^{(N)h}_{\sigma-Nm-j}(x,\xi,\lambda)| \leq c(|\xi|^m + |\lambda|)^{-N-1}|\xi|^{\sigma+m-j}, \tag{26}$$

being integrable at $\xi = 0$ for $j < n + \sigma + m$, $\lambda \neq 0$. Then the kernel and trace of $S_\lambda^{(N)}$ have expansions

$$\begin{aligned} K(S_\lambda^{(N)}, x, x) &= \sum_{j < \sigma + m + n} \tilde{s}_j^{(N)}(x)(-\lambda)^{\frac{n+\sigma-j}{m} - N} + O(|\lambda|^{-N-1+\varepsilon}), \\ \operatorname{Tr} S_\lambda^{(N)} &= \sum_{j < \sigma + m + n} \tilde{s}_j^{(N)}(-\lambda)^{\frac{n+\sigma-j}{m} - N} + O(|\lambda|^{-N-1+\varepsilon}), \end{aligned} \tag{27}$$

where the terms for $j < \sigma + m + n$ are calculated from the strictly homogeneous symbols (for $\lambda \in \mathbb{R}_-$):

$$\int_{\mathbb{R}^n} s^{(N)h}_{\sigma - Nm - j}(x, \xi, \lambda)\, đ\xi = (-\lambda)^{\frac{n+\sigma-j}{m} - N} \int_{\mathbb{R}^n} s^{(N)h}_{\sigma - Nm - j}(x, \eta, -1)\, đ\eta, \tag{28}$$

so that

$$\tilde{s}_j^{(N)}(x) = \int_{\mathbb{R}^n} s^{(N)h}_{\sigma - Nm - j}(x, \xi, -1)\, đ\xi, \quad \tilde{s}_j^{(N)} = \int \operatorname{tr} \tilde{s}_j^{(N)}(x)\, dx. \tag{29}$$

When $\sigma + n \notin \mathbb{N}$, there is no term with $(-\lambda)^{-N}$ in the expansion of $\operatorname{Tr} S_\lambda^{(N)}$, and (22) holds trivially.

When $\sigma + n \in \mathbb{N}$, the coefficient of $(-\lambda)^{-N}$ in $\operatorname{Tr} S_\lambda^{(N)}$ equals

$$C_0(A, P) - C_0(A, P') = \tilde{s}^{(N)}_{\sigma+n} = \int \int_{\mathbb{R}^n} \operatorname{tr} s^{(N)h}_{-Nm-n}(x, \xi, -1)\, đ\xi dx. \tag{30}$$

This demonstrates that the term is local, and gives a means to calculate it (as indicated in [G6], Rem. 3.12): Note that

$$s^{(N)h}_{-Nm-n}(x, \xi, \lambda) = \tfrac{\partial_\lambda^{N-1}}{(N-1)!} s^h_{-m-n}(x, \xi, \lambda).$$

Since s^h_{-m-n} satisfies (26) with $N = 1$, $j = \sigma + n$, it is integrable over $\mathbb{R}^n$ when $\lambda \neq 0$. Here

$$(-\lambda)^{-1} \int_{\mathbb{R}^n} s^h_{-m-n}(x, \xi, -1)\, đ\xi = \int_{\mathbb{R}^n} s^h_{-m-n}(x, \xi, \lambda)\, đ\xi \tag{31}$$

for $\lambda \in \mathbb{R}_-$. Moreover, the integral from (28) satisfies

$$\begin{aligned} \int_{\mathbb{R}^n} s^{(N)h}_{-Nm-n}(x, \xi, \lambda)\, đ\xi &= \tfrac{\partial_\lambda^{N-1}}{(N-1)!} \int_{\mathbb{R}^n} s^h_{-m-n}(x, \xi, \lambda)\, đ\xi \\ &= \tfrac{\partial_\lambda^{N-1}}{(N-1)!} \big[(-\lambda)^{-1} \int_{\mathbb{R}^n} s^h_{-m-n}(x, \xi, -1)\, đ\xi\big] \\ &= (-\lambda)^{-N} \int_{\mathbb{R}^n} s^h_{-m-n}(x, \xi, -1)\, đ\xi, \end{aligned} \tag{32}$$

which implies

$$\tilde{s}^{(N)}_{\sigma+n}(x) = \int_{\mathbb{R}^n} s^{(N)h}_{-Nm-n}(x,\xi,-1)\, đ\xi = \int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,-1)\, đ\xi. \tag{33}$$

The latter in turned into the residue integrand for $-\frac{1}{m}\operatorname{res}(A(\log P - \log P'))$ by Lemmas 1.2 and 1.3 of [G6], as already done in Section 2 there.

We conclude:

Theorem 3.1. *Let P and P' be classical elliptic ψdo's of order $m \in \mathbb{R}_+$ and such that the principal symbol has no eigenvalues on $\mathbb{R}_-$, let A be a classical ψdo of order σ, and let $S_\lambda = A((P-\lambda)^{-1} - (P'-\lambda)^{-1})$ and $F = A(\log P - \log P')$ with symbols s resp. f.*

Consider the case $\sigma + n \in \mathbb{N}$. Then

$$C_0(A,P) - C_0(A,P') = \int_X \operatorname{tr} \tilde{s}_{\sigma+n}(x)\, dx = -\tfrac{1}{m}\operatorname{res}(A(\log P - \log P')) \tag{34}$$

where, for each x, in local coordinates,

$$\tilde{s}_{\sigma+n}(x) = \int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,-1)\, đ\xi = -\tfrac{1}{m}\int_{|\xi|=1} f_{-n}(x,\xi)\, dS(\xi). \tag{35}$$

When $\sigma + n \notin \mathbb{N}$, the identities hold trivially (with zero values).

It follows moreover:

Corollary 3.1. *If, in Theorem 3.1, P' is replaced by an operator of a different order $m' > 0$, then one has:*

$$C_0(A,P) - C_0(A,P') = -\operatorname{res}(A(\tfrac{1}{m}\log P - \tfrac{1}{m'}\log P')). \tag{36}$$

Proof. Let P_0 be an elliptic, selfadjoint positive ψdo of order m, and define $P_0^{m'/m}$ by spectral calculus; it is an elliptic, selfadjoint positive ψdo of order m'. Then by the definition of the zeta function, $C_0(A,P_0) = C_0(A,P_0^{m'/m})$. Applications of Theorem 3.1 with P, P_0 and with $P', P_0^{m'/m}$ give:

$$\begin{aligned} C_0(A,P) - C_0(A,P') &= (C_0(A,P) - C_0(A,P_0)) \\ &\quad + (C_0(A,P_0) - C_0(A,P_0^{m'/m})) + (C_0(A,P_0^{m'/m}) - C_0(A,P')) \\ &= -\operatorname{res}(A(\tfrac{1}{m}\log P - \tfrac{1}{m}\log P_0)) - \operatorname{res}(A(\tfrac{1}{m'}\log P_0^{m'/m} - \tfrac{1}{m'}\log P')) \\ &= -\operatorname{res}(A(\tfrac{1}{m}\log P - \tfrac{1}{m'}\log P')), \end{aligned}$$

since $\frac{1}{m'}\log P_0^{m'/m} = \frac{1}{m}\log P_0$. ◻

4. A formula for the zero'th coefficient

Our strategy for calculating $C_0(A,P)$ is to use (36) in combination with an exact calculation for a very special choice P_0 of P, namely

$$P_0 = ((-\Delta)^{m/2} + 1)I_M \text{ with symbol } p_0 = (|\xi|^m + 1)I_M, \tag{37}$$

in suitable local coordinates; here I_M is the $M \times M$ identity matrix (understood in the following), $M = \dim E$, and m is even.

Let A be given, of order $\sigma \in \mathbb{R}$, then we take $m > \sigma + n$. Let $\Phi_j : E|_{U_j} \to V_j \times \mathbb{C}^M$, $j = 1, \dots, J$, be an atlas of trivializations with base maps κ_j from $U_j \subset X$ to $V_j \subset \mathbb{R}^n$, let $\{\psi_j\}_{1\le j\le J}$ be an associated partition of unity (with $\psi_j \in C_0^\infty(U_j)$), and let $\varphi_j \in C_0^\infty(U_j)$ with $\varphi_j = 1$ on $\operatorname{supp}\psi_j$. Then

$$A = \sum_{1\le j\le J} \psi_j A = \sum_{1\le j\le J} \psi_j A \varphi_j + \sum_{1\le j\le J} \psi_j A(1-\varphi_j), \tag{38}$$

where the last sum is a ψdo of order $-\infty$; for this the formula (1) is well-known, since $C_0(B,P) = \operatorname{Tr} B$ when B is of order $< -n$. So it remains to treat each of the terms $\psi_j A \varphi_j$. Consider e.g. $\psi_1 A \varphi_1$. We could have assumed from the start that X was already covered by a family of open subsets $U_{j0} \subset\subset U_j$. Thus it is no restriction to assume that ψ_1 and φ_1 are supported in $U_{10} \subset\subset U_1$, where $U_{10}, U_2, \dots, U_J$ cover X.

Replace U_j by $U_j' = U_j \setminus \overline{U}_{10}$ for $j \ge 2$, and write $U_1 = U_1'$, then $\{U_j'\}_{1\le j\le J}$ also covers X. Let $\{\psi_j'\}_{1\le j\le J}$ be an associated partition of unity, and let $\varphi_j' \in C_0^\infty(U_j')$ with $\varphi_j' = 1$ on $\operatorname{supp}\psi_j'$. By construction, $\psi_1' = 1$ on U_{10}. We use the Φ_j and κ_j on these subsets (setting $\kappa_j(U_j') = V_j'$, $\kappa_j(U_{10}) = V_{10}$), and denote the induced mappings for sections by Φ_j^*.

Now the auxiliary operator P is taken to act as follows:

$$Pu = \sum_{1\le j\le J} \varphi_j' [P_0((\psi_j' u) \circ \Phi_j^{*-1})] \circ \Phi_j^*. \tag{39}$$

It is elliptic with positive definite principal symbol, and for sections supported in U_{10}, it acts like P_0 when carried over to V_{10} (being a differential operator, it is local). The resolvent $(P-\lambda)^{-1}$, defined for large λ on the rays in $\mathbb{C} \setminus \mathbb{R}_+$, is of course not local, but its symbol in the local chart $V_1 \times \mathbb{C}^M$ is, for $x \in V_{10}$, equivalent with the symbol $(|\xi|^m + 1 - \lambda)^{-1} I_M$ of $(P_0 - \lambda)^{-1}$. For resolvents of differential operators, $q(x,\xi,\lambda) \sim q_0(x,\xi,\lambda)$ means that the difference is of order $-\infty$ and $O(\lambda^{-N})$ for any N (the symbols are strongly polyhomogeneous). This difference does not affect the coefficient of $(-\lambda)^{-1}$ that we are after.

Let $a(x,\xi)$ denote the symbol of $\psi_1 A\varphi_1$ carried over to $V_1 \times \mathbb{C}^M$; it vanishes for $x \notin V_{10}$. Then the symbol of $\psi_1 A\varphi_1(P-\lambda)^{-1}$ on V_1 is equivalent with $a(x,\xi)(|\xi|^m+1-\lambda)^{-1}$ (with an error that is $O(\lambda^{-N})$, any N); we use that the symbol composition here gives only one (product) term.

To find the coefficient of $(-\lambda)^{-1}$ in the expansion of $\operatorname{Tr}(\psi_1 A\varphi_1(P-\lambda)^{-1})$, we now just have to analyze the diagonal kernel calculated in V_1:

$$\begin{aligned} K(\psi_1 A\varphi_1(P-\lambda)^{-1},x,x) &\sim \int_{\mathbb{R}^n} a(x,\xi)(|\xi|^m+1-\lambda)^{-1}\,đ\xi \\ &\sim \sum_{j\in\mathbb{N}} \tilde c_j(x)(-\lambda)^{\frac{\sigma+n-j}{m}-1} + \sum_{k\in\mathbb{N}} \big(\tilde c'_k(x)\log(-\lambda)+\tilde c''_k(x)\big)(-\lambda)^{-k-1}. \end{aligned} \tag{40}$$

Here the value can be found explicitly, as follows.

Set $a_{-n}=0$ if $\sigma+n\notin\mathbb{N}$, and decompose the symbol a in three pieces $a_{>-n}$, a_{-n} and $a_{<-n}$, where

$$\begin{aligned} a_{>-n}(x,\xi) &= \sum_{0\le j<\sigma+n} a_{\sigma-j}(x,\xi), \\ a_{<-n}(x,\xi) &= a(x,\xi)-a_{-n}(x,\xi)-a_{>-n}(x,\xi). \end{aligned} \tag{41}$$

The symbol terms $a_{\sigma-j}(x,\xi)$ are assumed to be C^∞ in (x,ξ) and homogeneous of degree $\sigma-j$ in ξ for $|\xi|\ge 1$. The strictly homogeneous version $a^h_{\sigma-j}$ is homogeneous for $\xi\ne 0$ and coincides with $a_{\sigma-j}$ for $|\xi|\ge 1$. For the terms in $a_{>-n}$, the strictly homogeneous versions are integrable in ξ at $\xi=0$.

We recall that $⨍ a(x,\xi)\,đ\xi$ is defined for each x as a finite part integral (in the sense of Hadamard), namely the constant term in the asymptotic expansion of $\int_{|\xi|\le R} a(x,\xi)\,đ\xi$ in powers R^{-m_j} and $\log R$, for $R\to\infty$. Here

$$\begin{aligned} ⨍ a_{\sigma-j}(x,\xi)\,đ\xi &= \int_{|\xi|\le 1} (a_{\sigma-j}(x,\xi)-a^h_{\sigma-j}(x,\xi))\,đ\xi, \text{ for } \sigma-j>-n, \\ ⨍ a_{-n}(x,\xi)\,đ\xi &= \int_{|\xi|\le 1} a_{-n}(x,\xi)\,đ\xi, \\ ⨍ a_{<-n}(x,\xi)\,đ\xi &= \int_{\mathbb{R}^n} a_{<-n}(x,\xi)\,đ\xi; \end{aligned} \tag{42}$$

as one can check using polar coordinates (the formulas are special cases of [G4], (1.18)).

Lemma 4.1. *For $a_{\sigma-j}(x,\xi)$ with $\sigma - j > -n$,*

$$\begin{aligned}\int_{\mathbb{R}^n} & a_{\sigma-j}(x,\xi)(|\xi|^m - \lambda)^{-1}\,d\!\!\bar{}\,\xi \\ &= (-\lambda)^{\frac{\sigma+n-j}{m}-1}\int_{\mathbb{R}^n} a^h_{\sigma-j}(x,\xi)(|\xi|^m+1)^{-1}\,d\!\!\bar{}\,\xi \\ &\quad + (-\lambda)^{-1}⨍ a_{\sigma-j}(x,\xi)\,d\!\!\bar{}\,\xi + O(\lambda^{-2}),\end{aligned} \tag{43}$$

for $\lambda \to \infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. Also $\int_{\mathbb{R}^n} a_{\sigma-j}(x,\xi)(|\xi|^m+1-\lambda)^{-1}\,d\!\!\bar{}\,\xi$ has an expansion in powers of $(-\lambda)$ plus $o(\lambda^{-1})$; here the coefficient of $(-\lambda)^{-1}$ is likewise $⨍ a_{\sigma-j}(x,\xi)\,d\!\!\bar{}\,\xi$.

For $a_{-n}(x,\xi)$ one has:

$$\begin{aligned}\int_{\mathbb{R}^n} & a_{-n}(x,\xi)(|\xi|^m - \lambda)^{-1}\,d\!\!\bar{}\,\xi \\ &= \tfrac{1}{m}(-\lambda)^{-1}\log(-\lambda)\int_{|\xi|=1} a_{-n}(x,\xi)\,d\!\!\bar{}\,S(\xi) \\ &\quad + (-\lambda)^{-1}⨍ a_{-n}(x,\xi)\,d\!\!\bar{}\,\xi + O(\lambda^{-2}),\end{aligned} \tag{44}$$

for $\lambda \to \infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. $\int_{\mathbb{R}^n} a_{-n}(x,\xi)(|\xi|^m+1-\lambda)^{-1}\,d\!\!\bar{}\,\xi$ has a similar expansion, the coefficient of $(-\lambda)^{-1}$ again being $⨍ a_{-n}(x,\xi)\,d\!\!\bar{}\,\xi$.

Proof. By homogeneity, we have for $\lambda \in \mathbb{C}\setminus\overline{\mathbb{R}}_+$, writing $\lambda = -|\lambda|e^{i\theta}$, $|\theta| < \pi$,

$$\begin{aligned}\int_{\mathbb{R}^n} & a^h_{\sigma-j}(x,\xi)(|\xi|^m + |\lambda|e^{i\theta})^{-1}\,d\!\!\bar{}\,\xi \\ &= |\lambda|^{\frac{\sigma-j+n}{m}-1}\int_{\mathbb{R}^n} a^h_{\sigma-j}(x,\eta)(|\eta|^m+e^{i\theta})^{-1}\,d\!\!\bar{}\,\eta.\end{aligned} \tag{45}$$

This equals the first term in the right hand side of (43) if $\theta = 0$, and the identity extends analytically to general λ. Moreover, since

$$\begin{aligned}(|\xi|^m-\lambda)^{-1} &= (-\lambda)^{-1}(1-|\xi|^m/\lambda) \\ &= (-\lambda)^{-1}\sum_{k\in\mathbb{N}}(|\xi|^m/\lambda)^k \text{ for } |\lambda|\ge 2,\ |\xi|\le 1,\end{aligned} \tag{46}$$

we find that for $|\lambda| \geq 2$,

$$\begin{aligned}\int_{\mathbb{R}^n} &(a_{\sigma-j}(x,\xi) - a^h_{\sigma-j}(x,\xi))(|\xi|^m - \lambda)^{-1}\, d\!\!{}^-\xi \\ &= \int_{|\xi|\leq 1} (a_{\sigma-j}(x,\xi) - a^h_{\sigma-j}(x,\xi))(|\xi|^m - \lambda)^{-1}\, d\!\!{}^-\xi \\ &= (-\lambda)^{-1} \int_{|\xi|\leq 1} (a_{\sigma-j}(x,\xi) - a^h_{\sigma-j}(x,\xi))\, d\!\!{}^-\xi + O(\lambda^{-2}).\end{aligned} \tag{47}$$

This shows (43), in view of (42).

For the next observation, we use that the preceding results give an expansion in powers of $1-\lambda$; then since

$$\begin{aligned}(1-\lambda)^k &= \sum_{0\leq l\leq k} b_l \lambda^l \text{ when } k \in \mathbb{N}, \\ (1-\lambda)^s &= (-\lambda)^s + \sum_{l\geq 1} b_l \lambda^{s-l} \text{ when } |\lambda| \geq 2,\ s \notin \mathbb{N},\end{aligned} \tag{48}$$

only $⨍ a_{\sigma-j}$ contributes to the coefficient of $(-\lambda)^{-1}$.

Now consider $a_{-n}(x,\xi)$; again we can let $\theta = 0$. Here we write

$$\begin{aligned}\int_{\mathbb{R}^n} &a_{-n}(x,\xi)(|\xi|^m - \lambda)^{-1}\, d\!\!{}^-\xi \\ &= \int_{|\xi|\geq 1} a^h_{-n}(x,\xi)(|\xi|^m + |\lambda|)^{-1}\, d\!\!{}^-\xi \\ &\quad + \int_{|\xi|\leq 1} a_{-n}(x,\xi)(|\xi|^m - \lambda)^{-1}\, d\!\!{}^-\xi.\end{aligned} \tag{49}$$

The first term gives

$$\begin{aligned}\int_{|\xi|\geq 1} &a^h_{-n}(x,\xi)(|\xi|^m + |\lambda|)^{-1}\, d\!\!{}^-\xi \\ &= |\lambda|^{-1} \int_{|\eta|\geq |\lambda|^{-1/m}} a^h_{-n}(x,\eta)(|\eta|^m + 1)^{-1}\, d\!\!{}^-\eta \\ &= |\lambda|^{-1} \int_{r\geq |\lambda|^{-1/m}} r^{-1}(r^m+1)^{-1}\, dr \int_{|\xi|=1} a_{-n}(x,\xi)\, d\!\!{}^-S(\xi) \\ &= \tfrac{1}{m}(-\lambda)^{-1}\log(-\lambda) \int_{|\xi|=1} a_{-n}(x,\xi)\, d\!\!{}^-S(\xi) + O(\lambda^{-2}),\end{aligned} \tag{50}$$

since, with $s = r^m$,

$$\int r^{-1}(r^m+1)^{-1}\, dr = \tfrac{1}{m}\int s^{-1}(s+1)^{-1}\, ds = \tfrac{1}{m}(\log s - \log(s+1)),$$

where $\log(s+1) = O(s)$. The second term gives, as in (47),

$$\int_{|\xi|\le 1} a_{-n}(x,\xi)(|\xi|^m - \lambda)^{-1}\, đ\xi = (-\lambda)^{-1}\int_{|\xi|\le 1} a_{-n}(x,\xi)\, đ\xi + O(\lambda^{-2}). \tag{51}$$

This implies (44), in view of (42). The last statement follows using (48). □

For $a_{<-n}$, it is very well known that

$$\begin{aligned}\int_{\mathbb{R}^n} a_{<-n}(x,\xi)(|\xi|^m - \lambda)^{-1}\, đ\xi &= (-\lambda)^{-1}\int_{\mathbb{R}^n} a_{<-n}(x,\xi)\, đ\xi + o(\lambda^{-1}),\\ &= (-\lambda)^{-1} ⨍ a_{<-n}(x,\xi)\, đ\xi + o(\lambda^{-1});\end{aligned} \tag{52}$$

also here,

$$\int_{\mathbb{R}^n} a_{<-n}(x,\xi)(|\xi|^m + 1 - \lambda)^{-1}\, đ\xi = (-\lambda)^{-1} ⨍ a_{<-n}(x,\xi)\, đ\xi + o(\lambda^{-1}) \tag{53}$$

follows by use of (48).

Collecting the informations, we have:

Proposition 4.1. *The coefficient of* $(-\lambda)^{-1}$ *in the expansion (40) for* $K(\psi_1 A\varphi_1(P-\lambda)^{-1}, x, x)$ *equals*

$$\tilde{c}_n(x) + \tilde{c}''_0(x) = ⨍ a(x,\xi)\, đ\xi. \tag{54}$$

In the same localization, when we calculate $\psi_1 A\varphi_1 \log P$ by a Cauchy integral (as in [G6]), the localized piece will give $\psi_1 A\varphi_1\, \mathrm{OP}(\log(|\xi|^m+1))$. The symbol of this operator is

$$r(x,\xi) = a(x,\xi)\log(|\xi|^m + 1), \tag{55}$$

which has an expansion

$$r(x,\xi) = a(x,\xi)(m\log|\xi| - |\xi|^{-m} - \sum_{j\ge 2} d_j|\xi|^{-jm}), \tag{56}$$

convergent for $|\xi| \ge 2$. Inserting the expansion of a in homogeneous terms, we find since $m > \sigma + n$ that the full term of order $-n$ in $r(x,\xi)$ is $a_{-n} m\log|\xi|$, with no log-free part. So

$$\mathrm{res}_{x,0}\, r = \int_{|\xi|=1} \mathrm{tr}\, r_{-n,0}(x,\xi) đS(\xi) = 0. \tag{57}$$

It follows that the coefficient of $(-\lambda)^{-1}$ in the trace expansion of $\psi_1 A\varphi_1(P-\lambda)^{-1}$ is

$$\int_{\mathbb{R}^n} ⨍ \operatorname{tr} a(x,\xi)\, đ\xi dx = \int_{\mathbb{R}^n} (\mathrm{TR}_x(\psi_1 A\varphi_1) - \tfrac{1}{m}\operatorname{res}_{x,0}(\psi_1 A\varphi_1 \log P))\, dx, \tag{58}$$

using that $\operatorname{res}_{x,0}(\psi_1 A\varphi_1 \log P)$ is 0. This shows formula (1) in this very particular case:

$$C_0(\psi_1 A\varphi_1, P) = \int_{\mathbb{R}^n} (\mathrm{TR}_x(\psi_1 A\varphi_1) - \tfrac{1}{m}\operatorname{res}_{x,0}(\psi_1 A\varphi_1 \log P))\, dx. \tag{59}$$

Now, to find $C_0(\psi_1 A\varphi_1, P')$ for a general operator P' of order $m' \in \mathbb{R}_+$, we combine (59) with the trace defect formula (36). This gives, in the considered local coordinates:

$$\begin{aligned} C_0(\psi_1 A\varphi_1, P') &= C_0(\psi_1 A\varphi_1, P) + C_0(\psi_1 A\varphi_1, P') - C_0(\psi_1 A\varphi_1, P) \\ &= \int_{\mathbb{R}^n} (\mathrm{TR}_x(\psi_1 A\varphi_1) - \tfrac{1}{m}\operatorname{res}_{x,0}(\psi_1 A\varphi_1 \log P))\, dx \\ &\quad - \operatorname{res}(\psi_1 A\varphi_1(\tfrac{1}{m'}\log P' - \tfrac{1}{m}\log P)) \\ &= \int_{\mathbb{R}^n} (\mathrm{TR}_x(\psi_1 A\varphi_1) - \tfrac{1}{m'}\operatorname{res}_{x,0}(\psi_1 A\varphi_1 \log P'))\, dx. \end{aligned} \tag{60}$$

To this we can add:

$$\begin{aligned} &C_0(\psi_1 A(1-\varphi_1), P') = \operatorname{Tr}(\psi_1 A(1-\varphi_1)) \\ &\quad = \int (\mathrm{TR}_x(\psi_1 A(1-\varphi_1)) - \tfrac{1}{m'}\operatorname{res}_{x,0}(\psi_1 A(1-\varphi_1)\log P'))\, dx, \end{aligned} \tag{61}$$

where both terms have a meaning on X; TR_x defines the ordinary trace integral and $\operatorname{res}_{x,0}$ is zero.

The method applies likewise to all the other terms $\psi_j A\varphi_j$. Collecting the terms, and relabelling P' of order m' as P of order m, we have found:

Theorem 4.1. *Let A be a classical ψdo of order $\sigma \in \mathbb{R}$, and let P be a classical elliptic ψdo's of order $m \in \mathbb{R}_+$ such that the principal symbol has no eigenvalues on $\mathbb{R}_-$. We have in local coordinates as used above:*

$$\begin{aligned} C_0(A,P) &= \sum_{1\le j\le J} C_0(\psi_j A, P), \ \text{ where} \\ C_0(\psi_j A, P) &= \int (\mathrm{TR}_x(\psi_j A) - \tfrac{1}{m}\operatorname{res}_{x,0}(\psi_j A \log P))\, dx \end{aligned} \tag{62}$$

(the contribution from $\psi_j A\varphi_j$ defined in the corresponding local chart and that from $\psi_j A(1-\varphi_j)$ defined as an ordinary trace).

Note that $C_0(A,P)$ is independent of how we localize, so the expression resulting from (62) is independent of the choice of localization.

The logarithm is here defined by cutting the complex plane along $\mathbb{R}_-$. If P is given with another ray free of eigenvalues, the formulas hold with the logarithm defined to be cut along this ray. (We do not here study the issue of how these expressions depend on the ray.)

The invariance of the density $\big(\mathrm{TR}_x(A) - \frac{1}{m}\operatorname{res}_{x,0}(A\log P)\big)\,dx$ in the formula (1) is verified in [PS] also by a direct calculation. We note that (1) gives back the known formula

$$C_0(A,P) = \mathrm{TR}(A) \tag{63}$$

in cases where $\operatorname{res}_{x,0}(A\log P)$ vanishes. This is so when $\sigma + n \notin \mathbb{N}$ ([KV], [L]), and also in cases $\sigma + n \in \mathbb{N}$ with parity properties ([KV], [G4]): When $\sigma \in \mathbb{Z}$, we say that A has even-even alternating parity (in short: is even-even), resp. has even-odd alternating parity (in short: is even-odd), when

$$\begin{aligned} a_{\sigma-j}(x,-\xi) &= (-1)^{\sigma-j} a_{\sigma-j}(x,\xi), \text{ resp.} \\ a_{\sigma-j}(x,-\xi) &= (-1)^{\sigma-j-1} a_{\sigma-j}(x,\xi), \end{aligned} \tag{64}$$

for $|\xi| \ge 1$, all j. When P is even-even of even order m, then the classical part of $\log P$ is even-even. Then if (a) or (b) is satisfied:

(a) A is even-even and n is odd,

(b) A is even-odd and n is even,

$\operatorname{res}_{x,0}(A\log P)$ vanishes, $\mathrm{TR}_x A\,dx$ is a globally defined density, and (63) holds. [KV] treats the case (a), calling the even-even operators odd-class (perhaps because they have a canonical trace in odd dimension). The statements on $\mathrm{TR}_x A\,dx$ are extended to log-polyhomogeneous operators in [G4].

Observe a general consequence:

Corollary 4.1. *When A has order $\sigma \in \mathbb{Z}$ and satisfies (a) or (b), then $\operatorname{res}_{x,0}(A\log P)\,dx$ defines a global density for any P.*

Proof. In these cases, since $\mathrm{TR}_x A\,dx$ defines a global density, the other summand in $\big(\mathrm{TR}_x(A) - \frac{1}{m}\operatorname{res}_{x,0}(A\log P)\big)\,dx$ must do so too. (Note that P is not subject to order or parity restrictions here.) □

Appendix A. Corrections to earlier papers

Correction to [GS2]: In Corollary 2.10 on page 45, the formulas in (2.38) for the expansion coefficients $\tilde a_{j,l}$ are true only for $l = m_j$. For $l < m_j$, the $\tilde{a}_{j,l}$ depend on the full set $\{a_{j,l} \mid 0 \le j \le m_j\}$. This is so, because the Taylor expansion of $\Gamma(1-s)^{-1}$ must be taken into account.

Hence in the comparison of the expansion of $\operatorname{Tr}(A(P-\lambda)^{-1})$ with $\zeta(A,P,s)$, only the primary coefficient at each pole of $\Gamma(s)\zeta(A,P,s)$ is directly proportional to a coefficient in $\operatorname{Tr}(A(P-\lambda)^{-1})$. Similar statements hold for comparisons with $\operatorname{Tr}(A(P-\lambda)^{-N})$. This has lead to systematic inaccuracies in a number of subsequent works, however without substantial damage to the results in general.

We explain the needed correction in detail for [G4] and then list the related modifications needed in other papers (including a few additional corrections).

Corrections to [G4]:

1) The statements on page 69 linking the coefficients in (1.1) with the coefficients in (1.2) with the same index by universal proportionality factors is incorrect if $\nu+n \in \mathbb{N}$; the direct proportionality holds only for the primary pole coefficients, not for the next Laurent coefficient at each pole. Instead, at the second-order poles $-k, k \in \mathbb{N}$, there are universal linear transition formulas linking the coefficient set for $(-\lambda)^{-k-N}\log(-\lambda)$ and $(-\lambda)^{-k-N}$ with the coefficient set for $(s+k)^{-2}$ and $(s+k)^{-1}$.

This follows from [GS2, Prop. 2.9], (3.21), as explained in Lemma 2.1 of the present paper. The coefficients of $\operatorname{Tr}(A(P-\lambda)^{-N})$ at integer powers are directly proportional to the Laurent coefficients of the meromorphic function $\psi(s)$, where (with $N-1$ denoted M)

$$\begin{aligned} \zeta(A,P,s) &= \tfrac{M!}{(s-M)\dots(s-1)}\tfrac{1}{\pi}\sin(\pi(s-M))\psi(s-M),\\ \Gamma(s)\zeta(A,P,s) &= \tfrac{M!}{\Gamma(M-s)}\psi(s-M). \end{aligned} \tag{A.1}$$

(Cf. (18), use that $\frac{1}{\pi}\sin(\pi(s-M)) = (-1)^M/[\Gamma(s-1)\Gamma(1-s)]$.) In calculations of Laurent series at the poles, the Taylor expansion of the factor in front of $\psi(s)$ effects the higher terms.

Specifically in [G4], the sentence "The coefficients $\tilde{c}_j$ and c_j, $\tilde{c}'_k$ and c'_k, resp. $\tilde{c}''_k$ and c''_k are proportional by universal nonzero constants." should be replaced by: "The coefficients $\tilde{c}_j$ and c_j, resp. $\tilde{c}'_k$ and c'_k, are proportional by universal nonzero constants. When the c'_k vanish (e.g., when $\nu+n \notin \mathbb{N}$), the same holds for $\tilde{c}''_k$ and c''_k. More generally, the pair $\{\tilde{c}'_k, \tilde{c}''_k\}$ is for each k universally related to the pair $\{c'_k, c''_k\}$ in a linear way." The statement "$\tilde{c}''_0 = c''_0$" should be replaced by "$\tilde{c}''_0 = c''_0$ when $c'_0 = 0$", and the description of $C_0(A,P)$ in terms of resolvent trace expansion coefficients should be replaced by the description given in the present paper in Section 2.

However, since this changes the formula for $C_0(A,P)$ only by a multiple of $\operatorname{res} A$, the results of [G4] on $C_0(A,P)$ remain valid, because they are concerned with cases where $\operatorname{res} A = 0$. The statements Th. 1.3 (ii), Cor. 1.5 (ii) on the vanishing of all log-coefficients in parity cases still imply the vanishing of all double poles in (1.2).

In Section 3, the coefficients in (3.32) are linked with those in (3.30) in a more complicated way than stated, where only the leading coefficient at a pole is directly proportional to a coefficient in (3.30). But again, the results for parity cases remain valid since the needed correction terms vanish in these cases.

2) Page 79, remove the factor 2 (twice) in formula (1.44).

3) Page 84, formulas (3.9) and (3.10): The sums over k' should be removed, and so should the additional term $-P^{-s-1}$ in the first line. So $\mathcal{P}_l(P) = (-\log P)^l$ for all l.

4) Page 91, in formula (3.47), $-P^{-1}$ should be removed.

Correction to [G1]: Page 92, lines 7–8 from below, replace "The coefficients in (9.10) are proportional to those in (9.9) by universal factors." by "The unprimed coefficients in (9.10) are proportional to those in (9.9) by universal factors. For each k, the pair $\{\tilde a_{i,k}, \tilde a'_{i,k}\}$ (resp. $\{\tilde b_{i,k}, \tilde b'_{i,k}\}$) is universally related to the pair $\{a_{i,k}, a'_{i,k}\}$ (resp. $\{b_{i,k}, b'_{i,k}\}$) in a linear way."

Corrections to [G2]: Page 4, lines 5–7 from below, replace "The coefficients $\tilde a_k$, $\tilde a'_k$ and $\tilde a''_k$ are proportional to the coefficients a_k, a'_k and a''_k in (0.1) (respectively) by universal nonzero proportionality factor (depending on r)." should be replaced by: "The coefficients $\tilde a_k$ and $\tilde a'_k$ are proportional to the coefficients a_k and a'_k in (0.1) (respectively) by universal nonzero proportionality factor (depending on r). For each $k \ge 0$, the pair $\{\tilde a'_k, \tilde a''_k\}$ is universally related to the pair $\{a'_k, a''_k\}$ in a linear way."

Corrections to [G3]: Page 262, lines 13–14 should be changed as for [G2] above. In line 15, "$a''_0(F) = \tilde a''_0(F)$" should be replaced by: "$a''_0(F) = \tilde a''_0(F)$ if $\tilde a'_0(F) = 0$". There are some consequential reformulations in Sections 4 and 5, which do not endanger the results since the a''_0 terms are characterized in general modulo local contributions (and a'_0 is such), with precise statements only when $a'_0 = 0$.

Corrections to [G5]: The statement on Page 44, lines 11–12 from below "There are some universal proportionality factors linking the coefficients $\tilde c_j$ and c_j, $\tilde c'_k$ and c'_k, resp. $\tilde c''_k$ and c''_k" should be changed as indicated for [G4].

Correction to [GSc1]: Page 171, line 1, "The coefficients $\tilde{c}_j$, $\tilde{c}'_l$, $\tilde{c}''_l$ are proportional to the coefficients c_j, c'_l , c''_l by universal constants." should be replaced by "The coefficients $\tilde{c}_j$, $\tilde{c}'_l$ are proportional to the coefficients c_j, c'_l by universal constants."

Corrections to [GSc2]: The definition of $C_0(A, P)$ on page 1644 and the statements in lines 6–8 on page 1645 should be modified as for [G4]. This has no consequences for the results, which are mainly concerned with the trace definitions *modulo local contributions*, with exact formulas established only when the residue corrections vanish.

Corrections to [G6]: When the order m of the auxiliary operator P_1 is odd, the classical part of the symbol of $\log P_1$ does not satisfy the transmission condition, so the formulas referring to the residue definition of [FGLS] are only valid when m is even. This goes for the right-hand side in formula (3.41) of Theorem 3.10, which can however be interpreted in a more general sense when m is odd, since the local estimates in the proof remain valid. Similar remarks hold for formula (4.32) in Theorem 4.5 and (5.9) in Theorem 5.2.

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