HEAT TRACE EXPANSIONS FOR ELLIPTIC SYSTEMS WITH PSEUDODIFFERENTIAL BOUNDARY CONDITIONS

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1. Introduction.

One of the purposes of this paper is to prove asymptotic expansions of heat traces

$$
\begin{gathered}
\operatorname{Tr}(\varphi e^{-t\Delta_i}) \sim \sum_{-n\le k<0} a_{i,k} t^{k/2} + \sum_{k=0}^{\infty} (a_{i,k}\log t + a'_{i,k}) t^{k/2}, \text{ for } t\to 0,\\
\Delta_1 = {D_B}^* D_B, \quad \Delta_2 = D_B {D_B}^*,
\end{gathered} \tag{1.1}
$$

for general realizations D_B of first-order differential operators D (e.g. Dirac-type operators) on a manifold X with pseudodifferential boundary conditions: $B(u|_{X'}) = 0$ at the boundary $\partial X = X'$. In (1.1), φ denotes a compactly supported morphism. The unprimed coefficients are locally determined, the primed coefficients global.

Such realizations were considered first by Atiyah, Patodi and Singer in [APS75] who showed an interesting index formula in the so-called product case, when X is compact. We say that D is *of Dirac-type* when $D = \sigma(\partial_{x_n} + A_1)$ on a collar neighborhood of X', with a unitary morphism σ and a first-order differential operator A_1 such that $A_1 = A + x_n P_1 + P_0$ with A selfadjoint on X' and constant in x_n and the P_j of order j; the *product case* is where $P_1 = P_0 = 0$. B was in [APS75] taken equal to the orthogonal projection $\Pi_\ge$ onto the eigenspace for A associated with eigenvalues ≥ 0.

For Dirac-type operators on compact manifolds, finite expansions (1.1) (up to $k = 0$, with $\varphi = 1$ and $a_{i,0} = 0$) were shown in [G92], implying the index formula

$$
\operatorname{index} D_B = a'_{1,0} - a'_{2,0}, \quad \text{when } \varphi = 1 \text{ and } X \text{ is compact.} \tag{1.2}
$$

Full expansions were established in Grubb and Seeley [GS95], with precisions for the product case in [GS96]. Here $B = \Pi_\ge + B_0$ with special finite rank perturbations B_0.

Booss-Bavnbek and Wojciechowski studied, for the compact product case, the index of D_B in [BW93] and other works with $B = C^+ + S$, where C^+ is the Calderón projector for D (having the same principal part as $\Pi_\ge$) and S is a pseudodifferential operator (ψdo) of order -1. One of our motivations for the present work was to establish (1.1) for such problems too. A different type of boundary condition was introduced by Brüning and Lesch in [BL97] (in a study of the gluing problem for the eta invariant), showing heat trace expansions in the product case but with B principally different from $\Pi_\ge$ (Example 3.6 below). For this type, we obtain (1.1) without the product assumption.

Actually, we find that there are many more boundary conditions, different from the above, for which (1.1) can be obtained. In fact, D need not even be of Dirac-type, but can be *any first-order elliptic differential operator*. B need not be closely linked to the

Calderón projector but can be *any ψdo that is well-posed for D* in the sense defined by Seeley in [S69, Ch. VI]. We obtain (1.1) (and consequently also the index formula (1.2) when X is compact and $\varphi = I$) in all these cases, including the previously known cases.

The freedom to choose more general B seems to be useful e.g. for variational studies. It is also interesting to allow general D that are not tied, by the requirement of (principal) selfadjointness of the tangential part, to a specific choice of Hermitian structures.

In our method to establish (1.1), we imbed D_B and $D_B{}^*$, which are in themselves only injectively elliptic, into a truly elliptic system $\mathcal{D}_{\mathcal{B}}$, which we treat by use of the Calderón projector for $\mathcal{D}+\mu$ and by an elaboration of the calculus of weakly polyhomogeneous ψdo's introduced in [GS95]. This treatment works also for general elliptic systems P of order $d \geq 1$ with appropriate pseudo-normal ψdo boundary conditions $S\varrho u = 0$. We show a general result on resolvent expansions and heat trace expansions for such realizations:

$$\begin{aligned} \operatorname{Tr} \varphi \partial_\lambda^m (P_S - \lambda)^{-1} &\sim \sum_{-n\le k<0} \tilde{c}_k (-\lambda)^{\frac{k}{d}-m-1} + \sum_{k=0}^{\infty} (\tilde{c}_k \log(-\lambda) + \tilde{c}_k')(-\lambda)^{\frac{k}{d}-m-1}, \\ \operatorname{Tr} \varphi e^{-tP_S} &\sim \sum_{-n\le k<0} c_k t^{\frac{k}{d}} + \sum_{k=0}^{\infty} (c_k \log t + c_k') t^{\frac{k}{d}}, \text{ for } t \to 0; \end{aligned} \tag{1.3}$$

in the first formula, $\lambda \to \infty$ on a ray in $\mathbb{C}$, and the second formula follows when this holds on all rays with argument in $[\frac{\pi}{2}, \frac{3\pi}{2}]$. Such expansions were shown in cases where S is a differential operator by Seeley [S69,71] and Greiner [Gre71]; then there are no logarithmic terms and all the coefficients are locally defined. The crucial step in the analysis is to find the symbol structure of the resolvent. We do this not only for compact manifolds but also in noncompact situations with spatially uniform estimates; here we use the calculi established in [GK93] (with Kokholm), [G95], [G96].

The plan of the paper is as follows: We explain the general set-up in detail in Section 2, and the special definitions and adaptations for first-order problems in Section 3, referring also to the Appendix where the main properties of the Calderón projector are explained. In Section 4 we recall the calculus of weakly polyhomogeneous ψdo's introduced in [GS95] and show a needed result on spectral invariance, also for one-sided elliptic cases and noncompact manifolds, drawing on results from [G95]. In Section 5, we apply the various results to establish a decomposition of the resolvent in a sum of compositions with strongly and weakly polyhomogeneous factors. In Section 6 we derive trace results from this by use of methods from [GS95] and [GS96], obtaining in particular (1.1) and (1.2) for first-order operators with well-posed boundary conditions.

2. The general set-up.

On an n-dimensional C^∞ manifold X with boundary $\partial X = X'$ we consider an elliptic differential operator of order d, $P\colon C^\infty(X, E_1) \to C^\infty(X, E_2)$, between sections of Hermitian C^∞ vector bundles E_1 and E_2 of dimension N. X is provided with a smooth volume element $v(x)dx$ defining a Hilbert space structure on the sections.

In order to include noncompact manifolds such as $\mathbb{R}^n$, $\overline{\mathbb{R}}^n_+$ and exterior domains $\mathbb{R}^n \setminus Y$, $\overline{\mathbb{R}}^n_+ \setminus Y$ (Y smooth compact), we take X to be *admissible* as defined in [GK93], [G96]; this means that X is the union of a compact piece and finitely many conical pieces of the form $\{x = tx_0 \mid x_0 \in M \subset S^{n-1}, t > r\}$. X is covered by a finite system of local coordinate patches diffeomorphic to either bounded or conical open subsets of $\mathbb{R}^n$. We refer the reader to the references for detailed descriptions; the crucial assumption is that the admissible coordinate changes κ are such that $|\kappa(x)-\kappa(y)|/|x-y|$ is bounded above and below by positive constants, and all derivatives of κ and κ^{-1} are bounded. Admissible vector bundles are likewise defined. The differential operators and ψdo's considered in this context are defined by reference to the admissible local coordinate systems; their symbols are assumed to have global estimates in the space variable x, as in Hörmander [H85, Sect. 18.1]. This allows precise rules of calculus, with the usual composition formulas; the concepts are extended to pseudodifferential boundary operators in [G96] (and [GK93]). For brevity, we shall call such operators admissible (in [G96] they are called uniformly estimated or globally estimated), and we always assume in the following when working with admissible manifolds that the operators are of this type. A reader who is mainly interested in the case of compact manifolds can just disregard this generality.

We denote by $H^s(X,E_1)$ or just $H^s(E_1)$ the Sobolev space of sections of E_1 of order s, defined in terms of admissible local coordinates; a similar notation will be used for other manifolds and vector bundles.

The restrictions of the E_i to the boundary X' are denoted E_i'. We assume that a normal coordinate x_n has been chosen in a neighborhood U of the boundary X' such that the points are represented as $x=(x',x_n)$ there with $x'\in X'$, $x_n\in[0,c(x')[$, the E_i are isomorphic to the pull-backs of the E_i' there, and there is a normal derivative ∂_{x_n}. X' is provided with the volume element $v(x',0)dx'$ induced by $v(x',x_n)dx'dx_n$ on U. For a compact manifold, we take U as a collar neighborhood $X_c=X'\times[0,c[$; more generally this is used for the compact part and extended conically in the conical parts (cf. [G96, Sect. A.5]).

Let $\varrho=\{\gamma_0,\dots,\gamma_{d-1}\}$ with $\gamma_j u=(-\mathrm{i}\partial_{x_n})^j u|_{x_n=0}$ (i denotes the imaginary unit $\sqrt{-1}$). For $s>d-\frac12$, ϱ maps $H^s(E_i)$ into $\mathcal{H}^s(E_i'^d)=\prod_{0\le j<d}H^{s-j-\frac12}(E_i')$ $(E_i'^d=\bigoplus_{0\le j<d}E_i')$. The sections u of E_1 and w of E_2 in H^s $(s>d-\frac12)$ satisfy the Green's formula

$$(2.1)\qquad \begin{gathered}(Pu,w)_X-(u,P^*w)_X=(\mathcal{A}\varrho u,\varrho w)_{X'},\\ \mathcal{A}=(\mathcal{A}_{jk})_{j,k=0,\dots d-1}\text{ with }\mathcal{A}_{jk}\text{ of order }d-1-j-k.\end{gathered}$$

Here the $\mathcal{A}_{jk}$ are differential operators; those with $k>d-1-j$ are 0 ($\mathcal{A}$ is upper skew-triangular) and those with $k=d-1-j$ are isomorphisms, so $\mathcal{A}$ has an inverse of a similar type, just lower skew-triangular.

When S is an operator on $\mathcal{H}^d(E_1'^d)$, the boundary condition

$$(2.2)\qquad S\varrho u=0$$

determines the realization P_S of P, defined as the operator acting like P and with domain

$$(2.3)\qquad D(P_S)=\{u\in H^d(X,E_1)\mid S\varrho u=0\}.$$

We shall study boundary conditions that are *pseudo-normal* in the following sense:

Assumption 2.1. (Pseudo-normality) *S is a matrix of admissible classical ψdo's S_{jk} going from E_1' to admissible bundles F_j over X' such that*

$$(2.4)\qquad \begin{gathered}S=(S_{jk})_{j,k=0,\dots,d-1},\quad\text{with } S_{jk}\text{ of order } j-k,\quad S_{jk}=0\text{ for } j<k,\\ S_{jj}\text{ surjective and uniformly surjectively elliptic.}\end{gathered}$$

For convenience of notation, we here include bundles F_j of dimension 0. We denote $\bigoplus_{0\le j<d} F_j = F$. That symbols and operators are taken admissible when the manifolds and bundles are so, will often be tacitly understood.

The new generality in comparison with the *normal* boundary conditions considered in [G96] (for compact manifolds, one can also find the information in [G86], this will not be repeated), is that the S_{jj} are now allowed to be ψdo's; this is needed in our application to first-order operators. The normal boundary conditions have just surjective *morphisms* as the S_{jj}, hence regularity $\nu > 0$, whereas the present boundary conditions have regularity $\nu = 0$, in the sense of the regularity concept from [G96]. (There is a discussion in [G96, Remark 1.5.8]. Note that the book also allows nonlocal terms in the interior, excluded here.)

Our basic hypothesis for the resolvent analysis is the following:

Assumption 2.2. (Resolvent growth condition) *Let $E_1 = E_2 = E$. There is an open sector $\Gamma = \{\lambda \in \mathbb{C}\setminus\{0\} \mid \arg\lambda \in J\}$ (for an open interval $J \subset [0, 2\pi]$) such that the following holds:*

1° P is elliptic, and for the principal symbol p^0 of P, $p^0(x,\xi) - \lambda$ is invertible for all (x,ξ,λ) with $\lambda \in \Gamma \cup \{0\}$, $|\xi|^2 + |\lambda|^{2/d} \ge 1$, the inverse being $O((|\xi|^d + |\lambda|)^{-1})$ on closed subsectors Γ', uniformly in x.

2° F has dimension $Nd/2$, the system $\{P, S\varrho\}$ is elliptic, and for any closed subsector Γ' there is an $r \ge 0$ such that the resolvent $R_\lambda = (P_S - \lambda)^{-1}$ exists as a bounded operator in L_2 and is $O(\lambda^{-1})$ for $\lambda \in \Gamma'_r$;

$$\Gamma'_r = \{\lambda \in \Gamma' \mid |\lambda| \ge r\}. \tag{2.5}$$

The first condition means uniform parameter-ellipticity of $P - \lambda$, as defined in [G96, Sect. 3.1].

The second condition contains a global requirement of invertibility. If $S\varrho$ is *normal*, such invertibility for large λ is assured by a condition on principal symbols, namely uniform parameter-ellipticity of $\{P - \lambda, S\varrho\}$ as defined in [G96, Sect. 3.1]. This means that the associated model problem on $\mathbb{R}_+$ for each (x', ξ', λ) with $|\xi'|^2 + |\lambda|^{2/d} = 1$ is uniquely solvable with uniform bounds in x' for the solution operator, for λ in closed subsectors of Γ. Then the results of [G96, Sect. 3.3] imply invertibility with the $O(\lambda^{-1})$ estimate for large λ. When S is just pseudo-normal, condition 2° is more general.

R_λ will now be supplied with a Poisson operator K_λ to define an inverse of the full system $\{P - \lambda, S\varrho\}$. In the following lemma, $K_{\varrho,\lambda}$ denotes an auxiliary Poisson operator such that $\varrho K_{\varrho,\lambda} = I$, constructed e.g. as in [G96, Lemma 1.6.4] with $\langle\xi\rangle$ replaced by $\langle(\xi, |\lambda|^{1/d})\rangle$. (We use the notation $\langle x\rangle = (|x_1|^2 + \dots + |x_\nu|^2 + 1)^{\frac12}$ for $x = (x_1, \dots, x_\nu)$.) In its dependence on $\mu = |\lambda|^{1/d}$, $K_{\varrho,\lambda}$ is strongly polyhomogeneous on all rays, cf. Section 4, [GS95, App.]. If holomorphy in λ is desired, one can instead take the Poisson operator $K_{\varrho,\lambda}\colon \varphi \mapsto u$ solving the following Dirichlet problem, where Λ^{2d} is a positive differential operator with principal symbol $\langle\xi\rangle^{2d}$ and $|\arg\lambda - \omega| < \pi/2$:

$$(\Lambda^{2d} + (e^{-i\omega}\lambda)^2)u = 0 \text{ on } X, \quad \varrho u = \varphi \text{ on } X'.$$

Lemma 2.3. *Let Assumptions 2.1 and 2.2 hold. For the λ such that R_λ is defined, there exists a unique Poisson operator K_λ such that*

$$\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}^{-1} = \begin{pmatrix} R_\lambda & K_\lambda \end{pmatrix}. \tag{2.6}$$

In a neighborhood of each ray in Γ, K_λ equals

$$K_\lambda = [I - R_\lambda(P-\lambda)]K_{\varrho,\lambda}S'; \tag{2.7}$$

here $S' = (S'_{jk})_{j,k=0,\dots,d-1}$ is a right inverse of S, constructed such that for all j,k, S'_{jk} is a classical ψdo of order $j-k$, $S'_{jk}=0$ for $j<k$, and S'_{jj} is injective and injectively elliptic; and $K_{\varrho,\lambda}$ is an auxiliary right inverse of ϱ as described above.

Proof. Let us first explain the construction of S'. We can write $S = S_{\mathrm{diag}} + S_{\mathrm{sub}}$, where $S_{\mathrm{diag}} = (\delta_{jk}S_{jk})_{j,k=0,\dots,d-1}$ and S_{sub} is subtriangular (has zero entries in and above the diagonal). Here S_{diag} is surjective and surjectively elliptic of order 0 from $E_1'^d$ to F, hence $S_{\mathrm{diag}}S_{\mathrm{diag}}{}^*$ is bijective and elliptic of order 0 in F and therefore has an (elliptic) inverse $[S_{\mathrm{diag}}S_{\mathrm{diag}}{}^*]^{-1}$. Then S_{diag} has the right inverse $S'_{\mathrm{diag}} = S_{\mathrm{diag}}{}^*[S_{\mathrm{diag}}S_{\mathrm{diag}}{}^*]^{-1}$; again a classical ψdo of order 0. Finally, since $SS'_{\mathrm{diag}} = I + S_{\mathrm{sub}}S'_{\mathrm{diag}}$, where $S_{\mathrm{sub}}S'_{\mathrm{diag}}$ is subdiagonal and hence nilpotent, S has the right inverse

$$S' = S'_{\mathrm{diag}}(I + S_{\mathrm{sub}}S'_{\mathrm{diag}})^{-1} = S'_{\mathrm{diag}} \textstyle\sum_{0\le l<d}(-S_{\mathrm{sub}}S'_{\mathrm{diag}})^l;$$

it is of the asserted form.

The operator K_λ required in (2.6) is the solution operator for the problem

$$(P-\lambda)u = 0 \text{ on } X, \quad S\varrho u = \varphi \text{ on } X'. \tag{2.8}$$

First note that since R_λ is injective, the problem has at most one solution u for any φ. Define K_λ by (2.7); then check that $u = K_\lambda\varphi$ solves (2.8) by observing:

$$(P-\lambda)[I - R_\lambda(P-\lambda)] = 0 \text{ since } (P-\lambda)R_\lambda = I,$$

and, using that $S\varrho R_\lambda = 0$,

$$S\varrho K_\lambda = S\varrho K_{\varrho,\lambda}S' = I. \quad \square$$

For each fixed λ, the inverse $\begin{pmatrix} R_\lambda & K_\lambda \end{pmatrix}$ belongs to the pseudodifferential boundary operator calculus, but to start with, we in general only have a rough information on the behavior of R_λ with respect to λ that comes from its definition as a resolvent. Before showing this in an elementary lemma, let us recall the definition of parameter-dependent Sobolev spaces (used e.g. in [G96], [GS95]):

For $s\in\mathbb{R}$, the space $H^{s,\mu}(\mathbb{R}^n)$ is the Sobolev space provided with the norm

$$\|u\|_{H^{s,\mu}} = \|\langle(\xi,\mu)\rangle^s\hat u(\xi)\|_{L_2(\mathbb{R}^n)}. \tag{2.9}$$

The notion is extended to sections of a Hermitian bundle F over X by use of a finite family of admissible local coordinate systems (the space is then denoted $H^{s,\mu}(X,F)$ or $H^{s,\mu}(F)$). Note that $H^{s,0}(F)\simeq L_2(F)$, and that for $s\ge 0$, the norm is equivalent with $(\|u\|_{H^s}^2 + \langle\mu\rangle^{2s}\|u\|_{L_2}^2)^{\frac12}$.

Lemma 2.4. *Let R_λ and K_λ be as in Lemma 2.3. For any $s \ge 0$, R_λ and K_λ define continuous mappings (where $\mu = |\lambda|^{1/d}$, $\mathcal{H}^{s+d,\mu}(F) = \prod_{0\le j<d} H^{s+d-j-\frac12,\mu}(F_j)$)*

$$\begin{aligned} R_\lambda &: H^{s,\mu}(E) \to H^{s+d,\mu}(E), \\ K_\lambda &: \mathcal{H}^{s+d,\mu}(F) \to H^{s+d,\mu}(E), \end{aligned} \tag{2.10}$$

uniformly for λ in subsectors Γ'_r (as in Assumption 2.2).

Proof. From the elliptic regularity for the λ-independent system $\{P, S\varrho\}$ and from the resolvent growth condition follows that for $k \ge 1$, $v \in D(P_S) \cap H^{kd}(E_1)$,

$$\|v\|_{H^{kd}} \le c_{1,k}(\|P_S v\|_{H^{(k-1)d}} + \|v\|_{H^{(k-1)d}}), \quad |\lambda|\,\|R_\lambda f\|_{L_2} \le c_2\|f\|_{L_2}, \tag{2.11}$$

uniformly for $\lambda \in \Gamma'_r$. We use this first with $v = R_\lambda f$ and $k = 0$ to see that on the ray $\lambda = \mu^d e^{i\theta}$, $\mu \ge r^{1/d}$,

$$\begin{aligned} \|R_\lambda f\|_{H^{d,\mu}} &\le c_3(\|R_\lambda f\|_{H^d} + \langle\lambda\rangle\|R_\lambda f\|_{L_2}) \\ &\le c_4(\|(P_S - \lambda)R_\lambda f\|_{L_2} + \langle\lambda\rangle|\,\|R_\lambda f\|_{L_2} + \|R_\lambda f\|_{L_2}) \le c_5\|f\|_{L_2}; \end{aligned} \tag{2.12}$$

in other words, R_λ is continuous from $L_2(E)$ to $H^{d,\mu}(E)$, uniformly for $\mu \ge r^{1/d}$.

Next, combining (2.11) with (2.12) we find for $k = 1$:

$$\begin{aligned} \|R_\lambda f\|_{H^{2d,\mu}} &\le c'_3(\|R_\lambda f\|_{H^2 d} + \langle\lambda\rangle^2\|R_\lambda f\|_{L_2}) \\ &\le c'_4(\|(P_S - \lambda)R_\lambda f\|_{H^d} + |\lambda|\,\|R_\lambda f\|_{H^d} + \|R_\lambda f\|_{H^d} + \langle\lambda\rangle^2\|R_\lambda f\|_{L_2}) \\ &\le c'_5(\|f\|_{H^d} + \langle\lambda\rangle\|f\|_{L_2}) \le c_6\|f\|_{H^{d,\mu}}. \end{aligned}$$

This can be continued to give $H^{(k+1)d,\mu}$ estimates of $R_\lambda f$ in terms of $H^{kd,\mu}$ estimates of f for $k = 2, 3, \dots$, and we conclude that the first line in (2.10) holds for $s = dk$, $k = 0, 1, 2, \dots$. The remaining values of $s \ge 0$ are included by interpolation.

For the second line, we observe: When C is a parameter-independent ψdo on X' of order $l \ge 0$, it is bounded from $H^{s,\mu}$ to $H^{s-l,\mu}$ for all $s \in \mathbb{R}$, uniformly in μ; cf. Section 2.5 in [G96] (using that C is of regularity $\nu = l \ge 0$). It follows that S' maps $\mathcal{H}^{s,\mu}(E'^d) = \prod_{0\le j<d} H^{s-j-\frac12,\mu}(E')$ into $\mathcal{H}^{s,\mu}(F)$ with uniform bounds in μ, for $s \in \mathbb{R}$. [G96] also shows that ϱ maps $H^{s,\mu}(E)$ into $\mathcal{H}^{s,\mu}(E'^d)$ for $s > d - \frac12$ and that $K_{\varrho,\lambda}$ is continuous in the opposite direction, with uniform bounds in μ. Applying these facts to the factors in (2.7) and using what we just found for R_λ, we obtain the statement for K_λ in (2.10). □

Remark 2.5. There do exist boundary conditions other than those satisfying the assumption of pseudo-normality, for which the resolvent is $O(\lambda^{-1})$ on rays in $\mathbb{C}$. One example is the condition $\Lambda'^{-1}D_{x_1}\gamma_1 u + \Lambda'\gamma_0 u = 0$ for Δ on $\mathbb{R}^n_+$ studied in [G96, Ex. 1.7.17] (here $\Lambda' = (I - \Delta_{x'})^{\frac12}$); the coefficient of γ_1 is not surjective. For another type of example containing negative-order ψdo's on X' and defining a realization P_S that skew-selfadjoint and hence has many rays where the resolvent is $O(\lambda^{-1})$, see Remark 3.12 later. We expect that such cases may still be handled by variants of the present methods, but will give extra log terms at some of the negative powers of t in (1.2).

A third example is ${D_B}^*D_B$ considered below; here the surjectiveness is missing in the boundary condition $B\gamma_0 u = 0$, $(I - B^*)\sigma^*\gamma_0(\partial_{x_n} + A_1)u = 0$; but the questions for this operator are dealt with in a different way, as will be shown.

3. First order well-posed boundary problems.

For first-order operators (and odd-order operators more generally) it may not be possible to fulfill Assumptions 2.1 and 2.2 that lead to good resolvents — already the condition in Assumption 2.2 that Nd be even need not hold. However, for compact manifolds there do exist ψdo boundary conditions (not pseudo-normal)

$$B\gamma_0 u = 0, \tag{3.1}$$

such that the realization P_B is a Fredholm operator with a similar adjoint ${P_B}^*$. In this case there is an interest in studying the positive selfadjoint operator ${P_B}^* P_B$, which does have a resolvent. We now consider such problems in detail.

To begin with, *let X be compact* and let D be a first-order elliptic operator on X;

$$D\colon C^\infty(E_1) \to C^\infty(E_2), \tag{3.2}$$

where E_1 and E_2 are N-dimensional Hermitian vector bundles over X. D can be represented on $U = X_c$ as

$$D = \sigma(\tfrac{\partial}{\partial x_n} + A_1), \tag{3.3}$$

where σ is an isomorphism from $E_1|_U$ to $E_2|_U$ and A_1 is a first order differential operator that acts in the x' variable at $x_n = 0$. $A_1|_{x_n=0}$ has the principal symbol $a_1^0(x',\xi')$. For these operators,

$$\mathcal{A} = -\sigma \text{ on } X' \text{ and } \varrho = \gamma_0 \quad \text{in (2.1).} \tag{3.4}$$

A generalization to admissible manifolds will be included at the end of this section.

Definition 3.1. 1° We say that D is *"of Dirac-type"* when σ is a unitary morphism, and

$$A_1 = A + x_n P_1 + P_0, \tag{3.5}$$

where A is an elliptic first-order differential in $C^\infty(E_1')$ which is *selfadjoint* with respect to the Hermitian metric in E_1', and the P_j are differential operators of order $\le j$,

2° The *product case* is the case where D is of Dirac-type and, moreover, $v(x)dx = v(x',0)dx'dx_n$ on U, σ is constant in x_n, and $P_1 = P_0 = 0$.

As explained in [G92, p. 2036], unitarity of σ in (3.3) can be obtained by a simple homotopy near X', whereas the assumption on A_1 in 1° is an essential restriction in comparison with arbitrary first-order elliptic systems; it means that the principal symbol $a_1^0(x',\xi')$ of A_1 at $x_n = 0$ is Hermitian symmetric. P_1 and P_0 can be taken arbitrary near X', but for larger x_n, P_1 is subject to the requirement that D be elliptic.

When 1° holds, $a_1^0(x',\xi')$ equals the principal symbol $a^0(x',\xi')$ of A. Along with A one often considers the orthogonal projections $\Pi_\ge, \Pi_>, \Pi_\le, \Pi_<$ and Π_λ onto the closed spaces $V_\ge, V_>, V_\le, V_<$ and V_λ spanned by the eigenvalues of A in $L_2(E_1')$ that are $\ge 0, > 0, \le 0, < 0$ resp. $= \lambda$. (Since A is selfadjoint and elliptic of order 1, it has a discrete spectrum consisting of eigenvalues of finite multiplicity going to $\pm\infty$.) These operators are classical ψdo's of order 0; Π_λ is of order $-\infty$.

Atiyah, Patodi and Singer considered in [APS75] the product case. It is also studied e.g., in [GS96], [BW93], [BL97], whereas the case where only 1° holds is studied in [G92], [GS95] and other works. Cases where not even 1° holds, have to our knowledge not been studied for the purpose of heat trace expansions for boundary problems before.

We shall study boundary problems satisfying the condition of well-posedness introduced by Seeley in [S69]. The reader is encouraged to consult the Appendix, where the relevant material on the Calderón projector C^+ is collected. Let us here just recall that C^+ is a classical ψdo of order 0 in E_1' that projects $H^{s-\frac{1}{2}}(E_1')$ onto the space N_+^s of boundary values of null-solutions, for all $s \in \mathbb{R}$;

$$N_+^s = \gamma_0 Z_+^s \subset H^{s-\frac{1}{2}}(E_1'), \quad Z_+^s = \{\, z \in H^s(X, E_1) \mid Dz = 0 \text{ on } X \}. \tag{3.6}$$

We denote $I - C^+ = C^-$. The principal symbol $c^+(x', \xi')$ of C^+ is a projection in $\mathbb{C}^N$ onto the space $N_+(x', \xi')$ of boundary values of bounded solutions of $p^0(x', 0, \xi', D_{x_n})z(x_n) = 0$ on $\mathbb{R}_+$, such that the complementing projection $c^-(x', \xi')$ (the principal symbol of C^-) maps $\mathbb{C}^N$ onto the space $N_-(x', \xi')$ of boundary values of bounded solutions of $p^0(x', 0, \xi', D_{x_n})z(x_n) = 0$ on $\mathbb{R}_-$; cf. (A.11)ff. In relation to $a_1^0(x', \xi')$, $N_\pm(x', \xi')$ are the generalized eigenspaces for $a_1^0(x', \xi')$ associated with eigenvalues having real part $\gtrless 0$.

Remark 3.2. When D is of Dirac-type, so that $a_1^0(x', \xi')$ equals $a^0(x', \xi')$, $N_+(x', \xi')$ and $N_-(x', \xi')$ are orthogonal complements and are spanned by the eigenvectors belonging to the positive, resp. negative eigenvalues of $a^0(x', \xi')$. The projections $c^\pm(x', \xi')$ onto $N_\pm(X', \xi')$ along $N_\mp(x', \xi')$ are then orthogonal, and they are the principal symbols of $\Pi_\geq$ resp. $\Pi_<$. Thus for Dirac-type operators,

$$C^+ - \Pi_\geq \text{ is a classical } \psi\text{do of order } -1. \tag{3.7}$$

Definition 3.3. (Well-posedness) Let X be compact and let D be an elliptic first-order differential operator from $C^\infty(E_1)$ to $C^\infty(E_2)$. A classical ψdo B in E_1' of order 0 is *well-posed* for D when:

(i) The mapping defined by B in $H^s(E_1')$ has closed range for each $s \in \mathbb{R}$.
(ii) For each (x', ξ') with $|\xi'| = 1$, the principal symbol $b^0(x', \xi')$ maps $N_+(x', \xi')$ injectively onto the range of $b^0(x', \xi')$ in $\mathbb{C}^N$.

In comparison with the general choices of $B\colon H^s(E_1') \to H^s(F)$ (for $d = 1$) discussed in the Appendix, $F = E_1'$ here, so $M = N$. Condition (ii) assures that the system $\{D, B\gamma_0\}$ is *injectively elliptic*; see the explanation around (A.17). But (ii) is stronger than injective ellipticity, since the range of $b^0(x', \xi')$ can in general have a larger dimension than $b^0(x', \xi')N_+(x', \xi')$ has. (One can say that (ii) means injective ellipticity with smallest possible range dimension for b^0.)

Observe that when B satisfies Definition 3.3, $\{D, B\gamma_0\}$ cannot be surjectively elliptic if $n \geq 3$, since N is even and strictly larger that $\dim N_+(x', \xi') = N/2$, cf. (A.20). (If $n = 2$, this lack of surjective ellipticity holds when $\dim N_+(x', \xi') < N$.) Therefore, the system $\{D, B\gamma_0\}$ is *not* elliptic in the standard terminology, and, for example, its range does not have a smooth complement. The word "well-posed" does not conflict with this and was well chosen by Seeley. (Some authors use the dangerous notation "globally elliptic" for these boundary problems — sometimes even abbreviated to "elliptic".)

It is shown in [S69] that when Definition 3.3 is satisfied, one can always replace (3.1) by an equivalent condition

$$B_1 \gamma_0 u = 0, \tag{3.8}$$

where B_1 is a *projection* in the H^s-spaces, in addition to being well-posed for D. The range of B_1 in $H^s(E_1')$ is closed for each s, since it is the nullspace of the complementing projection $I - B_1$ which is likewise a ψdo of order 0. Thus it is no restriction to assume that B in (3.1) is a projection; we shall often do that.

Seeley shows in [S69] that for each boundary condition (3.1) with B well-posed for D, the realization D_B defined as in (2.3) (with domain $D(D_B) = \{u \in H^1(X, E_1) \mid B\gamma_0 u = 0\}$) is a Fredholm operator from $D(D_B)$ to $L_2(E_2)$. Moreover, when B is a projection, the adjoint $D_B{}^*$ (when D_B is considered as an unbounded operator from $L_2(E_1)$ to $L_2(E_2)$) is the realization of D^* with domain

$$D(D_B{}^*) = \{u \in H^1(X, E_2) \mid (I - B^*)\sigma^* \gamma_0 u = 0\} = D((D^*)_{(I-B^*)\sigma^*}); \tag{3.9}$$

here $(I - B^*)\sigma^*$ is well-posed for D^*. The nullspaces $Z(D_B)$ and $Z(D_B{}^*)$ are finite dimensional spaces of C^∞ sections, defining $\operatorname{index} D_B = \dim Z(D_B) - Z(D_B{}^*)$.

It is useful to know that when B has been replaced by a projection B_1, then furthermore, B_1 can be replaced by a projection B_2 that is *orthogonal* in $L_2(E_1')$. This may possibly be inferred from [S69] which leaves out details on the proof of the relevant Lemma VI.3, but it certainly follows from [BW93, Lemma 12.8], recalled as Lemma A.7 in the Appendix. Lemma A.7 and Remark A.8 imply that when R is a classical ψdo in E_1' which acts as a projection in $H^s(E_1')$, then R_{ort} defined by (A.35) is a projection which is orthogonal in $L_2(E_1')$ and has the same range as R in $H^s(E_1')$ for all s. When we apply this construction to $R = I - B_1$, (3.8) can be replaced by the condition $B_2\gamma_0 u = 0$ with the orthogonal projection $B_2 = I - R_{\text{ort}}$. On the principal symbol level, since the range of $r^0(x', \xi')$ equals the range of $r^0_{\text{ort}}(x', \xi')$, the operators $b_1^0(x', \xi')$ and $b_2^0(x', \xi')$ have the same nullspace, so (A.18) for one of them implies (A.18) for the other. Moreover, the range dimensions for $b_1^0(x', \xi')$ and $b_2^0(x', \xi')$ must be the same (equal to N minus the dimension of the nullspace), so also the surjectiveness required in (ii) carries over from b_1^0 to b_2^0. So also B_2 is well-posed for P. Only the *orthogonal* projection defining a boundary condition is uniquely determined from it; without the orthogonality there can be many choices of projection that give the same condition.

We now consider some examples.

Example 3.4. Clearly, the choice $B = C^+$ is well-posed, and so is $B = \Pi_\geq$ when D is of Dirac-type, in view of Remark 3.2. The first situation that was considered for index questions, in [APS75], was the choice $B = \Pi_\geq$ in the product case. This choice is convenient because it permits construction of the heat trace operators (in a good approximation) by easy functional calculus for the selfadjoint operator A.

Grubb and Seeley consider in [GS96] the product case with $B - \Pi_\geq$ ranging in the nullspace of A, and in [GS95] Dirac-type operators with $B - \Pi_\geq$ ranging in the eigenspace for eigenvalues of A of modulus $\leq a$ (some $a > 0$), showing full heat trace expansions.

Booss-Bavnbek and Wojciechowski [BW93] consider, for the product case, index questions for the full set of projections B of the form

$$B = C^+ + S, \quad S \text{ of order } -1; \tag{3.10}$$

likewise well-posed. This includes the preceding cases, and moreover allows infinite rank perturbations of $\Pi_{\geq}$.

For our heat trace estimates later, it is important to observe:

Proposition 3.5. *In the product case, when X is compact,*

$$C^+ - \Pi_{\geq} \text{ is a } \psi\text{do of order } -\infty. \tag{3.11}$$

Proof. We shall compare D, extended as $\sigma(\partial_{x_n} + A)$ on $X' \times\,]-c, 0]$, with the operator σD^0, where

$$D^0 = \partial_{x_n} + A', \quad A' = A + \Pi_0, \tag{3.12}$$

on $X^0 = X' \times \mathbb{R}_+$ and on $\widetilde{X}^0 = X' \times \mathbb{R}$, provided with the volume element $v(x', 0)\, dx' dx_n$. D^0 acts in E_1^0 and in $\widetilde{E}_1^0$, the pull-backs of E_1' to X^0 and $\widetilde{X}^0$; it satisfies the Green's formula:

$$(D^0 u, w)_{X^0} - (u, D^{0*} w)_{X^0} = -(\gamma_0 u, \gamma_0 w)_{X'}.$$

D^0 has an inverse Q^0 on $\widetilde{X}^0$, easily described by its action on functions of x_n taking values in the eigenspaces V_λ' of A' (here $V_0' = \{0\}$, $V_1' = V_1 \oplus V_0$, $V_\lambda' = V_\lambda$ for $\lambda \neq 0, 1$): When $f(x_n)$ has values in V_λ', Q^0 acts on f as the ψdo in x_n with symbol $(\mathrm{i}\xi_n + \lambda)^{-1}$; more generally when f has an expansion $f(x) = \sum_{\lambda \in \operatorname{spec} A'} f_\lambda(x_n) u_\lambda(x')$ in terms of eigenfunctions u_λ, then $Q^0 f = \sum_\lambda \mathcal{F}^{-1}_{\xi_n \to x_n}[(\mathrm{i}\xi_n + \lambda)^{-1} \hat{f}_\lambda(\xi_n)] u_\lambda(x')$.

For D^0, the Calderón projector is constructed exactly as in the differential operator case; it equals $\gamma_0^+ Q^0 \widetilde{\gamma}_0^*$ as in (A.10). It acts on a $\varphi \in V_\lambda'$ like the Calderón projector for $\partial_{x_n} + \lambda$, so

$$\gamma_0^+ Q^0 \widetilde{\gamma}_0^* \varphi = \begin{cases} \varphi \text{ if } \lambda \geq 0 \\ 0 \text{ if } \lambda < 0 \end{cases}$$

(one may also consult (A.12)). This implies that $\gamma_0^+ Q^0 \widetilde{\gamma}_0^* = \Pi_{\geq}$.

Now σD^0 and D differ only by the term $\sigma \Pi_0$ on $\widetilde{X}_c = X' \times\,]-c, c[$. Let Q be a parametrix of D on $\widetilde{X} = X \cup \widetilde{X}_c$. Let χ and $\chi_1 \in C_0^\infty(\,]-c, c[\,)$, equal to 1 on a neighborhood of 0 and satisfying $\chi\chi_1 = \chi$. Then, in view of (A.1), we have on $\widetilde{X}_c$:

$$\begin{aligned} \chi(Q - (\sigma D^0)^{-1})\chi &= \chi Q \chi_1 \sigma D^0 Q^0 \sigma^{-1} \chi_1 \chi - \chi\chi_1 (QD - \mathcal{T}_2) \chi_1 Q^0 \sigma^{-1} \chi \\ &= \chi Q [\chi_1 \sigma D^0 - D \chi_1] Q^0 \sigma^{-1} \chi + \chi \mathcal{T}_2 \chi_1 Q^0 \sigma^{-1} \chi \\ &= \chi Q [\chi_1 \sigma \Pi_0 - (\partial_{x_n} \chi_1) \sigma] Q^0 \sigma^{-1} \chi + \chi \mathcal{T}_2 \chi_1 Q^0 \sigma^{-1} \chi. \end{aligned} \tag{3.13}$$

Define the anisotropic spaces $H^{(s,t)}(X' \times \mathbb{R})$ and $H^{(s,t)}(X' \times\,]-c, c[\,)$, via local coordinates and a partition of unity on X', from the spaces $H^{(s,t)}(\mathbb{R}^{n-1} \times \mathbb{R})$ with norm $\|\langle\xi\rangle^s \langle\xi'\rangle^t \hat{u}(\xi)\|$. The operators have the continuity properties:

$$\begin{aligned} \chi Q \chi_1 &\colon H^{(s,t)}(E_2|_{\widetilde{X}_c}) \to H^{(s+1,t)}(E_1|_{\widetilde{X}_c}), & Q^0 &\colon H^{(s,t)}(\widetilde{E}_1^0) \to H^{(s+1,t)}(\widetilde{E}_1^0), \\ \chi \mathcal{T}_2 \chi_1 &\colon H^{(s,t)}(E_2|_{\widetilde{X}_c}) \to H^{(s_1,t_1)}(E_1|_{\widetilde{X}_c}), & \Pi_0 &\colon H^{(s,t)}(\widetilde{E}_1^0) \to H^{(s,t_1)}(\widetilde{E}_1^0), \\ \gamma_0^+ &\colon H^{(1,t)}(X_c) \to H^{\frac12 + t}(X'), & \widetilde{\gamma}_0^* &\colon H^{-\frac12 + t}(X') \to H^{(-1,t)}(\widetilde{X}_c), \end{aligned}$$

for all $s, s_1, t, t_1 \in \mathbb{R}$. Such properties are easy to show and are e.g. dealt with in [G86,G96, Sect. 2.5] (that can be used with fixed μ); for the statement on Q^0 one can generalize those proofs or use functional calculus, observing that $A'^{-1}\colon H^t(E_1') \xrightarrow{\sim} H^{t+1}(E_1')$, where the norm in $H^t(E_1')$ of $u = \sum_{\lambda\in\operatorname{spec} A'} c_\lambda u_\lambda$ is equivalent with $(\sum_\lambda |c_\lambda|^{2t})^{\frac12}$. Then the operator in (3.13) is continuous from $H^{(-1,t)}(E_1|_{\widetilde{X}_c})$ to $H^{(1,t_1)}(E_1|_{\widetilde{X}_c})$ for all $t, t_1 \in \mathbb{R}$, and when we compose it to the left with γ_0^+ and to the right with $\widetilde{\gamma}_0^*$, we get an operator that is continuous from $H^t(E_1')$ to $H^{t_1}(E_1')$ for all $t, t_1 \in \mathbb{R}$. Then this is a ψdo of order $-\infty$ on X'. Here

$$\gamma_0^+\chi(Q - (\sigma D^0)^{-1})\chi\widetilde{\gamma}_0^*\sigma = \gamma_0^+Q\widetilde{\gamma}_0^*\sigma - \gamma_0^*Q^0\widetilde{\gamma}_0^* = C^+ - \gamma_0^+\mathcal{T}_3 - \Pi_\geq,$$

cf. (A.7), so $C^+ - \Pi_\geq$ is a ψdo of order $-\infty$ on X'. $\square$

Example 3.6. Well-posed B need not be of the type (3.10). One example was introduced by Brüning and Lesch [BL97], in the product case and under the additional hypotheses that D is formally selfadjoint and

$$\sigma A = -A\sigma, \quad \sigma^2 = -I, \quad \tau A = -A\tau, \quad \tau^2 = I, \quad \tau\sigma = -\sigma\tau, \tag{3.14}$$

where τ is an auxiliary morphism or ψdo of order 0. The prototype is, for $\cos\theta \neq 0$,

$$B_\theta = \cos^2\theta\,\Pi_> + \sin^2\theta\,\Pi_< - \cos\theta\sin\theta\,\tau(\Pi_> + \Pi_<) + B', \tag{3.15}$$

with a suitable projection B' in V_0. Here B_θ is principally different from $\Pi_\geq$ when $\cos^2\theta \neq 1$. D_{B_θ} is selfadjoint.

For the analysis it is useful to observe that (3.14) implies a spectral symmetry of A; in fact τ (as well as σ) defines isometries of the eigenspaces V_j^+ for positive eigenvalues λ_j^+ (ordered increasingly) onto the eigenspaces V_j^- for negative eigenvalues $\lambda_j^- = -\lambda_j^+$ and vice versa (in particular, $\eta(A,s) \equiv 0$). Then the nullspace of B_θ in V_0^- is a "shifted version" of $V_<$:

$$\overline{\operatorname{span}}\{e_{j,k}^- + \tan\theta\, e_{j,k}^+ \mid j > 0,\ k = 1,\dots,\nu_j\}; \tag{3.16}$$

here the $e_{j,k}^-$, $1 \leq k \leq \nu_j$, are an orthonormal basis of V_j^-, and $e_{j,k}^+ = \tau e_{j,k}^-$.

For $B = B_\theta$, [BL97] shows a precise version of (1.1), related to that of [GS96] (see also Grubb [G97, Remark 4.14]). The present study allows generalizations to the non-product case and perturbations of order -1. The same holds for the more abstractly formulated well-posed conditions in [BL97].

Example 3.7. Without assuming spectral symmetry, we can give general examples of well-posed B for Dirac-type operators by taking

$$B = \Pi_\geq + \Pi_\geq S \Pi_<, \tag{3.17}$$

where S is a classical ψdo of order 0 in E_1'. B is a projection, since $\Pi_<\Pi_\geq = 0$; so (i) in Definition 3.3 is satisfied. For the principal symbols, the injectiveness (A.18) is obvious for $b^0(x',\xi') = c^+(x',\xi') + c^+(x',\xi')s^0(x',\xi')c^-(x',\xi')$. Moreover,

$$b^0(x',\xi')N_+(x',\xi') \subset b^0(x',\xi')\mathbb{C}^N \subset N_+(x',\xi'),$$

so since the former has the same dimension as $N_+(x',\xi')$, there must be equality. Then also (ii) of Definition 3.3 is satisfied.

By use of Lemma A.7ff, B may be replaced by the orthogonal projection $B_1 = I - (I-B)_{\mathrm{ort}}$, defining the same boundary condition. To calculate B_1, write S and B in blocks according to the decomposition $L_2(E_1') = V_\geq \oplus V_<$:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & S_{12} \\ 0 & 0 \end{pmatrix}.$$

Then with $R = I - B = \begin{pmatrix} 0 & -S_{12} \\ 0 & 1 \end{pmatrix}$, we find from (A.35) that

$$R_{\mathrm{ort}} = \begin{pmatrix} S_{12}S_{12}^*(I+S_{12}S_{12}^*)^{-1} & -S_{12}(I+S_{12}^*S_{12})^{-1} \\ -S_{12}^*(I+S_{12}S_{12}^*)^{-1} & (I+S_{12}^*S_{12})^{-1} \end{pmatrix}. \tag{3.18}$$

This is principally different from $\Pi_< = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ as soon as S_{12} has nonvanishing principal symbol, which is the generic case (when $0 < \dim N_+(x',\xi') < N$, in particular when $n \geq 3$). Thus $B_1 = I - R_{\mathrm{ort}}$ is an orthogonal projection that satisfies Definition 3.3 and differs principally from $\Pi_\geq$. More generally, we can take B to have principal part (3.17).

— Let us remark that if there is a spectral symmetry: $A\tau = -\tau A$ for some zero-order ψdo τ with $\tau^2 = I$, then the following choice:

$$B = \Pi_\geq + \beta\tau\Pi_<, \text{ some } \beta \in \mathbb{R}, \tag{3.19}$$

is of the above type with $S = \beta\tau$, since $\tau\Pi_< = \tau\Pi_<\Pi_< = \Pi_>\tau\Pi_<$. The condition defined by this B is similar to that defined by (3.15); in fact the nullspace of B in V_0^- equals (3.16) with $\tan\theta = -\beta$.

Since C^+ is not in general an *orthogonal* projection, it may be of interest to consider also the orthogonalized version C^+_{ort}, called the *orthogonal Calderón projector*; cf. Lemma A.7ff. When P is of Dirac-type, the principal symbol of C^+ is the orthogonal projection c^+ (cf. Remark 3.2), so a replacement of C^+ by C^+_{ort} changes only the lower order part;

$$C^+ - C^+_{\mathrm{ort}} \text{ is of order } -1 \text{ when } P \text{ is of Dirac-type.} \tag{3.20}$$

Remark 3.8. When c^+ is not symmetric, $C^+ - C^+_{\mathrm{ort}}$ is of order 0, not -1. For a simple example with c^+ non-symmetric, take e.g. (for a neighborhood of the boundary represented as $\mathbb{R}^2_+$)

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_{x_2} + \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}\partial_{x_1}; \quad \text{here } c^+(\xi_1) = \begin{pmatrix} \frac12 & \mathrm{i} \\ -\frac14\mathrm{i} & \frac12 \end{pmatrix}\xi_1.$$

(The formula is easily shown using (A.12).) This c^+ is a projection but not an orthogonal one.

Example 3.9. Example 3.7 can be generalized to arbitrary D as follows: Consider $C^+_{\rm ort}$ and its complementing projection $C^{+-}_{\rm ort} = I - C^+_{\rm ort}$. Let us denote their principal symbols and range spaces

$$c^+_{\rm ort}, \quad I - c^+_{\rm ort} = c^{+-}_{\rm ort}, \quad N_+(x',\xi'), \quad \mathbb{C}^N \ominus N_+(x',\xi') = N_+^-(x',\xi'). \tag{3.21}$$

Then the whole discussion in Example 3.7 is valid with $\Pi_\geq$ and $\Pi_<$ replaced by $C^+_{\rm ort}$ and $C^{+-}_{\rm ort}$, giving well-posed operators (where we can add S_1 of order -1):

$$B = C^+_{\rm ort} + C^+_{\rm ort} S C^{+-}_{\rm ort} + S_1. \tag{3.22}$$

Example 3.10. Examples 3.7 and 3.9 are, in a *microlocal* sense, the most general possible. When B defines the condition $B\gamma_0 u = 0$, so does CB for any invertible classical elliptic ψdo C of order 0; in this sense, B and CB can be regarded as equivalent. Now if B satisfies Definition 3.3, we can *for* (x',ξ') *in a neighborhood of each* (x_0',ξ_0') ($|\xi'|=1$) find a smooth family of bijective matrices $c(x',\xi')$ such that $c(x',\xi')b^0(x',\xi')$ is of the form $c^+_{\rm ort}(x',\xi') + c^+_{\rm ort}(x',\xi')s(x',\xi')c^{+-}_{\rm ort}(x',\xi')$, as follows: Note that $\mathbb{C}^N$ has the two decompositions (depending smoothly on (x',ξ'))

$$\mathbb{C}^N = N_+(x',\xi') \dot{+} N_+^-(x',\xi') = R(b^0(x',\xi')) \dot{+} Z(b^0(x',\xi')); \tag{3.23}$$

the latter denote the range and nullspace of b^0 (we now omit the indication (x',ξ')). Here b^0 defines a homeomorphism c_1 of N_+ onto $R(b^0)$. Let $c_2 = c_1^{-1}$ and let c_3 be a homeomorphism of $Z(b^0)$ onto N_+^- (it can be chosen to depend smoothly on (x',ξ') in a neighborhood of (x_0',ξ_0')); then

$$c_4 = c_2 b^0 + c_3(I - b^0) \tag{3.24}$$

is a bijection in $\mathbb{C}^N$. Now its inverse $c = c_4^{-1}$ does the job: It is a bijection in $\mathbb{C}^N$ that maps $R(b^0)$ to N_+ as an inverse of b^0 from N_+ to $R(b^0)$. So $c\,b^0$ ranges in N_+ and is the identity on N_+, and hence

$$cb^0 = c^+_{\rm ort} c\, b^0 (c^+_{\rm ort} + c^{+-}_{\rm ort}) = c^+_{\rm ort} + c^+_{\rm ort} c\, b^0 c^{+-}_{\rm ort}; \tag{3.25}$$

it is of the desired form and is equivalent with b^0.

We shall now show how the resolvents of the operators

$$(\Delta_1 - \lambda)^{-1}, \quad (\Delta_2 - \lambda)^{-1}, \text{ where } \Delta_1 = D_B{}^* D_B, \quad \Delta_2 = D_B D_B{}^*, \tag{3.26}$$

can be treated within the framework of Section 2. In fact, there is a nice trick of replacing the study of the injectively elliptic first-order system $\{D, B\gamma_0\}$ by a truly elliptic first-order system $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ satisfying the resolvent growth condition, in such a way that the second-order resolvents (3.26) are found from the resolvent construction for $\mathcal{D}_{\mathcal{B}}$:

Let B be a well-posed projection and let us denote

$$\mathcal{D} = \begin{pmatrix} 0 & -D^* \\ D & 0 \end{pmatrix}, \quad \mathcal{D}_{\mathcal{B}} = \begin{pmatrix} 0 & -D_B{}^* \\ D_B & 0 \end{pmatrix}. \tag{3.27}$$

The operator $\mathcal{D}$ in (3.27) is formally skew-selfadjoint on X. The operator $\mathcal{D}_{\mathcal{B}}$ is skew-selfadjoint as an unbounded operator in $L_2(E)$, $E = E_1 \oplus E_2$. It then has a resolvent $\mathcal{R}_\mu = (\mathcal{D}_{\mathcal{B}} + \mu)^{-1}$ for $\mu \in \mathbb{C} \setminus i\mathbb{R}$. A calculation shows that

$$\mathcal{R}_\mu = (\mathcal{D}_{\mathcal{B}} + \mu)^{-1} = \begin{pmatrix} \mu R_{1,\mu} & {D_B}^* R_{2,\mu} \\ -D_B R_{1,\mu} & \mu R_{2,\mu} \end{pmatrix}, \text{ where} \quad R_{1,\mu} = (\Delta_1 + \mu^2)^{-1}, \quad R_{2,\mu} = (\Delta_2 + \mu^2)^{-1}; \tag{3.28}$$

this shows how the resolvents (3.26) can be recovered from $\mathcal{R}_\mu$. Also $D_B R_{1,\mu}$ and ${D_B}^* R_{2,\mu}$ are determined. When $\mu \in \Gamma_0$,

$$\Gamma_0 = \{ z \in \mathbb{C} \mid |\arg z| < \pi/2 \}, \tag{3.29}$$

then $\lambda = -\mu^2$ runs through $\mathbb{C} \setminus \overline{\mathbb{R}}_+$, so it suffices for (3.26) to let $\mu \in \Gamma_0$.

Now $\mathcal{D}_{\mathcal{B}}$ is the realization of $\mathcal{D}$ in $L_2(E)$ of the boundary condition

$$\mathcal{B}\gamma_0 u = 0, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3.30}$$

where $\mathcal{B}$ is the row matrix (cf. (3.9))

$$\mathcal{B} = \begin{pmatrix} B & (I - B^*)\sigma^* \end{pmatrix}, \tag{3.31}$$

going from $L_2(E_1') \times L_2(E_2')$ to $L_2(E_1')$. Since the ranges of B and $I - B^*$ are orthogonal complements in $L_2(E_1')$, $\mathcal{B}$ is *surjective*; note that the dimension N of E_1' is half of the dimension $2N$ of $E' = E_1' \oplus E_2'$. Moreover, $\mathcal{B}$ has as a right inverse the ψdo $\mathcal{C}$ of order 0,

$$\mathcal{C} = \begin{pmatrix} B^* \\ (\sigma^*)^{-1}(I - B) \end{pmatrix} [BB^* + (I - B^*)(I - B)]^{-1} \tag{3.32}$$

(cf. Lemma A.7); in particular, $\mathcal{B}$ is surjectively elliptic. Now $\{\mathcal{D} + 1, \mathcal{B}\gamma_0\}$ has the inverse $\begin{pmatrix} \mathcal{R}_1 & \mathcal{K}_1 \end{pmatrix}$ with $\mathcal{K}_1 = [I - \mathcal{R}_1(\mathcal{D} + 1)]K_{\gamma_0,1}\mathcal{C}$ as in (2.7). Since the inverse is continuous from $L_2(E) \times H^{\frac12}(E_1')$ to $H^1(E)$, $\{\mathcal{D} + 1, \mathcal{B}\gamma_0\}$ and hence also $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ is elliptic. Thus all the conditions in Assumption 2.1 and 2.2 are satisfied by $\{\mathcal{D}, \mathcal{B}\varrho\}$, with N replaced by $2N$, $d = 1$, $\varrho = \gamma_0$, $F = F_0 = E_1'$!

Then the consequences we draw later for the general systems in Section 2 apply in particular to $\mathcal{D}_{\mathcal{B}}$.

Example 3.11. By Theorem A.6, the adjoint of D_{C^+} is the realization of D^* determined by the analogous boundary condition ${C'}^+\gamma_0 u = 0$, where ${C'}^+$ is the Calderón projector for D^*, if D has an invertible extension to a closed neighborhood of X. More generally, the adjoint boundary condition is $({C'}^+ - \mathcal{T}_6)\gamma_0 u = 0$, where $\mathcal{T}_6$ is a ψdo of order $-\infty$. In view of (A.30), $\mathcal{B}$ is in this case the surjective operator

$$\mathcal{B} = \begin{pmatrix} C^+ & (I - {C^+}^*)\sigma^* \end{pmatrix} = \begin{pmatrix} C^+ & \sigma^*({C'}^+ - \mathcal{T}_6) \end{pmatrix}. \tag{3.33}$$

Remark 3.12. The trick of considering the "doubled-up" system (3.27) will be restricted to first-order operators in this paper. Well-posed boundary conditions can also be defined for higher order systems, cf. [S69]. But here when one takes the example of $B = C^+$, one gets an operator on the boundary with entries of negative order that are generally nontrivial, and these exist also in the doubled-up version and violate the requirement concerning order ≥ 0 in Assumption 2.1. Manipulations with order-reducing operators do not seem to help; they cannot at the same time remove a singularity in ξ' and be strongly polyhomogeneous in (ξ', μ). (See also Remark 2.5 and the calculations after (5.8).)

The analysis of (3.30)–(3.32) moreover tells us how to include admissible manifolds in the study of first-order systems. Here we need a uniformity in x' in the well-posedness condition. We restrict the attention to projections B.

Definition 3.14. (Uniform well-posedness) Let D be an admissible, uniformly elliptic first-order differential operator from E_1 to E_2 (admissible vector bundles over an admissible manifold X). Let B be an admissible classical ψdo of order 0 in E_1' with $B^2 = B$. We say that B is *uniformly well-posed* for D, when B satisfies Definition 3.2 (ii) and in addition, $\mathcal{B}$ defined by (3.31) is uniformly surjectively elliptic and $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ (cf. (3.27)) is uniformly elliptic.

When Definition 3.14 is satisfied, the realization $\mathcal{D}_{\mathcal{B}}$ is seen by Green's formula to be skew-symmetric. It is skew-selfadjoint since $(\mathcal{D}_{\mathcal{B}})^*$ acts like $\mathcal{D}^*$ and $u \in D((\mathcal{D}_{\mathcal{B}})^*)$ implies $u \in L_2(E)$ with $\mathcal{D}^* u \in L_2(E)$ and $\mathcal{B}\gamma_0 u = 0$ as an element of $H^{-\frac{1}{2}}(E_1')$, hence by use of a parametrix of $\{\mathcal{D}, \mathcal{B}\gamma_0\}$ it is seen that $u \in H^1(E)$ and thus $u \in D(\mathcal{D}_{\mathcal{B}})$.

It follows that Assumptions 2.1 and 2.2 are satisfied, with $\Gamma = \Gamma_0$; so (3.28) exists and gives the resolvents of the Δ_i as in the compact case.

Examples are constructed as in the preceding text, most easily when D has an invertible extension to a boundaryless manifold so that there is a precise Calderón projector as in Theorem A.1 (then $B = C^+$ is a particular example).

4. Elements of weakly polyhomogeneous ψdo calculus.

We here recall the more technical definitions of ψdo classes from Grubb and Seeley [GS95], now allowing non-compact admissible manifolds and globally estimated operators as in [G95], [G96].

First, the symbol space $S^m(\mathbb{R}^\nu \times \mathbb{R}^n)$ consists of the functions $p(x,\xi) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n)$ such that

$$\partial_x^\beta \partial_\xi^\alpha p = O(\langle \xi \rangle^{m-|\alpha|}) \text{ for all } \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^\nu; \tag{4.1}$$

$\mathbb{N} = \{\text{ integers } \geq 0\}$. The basic rules of calculus for this space are well-known from Hörmander [H85, Sect. 18.1]. (When we are only interested in symbols with estimates valid over compact subsets of $\mathbb{R}^n$, we can use the results of the global calculus by introducing suitable cut-off functions.) A symbol $p \in S^m(\mathbb{R}^\nu \times \mathbb{R}^n)$ is called classical (or classical polyhomogeneous) of degree m if it has an expansion $p \sim \sum_{j \in \mathbb{N}} p_j$, where the p_j are homogeneous in ξ of degree $m - j$ for $|\xi| \geq 1$, and $p - \sum_{j<J} p_j \in S^{m-J}(\mathbb{R}^\nu \times \mathbb{R}^n)$ for $J \in \mathbb{N}$.

Next, we define a class of symbols p depending on a parameter μ varying in a sector $\Gamma \subset \mathbb{C} \setminus \{0\}$. It is the behavior for $|\mu| \to \infty$ that is important here, and we often describe it in terms of the behavior of $p(x, \xi, \frac{1}{z})$ for $z \to 0$, $\frac{1}{z} = \mu \in \Gamma$. For brevity of notation, we write $\partial_z^j p(x, \xi, \frac{1}{z})$ (or just $\partial_z^j p$) for the j'th z-derivative of the composite function $z \mapsto p(x, \xi, \frac{1}{z})$.

Definition 4.1. Let n and ν be positive integers, and let m and $d \in \mathbb{R}$. Let Γ be a sector in $\mathbb{C}\setminus\{0\}$. The space $S^{m,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$ consists of the functions $p(x,\xi,\mu) \in C^\infty(\mathbb{R}^\nu \times \mathbb{R}^n \times \Gamma)$ that are holomorphic in $\mu \in \overset{\circ}{\Gamma}$ for $|(\xi,\mu)| \ge \varepsilon$ (some $\varepsilon > 0$) and satisfy, for all $j \in \mathbb{N}$,

$$\partial_z^j p(\cdot,\cdot,\tfrac{1}{z}) \text{ is in } S^{m+j}(\mathbb{R}^\nu \times \mathbb{R}^n) \text{ for } \tfrac{1}{z} \in \Gamma,$$
$$\text{with estimates valid uniformly for } |z| \le 1, \tfrac{1}{z} \in \text{closed subsectors of } \Gamma.$$

Moreover, we set $S^{m,d} = \mu^d S^{m,0}$; that is, $S^{m,d}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$ consists of the functions p (holomorphic in $\mu \in \overset{\circ}{\Gamma}$ for $|(\xi,\mu)| \ge \varepsilon$) such that for all $j \in \mathbb{N}$,

$$\partial_z^j (z^d p(\cdot,\cdot,\tfrac{1}{z})) \text{ is in } S^{m+j}(\mathbb{R}^\nu \times \mathbb{R}^n) \text{ for } \tfrac{1}{z} \in \Gamma,$$
$$\text{with estimates valid uniformly for } |z| \le 1, \tfrac{1}{z} \in \text{closed subsectors of } \Gamma.$$

Sometimes the symbols are only defined for $|\mu| \ge$ a constant depending on the subsector of Γ; this requires obvious modifications. We can identify

$$S^m(\mathbb{R}^\nu \times \mathbb{R}^n) \subset S^{m,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \mathbb{C}\setminus\{0\}). \tag{4.2}$$

Asymptotic expansions and polyhomogeneous subclasses are introduced as follows.

Definition 4.2. 1° Let $p \in S^{m-d,d}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$ and let p_j be a sequence of symbols in $S^{m-j-d,d}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$ such that

$$p - \textstyle\sum_{j<J} p_j \in S^{m-J-d,d}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \text{ for any } J \in \mathbb{N};$$

then we say that $p \sim \sum_{j\in\mathbb{N}} p_j$ in $S^{m-d,d}$.

2° If, moreover, the p_j are weakly homogeneous of degree $m-j$, i.e.,

$$p_j(x,t\xi,t\mu) = t^{m-j} p_j(x,\xi,\mu) \text{ for } |\xi| \ge 1,\ t \ge 1,\ (\xi,\mu) \in \mathbb{R}^n \times \Gamma, \tag{4.3}$$

we say that p is **weakly polyhomogeneous**.

3° If, furthermore, the p_j are strongly homogeneous of degree $m-j$, i.e.,

$$p_j(x,t\xi,t\mu) = t^{m-j} p_j(x,\xi,\mu) \text{ for } |\xi|^2 + |\mu|^2 \ge 1,\ t \ge 1,\ (\xi,\mu) \in \mathbb{R}^n \times \Gamma, \tag{4.4}$$

and the following estimates hold for all indices α, β, J:

$$\partial_x^\beta \partial_\xi^\alpha \partial_\mu^k (p - \textstyle\sum_{j<J} p_j) = O(\langle(\xi,\mu)\rangle^{m-J-|\alpha|-k}), \tag{4.5}$$

then we say that p is **strongly polyhomogeneous**.

(For simplicity, we leave out the possibility of noninteger steps between the degrees of the p_j, included in [GS95].) It is shown in [GS95] that the conditions in 3° imply those in 1° and 2°. Thus the strongly polyhomogeneous symbol can be thought of as the case where μ enters as an extra cotangent variable, on a par with the others, in a classical symbol. For example, for $m \in \mathbb{Z}$,

$$(|\xi|^2 + |\mu|^2 + 1)^{m/2} \in \begin{cases} S^{m,0} + S^{0,m} & \text{for } m \ge 0, \\ S^{m,0} \cap S^{0,m} & \text{for } m \le 0, \end{cases} \tag{4.6}$$

is strongly polyhomogeneous, whereas (with $n = 2$)

$$\Big(\frac{\xi_1^4 + \xi_2^4}{\xi_1^2 + \xi_2^2 + 1} + |\mu|^2\Big)^{-1} \in S^{-2,0} \cap S^{0,-2} \tag{4.7}$$

is weakly polyhomogeneous. (For (4.7), cf. [GS95, Th. 1.17].) We shall use a special name (as in [G97]) for symbols with this behavior:

Definition 4.3. Let r be an integer ≥ 0. A symbol $s(x,\xi,\mu)$ (and the operator it defines) is called **special parameter-dependent of order** $-r$, when

$$s(x,\xi,\mu) \in S^{-r,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \cap S^{0,-r}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \text{ with}$$
$$\partial_\mu^m s(x,\xi,\mu) \in S^{-r-m,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \cap S^{0,-r-m}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \quad (4.8)$$

for any m, all $\partial_\mu^m s(x,\xi,\mu)$ being weakly polyhomogeneous.

In particular, a strongly polyhomogeneous symbol of order $-r$ has this property, cf. [GS95, Th. 1.16].

The rules of calculus for the symbol spaces and the associated operators are described in detail in [GS95]. Let us here just recall a few elements: A symbol $p(x,\xi,\mu)$ with x and $\xi \in \mathbb{R}^n$ defines a family of ψdo's depending on $\mu \in \Gamma$,

$$P_\mu f(x) = \mathrm{OP}(p)f(x) = (2\pi)^{-n} \int e^{\mathrm{i}x\cdot\xi} p(x,\xi,\mu)\hat{f}(\xi)\,d\xi; \quad (4.9)$$

the indication sub-μ may be left out. There holds the composition rule:

$$P_\mu \in \mathrm{OP}(S^{m,d}),\ P'_\mu \in \mathrm{OP}(S^{m',d'}) \implies P_\mu P'_\mu \in \mathrm{OP}(S^{m+m',d+d'}), \quad (4.10)$$

with symbol

$$(p \circ p')(x,\xi,\mu) \sim \textstyle\sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,\xi,\mu)(-\mathrm{i}\partial_x)^\alpha p'(x,\xi,\mu) \text{ in } S^{m+m',d+d'}. \quad (4.11)$$

Theorem 1.23 in [GS95], formulated there for symbols with local estimates in x, extends without difficulty to symbols with global estimates in x (the proof is in fact simplified because the compositions can be carried out directly, without cut-off functions, in the global calculus):

Theorem 4.4. *Let $p(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma) \otimes \mathcal{L}(\mathbb{C}^N, \mathbb{C}^N)$ be such that $p = p_0 + p_{-1}$ with $p_{-1} \in S^{-1,0}$ and with $p_0^{-1} \in C^\infty$ bounded uniformly in $(x,\xi,\mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \Gamma'_1$, for any closed subsector Γ' of Γ and $\Gamma'_1 = \{\mu \in \Gamma' \mid |\mu| \geq 1\}$. Then there exists a parametrix symbol $q(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu \times \mathbb{R}^n, \Gamma)$ such that $p \circ q \sim I$ in $S^{0,0}$; here*

$$q \sim q_0 \circ \textstyle\sum_{k\in\mathbb{N}} r^{\circ k}, \text{ where}$$
$$q_0 = {p_0}^{-1},\ r = I - p\circ q_0,\ r^{\circ k} = r\circ r\circ\cdots\circ r \ (k \text{ factors}). \quad (4.12)$$

If p is weakly resp. strongly polyhomogeneous, so is q.

We shall not introduce a general ellipticity definition but just say that the operators with symbol satisfying the hypotheses of Theorem 4.4 are *uniformly parameter-elliptic in the sense of Theorem* 4.4.

It will be useful to observe that there are one-sided variants of Theorem 4.4:

Corollary 4.5.

1° *Let* $p(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu \times \mathbb{R}^n,\Gamma) \otimes \mathcal{L}(\mathbb{C}^N,\mathbb{C}^M)$ *be such that* $p = p_0 + p_1$ *with* $p_{-1} \in S^{-1,0}$ *and with* p_0 *having a right inverse* $q_0 \in C^\infty$ *that is bounded uniformly in* $(x,\xi,\mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \Gamma_1'$, *for any closed truncated subsector* Γ_1' *of* Γ. *Then there exists a right parametrix symbol* $q(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu \times \mathbb{R}^n,\Gamma) \otimes \mathcal{L}(\mathbb{C}^M,\mathbb{C}^N)$ *such that* $p \circ q \sim I$ *in* $S^{0,0}$; *here*

$$q \sim p^* \circ (p \circ p^*)^{\circ -1}, \tag{4.13}$$

where $(p \circ p^*)^{\circ -1}$ *is a parametrix symbol for* $p \circ p^*$ *according to Theorem 4.4.*

2° *When the assumptions in 1° hold with "right" replaced by "left," there exists a left parametrix symbol* $q(x,\xi,\mu) \in S^{0,0}(\mathbb{R}^\nu \times \mathbb{R}^n,\Gamma) \otimes \mathcal{L}(\mathbb{C}^M,\mathbb{C}^N)$ *such that* $q \circ p \sim I$ *in* $S^{0,0}$; *here* $q \sim (p^* \circ p)^{\circ -1} \circ p^*$, *where* $(p^* \circ p)^{\circ -1}$ *is a parametrix symbol for* $p^* \circ p$ *according to Theorem 4.4.*

Proof. This follows immediately from Theorem 4.4, when we note that $p^* \circ p$ in case 1°, resp. $p \circ p^*$ in case 2°, satisfies the hypotheses of Theorem 4.4. □

We say that symbols satisfying the hypotheses in 1° resp. 2° are *uniformly surjectively, resp. injectively, parameter-elliptic* in the sense of Corollary 4.5.

In the previous works [GK93], [G95,G96], results were shown both for parameter-independent ψdo's and for parameter-dependent ψdo's of a slightly different type than here; it is the parameter-independent results from [G95] that are most fundamental for the next theorem.

An important step in the resolvent construction in Section 5 is to show that when a family of ψdo's P_μ is weakly polyhomogeneous of order 0 and is such that P_μ has an inverse P_μ^{-1} that is bounded in some $H^{s,\mu}$-norm uniformly in μ, then the inverse P_μ^{-1} is again a weakly polyhomogeneous ψdo family of order 0, and symbol estimates of μ-derivatives for P_μ carry over to P_μ^{-1}. In fact we need a result of this kind when there is merely a right inverse. When $P_\mu = I - S_\mu$ with S_μ of suitably small norm, such results can be shown by use the Neumann series expansion, and entered already in [GS95]. For more general P_μ, more efforts are needed, and the question is closely related to the question of *spectral invariance* — briefly expressed this means that when a ψdo in a specific class has an inverse in some operator sense, then the inverse is a ψdo belonging to the calculus too, and both operators are elliptic.

First we show the spectral invariance property for weakly polyhomogeneous ψdo's with global estimates in x, using techniques from [G95] and [GS95].

Theorem 4.6. *Let* E_1 *and* E_2 *be admissible vector bundles of dimension* N *over an admissible boundaryless manifold* $\widetilde{X}$, *and let* P_μ *(depending on* μ *in a sector* Γ *of* $\mathbb{C}$*) be a weakly polyhomogeneous* ψ*do with symbol in* $S^{0,0}$ *in admissible coordinate systems, such that for some* $l \in \mathbb{Z}$, $P_\mu \colon H^{l,\mu}(E_1) \to H^{l,\mu}(E_2)$ *(which is bounded uniformly for* μ *in closed truncated subsectors* Γ_r'*) has an inverse* P_μ^{-1} *that is likewise* $H^{l,\mu}$*-bounded uniformly for* μ *in subsectors* Γ_r'. *Then* P_μ^{-1} *is a weakly polyhomogeneous* ψ*do with symbol in* $S^{0,0}$. *Moreover,* P_μ *and* P_μ^{-1} *are uniformly parameter-elliptic in the sense of Theorem 4.4.*

If P_μ *is strongly polyhomogeneous, then so is* P_μ^{-1}. *If* P_μ *is special parameter-dependent of order* 0 *(cf. Definition 4.3), then so is* P_μ^{-1}.

Proof. Consider a Γ'_r. First let $l = 0$, so that $H^{l,\mu}$ is simply L_2. We begin by reducing (as in [9, Th. 1.14] or [7, Lemma 3.1.6]) to a consideration of operators of the form $I - Q_\mu$ with Q_μ small: Since P_μ and P_μ^{-1} are uniformly bounded, there exist positive constants $c \le C$ such that

$$c\|u\|^2_{L_2(E_1)} \le \|P_\mu u\|^2_{L_2(E_2)} \le C\|u\|^2_{L_2(E_1)}, \text{ for all } u \in L_2(E_1), \mu \in \Gamma'_r. \tag{4.14}$$

Then since $\|P_\mu u\|^2_{L_2(E_2)} = (P_\mu^* P_\mu u, u)_{L_2(E_1)}$,

$$c\|u\|^2_{L_2(E_1)} \le (P_\mu^* P_\mu u, u)_{L_2(E_1)} \le C\|u\|^2_{L_2(E_1)}, \text{ for all } u \in L_2(E_1), \mu \in \Gamma'_r.$$

It follows that

$$0 \le ((I - C^{-1}P_\mu^* P_\mu)u, u)_{L_2(E_1)} \le (\tfrac{C-c}{C}u, u)_{L_2(E_1)}, \text{ for all } u \in L_2(E_1), \mu \in \Gamma'_r, \tag{4.15}$$

and hence when we introduce the selfadjoint operator $Q_\mu = I - C^{-1}P_\mu^* P_\mu$,

$$C^{-1}P_\mu^* P_\mu = I - Q_\mu, \text{ with } \|Q_\mu\|_{\mathcal{L}(L_2(E_1))} \le \tfrac{C-c}{C} = \delta < 1, \quad Q_\mu \ge 0. \tag{4.16}$$

Since $\delta < 1$ and $\|Q_\mu^k\| \le \delta^k$ for $k \in \mathbb{N}$, the inverse $(I - Q_\mu)^{-1}$ exists as $\sum_{k\in\mathbb{N}} Q_\mu^k$ (the Neumann series) with convergence in operator norm, uniformly in $\mu \in \Gamma'_r$. Composition with $(I - Q_\mu)^{-1}$ in (4.16) shows that

$$P_\mu^{-1} = (I - Q_\mu)^{-1}C^{-1}P_\mu^*. \tag{4.17}$$

We now study $(I - Q_\mu)^{-1}$. Since Q_μ has L_2-operator norm $\le \delta < 1$ by (4.16), it follows from a classically known fact (see e.g. the references around [7, Lemma 3.1.5]) that the principal symbol $q^0(x, \xi, \mu)$ must have norm $\le \delta$. (In fact, when $\chi(x) \in C_0^\infty$, the essential spectrum of $\chi Q_\mu \chi$ for each μ equals the union over x and $|\xi| \ge 1$ of the spectra of $\chi(x)^2 q^0(x, \xi, \mu)$.) Thus $I - q^0$ has an inverse bounded uniformly in $(x, \xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^n \times \Gamma'_r$, so $I - Q_\mu$ is parameter-elliptic in the sense of Theorem 4.4 (the lower order symbol $q - q^0$ is in $S^{-1,0}$ in admissible local coordinates, since this holds for P). Thus $I - Q_\mu$ has a parametrix belonging to the calculus. Hence so does $P_\mu^* P_\mu = C(I - Q_\mu)$, and then also P_μ. (The parametrices are again u.p.-elliptic in the sense of Theorem 4.4.)

To see that the true inverse of $I - Q_\mu$ belongs to the calculus, we can for operators on compact manifolds appeal to a well-known result for standard ψdo's and use the uniformity in μ for the symbol and its derivatives, as in [GS95, Th. 3.8]. To include operators on noncompact (admissible) manifolds, we appeal to a result of [G95]. Theorem 1.12 (1) there implies that when P_0 is a single (parameter-independent) ψdo of order 0, belonging to the global calculus and elliptic uniformly in x, then if $P_0 \colon L_2(E_1) \to L_2(E_2)$ has a bounded inverse P_0^{-1}, this inverse belongs to the calculus and is also a parametrix of P_0. In particular, it is of order 0 and elliptic uniformly in x, and its symbol expansion is found by the standard parametrix construction. Now when we consider the family $I - Q_\mu$ depending on $\mu \in \Gamma'_r$, we use this result for each μ, and note *moreover* that the analysis used in the proof of [G95, Th. 1.12] relies on estimates that for $I - Q_\mu$ hold uniformly in $\mu \in \Gamma'_r$. Thus $(I - Q_\mu)^{-1}$ will have its symbol belonging to S^0 *uniformly in* $\mu \in \Gamma'_r$ (in admissible coordinate systems). This shows the first requirement for having the symbol in $S^{0,0}$. For the remaining requirements on higher z-derivatives ($z = \frac{1}{\mu}$, cf. Definition 4.1), we use successively the formulas

$$\partial_z^j (I - Q_\mu)^{-1} = (I - Q_\mu)^{-1} \sum_{l<j} \binom{j}{l} \partial_z^{j-l} Q_\mu \, \partial_z^l (I - Q_\mu)^{-1}, \quad j > 0 \tag{4.18}$$

(that follow from $\partial_z^j [(I - Q_\mu)(I - Q_\mu)^{-1}] = 0$ by the Leibniz formula); they allow the conclusion that $\partial_z^j (I - Q_\mu)^{-1}$ is in S^j uniformly in $\mu \in \Gamma'_r$.

This shows that $(I - Q_\mu)^{-1}$ has symbol in $S^{0,0}$. It is weakly polyhomogeneous there, since a parametrix of $I - Q_\mu$ is so by Theorem 4.4. Finally, since P_μ^* is also weakly polyhomogeneous with symbol in $S^{0,0}$, the formula (4.17) allows us to conclude, by the composition rules, that P_μ^{-1} is a weakly polyhomogeneous ψdo with symbol in $S^{0,0}$. This shows the main part of the theorem when $s = 0$. In this case the last statements follow by use of a version of (4.18) with derivatives in μ and $I - Q_\mu$ replaced by P_μ; this shows that the relevant estimates of the symbol of P_μ carry over to the symbol of the inverse.

If $l \neq 0$, we reduce to the preceding case as follows: For any admissible vector bundle F over $\widetilde{X}$ there exists a family of isomorphisms $\Lambda^m_{F,\mu}$ from $H^{r,\mu}(F)$ to $H^{r-m,\mu}(F)$ ($m \in \mathbb{Z}$) with principal symbol essentially $\langle(\xi,\mu)\rangle^m I$ and $\Lambda^0_{F,\mu} = I$, $\Lambda^{-m}_{F,\mu} = (\Lambda^m_{F,\mu})^{-1}$, such that the operator norm of $\Lambda^m_{F,\mu}$ for any s is uniformly bounded in μ, for $\arg\mu$ in an interval $]\theta_1, \theta_2[$. (These order-reducing operators are a standard tool in [G86,G96,G95]; to get holomorphicness in μ for $|\arg\mu - \omega| < \delta$, say, one can for $m > 0$ take an operator as in [G96, Corollary 3.2.12] with $\langle(\xi,\mu)\rangle$ replaced by $(|\xi|^{2m} + (e^{-i\omega}\mu)^{2m} + 1)^{\frac{1}{2}}$ that is well-defined when $\delta \le \pi/2m$; for $-m$ one takes the inverse). Then we replace P_μ and P_μ^{-1} on suitable subsectors by

$$P_{1,\mu} = \Lambda^l_{E_2,\mu} P_\mu \Lambda^{-l}_{E_1,\mu}, \quad P^{-1}_{1,\mu} = \Lambda^l_{E_1,\mu} P_\mu^{-1} \Lambda^{-l}_{E_2,\mu}. \tag{4.19}$$

Here $P_{1,\mu}$ and $P^{-1}_{1,\mu}$ are uniformly bounded with respect to L_2 norms. Assume e.g. that $l > 0$. In view of (4.6) and (4.10), $P_\mu \Lambda^{-l}_{E_1,\mu}$ has symbol in $S^{-l,0} \cap S^{0,-l}$; subsequently $P_{1,\mu} = \Lambda^l_{E_2,\mu} P_\mu \Lambda^{-l}_{E_1,\mu}$ has symbol in

$$(S^{l,0} + S^{0,l}) \circ (S^{-l,0} \cap S^{0,-l}) \subset (S^{0,0} \cap S^{l,-l}) + (S^{-l,l} \cap S^{0,0}) \subset S^{0,0}. \tag{4.20}$$

It is seen in a similar way that the m'th μ-derivative of $P_{1,\mu}$ has symbol in $S^{-m,0} \cap S^{0,-m}$. This $P_{1,\mu}$ satisfies the hypotheses with $l = 0$, so the already proved part of the theorem shows that $P^{-1}_{1,\mu}$ is as asserted. We get back to P_μ^{-1} by considerations as in (4.20). □

When there is merely a one-sided inverse — right or left — of a given ψdo, one cannot in general expect to show that that particular operator belongs to the calculus, simply because it is generally not uniquely determined. However, one can show in such cases that there *exists* a right resp. left inverse with the expected symbol properties. (This seems to be a new observation in general.)

Theorem 4.7.

1° *Let E and F be admissible vector bundles of dimension N resp. M over an admissible boundaryless manifold $\widetilde{X}$, and let P_μ (depending on μ in a sector Γ of $\mathbb{C}$) be a weakly* *polyhomogeneous ψdo with symbol in $S^{0,0}$ in admissible coordinate systems, such that for some $l \in \mathbb{Z}$, $P_\mu \colon H^{l,\mu}(E) \to H^{l,\mu}(F)$ has a right inverse R_μ that is likewise bounded uniformly for μ in truncated closed subsectors Γ'_r. Then P_μ has a right inverse R'_μ that is a weakly polyhomogeneous ψdo with symbol in $S^{0,0}$.*

If P_μ is strongly polyhomogeneous, then so is R'_μ. If P_μ is special parameter-dependent of order 0, then so is P'_μ.

2° A similar statement holds with "right" replaced by "left."

Proof. One can reduce to the case $l = 0$ in the same way as in the preceding proof. Consider a truncated closed subsector Γ'_r. The identity $P_\mu R_\mu = I$ implies $R^*_\mu P^*_\mu = I$. Since R_μ is uniformly L_2-bounded for $\mu \in \Gamma'_r$, so is its adjoint R^*_μ:

$$\|R^*_\mu u\|_{L_2(F)} \le C\|u\|_{L_2(E)} \text{ for } u \in L_2(E),\ \mu \in \Gamma'_r,$$

for some fixed $C > 0$. Insertion of $u = P^*_\mu v$ for an arbitrary $v \in L_2(F)$ gives

$$\|v\|^2_{L_2(F)} = \|R^*_\mu P^*_\mu v\|^2_{L_2(F)} \le C^2\|P^*_\mu v\|^2_{L_2(E)} = C^2(P_\mu P^*_\mu v, v)_{L_2(F)}.$$

This shows that the selfadjoint operator $P_\mu P^*_\mu$ in $L_2(F)$ has lower bound $\ge C^{-2}$, so it has an inverse $(P_\mu P^*_\mu)^{-1}$ with L_2-operator norm $\le C^{-2}$ for $\mu \in \Gamma'_r$.

Here Theorem 4.6 applies to $P_\mu P^*_\mu$, since it has symbol in $S^{0,0}$ by the composition rules (cf. (4.9)). Then $(P_\mu P^*_\mu)^{-1}$ is a weakly polyhomogeneous ψdo with symbol in $S^{0,0}$ (since Γ'_r was arbitrary). From the identity $P_\mu P^*_\mu (P_\mu P^*_\mu)^{-1} = I$ follows that

$$R'_\mu = P^*_\mu (P_\mu P^*_\mu)^{-1} \tag{4.21}$$

is a right inverse of P_μ; it is likewise a ψdo with symbol in $S^{0,0}$.

The statements on strong polyhomogeneity and special parameter-dependence follow in a similar way from Theorem 4.6 applied to $P_\mu P^*_\mu$. This shows 1°, and assertion 2° follows by obvious modifications of the proof. □

The theorem does not say anything about the structure of R_μ itself. However, we shall use it in Section 5 in a situation where we can also infer that the given right inverse is a weakly polyhomogeneous ψdo.

5. Analysis of the resolvent.

Consider P_S as defined in Section 2; in particular it can be equal to $\mathcal{D}_\mathcal{B}$ as introduced in Section 3. We shall find a constructive expression of its resolvent in a form that allows showing asymptotic expansions of traces.

The strategy in [GS95] for characterizing the resolvent $(\Delta_1 + \mu^2)^{-1}$ associated with a Dirac-type problem with a boundary condition $(\Pi_\ge + B_0)\gamma_0 u = 0$ was essentially to express the general resolvent as a suitable perturbation of the product case resolvent, by a term that is of lower order at the boundary. When P is not of Dirac-type, we do not have a simpler reference problem (like the product case) to depart from, so a new strategy is needed. Here we establish the analysis directly by use of a Calderón projector for $P - \lambda$.

For a general explanation of the Calderón projector and associated Poisson operator and their use, see the Appendix. As noted there, the Calderón projector is most manageable when the elliptic operator, one is dealing with, can be extended to a boundaryless manifold $\widetilde{X} \supset X$ such that the extension is invertible there. This cannot be achieved for all P, but in the present case, the resolvent assumption for $P-\lambda$ comes in useful. In fact, when Assumption 2.2 1° holds, we can extend $P-\lambda$ to a ψdo $\widetilde{P}_\lambda$ on a neighboring manifold $\widetilde{X}$, such that $\widetilde{P}_\lambda$ is invertible for large λ; then we can get a good definition of the Calderón projector for this operator for such large λ:

Theorem 5.1. *Let P be such that Assumption 2.2 1° is satisfied. Let*

$$Z^s_{\lambda,+} = \{\, z \in H^s(X,E) \mid (P-\lambda)z = 0 \text{ on } X \}, \quad N^s_{\lambda,+} = \varrho Z^s_{\lambda,+}, \tag{5.1}$$

for $s \in \mathbb{R}$. Let $\widetilde{X}$ be an admissible boundaryless n-dimensional manifold in which X is smoothly imbedded, the bundle E being extended to an admissible bundle $\widetilde{E}$ there; take $\widetilde{X}$ compact when X is compact.

Each ray $re^{i\theta_0}$ in Γ has a neighborhood $\Gamma' = \{\, \lambda = re^{i\theta} \mid |\theta-\theta_0| \le \varepsilon \,\}$ in Γ so that for $\lambda \in \Gamma'$, there is an extension $\widetilde{P}_\lambda$ of $P-\lambda$ to $\widetilde{E}$ (acting like $P-\lambda$ on X), which is a uniformly parameter-elliptic strongly polyhomogeneous ψdo of degree d with respect to $\mu \in \widetilde{\Gamma}' = (-\Gamma')^{1/d}$ and has a parametrix $\widetilde{Q}_\lambda$ for $\lambda \in \Gamma'$ which is an inverse for $|\lambda| \ge r'$ (some $r' \ge 0$). Then when we define (cf. (A.3)ff.)

$$K^+_\lambda = -r^+ \widetilde{Q}_\lambda \widetilde{\varrho}^* \mathcal{A}, \quad C^+_\lambda = \varrho K^+_\lambda, \quad C^-_\lambda = I - C^+_\lambda, \tag{5.2}$$

we have for $\lambda \in \Gamma'_{r'}$, all $s \in \mathbb{R}$, cf. (2.5), (A.2):

K^+_λ maps $\mathcal{H}^s(E'^d)$ onto $Z^s_{\lambda,+}$ with right inverse ϱ, and C^+_λ is a projection in $\mathcal{H}^s(E'^d)$ with range $N^s_{\lambda,+}$. Here C^+_λ is a matrix of classical ψdo's $C^+_\lambda = (C^+_{\lambda,jk})_{j,k=0,\dots,d-1}$ with $C^+_{\lambda,jk}$ strongly polyhomogeneous of order $j-k$ with respect to $\mu \in \widetilde{\Gamma}'$, and K^+_λ is row of Poisson operators $(K^+_{\lambda,j})_{j=0,\dots,d-1}$ with $K^+_{\lambda,j}$ strongly polyhomogeneous of order $-j$; all the operators are admissible.

Proof. We here use ideas from [S69], in particular from the appendix there. Denote

$$\Gamma_{(\alpha)} = \{\, re^{i\theta} \mid r>0,\ |\theta| \le \alpha \,\}. \tag{5.3}$$

Consider a ray $re^{i\theta_0}$ in Γ; multipying $P-\lambda$ by a complex constant we can obtain that $\theta_0 = \pi$ and that $\Gamma_{(\delta)} \subset -\Gamma$ for some $\delta > 0$. Then for $\varepsilon \le \delta/2$:

$$\begin{aligned} -\lambda \in \Gamma_{(\varepsilon)},\ -\tau \in \Gamma_{(\varepsilon)} \implies |\xi|^{2d} + \lambda^2 \in \Gamma_{(2\varepsilon)} \text{ and } -\lambda - \tau(|\xi|^{2d}+\lambda^2)^{\frac12} \in \Gamma_{(2\varepsilon)} \\ \implies p(x,\xi) - \lambda - \tau(|\xi|^{2d}+\lambda^2)^{\frac12} \text{ is invertible.} \end{aligned}$$

We can then, for $\lambda \in \Gamma' = -\Gamma_{(\varepsilon)}$ and $|\xi|^{2d} + |\lambda|^2 \ge 1$ define a homotopy of $p^0 - \lambda I$ to the symbol $\mathfrak{p}(\xi,\lambda) = (|\xi|^{2d}+\lambda^2)^{\frac12} I$: Set

$$\widetilde{p}^0(x,\xi,\lambda,\theta) = \mathfrak{p}(\xi,\lambda) \tfrac{i}{2\pi} \int_{\mathcal{C}} \lambda^\theta [\mathfrak{p}(\xi,\lambda)^{-1}(p^0(x,\xi) - \lambda I) - \tau I]^{-1}\, d\tau, \tag{5.4}$$

where $\mathcal{C}$ is a curve in $(-\Gamma_{(\varepsilon)} \cup \{|\tau| \le 1\}) \setminus \overline{\mathbb{R}}_-$ encircling the eigenvalues of $\mathfrak{p}(\xi,\lambda)^{-1}(p^0(x,\xi) - \lambda^d I)$ (note that λ^θ is well-defined on $\mathcal{C}$). Here $\widetilde{p}^0(x,\xi,\lambda,\theta)$ equals $\mathfrak{p}(\xi,\lambda)I$ for $\theta = 0$ and equals $p^0(x,\xi)-\lambda I$ for $\theta=1$, and it is homogeneous of degree d in $(\xi,|\lambda|^{1/d})$, holomorphic in λ, C^∞, and invertible for all $\theta\in[0,1]$, all $|\xi|^{2d}+|\lambda|^2\ge 1$ with $\lambda\in-\Gamma_{(\varepsilon)}$.

We can assume that $\widetilde X$ contains the neighboorhood $U\cup U_-$ of X' (described at the start of the Appendix), where we can identify $\widetilde E$ with the pull-back of E'. In view of the uniform parameter-ellipticity, there is a neighborhood V of X with $X\cup(X'\times[-c,0])\subset\overline{V}\subset X\cup U_-$ so that P extends to V as an admissible differential operator satisfying Assumption 2.2 1°. Moreover, we can deform the symbol $p^0(x,\xi)-\lambda$ smoothly through u.p.-elliptic ψdo symbols homogeneous in $(\xi,|\lambda|^{1/d})$ to $\mathfrak{p}(\xi,\lambda)I$ by use of (5.4) when x_n goes from $-\frac13 c$ to $-\frac23 c$, and then extend it as $\mathfrak{p}(\xi,\lambda)I$ on the rest of $\widetilde X$. This gives a principal symbol $p_1^0(x,\xi,\lambda)$ defined on all of $\widetilde X$, defining a u.p.-elliptic ψdo $\widetilde P_{1,\lambda}$ of order d; it is strongly polyhomogeneous for $\mu\in\widetilde\Gamma'$. Now take

$$\widetilde P_\lambda=\varphi(P-\lambda I)\varphi+\psi\widetilde P_{1,\lambda}\psi, \tag{5.5}$$

where φ and ψ are admissible (bounded with bounded derivatives) C^∞ functions on $\widetilde X$ with $\varphi^2+\psi^2=1$, such that φ is 1 on $X\cup(X'\times[-\frac19 c,0])$ and ψ is 1 on the complement of $X\cup(X'\times[-\frac29,0])$. This $\widetilde P_\lambda$ is a u.p.-elliptic and strongly polyhomogeneous ψdo of order d that acts like $P-\lambda$ on distributions supported in a neighborhood of X. $\widetilde P_{\lambda,+}$ has the same Green's formula as P, (2.1).

$\widetilde P_\lambda$ has a parametrix $\widetilde Q'_\lambda$ for $\lambda\in-\Gamma_{(\varepsilon)}$, u.p.-elliptic and strongly polyhomogeneous of order $-d$, by the usual formulas. Since $\widetilde P_\lambda\widetilde Q'_\lambda=I+\mathcal S_\lambda$ where $\mathcal S_\lambda$ is strongly polyhomogeneous of order -1, hence has an L_2 operator norm going to 0 for $|\lambda|\to\infty$ in $-\Gamma_{(\varepsilon)}$, $I+\mathcal S_\lambda$ can be inverted within the calculus (by a Neumann series) for sufficiently large λ; here $\widetilde Q'_\lambda$ can be modified to the true inverse $\widetilde Q_\lambda=\widetilde Q'_\lambda(I+\mathcal S_\lambda)^{-1}$. This is strongly polyhomogeneous with global spatial estimates, by Theorem 4.6. (A detailed account in a more general situation is given in [G96, Th. 3.2.11]; for compact manifolds, [G86, Remark 3.2.12] or Shubin [Sh87] suffice.)

We now simply define K^+_λ and C^+_λ by (5.2); then the verification that they have the mentioned mapping properties goes exactly as in Theorem A.1. The resulting operators are strongly polyhomogeneous by [GS95, Lemma A.1, Th. 1.16] and have uniform spatial estimates since $\widetilde Q_\lambda$ and $\mathcal A$ do so. □

For use later in Corollary 5.4 let us also note that $\varrho\widetilde Q_{\lambda,+}$ (as a function of $\mu=(-\lambda)^{1/d}\in\widetilde\Gamma'$) is a strongly polyhomogeneous trace operator of class 0, cf. [G95, Lemma A.1 (ii)].

Now Theorem A.4 is valid for $P-\lambda$ with $C^\pm$, K^+ and Q_+ replaced by $C^\pm_\lambda$, K^+_λ and $\widetilde Q_{\lambda,+}$, in the exact form since the extension $\widetilde P_\lambda$ of $P-\lambda$ has the inverse $\widetilde Q_\lambda$ on $\widetilde X$. Consider a system $\begin{pmatrix}P-\lambda\\ S\varrho\end{pmatrix}$ satisfying Assumptions 2.1 and 2.2. By Lemma 2.3, it is surjective from $H^d(E)$ to $L_2(E)\times\mathcal H^d(F)$ for each large $\lambda\in\Gamma$. Then we have in view of (A.14)–(A.16) (or Theorem A.4) that the ψdo SC^+_λ on X' is surjective for each λ. We shall show that SC^+_λ has a right inverse belonging to our weakly polyhomogeneous ψdo's.

Lemma 5.2. *Let $\lambda\in\Gamma'_r$ (with Γ' as in Theorem 5.1 and r so large that $\widetilde Q_\lambda=\widetilde P_\lambda^{-1}$ and Assumption 2.2 is satisfied). Then SC^+_λ has the right inverse, with K_λ defined by Lemma 2.3,*

$$S'_\lambda=\varrho K_\lambda; \tag{5.6}$$

it is a ψdo mapping $\mathcal{H}^{s,\mu}(F)$ onto $\mathcal{H}^{s,\mu}(E'^d)$ with uniform bounds in $\mu = |\lambda|^{1/d}$, for all $s \geq d$.

Proof. By the converse part of Theorem A.4 1°, (5.6) is a right inverse of SC^+_λ. The mapping property follows from the second line in (2.10) by composition with ϱ. □

We would like to use Theorem 4.7 to show that S'_λ is weakly polyhomogeneous in terms of $\mu = (-\lambda)^{1/d}$. One difficulty in this is that S'_λ is just a right inverse, not a two-sided inverse (and such right inverses are not uniquely determined). Another difficulty is that S and C^+_λ are multi-order systems. But these difficulties can be overcome, as shown in the following theorem.

To eliminate the effects of the multi-order, we conjugate the operators (in each subsector Γ'_r) with

$$\Theta_{F,\lambda} = \begin{pmatrix} \Lambda^{d-1}_{F_0,\mu} & 0 & \dots & 0 \\ 0 & \Lambda^{d-2}_{F_1,\mu} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_{F_{d-1}} \end{pmatrix}, \quad \mu = (-\lambda)^{1/d}, \tag{5.7}$$

and the analogous operator $\Theta_{E'^d,\lambda}$; the entries are defined as in the proof of Theorem 4.6. We set

$$\widetilde{S}_\lambda = \Theta_{F,\lambda} S \Theta^{-1}_{F,\lambda}, \quad \widetilde{C}^+_\lambda = \Theta_{E'^d,\lambda} C^+_\lambda \Theta^{-1}_{E'^d,\lambda}. \tag{5.8}$$

Here the entries are of order 0. $\widetilde{C}^+_\lambda$ is again strongly polyhomogeneous in terms of $\mu \in \widetilde{\Gamma}'$ since the $\Lambda^l_{E',\mu}$ are so; hence it is in fact special parameter-dependent of order 0. For $\widetilde{S}_\lambda$ is follows from the lower triangular form of S that $\widetilde{S}_\lambda$ is again lower triangular. The entries in and below the diagonal are of the form $\Lambda^{d-1-j}_{F_j,\mu} S_{jk} \Lambda^{k+1-d}_{F_k,\mu}$ with $j \geq k$ and thus, since $S_{jk} \in S^{j-k} \subset S^{j-k,0}$, they are seen to have symbols in $S^{0,0}$ with μ-derivatives of order m in $S^{-m,0} \cap S^{0,-m}$ for any m, by calculations as around (4.20). (For $k < j < d-1$ one needs the observation that $S^{j-k,k-j} \cap S^{j+1-d,d-1-j} \subset S^{0,0}$ by interpolation since $j-k > 0$, $j+1-d < 0$.) Thus $\widetilde{S}_\lambda$ is special parameter-dependent of order 0. We also define

$$\widetilde{S}'_\lambda = \Theta_{E'^d,\lambda} S'_\lambda \Theta^{-1}_{F,\lambda}. \tag{5.9}$$

Theorem 5.3. *Let P and S satisfy Assumptions* 2.1 *and* 2.2.

For λ in truncated subsectors Γ'_r of Γ (as in Lemma 5.2*), the operator SC^+_λ has a right inverse $S''_\lambda = \Theta^{-1}_{E'^d,\lambda} \widetilde{S}''_\lambda \Theta_{F,\lambda}$ where $\widetilde{S}''_\lambda$ is special parameter-dependent of order* 0 *(in terms of $\mu = (-\lambda)^{1/d}$).*

The right inverse S'_λ defined in Lemma 5.2 *equals $C^+_\lambda S''_\lambda$, and $\widetilde{S}'_\lambda$ defined by* (5.9) *is special parameter-dependent of order* 0.

Proof. The operator $\widetilde{S}_\lambda \widetilde{C}^+_\lambda$ is continuous from $H^{t,\mu}(E'^d)$ to $H^{t,\mu}(F)$ for any s, in particular for $s = 0$. It has the right inverse $\widetilde{S}'_\lambda$, which is continuous from $H^{t,\mu}(F)$ to $H^{t,\mu}(E'^d)$, uniformly in μ, for $t \geq \frac{1}{2}$, in view of (5.6), (2.10) and the mapping properties of the $\Lambda^l_{F_j,\mu}$. In particular, the continuity holds with $t = 1$. We can then apply Theorem 4.7 with $l = 1$, which shows the existence of a right inverse $\widetilde{S}''_\lambda$ that is special parameter-dependent of order 0.

The right inverse we have constructed in this way need not be the same as $\widetilde{S}'_\lambda$ defined after Lemma 5.2 in (5.9). However, since $\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}$ is *bijective*, we infer from the converse parts of 1° and 2° in Theorem A.4 that $\begin{pmatrix} S \\ C^-_\lambda \end{pmatrix}$ is injective and SC^+_λ is surjective, hence S defines a *bijection* of $N^s_{\lambda,+}$ onto $\mathcal{H}^s(F)$, and so does SC^+_λ. Then SC^+_λ has only one right inverse ranging in $N^s_{\lambda,+}$. Now S'_λ in (5.6) does map into $N^s_{\lambda,+}$ since $(P-\lambda)K_\lambda = 0$, so it is *the* right inverse of SC^+_λ ranging in $N^s_{\lambda,+}$. When S'''_λ is an arbitrary right inverse, then

$$I = SC^+_\lambda S'''_\lambda = SC^+_\lambda C^+_\lambda S'''_\lambda,$$

so $C^+_\lambda S'''_\lambda$ is a right inverse ranging in $N_{\lambda,+}$; hence it must equal S'_λ. In particular, for the right inverse S''_λ found above,

$$S'_\lambda = C^+_\lambda S''_\lambda.$$

It then follows from the rules of calculus that also $\widetilde{S}'_\lambda = \Theta_{E'^d,\lambda} S'_\lambda \Theta^{-1}_{F,\lambda} = \widetilde{C}^+_\lambda \widetilde{S}''_\lambda$ is a special parameter-dependent ψdo of order 0. □

Since $\widetilde{Q}_\lambda$ is the inverse of $\widetilde{P}_\lambda$, we can now apply the direct part of Theorem A.4 1° to describe the inverse of $\begin{pmatrix} P-\lambda \\ S\varrho \end{pmatrix}$. This gives as an immediate corollary:

Corollary 5.4. *For λ in truncated subsectors Γ'_r of Γ (as in Lemma 5.2), the resolvent $R_\lambda = (P_S - \lambda)^{-1}$ and the Poisson solution operator K_λ in (2.6) satisfy*

$$\begin{aligned} R_\lambda &= \widetilde{Q}_{\lambda,+} - G_\lambda \text{ with } G_\lambda = K^+_\lambda S'_\lambda S\varrho \widetilde{Q}_{\lambda,+}, \\ K_\lambda &= K^+_\lambda S'_\lambda, \end{aligned} \tag{5.10}$$

where S'_λ is as in Theorem 5.3.

In terms of $\mu = (-\lambda)^{1/d}$, K^+_λ resp. $\varrho\widetilde{Q}_{\lambda,+}$ are a strongly polyhomogeneous Poisson resp. trace operator, and $\Theta_{E'^d,\lambda} S'_\lambda \Theta^{-1}_{F,\lambda}$ and $\Theta_{E'^d,\lambda} S'_\lambda S \Theta^{-1}_{F,\lambda}$ are special parameter-dependent ψdo's of order 0. In particular, we can write

$$G_\lambda = \mathcal{K}_\lambda \mathcal{S}_\lambda \mathcal{T}_\lambda \text{ with } \mathcal{K}_\lambda = K^+_\lambda \Theta^{-1}_{E'^d,\lambda},\ \mathcal{S}_\lambda = \Theta_{E'^d,\lambda} S'_\lambda S \Theta^{-1}_{E'^d,\lambda},\ \mathcal{T}_\lambda = \Theta_{E'^d,\lambda} \varrho \widetilde{Q}_{\lambda,+}, \tag{5.11}$$

where $\mathcal{K}_\lambda$ is a strongly polyhomogeneous Poisson operator of order $1-d$, $\mathcal{S}_\lambda$ is a special parameter-dependent ψdo on X' of order 0, and $\mathcal{T}_\lambda$ is a strongly polyhomogeneous trace operator of order -1.

Here S'_λ and $S'_\lambda S$ are covered by the analysis in Theorem 5.3, whereas K^+_λ and $\varrho\widetilde{Q}_{\lambda,+}$ were described in Theorem 5.1ff; see also (5.7).

6. Trace formulas.

We can finally obtain trace formulas, by the methods of [GS95].

Theorem 6.1. *Let P_S be the realization (2.3) defined from a differential operator P of order d in a bundle E over a manifold X together with a boundary condition (2.2) (all admissible), such that Assumptions 2.1 and 2.2 are satisfied. When $(m+1)d > n = \dim X$, the resolvent $R_\lambda = (P_S - \lambda)^{-1}$ satisfies for any compactly supported morphism φ in E:*

$$\operatorname{Tr}(\varphi \partial_\lambda^m (P_S - \lambda)^{-1}) \sim a_0(-\lambda)^{\frac{n}{d}-m-1} + \sum_{j=1}^{\infty} (a_j + b_j)(-\lambda)^{\frac{n-j}{d}-m-1} + \sum_{k=0}^{\infty} (c_k \log(-\lambda) + c'_k)(-\lambda)^{-\frac{k}{d}-m-1}, \tag{6.1}$$

for $\lambda \to \infty$ in closed subsectors of Γ. The coefficients a_j, b_j and c_k are integrals, $\int_{X_1} a_j(x)\,dx$, $\int_{X'_1} b_j(x')\,dx'$ and $\int_{X'_1} c_k(x')dx'$, of densities a_j locally determined by the symbols of P, resp. b_j and c_k locally determined by the symbols of P and S at X'; here X_1 is a smooth compact neighborhood of $\operatorname{supp}\varphi$ in X such that $X'_1 = X_1 \cap X'$ is a neighborhood of $\operatorname{supp}\varphi \cap X'$ in X'. The c'_k are in general globally determined.

Proof. $\varphi \partial_\lambda^m R_\lambda$ is trace class since it maps $L_2(E)$ into $H^{(m+1)d}(E|_{X_1})$ and the injection $H^{(m+1)d}(E|_{X_1}) \hookrightarrow L_2(E|_{X_1})$ is trace class. The kernel is continuous and the trace is the integral of the fiber trace of the kernel on the diagonal, so one only has to integrate over X_1. Consider a truncated subsector Γ'_r as in Lemma 5.2. From Corollary 5.4 follows that

$$\begin{aligned} \partial_\lambda^m R_\lambda &= \partial_\lambda^m (P_S - \lambda)^{-1} = m!(P_S - \lambda)^{-m-1} = m!(\widetilde{Q}_{\lambda,+} - G_\lambda)^{m+1} \\ &= m!(\widetilde{Q}_{\lambda,+})^{m+1} + \textstyle\sum_{k=1}^{m+1} \operatorname{pol}_k(\widetilde{Q}_{\lambda,+}, G_\lambda) \\ &= m!(\widetilde{Q}_\lambda^{m+1})_+ + \widetilde{G}_\lambda + \textstyle\sum_{k=1}^{m+1} \operatorname{pol}_k(\widetilde{Q}_{\lambda,+}, G_\lambda), \end{aligned} \tag{6.2}$$

where the expressions pol_k are "polynomials" in the two (non-commuting) terms in R_λ, in the sense that they are linear combinations of compositions with $m-k$ factors $\widetilde{Q}_{\lambda,+}$ and k factors G_λ. The term $\widetilde{G}_\lambda$ is the singular Green operator (cf. e.g. [G96, (1.2.35)])

$$\widetilde{G}_\lambda = m!((\widetilde{Q}_{\lambda,+})^{m+1} - (\widetilde{Q}_\lambda^{m+1})_+). \tag{6.3}$$

In the dependence on $\mu = (-\lambda)^{1/d}$, we have in view of the rules of calculus of [GS95], [G96] that $\widetilde{Q}_\lambda^{m+1}$ is a strongly polyhomogeneous ψdo of order $-(m+1)d$ on $\widetilde{X}$, $\widetilde{G}_\lambda$ is a strongly polyhomogeneous singular Green operator of order $-(m+1)d$ on X, and the sum over k is a sum of compositions containing strongly polyhomogeneous operators (of all types) together with the special parameter-dependent ψdo $\widetilde{\mathcal{S}}_\lambda$.

Consider the trace

$$\operatorname{Tr}_X \varphi \partial_\lambda^m R_\lambda = \operatorname{Tr}_X \varphi m!(\widetilde{Q}_\lambda)_+^{m+1} + \operatorname{Tr}_X \varphi[\widetilde{G}_\lambda + \sum_{k=1}^{m+1} \operatorname{pol}_k(\widetilde{Q}_{\lambda,+}, G_\lambda)].$$

By the construction of $\widetilde{P}_\lambda$ in Theorem 5.1, the restriction $(\widetilde{Q}_\lambda^{m+1})_+$ of $\widetilde{Q}_\lambda^{m+1}$ is the restriction of a strongly polyhomogeneous parametrix of $(P-\lambda)^{m+1}$ defined on a neighborhood of X, so $\operatorname{Tr}_X \varphi m!(\widetilde{Q}_\lambda^{m+1})_+$ contributes a well-known expansion $\sum_0^\infty a_j(-\lambda)^{\frac{n-j}{d}-m-1}$.

The singular Green operator $\varphi\widetilde{G}_\lambda$ is strongly polyhomogeneous of order $-(m+1)d$ and hence of regularity $+\infty$ in the sense of [G86,G96], so it contributes an expansion $\sum_1^\infty b_{0,j}(-\lambda)^{\frac{n-j}{d}-m-1}$, by the proof of [G86, Th. 3.3.10ff.] or [G96, Th. 3.3.9ff.], also recalled in [G92, App.].

In view of (5.11), the terms in the polynomials pol_k contain $\mathcal{S}_\lambda$ as one or several factors. Here we use the invariance of the trace under cyclic permutation of the operators, to reduce to the study of an operator on X'. Since $\widetilde{Q}_{\lambda,+}$ composes with strongly polyhomogeneous Poisson and trace operators to give Poisson resp. trace operators that are again strongly polyhomogeneous, each term in pol_k has the structure

$$\mathcal{G}_\lambda = \varphi\mathcal{K}_{1,\lambda}\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda}, \tag{6.4}$$

with $\mathcal{G}_\lambda$ of total order $-(m+1)d$ and the $\mathcal{K}_{j,\lambda}$ and $\mathcal{T}_{j,\lambda}$ strongly polyhomogeneous Poisson and trace operators of order ≤ 0. Let ψ denote a morphism over X' that is the identity over a neighborhood of $\operatorname{supp}\varphi\cap X'$ and is supported in X'_1; then $\varphi\mathcal{K}_{1,\lambda}(I-\psi)$ is strongly polyhomogeneous of order $-\infty$, so its norm in Sobolev spaces is $O(\langle\lambda\rangle^{-M})$, any M, and $\operatorname{Tr}\varphi\mathcal{K}_{1,\lambda}(I-\psi)\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda}$ is $O(\langle\lambda\rangle^{-M})$, any M. For the remaining part,

$$\begin{aligned}\operatorname{Tr}_X \varphi\mathcal{K}_{1,\lambda}\psi\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}\mathcal{S}_\lambda\mathcal{T}_{J,\lambda} &= \operatorname{Tr}_{X'}\mathcal{S}'_\lambda, \text{ with}\\ \mathcal{S}'_\lambda = \psi\mathcal{S}_\lambda\mathcal{T}_{1,\lambda}\mathcal{K}_{2,\lambda}\mathcal{S}_\lambda\mathcal{T}_{2,\lambda}\dots\mathcal{K}_{J,\lambda}&\mathcal{S}_\lambda\mathcal{T}_{J,\lambda}\varphi\mathcal{K}_{1,\lambda};\end{aligned} \tag{6.5}$$

here the factors $\mathcal{T}_{j,\lambda}\mathcal{K}_{j+1,\lambda}$ and $\mathcal{T}_{J,\lambda}\varphi\mathcal{K}_{1,\lambda}$ are strongly polyhomogeneous ψdo's on X' of orders ≤ 0. It follows that the ψdo $\mathcal{S}'_\lambda$ is a special parameter-dependent ψdo of order $-(m+1)d$. We can now apply [GS95, Th. 2.1] to this by integration over X'_1, using a reduction to local trivializations and a partition of unity. Since X' has dimension $n-1$ and the symbol has degrees $-(m+1)d-k$, $k\ge 0$, and μ-exponent $-(m+1)d$, we get an expansion in a series of locally determined terms $\tilde{b}_k(-\lambda)^{\frac{n-k}{d}-m-1}$, $k\ge 1$, together with a series of terms $(\tilde{c}_k\log(-\lambda)+\tilde{c}'_k)(-\lambda)^{\frac{k}{d}-m-1}$, $k\ge 0$, with $\tilde{c}_k$ locally determined.

Collecting all the contributions, we find (6.1). □

We have as an immediate consequence:

Corollary 6.2. *When J in Assumption 2.2 contains $[\theta_1,\theta_2]$ with $]\theta_1,\theta_2[\supset[\frac{\pi}{2},\frac{3\pi}{2}]$, so that the heat operator e^{-tP_S} can be defined for $t>0$ by*

$$\begin{gathered} e^{-tP_S} = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}} e^{-t\lambda}(P_S-\lambda)^{-1}\,d\lambda, \text{ with}\\ \mathcal{C} = \{\lambda = e^{\mathrm{i}\theta_2}r \mid r\ge r_0\} + \{\lambda = e^{\mathrm{i}\theta}r_0 \mid \theta_2>\theta>\theta_1\} + \{\lambda = e^{\mathrm{i}\theta_1}r \mid r_0\le r\},\end{gathered} \tag{6.6}$$

then there are trace expansions for $t\to 0$, when φ has compact support:

$$\operatorname{Tr}(\varphi e^{-tP_S}) \sim \bar{a}_0 t^{-\frac{n}{d}} + \sum_{j=1}^{\infty}(\bar{a}_j+\bar{b}_j)t^{\frac{j-n}{d}} + \sum_{k=0}^{\infty}(\bar{c}_k\log t + \bar{c}'_k)t^{\frac{k}{d}}; \tag{6.7}$$

here the coefficients are proportional to those in (6.1) by universal factors.

Proof. (6.6) implies

$$\operatorname{Tr}\varphi e^{-tP_S} = \tfrac{\mathrm{i}}{2\pi}\int_{\mathcal{C}}(-t)^{-m}e^{-t\lambda}\operatorname{Tr}\varphi\partial_\lambda^m(P_S-\lambda)^{-1}\,d\lambda.$$

The expansion (6.7) is shown by insertion of sums of terms from (6.1) down to a given order plus a remainder $O(\langle\lambda\rangle^{-N})$, and letting the order $\to -\infty$. Here one uses simple calculations such as:

$$\begin{aligned}\int_{\mathcal C}(-t)^{-m}e^{-t\lambda}(-\lambda)^s\log(-\lambda)\,d\lambda &= -\tfrac{d}{ds}\int_{\mathcal C}(-t)^{-m}e^{-t\lambda}(-\lambda)^s\,d\lambda\\ &= -\tfrac{d}{ds}(-t)^{-m}t^{-s-1}\int_{\mathcal C'}e^{-\varrho}(-\varrho)^s\,d\varrho = \text{const.}\ t^{-m-s-1}\log t. \qquad \square\end{aligned}$$

Theorem 6.1 holds in particular for $(\mathcal D_{\mathcal B}+\mu)^{-1}$, giving expansions of the form

$$\operatorname{Tr}(\varphi\partial_\mu^m(\mathcal D_{\mathcal B}+\mu)^{-1}) \sim \sum_{j=0}^{n-1} c_{j-n}\mu^{n-j-m-1} + \sum_{k=0}^{\infty}(c_k\log\mu + c'_k)\mu^{-k-m-1}, \tag{6.8}$$

for $\mu\to\infty$ in closed subsectors of Γ_0. We apply this to (3.26) by use of (3.28) as in [GS95, Sect. 3.4]: Take $\varphi = (\varphi_{kl})_{k,l=1,2}$ with just one block different from zero in order to get the traces of the individual blocks in (3.28), and set $\lambda = -\mu^2$. This gives trace expansions of the m'th derivatives of $\varphi(\Delta_i-\lambda)^{-1}$ $(i=1,2)$, $\psi D_B(\Delta_1-\lambda)^{-1}$ and $\psi {D_B}^*(\Delta_2-\lambda)^{-1}$, and consequences for heat trace expansions as in Corollary 6.2:

Theorem 6.3. *Let D_B be the realization of a first-order uniformly elliptic differential operator D from E_1 to E_2 with a uniformly well-posed boundary condition $B\gamma_0u=0$ (manifolds, bundles and operators being admissible). Then when φ and ψ are compactly supported morphisms (in E_i resp. from E_j to E_i, $i,j=1,2$), there are resolvent trace expansions in closed truncated subsectors of $\mathbb C\setminus\overline{\mathbb R}_+$, for $m\ge n$:*

$$\begin{aligned}&\operatorname{Tr}(\varphi\partial_\lambda^m(\Delta_i-\lambda)^{-1}) \sim \sum_{j=0}^{n-1}\tilde a_{i,j-n}(-\lambda)^{\frac{n-j}{2}-m-1} + \sum_{k=0}^{\infty}\big(\tilde a_{i,k}\log(-\lambda)+\tilde a'_{i,k}\big)(-\lambda)^{\frac{-k}{2}-m-1},\\ &\operatorname{Tr}(\psi D_B\partial_\lambda^m(\Delta_1-\lambda)^{-1}) \sim \sum_{j=1}^{n-1}\tilde b_{1,j-n}(-\lambda)^{\frac{n-j+1}{2}-m-1}\\ &\qquad\qquad + \sum_{k=0}^{\infty}\big(\tilde b_{1,k}\log(-\lambda)+\tilde b'_{1,k}\big)(-\lambda)^{\frac{-k+1}{2}-m-1},\end{aligned} \tag{6.9}$$

with a similar formula for $\operatorname{Tr}(\psi {D_B}^*\partial_\lambda^m(\Delta_2-\lambda)^{-1})$ with coefficients $\tilde b_{2,k}$ and $\tilde b'_{2,k}$. Moreover, there are heat trace expansions when $t\to 0+$:

$$\begin{aligned}\operatorname{Tr}(\varphi e^{-t\Delta_i}) &\sim \sum_{j=0}^{n-1}a_{i,j-n}t^{\frac{j-n}{2}} + \sum_{k=0}^{\infty}\big(a_{i,k}\log t + a'_{i,k}\big)t^{\frac k2}, \quad i=1,2;\\ \operatorname{Tr}(\psi D_B e^{-t\Delta_1}) &\sim \sum_{j=1}^{n-1}b_{1,j-n}t^{\frac{j-n-1}{2}} + \sum_{k=0}^{\infty}\big(b_{1,k}\log t + b'_{1,k}\big)t^{\frac{k-1}{2}},\end{aligned} \tag{6.10}$$

with a similar formula for $\operatorname{Tr}(\psi D_B{}^* e^{-t\Delta_2})$ *with coefficients* $b_{2,k}$ *and* $b'_{2,k}$. *The coefficients in* (6.10) *are proportional to those in* (6.9) *by universal factors. The unprimed coefficients are locally determined; the primed coefficients depend on the operators in a global way.*

The terms $\tilde b_{i,-n}(-\lambda)^{\frac12-m-1}$ and $b_{i,-n}t^{\frac{n+1}{2}}$ have been left out, since their coefficients are formed by integration in ξ of functions that are odd in ξ, which gives zero.

When X is compact, one can also pass via the zeta function as in [GS95]. One then gets, with the same $a_{i,k}$, $a'_{i,k}$, $b_{i,k}$ and $b'_{i,k}$ as in (6.10):

$$\begin{aligned}\Gamma(s)\operatorname{Tr}(\varphi\Delta_i^{-s}) &\sim \sum_{j=0}^{n-1}\frac{a_{i,j-n}}{s-\frac{j-n}{2}}+\frac{\operatorname{Tr}\varphi\Pi_0(D_B)}{s}+\sum_{k=0}^{\infty}\Big(\frac{-a_{i,k}}{(s-\frac{k}{2})^2}+\frac{a'_{i,k}}{s-\frac{k}{2}}\Big),\\ \Gamma(s)\operatorname{Tr}(\psi D_B\Delta_1^{-s}) &\sim \sum_{j=1}^{n-1}\frac{b_{1,j-n}}{s-\frac{j-n-1}{2}}+\sum_{k=0}^{\infty}\Big(\frac{-b_{1,k}}{(s-\frac{k-1}{2})^2}+\frac{b'_{1,k}}{s-\frac{k-1}{2}}\Big),\end{aligned}\tag{6.11}$$

with a similar formula for $\operatorname{Tr}(\psi D_B{}^*\Delta_2^{-s})$ with coefficients $b_{2,k}$ and $b'_{2,k}$. (The left-hand side is meromorphic on $\mathbb{C}$ and the right-hand side gives the full pole structure.)

The results apply of course to all the cases presented in the examples in Section 3. For comparison with earlier results it is of interest to see how the expansions vary under perturbations of B.

Theorem 6.4. *Consider two choices* B_1 *and* B_2 *of* B *as in Theorem* 6.3, *such that* $B' = B_2 - B_1$ *is a* ψ*do of order* -1. *Denote*

$$\begin{aligned}&\mathcal{B}_i = (\,B_i\quad (I-B_i^*)\sigma^*\,),\ i=1,2,\quad \mathcal{B}'=\mathcal{B}_2-\mathcal{B}_1,\\ &\begin{pmatrix}\mathcal{D}+\mu\\ \mathcal{B}_i\gamma_0\end{pmatrix}^{-1} = (\,\mathcal{R}_{i,\mu}\quad \mathcal{K}_{i,\mu}\,)\ \text{for } \mu\in\mathbb{C}\setminus i\mathbb{R},\ i=1,2.\end{aligned}\tag{6.12}$$

Then

$$\mathcal{R}_{2,\mu}=\mathcal{R}_{1,\mu}-\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu},\quad \mathcal{K}_{2,\mu}=\mathcal{K}_{1,\mu}-\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{K}_{2,\mu}.\tag{6.13}$$

Here when $m\ge n$ *and* φ *has compact support,* $\operatorname{Tr}\varphi\partial_\mu^m(\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu})$ *has an asymptotic expansion for* $\mu\to\infty$ *in closed subsectors of* Γ_0*:*

$$\operatorname{Tr}\varphi\partial_\mu^m(\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu})\sim\sum_{j=2}^{n-1}c_{j-n}\mu^{n-m-1-j}+\sum_{k=0}^{\infty}(c_k\log\mu+c'_k)\mu^{-m-1-k},\tag{6.14}$$

so the first two terms in the expansions (6.9)–(6.11) *are the same for* D_{B_1} *and* D_{B_2}. *If* B' *is of order* $-\infty$, *the series* (6.14) *reduces to*

$$\operatorname{Tr}\varphi\partial_\mu^m(\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu})\sim\sum_{k=0}^{\infty}c'_k\mu^{-m-1-k},\tag{6.15}$$

so all the unprimed terms in the expansions (6.9)–(6.11) *are the same for* D_{B_1} *and* D_{B_2}.

Proof. By the definition of the inverses,

$$\begin{pmatrix}\mathcal{D}+\mu\\ \mathcal{B}_1\gamma_0\end{pmatrix}^{-1}(\,\mathcal{R}_{2,\mu}\quad\mathcal{K}_{2,\mu}\,)=\begin{pmatrix}I & 0\\ -\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu} & I-\mathcal{B}'\gamma_0\mathcal{K}_{2,\mu}\end{pmatrix}.$$

Composition with $(\,\mathcal{R}_{1,\mu} \quad \mathcal{K}_{1,\mu}\,)$ gives

$$(\,\mathcal{R}_{2,\mu} \quad \mathcal{K}_{2,\mu}\,) = (\,\mathcal{R}_{1,\mu} \quad \mathcal{K}_{1,\mu}\,)\begin{pmatrix} I & 0 \\ -\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu} & I-\mathcal{B}'\gamma_0\mathcal{K}_{2,\mu}\end{pmatrix},$$

which implies (6.13). Now by use of circular permutation as in the proof of Theorem 6.1, the Leibniz formula and the explicit formulas in Corollary 5.4,

$$\begin{aligned}\operatorname{Tr}_X \varphi\partial_\mu^m(\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu}) &= \operatorname{Tr}_X \textstyle\sum_{k\le m}\binom{m}{k}\varphi\partial_\mu^k\mathcal{K}_{1,\mu}\mathcal{B}'\gamma_0\partial_\mu^{m-k}\mathcal{R}_{2,\mu}\\ &= \operatorname{Tr}_{X'}\partial_\mu^m(\mathcal{B}'\gamma_0\mathcal{R}_{2,\mu}\varphi\mathcal{K}_{1,\mu}) = \operatorname{Tr}_{X'} S'_\mu,\\ \text{where } S'_\mu &= \partial_\mu^m(\mathcal{B}'\gamma_0(\widetilde{Q}_{\mu,+} - K^+_\mu S_{2,\mu}\mathcal{B}_2\gamma_0\widetilde{Q}_{\mu,+})\varphi K^+_\mu S_{1,\mu});\end{aligned}$$

here we denote by $S_{i,\mu}$ the right inverses of $\mathcal{B}_iC^+_\mu$ constructed for the respective problems in Lemma 5.2 and Theorem 5.3. As shown earlier, $\gamma_0\widetilde{Q}_{\mu,+}\varphi K^+_\mu$ and $\gamma_0K^+_\mu = C^+_\mu$ are strongly polyhomogeneous ψdo's on X' of orders -1 and 0, and the $S_{i,\mu}$ are special parameter-dependent of order 0. Since $\mathcal{B}'$ is independent of μ and of order -1, it follows that S'_μ has symbol in $S^{-2-m,0}\cap S^{-1,-1-m}$. Then [GS95, Th. 2.1] implies (6.14).

If B' is of order $-\infty$, so is $\mathcal{B}'$; then S'_μ has symbol in $S^{-\infty,-1-m}$, and [GS95, Th. 2.1] or just [GS95, Prop. 1.21] implies (6.15). ☐

In the case with X compact and a product structure near X', the Calderón projector differs from $\Pi_\ge$ by an operator of order $-\infty$ by Proposition 3.5, so for $B = C^+$, the expansions (6.9)–(6.11) only differ in the primed coefficients from the expansions known for $B = \Pi_\ge$. Here it was shown in [GS96] that all the logarithmic terms vanish when $n = \dim X$ is odd; when n is even, the logarithmic terms with k even > 0 vanish, and the logarithm at the power zero vanishes if in addition $\varphi = I$ (exact formulas were also given). So we find:

Corollary 6.5. *Consider the product case with X compact, $B = C^+$. Then the expansions* (6.9)–(6.11) *differ from those known for $B = \Pi_\ge$ only in the primed coefficients. In particular: When n is odd, all the logarithmic terms vanish. When n is even, the logarithmic terms with k even > 0 vanish in* (6.9)–(6.10)*; also the $\tilde{a}_{i,0}$ and $a_{i,0}$ vanish if $\varphi = I$.*

Note that it is the global coefficients that may be changed when we replace $\Pi_\ge$ by C^+ in the product case, whereas the locally determined coefficients are unchanged. More precise statements can be inferred from the precise formulas in [GS96], showing that the local coefficients resulting from the boundary condition are proportional, by certain universal constants, to specific coefficients in the zeta and eta function expansions (or heat trace expansions) for A. It is shown in Gilkey and Grubb [GG97] that these coefficients are generically nonzero.

Remark 6.6. Our results show that the boundary conditions considered in [BL97] give heat operators with trace expansions (6.10) also when the structure is not of product type near X'; this is a new result. One can moreover use Theorem 6.4 to conclude in the product case that conditions that differ from those in [BL97] by an operator of order $-\infty$ have similar locally determined coefficients, in the same way as in the comparison with the case $B = \Pi_\ge$ in Corollary 6.5.

Let us finally observe the resulting index formula:

Corollary 6.7. *Let X be compact and let B be well-posed for D. Then the index of D_B equals*

$$\operatorname{index} D_B = a'_{1,0} - a'_{2,0} \tag{6.16}$$

where the $a'_{i,0}$ are the coefficients entering in (6.10) with $\varphi = 1$.

Moreover, when $\varphi = 1$, all the other coefficents coincide for $i = 1$ and 2:

$$a_{1,k} = a_{2,k} \text{ for all } k \geq -n \text{ and } a'_{1,k} = a'_{2,k} \text{ for all } k > 0. \tag{6.17}$$

Proof. This follows from the well-known fact (cf. e.g. [G86,G96, Sect. 4.3]) that

$$\operatorname{index} D_B = \operatorname{Tr} e^{-t\Delta_1} - \operatorname{Tr} e^{-t\Delta_2} \quad \text{for } t > 0; \tag{6.18}$$

since this expression is constant in t, the variable terms must vanish. (One can make a successive elimination of the terms $(a_{1,-n} - a_{2,-n})t^{-\frac{n}{2}}$, $(a_{1,1-n} - a_{2,1-n})t^{-\frac{n-1}{2}}$, etc., by order of magnitude.) □

A. Appendix.

We here recall, and extend to admissible manifolds, the definition and application of the Calderón projector C^+ for an elliptic differential operator $P\colon C^\infty(X, E_1) \to C^\infty(X, E_2)$ of order d, as introduced by Calderón [C63], Seeley [S66,S69], see also Hörmander [H66], Boutet de Monvel [BM66], Grubb [G77].

The manifold X is taken to be compact or, more generally, admissible as defined in [GK93], [G96], see the introduction to Section 2; P is assumed to be admissible and uniformly elliptic. We can assume that X is smoothly imbedded in an n-dimensional admissible boundaryless manifold $\widetilde{X}$ such that X' is an $(n-1)$-dimensional hypersurface in $\widetilde{X}$ and E_1 and E_2 are restrictions to X of N-dimensional bundles $\widetilde{E}_1$ and $\widetilde{E}_2$ over $\widetilde{X}$; one such choice is to double up the neighborhood U along X', augmenting X by the reflected piece U_-. In $U \cup U_-$ we write $x = (x', x_n)$, where $|x_n| < c(x')$, $c(x') \geq c > 0$. In the compact case one can add another piece to $X \cup U_-$ to get a compact $\widetilde{X}$.

If P extends to a uniformly elliptic operator (also denoted P) from $C^\infty(\widetilde{E}_1)$ to $C^\infty(\widetilde{E}_2)$, we let Q denote an admissible parametrix of P on $\widetilde{X}$; then

$$PQ = I + \mathcal{T}_1, \quad QP = I + \mathcal{T}_2 \quad \text{on } \widetilde{X}, \tag{A.1}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are admissible ψdo's on $\widetilde{X}$ of order $-\infty$. The use of Calderón projectors is simplest if $\widetilde{X}$ and P can be chosen so that P is *invertible* on $\widetilde{X}$; then Q stands for the inverse (necessarily admissible by the spectral invariance proved in [G95]), and $\mathcal{T}_1$ and $\mathcal{T}_2$ are zero.

Let us denote $X^\circ = X_+$, $\widetilde{X} \setminus X = X_-$, $\widetilde{E}_i|_{X_\pm} = E_{i,\pm}$. The mapping $\varrho = \{\gamma_0, \dots, \gamma_{d-1}\}$ ($\gamma_j u = (D^j_{x_n} u)|_{x_n=0}$) can be regarded as a mapping either from functions on $\overline{X}_+$, or from functions on $\overline{X}_-$, or from functions on $\widetilde{X}$, to functions on X'; to distinguish between the three versions, we denote them ϱ^+, ϱ^- resp. $\widetilde{\varrho}$ (so $\varrho = \varrho^+$). When $F = F_0 \oplus \cdots \oplus F_{d-1}$ are vector bundles over X' we denote

$$\begin{aligned} \mathcal{H}^s(F) &= \textstyle\prod_{0\leq j<d} H^{s-j-\frac{1}{2}}(X', F_j)\,, \\ \widetilde{\mathcal{H}}^s(F) &= \textstyle\prod_{0\leq j<d} H^{s+j+\frac{1}{2}}(X', F_j) = (\mathcal{H}^{-s}(F))'\,. \end{aligned} \tag{A.2}$$

Indication of manifolds will often be left out. Writing $\bigoplus_{0\le j<d} E_i' = E_i'^d$, we have that $\varrho^\pm$ and $\widetilde{\varrho}$ map the respective H^s spaces into $\mathcal{H}^s(E_i'^d)$ for $s > d-\frac12$. The mapping $\widetilde{\varrho}\colon H^s(\widetilde{E}_i) \to \mathcal{H}^s(E_i'^d)$ has the adjoint $\widetilde{\rho}^* : \widetilde{\mathcal{H}}^{-s}(E_i'^d) \to H^{-s}(\widetilde{E}_i)$ for $s > d-\frac12$; it ranges in distributions supported in X'. (For further explanation, note that $\widetilde{\varrho}^*$ is the row vector $\{I, D_{x_n}, \dots, D_{x_n}^{d-1}\}\widetilde{\gamma}_0^*$, where $\widetilde{\gamma}_0^*$ in local coordinates where X' is replaced by $\mathbb{R}^{n-1}$ sends a function $\varphi(x')$ on $\mathbb{R}^{n-1}$ into the distribution $\varphi(x')\otimes\delta(x_n)$.) We use the notation $A_\pm$ for the truncation of a ψdo A on $\widetilde{X}$ to $X_\pm$:

$$A_\pm = r^\pm A e^\pm, \text{ when } A \text{ is a } \psi\text{do on } \widetilde{X}; \tag{A.3}$$

here $r^\pm$ means restriction to $X_\pm$ and $e^\pm$ means extension by zero on $X_\mp$.

Define the spaces

$$\begin{aligned} Z_\pm^s &= \{\, z \in H^s(X_\pm, E_{1,\pm}) \mid Pz = 0 \text{ on } X_\pm \}, \quad s\in\mathbb{R},\\ N_\pm^s &= \varrho^\pm Z_\pm^s \subset \mathcal{H}^s(E_1'^d),\\ Z_0 &= \{\, z \in C^\infty(\widetilde{X},\widetilde{E}_1)\cap H^d(\widetilde{X},\widetilde{E}_1) \mid Pz = 0,\ \operatorname{supp} z \subset X \}; \end{aligned} \tag{A.4}$$

here Z_0 is identified with a subspace of the Z_+^s and has finite dimension when X is compact. Although the trace operator ϱ is defined on $H^s(E_{1,\pm})$ for $s > d-\frac12$ only, the definition of the spaces $N_\pm^s$ of Cauchy data for null solutions can be extended to all $s\in\mathbb{R}$, by results in Lions and Magenes [LM68] or by the arguments in [S66,S69]. Seeley showed in [S69], in the case where X is compact, that there exist continuous mappings

$$K^+ \colon \mathcal{H}^s(E_1'^d) \to H^s(E_{1,+}), \quad C^+ = \varrho^+K^+ \colon \mathcal{H}^s(E_1'^d) \to \mathcal{H}^s(E_1'^d) \tag{A.5}$$

(defined consistently for all $s\in\mathbb{R}$) with the properties:

(A.i) For each $s\in\mathbb{R}$, K^+ maps $\mathcal{H}^s(E_1'^d)$ into Z_+^s, such that

$$Z_+^s = K^+(\mathcal{H}^s)\dot{+}Z_0, \quad \varrho^+K^+\varphi = \varphi \text{ for } \varphi\in N_+^s, \quad K^+\varrho^+ z = z \text{ for } z \in K^+(\mathcal{H}^s). \tag{A.6}$$

(A.ii) $C^+ = \varrho^+K^+$ is a projection in $\mathcal{H}^s(E_1'^d)$ with range N_+^s.

(A.iii) The operators satisfy:

$$K^+ = -r^+Q\widetilde{\varrho}^*\mathcal{A} + \mathcal{T}_3, \quad C^+ = -\varrho^+Q\widetilde{\varrho}^*\mathcal{A} + \varrho^+\mathcal{T}_3, \tag{A.7}$$

where $\mathcal{T}_3$ and $\varrho^+\mathcal{T}_3$ are integral operators from X' to X resp. X' with C^∞ kernels; here $\mathcal{T}_3 = 0$ when Q is the inverse of P on $\widetilde{X}$.

C^+ is a matrix of classical ψdo's, $C^+ = (C_{jk}^+)_{j,k=0,\dots,d-1}$ with C_{jk}^+ of order $j-k$; it is called the *Calderón projector* for P. We also define the complementing Calderón projector

$$C^- = I - C^+. \tag{A.8}$$

In the terminology of [BM71], K^+ is a Poisson operator. Because of the presence of the mapping $\widetilde{\varrho}^*$, the full symbols of K^+ and C^+ are determined from the symbol of P and its derivatives at X' (modulo symbols of order $-\infty$).

Although the result is independent of the existence of convenient extensions of X and P, the deduction of it is easiest to explain when P has an invertible extension to $\widetilde{X}$. Then it also has a nice generalization to non-compact cases:

Theorem A.1. *In the case of admissible manifolds, bundles and operators, assume that P has the inverse Q on $\widetilde{X}$. Then the spaces $N^s_\pm$ are complementing subspaces of $\mathcal{H}^s(E_1'^d)$:*

$$\mathcal{H}^s(E_1'^d) = N^s_+ \dot{+} N^s_-. \tag{A.9}$$

When we define

$$K^\pm = \mp r^\pm Q \widetilde{\varrho}^* \mathcal{A}, \quad C^\pm = \varrho^\pm K^\pm = \mp \varrho^\pm r^\pm Q \widetilde{\varrho}^* \mathcal{A}, \tag{A.10}$$

the Poisson operators $K^\pm \colon \mathcal{H}^s(E_1'^d) \to H^s(E_{1,\pm})$ have range equal to $Z^s_\pm$ and provide right inverses of $\varrho^\pm$ on $Z^s_\pm$, respectively; and the ψdo's $C^\pm$ (the Calderón projectors for P) are the projections of $\mathcal{H}^s(E_1'^d)$ onto $N^s_\pm$ along $N^s_\mp$, respectively.

Proof. The proof is a generalization of the deduction in [S66], [S69] for the invertible case with $\widetilde{X}$ compact. In fact, the proof given in [G96, Ex. 1.3.5] carries over *verbatim* to the present admissible manifolds, when the operators are admissible (have uniformly x-estimated symbols; the calculus for such operators is worked out in [G96, Ch. 2–3]), and one allows the range bundle for P to be different from the initial bundle E. To save space, we refrain from repeating the details here. □

When P merely satisfies (A.1), one can still define operators K^+ and C^+ by formulas similar to (A.7); then they have the desired mappping properties only modulo smoothing operators. The properties (A.i)–(A.ii) achieved in [S69] for the compact case require more precision. A construction taking account of smoothing operators is worked out in [G77] for general multi-order operators P on compact manifolds, with applications. The book of Booss-Bavnbek and Wojciechowski [BW93] goes through the proof of Theorem A.1 for first-order operators in the product case, cf. Definition 3.1.

The principal symbols are determined by the analogous (exact) construction for the model operator $p^0(x', 0, \xi', D_{x_n})$ in $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ for $|\xi'| = 1$; here $\mathcal{S}(\overline{\mathbb{R}}_\pm) = r^\pm \mathcal{S}(\mathbb{R})$. The nullspaces

$$Z_\pm(x', \xi') = \{ z(x_n) \in \mathcal{S}(\overline{\mathbb{R}}_\pm)^N \mid p^0(x', 0, \xi', D_{x_n}) z = 0 \text{ on } \mathbb{R}_\pm \}, \tag{A.11}$$

are finite dimensional subspaces of $\mathcal{S}(\overline{\mathbb{R}}_\pm)^N$ consisting of exponential polynomials decreasing for $x_n \to \pm\infty$, respectively, and the corresponding Cauchy data spaces $N_\pm(x', \xi') = \varrho^\pm Z_\pm(x', \xi')$ are complementing subspaces of $\prod_{0 \le j < d} \mathbb{C}^N = \mathbb{C}^{Nd}$. The dimension of $N_\pm(x', \xi')$ equals the sum of the multiplicities of the roots in $\det p^0(x', 0, \xi', \tau)$ (considered as a polynomial in τ) with imaginary part $\gtrless 0$, respectively.

Example A.2. When $d = 1$ and $P = D$ is written as in (3.3), the model operator is $d^0(x', 0, \xi', D_{x_n}) = \sigma(x')(\frac{d}{dx_n} + a_1^0(x', \xi'))$. It is seen e.g. by changing $a_1^0(x', \xi')$ to Jordan normal form that the spaces $N_\pm(x', \xi') \subset \mathbb{C}^N$ are the generalized eigenspaces for $a_1^0(x', \xi')$ associated with the eigenvalues having real part $\gtrless 0$, respectively (i.e., the roots of the polynomial $\det(i\tau I + a_1^0(x', \xi'))$ in τ having imaginary part $\gtrless 0$, respectively). The corresponding Calderón projectors $c^\pm(x', \xi')$, projecting onto $N_\pm(x', \xi')$ along $N_\mp(x', \xi')$, respectively, can be found from the formulas:

$$c^\pm(x', \xi') = \tfrac{1}{2\pi} \int_{\mathcal{L}_\pm} (i\tau I + a_1^0(x', \xi'))^{-1} \, d\tau; \tag{A.12}$$

here the integration curve $\mathcal{L}_\pm$ lies in $\mathbb{C}_\pm = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau \gtrless 0\}$ and encircles the τ-roots of $\det(i\tau I + a_1^0(x',\xi'))$ (the poles of $(d^0)^{-1}$) there, respectively. $c^\pm$ is the principal symbol of $C^\pm$. The associated Poisson operator $k^\pm$ from $N_\pm(x',\xi')$ to $Z_\pm(x',\xi')$ is the multiplication by $k^\pm(x',\xi',x_n) = \pm r^\pm \mathcal{F}^{-1}_{\xi_n\to x_n}(i\xi_n I + a_1^0(x',\xi'))^{-1}$, where $\mathcal{F}$ is the Fourier transform.

When $a_1^0(x',\xi')$ is symmetric, equal to $a^0(x',\xi')$ as in (3.5)ff., $N_\pm(x',\xi')$ is the positive resp. negative eigenspace of $a^0(x',\xi')$ (here the roots of $\det(i\tau I + a_1^0(x',\xi'))$ lie on the imaginary axis, in $\mathbb{C}_+$ resp. $\mathbb{C}_-$), and the $c^\pm(x',\xi')$ are *orthogonal projections.*

Let us now explain the use of the Calderón projectors in the study of boundary value problems:

$$Pu = f \text{ on } X, \quad S\varrho u = \varphi \text{ on } X', \tag{A.13}$$

where S is a system of ψdo's S_{jk} of order $j-k$ ($j,k = 0,\dots,d-1$) going from E_1' to bundles F_j of dimension ≥ 0 over X'; $M = \sum_{0\leq j<d} \dim F_j$. (Say, $f \in H^{s-d}(E_2)$ and $\varphi \in \mathcal{H}^s(F)$ are given, and u is sought in $H^s(E_1)$, for some $s > d - \frac{1}{2}$.) Assume for simplicity in this explanation that Q is the inverse of P on $\widetilde{X}$. We can replace u by $z = u - Q_+f$ (cf. (A.3)) and φ by $\psi = \varphi - S\varrho Q_+ f$; this reduces (A.13) to the problem

$$Pz = 0 \text{ on } X, \quad S\varrho z = \psi \text{ on } X'. \tag{A.14}$$

Here $\psi \in \mathcal{H}^s(F)$ and z is sought in Z_+^s. If we set $\eta = \varrho z$, i.e., $z = K^+\eta$ (cf. Theorem A.1), the problem (A.14) is *equivalent with* the problem of finding $\eta \in \mathcal{H}^s(E_1'^d)$ such that

$$S\eta = \psi, \quad \eta \in N_+^s. \tag{A.15}$$

Since N_+^s is the nullspace for C^- as well as the range space for C^+ in $\mathcal{H}^s(E_1'^d)$, we now have the following two equivalent strategies to solve problem (A.15):

$$\begin{aligned} &\text{(a) Find } \eta \text{ such that } \begin{pmatrix} S \\ C^- \end{pmatrix}\eta = \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \\ &\text{(b) Find } \chi \text{ such that } SC^+\chi = \psi, \text{ then set } \eta = C^+\chi. \end{aligned} \tag{A.16}$$

It follows that the problem has *uniqueness of solution* if and only if $\begin{pmatrix} S \\ C^- \end{pmatrix}$ is injective; and the problem has *existence of solution* if and only if SC^+ is surjective. This discussion is followed up in Theorem A.4 below, after we have recalled the definitions of the appropriate ellipticity concepts.

The problem (A.13) is called *injectively resp. surjectively elliptic* when the model problem

$$\begin{aligned} p^0(x',0,\xi',D_{x_n})u &= 0 \text{ on } \mathbb{R}_+, \\ s^0(x',\xi')\varrho u &= v \text{ at } x_n = 0, \end{aligned} \tag{A.17}$$

for all x', all $|\xi'| = 1$ has *uniqueness, resp. existence* of solution $u \in \mathcal{S}(\overline{\mathbb{R}}_+)^N$ for all $v \in \mathbb{C}^M$. This is equivalent with injectiveness resp. surjectiveness of the operator $\begin{pmatrix} p^0 \\ s^0\varrho \end{pmatrix}$

from $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ to $\mathcal{S}(\overline{\mathbb{R}}_+)^N \times \mathbb{C}^M$. By the Calderón projector construction on the principal symbol level, the solutions in $\mathcal{S}(\overline{\mathbb{R}}_+)^N$ of the first line of (A.17) are mapped bijectively onto $N_+(x',\xi')$ by ϱ. Hence injective resp. surjective ellipticity is equivalent with *injectiveness resp. surjectiveness of the mapping* $s^0(x',\xi')$ *from* $N_+(x',\xi')$ *to* $\mathbb{C}^M$. Observe that injective ellipticity holds if and only if

$$v \in \mathbb{C}^{Nd},\ s^0(x',\xi')v = 0,\ c^-(x',\xi')v = 0 \implies v = 0; \tag{A.18}$$

i.e., the nullspaces of s^0 and c^- are *linearly independent*; this can also be stated as the property that $\begin{pmatrix} s^0(x',\xi') \\ c^-(x',\xi') \end{pmatrix}$ is injective for all x', all $|\xi'| = 1$. Surjective ellipticity of the boundary value problem holds if and only if $s^0(x',\xi')c^+(x',\xi')$ is surjective for all x', all $|\xi| = 1$. Thus, in other words:

$$\begin{aligned} &\begin{pmatrix} P \\ S\varrho \end{pmatrix} \text{ is injectively elliptic} \iff \begin{pmatrix} S \\ C^- \end{pmatrix} \text{ is injectively elliptic;} \\ &\begin{pmatrix} P \\ S\varrho \end{pmatrix} \text{ is surjectively elliptic} \iff SC^+ \text{ is surjectively elliptic.} \end{aligned} \tag{A.19}$$

Note in particular that injective resp. surjective ellipticity implies that $M \geq \dim N_+(x',\xi')$, resp. $M \leq \dim N_+(x',\xi')$.

Problems that are both injectively and surjectively elliptic are simply called *elliptic*; then $M = \dim N_+(x',\xi')$. When $M = \dim N_+(x',\xi')$, ellipticity is equivalent with injective ellipticity and with surjective ellipticity, for dimensional reasons. The property is a generalization of the Shapiro-Lopatinskiĭ condition.

For noncompact manifolds we need a spatial uniformity in the ellipticity hypotheses. Here P and S are assumed to be admissible, and when P is uniformly elliptic, the problem is called uniformly injectively resp. surjectively elliptic when there is a left resp. right inverse of the model problem at the boundary that is uniformly bounded in x'; this is equivalent with uniform injective resp. surjective ellipticity of $\begin{pmatrix} S \\ C^- \end{pmatrix}$ resp. SC^+.

Since p^0 satisfies $p^0(x,-\xi) = (-1)^d p^0(x,\xi)$, the polynomial $\det p^0(x',0,\xi',\tau)$ in τ has equally many roots in $\mathbb{C}_+$ and $\mathbb{C}_-$ when $n \geq 3$ (then ξ' can be connected to $-\xi'$ by a curve in $\{\eta' \in \mathbb{R}^{n-1} \mid |\eta'| = 1\}$), so Nd must be even and

$$\dim N_+(x',\xi') = \dim N_-(x',\xi') = Nd/2 \tag{A.20}$$

then (the so-called properly elliptic case). Here ellipticity of (A.13) requires $M = Nd/2$.

As shown in [G77, Th. 3.1, 3.2] for very general systems on compact manifolds, one can give explicit formulas for a left/right parametrix of the system $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ when injective/surjective ellipticity holds. We shall extend this to admissible manifolds where Theorem A.1 applies, and at the same time keep track of how much is needed to get exact formulas when Seeley's projector (A.i)–(A.iii) is used in the compact case. ([G77] treats systems P of mixed order; for such systems the formulas contain an extra block matrix $\mathcal{B}$. When P is of a single order, $\mathcal{B}$ is void — zero-dimensional — and the results hold with $\mathcal{B}$ and its effects omitted.)

First we show a preparatory lemma. All calculations in the following are justified within the extension of the calculus of Boutet de Monvel given in [G96]. Recall that operators are said to be "of class 0" when they are well-defined on $L_2(X)$ (do not involve γ_0).

Lemma A.3. *Let X be compact or admissible and let P be a uniformly elliptic differential operator of order d. In the compact case, define the Calderón projectors $C^\pm$ as in* (A.i)—(A.iii), (A.8)*; in the admissible case assume that P has an inverse Q on $\widetilde{X}$ and define $C^\pm$ as in Theorem* A.1*. The following formulas are valid on, respectively, $H^s(E_2)$ with $s > -\frac{1}{2}$, $H^s(E_1)$ with $s > \frac{1}{2}$, or $\mathcal{H}^s(E_1'^d)$ with $s \in \mathbb{R}$:*

$$\text{(A.21)} \qquad \begin{aligned} &\text{(i)} && PQ_+ = I + \mathcal{T}_{1,+},\\ &\text{(ii)} && Q_+P = I - K^+\varrho + \mathcal{T}_4, \ \textit{with } \mathcal{T}_4 = \mathcal{T}_{2,+} + \mathcal{T}_3\varrho,\\ &\text{(iii)} && K^+C^- = \mathcal{T}_5, \ \textit{with } \mathcal{T}_5 = \mathcal{T}_4K^+ = \mathcal{T}_{2,+}K^+ + \mathcal{T}_3C^+. \end{aligned}$$

Here the $\mathcal{T}_j$ come from (A.1), (A.7)*; they vanish when $Q = P^{-1}$.*

Proof. Formula (i) follows from the first formula in (A.1) by truncation to X (application of (A.3)), since $(PQ)_+ = PQ_+$. Next, we note that Green's formula (2.1) can be written in distributional form:

$$\text{(A.22)} \qquad e^+r^+P\tilde{u} = Pe^+r^+\tilde{u} + \widetilde{\varrho}^*(\mathcal{A}\varrho u) \text{ for } \tilde{u} \in H^d(\widetilde{E}_1), \ u = r^+\tilde{u}.$$

Formula (ii) follows from this by composition with r^+Q and use of (A.1) and (A.7). For (iii), we use (ii) and the facts that $\varrho K^+ = C^+$, $PK^+ = 0$, in the calculation:

$$\begin{aligned} K^+C^- &= K^+ - K^+C^+ = K^+ - K^+\varrho K^+\\ &= K^+ - (I - Q_+P - \mathcal{T}_{2,+} - \mathcal{T}_3\gamma_0)K^+ = \mathcal{T}_{2,+}K^+ + \mathcal{T}_3C^+. \quad \square \end{aligned}$$

Theorem A.4. *Assumptions as in Lemma* A.3*. Let $S = (S_{jk})_{j,k=0,\dots,d-1}$ be a system of admissible classical ψdo's S_{jk} of orders $j-k$ from E_1' to F_j.*

1° Assume that $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ (equivalently, SC^+) is uniformly surjectively elliptic.

When S_1 is a given right parametrix of SC^+, then

$$\text{(A.23)} \qquad (\, R_S \quad K_S \,) = (\, Q_+ - K^+S_1S\varrho Q_+ \quad K^+S_1 \,)$$

is a right parametrix of $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$, in the sense that

$$\text{(A.24)} \qquad \begin{pmatrix} P \\ S\varrho \end{pmatrix} (\, R_S \quad K_S \,) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \mathcal{T},$$

where $\mathcal{T}$ is of order $-\infty$ and class 0*. If, moreover, $PQ_+ = I$ and S_1 is a right* **inverse** *of SC^+, then $(\, R_S \quad K_S \,)$ is a right* **inverse** *of $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$.*

Conversely, when $(\, R_S \quad K_S \,)$ is a given right parametrix or inverse of $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$, then

$$\text{(A.25)} \qquad S_1 = \varrho K_S$$

is a right parametrix resp. inverse of SC^+.

2° Assume instead that $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ (equivalently $\begin{pmatrix} S \\ C^- \end{pmatrix}$) is uniformly injectively elliptic.

When $(\,S_1 \quad S_2\,)$ *is a given left parametrix of* $\begin{pmatrix} S \\ C^- \end{pmatrix}$, *then the operator defined in* (A.23) *is a left parametrix of* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$, *in the sense that*

$$(\,R_S \quad K_S\,)\begin{pmatrix} P \\ S\varrho \end{pmatrix} = I + \mathcal{T}', \tag{A.26}$$

where $\mathcal{T}' = \mathcal{T}'' + \mathcal{T}'''\varrho$ *with* $\mathcal{T}''$ *and* $\mathcal{T}'''$ *of order* $-\infty$, $\mathcal{T}''$ *of class* 0. *If, moreover,* $Q = P^{-1}$ *and* $(\,S_1 \quad S_2\,)$ *is a left* **inverse** *of* $\begin{pmatrix} S \\ C^- \end{pmatrix}$, *then* $(\,R_S \quad K_S\,)$ *is a left* **inverse** *of* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$.

Conversely, when $(\,R_S \quad K_S\,)$ *is a given left parametrix or inverse of* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$, *then*

$$(\,S_1 \quad S_2\,) = (\,\varrho K_S \quad I - \varrho K_S S\,) \tag{A.27}$$

is a left parametrix resp. inverse of $\begin{pmatrix} S \\ C^- \end{pmatrix}$.

3° *In the case where* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$ *is two-sided elliptic, each of the constructions in* 1° *or* 2°, *departing from a right parametrix of* SC^+ *resp. a left parametrix of* $\begin{pmatrix} S \\ C^- \end{pmatrix}$, *gives a two-sided parametrix of* $\begin{pmatrix} P \\ S\varrho \end{pmatrix}$.

Proof. For the first assertion in 1°, write $SC^+S_1 = I + \mathcal{R}_1$ where $\mathcal{R}_1$ is a ψdo on X' of order $-\infty$. Then by (A.21i) and the facts that $PK^+ = 0$ and $\varrho K^+ = C^+$,

$$\begin{aligned} P(Q_+ - K^+S_1S\varrho Q_+)u &= u + \mathcal{T}_{1,+}u, \\ S\varrho(Q_+ - K^+S_1S\varrho Q_+)u &= S\varrho Q_+u - SC^+S_1S\varrho Q_+u = -\mathcal{R}_1S\varrho Q_+u, \\ PK^+S_1\varphi &= 0, \\ S\varrho K^+S_1\varphi &= SC^+S_1\varphi = \varphi + \mathcal{R}_1\varphi; \end{aligned} \tag{A.28}$$

This shows (A.24). Since ϱQ_+ is well-defined on $L_2(X, E_2)$, it is a trace operator *of class* 0 (cf. [BM71] or e.g. [G96, pp. 27ff. and 279]); hence the composed trace operator $\mathcal{R}_1S\varrho Q_+$, which is of order $-\infty$, is of class 0.

Now if, furthermore, $\mathcal{T}_{1,+} = 0$ and $\mathcal{R}_1 = 0$, the smoothing terms in (A.28) are zero, so $(\,R_S \quad K_S\,)$ is a right *inverse*.

In the converse direction, when (A.24) holds, then

$$PK_S = \mathcal{T}_{12}, \quad S\varrho K_S = I + \mathcal{T}_{22},$$

with operators $\mathcal{T}_{12}$ and $\mathcal{T}_{22}$ of order $-\infty$. If $\mathcal{T}_{12}$ and $\mathcal{T}_{22}$ are 0, K_S maps into Z_+^s so that $C^-\varrho K_S = 0$ and consequently

$$SC^+\varrho K_S = S\varrho K_S - SC^-\varrho K_S = I;$$

so ϱK_S is a right *inverse* of SC^+. More generally, by (A.21ii),

$$\begin{aligned} SC^+\varrho K_S &= S\varrho K^+\varrho K_S = B\varrho(I - Q_+P - \mathcal{T}_4)K_S \\ &= I + \mathcal{T}_{22} - S\varrho Q_+\mathcal{T}_{12} - S\varrho\mathcal{T}_4K_S = I + \mathcal{R}_2, \end{aligned}$$

with $\mathcal{R}_2$ a ψdo on X' of order $-\infty$; so ϱK_S is a right parametrix of SC^+. This proves 1°.

For the first assertion in 2°, write $S_1 S + S_2 C^- = I + \mathcal{R}_3$ with $\mathcal{R}_3$ of order $-\infty$. Now let us check the composition (A.26). Using (A.21ii–iii) and the fact that $C^- C^+ = 0$, we find:

$$\begin{aligned}
(\, Q_+ - K^+ S_1 S \varrho Q_+ \quad K^+ S_1 \,) \begin{pmatrix} P \\ S\varrho \end{pmatrix} &= (I - K^+ S_1 S \varrho) Q_+ P + K^+ S_1 S \varrho \\
&= (I - K^+ S_1 S \varrho)(I - K^+ \varrho + \mathcal{T}_4) + K^+ S_1 S \varrho \\
&= I - K^+ (I - S_1 S C^+) \varrho + (I - K^+ S_1 S \varrho) \mathcal{T}_4 \\
&= I - K^+ (I - (I - S_2 C^- + \mathcal{R}_3) C^+) \varrho + (I - K^+ S_1 S \varrho) \mathcal{T}_4 \\
&= I - K^+ C^- \varrho - K^+ \mathcal{R}_3 C^+ \varrho + (I - K^+ S_1 S \varrho) \mathcal{T}_4 \\
&= I - \mathcal{T}_5 \varrho - K^+ \mathcal{R}_3 C^+ \varrho + (I - K^+ S_1 S \varrho) \mathcal{T}_4 ,
\end{aligned}$$

which is of the asserted form. Here if moreover Q is the inverse of P and $\mathcal{R}_3 = 0$, all smoothing terms vanish, so $(\, R_S \quad K_S \,)$ is a left *inverse*.

For the converse statement, define $(\, S_1 \quad S_2 \,)$ by (A.27) and check its left composition with $\begin{pmatrix} S \\ C^- \end{pmatrix}$:

$$\text{(A.29)} \quad (\, \varrho K_S \quad I - \varrho K_S S \,) \begin{pmatrix} S \\ C^- \end{pmatrix} = \varrho K_S S + C^- - \varrho K_S S C^- = \varrho K_S S C^+ + I - C^+ .$$

When $w = K^+ C^+ \varphi$ for some $\varphi \in C^\infty(E_1'^d)$, then $Pw = 0$, $\varrho w = C^+ C^+ \varphi = C^+ \varphi$ and $S \varrho w = S C^+ \varphi$, so in view of (A.26),

$$w + \mathcal{T}' w = K_S S \varrho w = K_S S C^+ \varphi .$$

It follows that

$$\varrho K_S S C^+ \varphi = \varrho w + \varrho \mathcal{T}' w = C^+ \varphi + \varrho \mathcal{T}' K^+ C^+ \varphi ,$$

for $\varphi \in C^\infty(E_1'^d)$. Then the expression in (A.29) equals

$$\varrho K_S S C^+ + I - C^+ = I + \varrho \mathcal{T}' K^+ C^+ = I + \mathcal{R}_4 ,$$

where $\mathcal{R}_4$ is a ψdo on X' of order $-\infty$. So $(\, S_1 \quad S_2 \,)$ is a left parametrix. It is a left *inverse* if $\mathcal{T}' = 0$. This ends the proof of 2°.

The statement in 3° is a standard consequence. □

When there is a left inverse, there is uniqueness of a solution $u \in H^s(E_1)$ for the boundary value problem (A.13) with data $f \in H^{s-d}(E_2)$, $\varphi \in \mathcal{H}^s(F)$, $s > d - \frac{1}{2}$. When there is a left parametrix, there is "best regularity of solutions," in the sense that if $u \in H^t(E_1)$ for some t, then $Pu \in H^{s-d}(E_2)$ and $S\varrho u \in \mathcal{H}^s(F)$ imply $u \in H^s(E_1)$ (since $u = R_S P u + K_S S \varrho u + \mathcal{T}' u$); $s, t > d - \frac{1}{2}$. Moreover, if X is compact, there is uniqueness modulo a finite dimensional smooth subspace.

When there is a right inverse, there is existence of solution for the boundary value problem (A.13); when there is a right parametrix and X is compact, there is existence of solution for data in the complement of a finite dimensional smooth space.

In the admissible case, when K^+ and C^+ are merely defined modulo smoothing operators, there is a version of Theorem A.4 with parametrices everywhere.

Example A.5. The systems $\begin{pmatrix} P \\ \varrho \end{pmatrix}$ and $\begin{pmatrix} P \\ C^+\varrho \end{pmatrix}$ are injectively elliptic; they both have the left parametrix $(Q_+ \quad K^+)$ (inverse when $Q = P^{-1}$). In fact, by (A.21ii–iii),

$$Q_+P + K^+\varrho = I + \mathcal{T}_4; \quad Q_+P + K^+C^+\varrho = I + \mathcal{T}_4 - \mathcal{T}_5\varrho.$$

This left parametrix/inverse is also found from (A.23), when we use that $\begin{pmatrix} I \\ C^- \end{pmatrix}$ and $\begin{pmatrix} C^+ \\ C^- \end{pmatrix}$ both have the left inverse $(C^+ \quad C^-)$. The case $S = C^+$ is studied in Section 3 in the case $d = 1$.

Formula (A.21i) shows that Q_+ is a right parametrix of P *without boundary conditions*; i.e., in the case $F = 0$. This is also confirmed by the formulas in the theorem.

When $S = C^+$, we have according to a result of Seeley [S69] (recalled in (3.9) for the case $d = 1$) that the adjoint of the realization P_S is the realization of P^* determined by the boundary condition $(I - C^{+*})\mathcal{A}^*\varrho u = 0$. For completeness, we now show that this boundary condition is the Calderón projector condition for P^* (up to a smoothing term, unless $Q = P^{-1}$).

Theorem A.6. *For P^* (provided with the parametrix Q^* on $\widetilde{X}$), denote by C'^+ the associated Calderón projector according to* (A.i)–(A.iii) *or Theorem* A.1. *Then*

$$C'^+ = (\mathcal{A}^*)^{-1}(I - C^{+*})\mathcal{A}^* + \mathcal{T}_6, \tag{A.30}$$

where $\mathcal{T}_6$ is a ψdo of order $-\infty$ that vanishes when $Q = P^{-1}$.

In particular,

$$(I - C^{+*})\mathcal{A}^*\varrho u = 0 \iff (C'^+ - \mathcal{T}_6)\varrho u = 0. \tag{A.31}$$

Proof. Since P^* has a Green's formula similar to (2.1) but with $\mathcal{A}$ replaced by $-\mathcal{A}^*$, the Calderón projector and associated Poisson operator for P^* satisfy formulas

$$K'^+ = r^+Q^*\widetilde{\varrho}^*\mathcal{A}^* + \mathcal{T}_3', \quad C'^+ = \varrho^+Q^*\widetilde{\varrho}^*\mathcal{A}^* + \varrho\mathcal{T}_3', \tag{A.32}$$

where $\mathcal{T}_3'$ is a ψdo of order $-\infty$, vanishing when $Q = P^{-1}$.

There is a Poisson operator K_ϱ lifting sections $\varphi \in \mathcal{H}^d(E_1'^d)$ to sections $u \in H^d(E_1)$ such that $\varrho K_\varrho u = \varphi$, cf. e.g. [G96, Lemma 1.6.4] or the text before Lemma 2.3 above. We have from (A.21ii), by application of ϱ:

$$\begin{aligned} K^+\varrho u &= u - Q_+Pu + \mathcal{T}_{2,+}u + \mathcal{T}_3\varrho u, \\ C^+\varphi &= \varphi - \varrho Q_+Pu + \varrho\mathcal{T}_{2,+}u + \varrho\mathcal{T}_3\varphi \\ &= \varphi - \varrho Q_+PK_\varrho\varphi + \varrho\mathcal{T}_{2,+}K_\varrho\varphi + \varrho\mathcal{T}_3\varphi. \end{aligned} \tag{A.33}$$

For the term ϱQ_+Pu we note that when $\psi \in \widetilde{\mathcal{H}}^0(E_1'^d)$ (cf. (A.2)):

$$\begin{aligned} (\varrho Q_+Pu, \psi)_{X'} &= (\widetilde{\varrho}Qe^+Pu, \psi)_{X'} = (e^+Pu, Q^*\widetilde{\varrho}^*\psi)_{\widetilde{X}} \\ &= (Pu, r^+Q^*\widetilde{\varrho}^*\psi)_X = (Pu, [K'^+ - \mathcal{T}_3'](\mathcal{A}^*)^{-1}\psi)_X \\ &= (Pu, K'^+(\mathcal{A}^*)^{-1}\psi)_X - (u, P^*K'^+(\mathcal{A}^*)^{-1}\psi)_X - (Pu, \mathcal{T}_3'(\mathcal{A}^*)^{-1}\psi)_X \\ &= (\varphi, \mathcal{A}^*C'^+(\mathcal{A}^*)^{-1}\psi)_{X'} - (\varphi, (PK_\varrho)^*\mathcal{T}_3'(\mathcal{A}^*)^{-1}\psi)_{X'}. \end{aligned}$$

It is used here that $Qe^+Pu \in H^d(\widetilde X)$ so that $\widetilde\varrho$ and ϱr^+ give the same result, that $P^*K'^+ = 0$, and that the Poisson operator PK_ϱ has as its adjoint a trace operator $(PK_\varrho)^*$ of class 0. Taking this together with (A.33), we find:

$$(C^+\varphi,\psi)_{X'} = (\varphi,\psi) - (\varphi, \mathcal{A}^*C'^+(\mathcal{A}^*)^{-1}\psi)_{X'} \\ + (\varphi,(PK_\varrho)^*\mathcal{T}_3'(\mathcal{A}^*)^{-1}\psi)_{X'} + (\varphi,(\varrho\mathcal{T}_{2,+}K_\varrho)^*\psi)_{X'} + (\varphi,(\varrho\mathcal{T}_3)^*\psi)_{X'},$$

which shows that

$$\text{(A.34)}\qquad \begin{aligned} C^{+*} &= I - \mathcal{A}^*C'^+(\mathcal{A}^*)^{-1} + \mathcal{T}_6', \text{ with} \\ \mathcal{T}_6' &= (PK_\varrho)^*\mathcal{T}_3'(\mathcal{A}^*)^{-1} + (\varrho\mathcal{T}_{2,+}K_\varrho)^* + (\varrho\mathcal{T}_3)^*. \end{aligned}$$

Then (A.30) holds with $\mathcal{T}_6 = (\mathcal{A}^*)^{-1}\mathcal{T}_6'\mathcal{A}^*$; $\mathcal{T}_6'$ and $\mathcal{T}_6$ are ψdo's on X' of order $-\infty$ by the rules of calculus. □

We end with some remarks for the case $d = 1$. Recall from the analysis of the boundary value problem, in particular (A.15), that it is really *the space* N_+^s that matters in the discussion of solvability, rather than a certain *projection* onto it. When $d = 1$, $\mathcal{H}^{\frac12}(E_1') = L_2(E_1')$. Here it may be convenient to replace C^+ by a projection in $\mathcal{H}^s(E_1') = H^{s-\frac12}(E_1')$ that has the same range N_+^s and is *orthogonal* for $s - \frac12 = 0$ (on $L_2(E_1')$), in particular if the L_2-structure has an important meaning in the context. This can indeed be obtained, by use of the following lemma shown for compact manifolds in [BW93, Lemma 12.8]:

Lemma A.7. *When R is a projection in a Hilbert space H, then $RR^* + (I - R^*)(I - R)$ is invertible and*

$$\text{(A.35)}\qquad R_{\text{ort}} = RR^*[RR^* + (I - R^*)(I - R)]^{-1}$$

is an orthogonal projection in H satisfying

$$\text{(A.36)}\qquad R(H) = R_{\text{ort}}(H).$$

Here if $H = L_2(F)$, where F is an admissible vector bundle over a manifold X', and R is an admissible classical ψdo of order 0 in F, then the same holds for R_{ort}, and the principal symbol is determined by a formula similar to (A.35) *on the principal symbol level.*

Proof. The formulas (A.35) and (A.36) are verified in detail in [BW93]. For the last statement, the invertibility of [ ] implies, by the spectral invariance shown in [G95], that it is uniformly elliptic and its inverse is likewise admissible, classical and uniformly elliptic of order 0. Then since the principal symbol of R is a projection, the formulas likewise hold on the principal symbol level. □

Remark A.8. Since the range of R in $H^s(F)$ equals the nullspace of $I - R$ there, it follows from the fact that $I - R$ and $I - R_{\text{ort}}$ have the same nullspace in $L_2(F)$ that they also have the same nullspace in $H^s(F)$, $s \ge 0$. Hence

$$\text{(A.37)}\qquad R(H^s(F)) = R_{\text{ort}}(H^s(F)),$$

for $s \geq 0$. This property extends to negative s by consideration of the adjoint R^*, which is likewise a projection and a classical ψdo of order 0. Indeed, the nullspace of $I - R$ in $H^{-s}(F)$ $(s \geq 0)$ is the annihilator of the range of $R' = I - R^*$ in $H^s(F)$. Here one finds from (A.35) that

$$R'_{\text{ort}} = I - R_{\text{ort}}. \tag{A.38}$$

Since $R'(H^s(F)) = R'_{\text{ort}}(H^s(F))$ for $s \geq 0$ as already shown, the annihilators, equal to the nullspaces of $I - R$ and $I - R_{\text{ort}}$ in $H^{-s}(F)$, are the same.

Let us apply the lemma and remark to C^+ in the case $d = 1$. This gives a pseudodifferential projection C^+_{ort} of $H^{s-\frac{1}{2}}(E'_1)$ onto N^s_+ (all $s \in \mathbb{R}$) that is an orthogonal projection of $L_2(E'_1)$ onto $N^{\frac{1}{2}}_+$. The complementing projection is $C^{+-}_{\text{ort}} = I - C^+_{\text{ort}}$; its range is a closed subspace of $H^{s-\frac{1}{2}}(E'_1)$ that equals $L_2(E'_1) \ominus N^{\frac{1}{2}}_+$ when $s = \frac{1}{2}$. It will be different from N^s_- whenever C^+ is not selfadjoint, which is the most usual case. (See Remark 3.8 for an example where C^+ and C^+_{ort} are even principally different.)

Together with C^+_{ort} we can consider the Poisson operator

$$K^+_{\text{ort}} = K^+ C^+_{\text{ort}}, \tag{A.39}$$

it clearly maps N^s_+ into Z^s_+ with the same range as K^+, hence with range complement Z_0; and

$$\gamma^+_0 K^+_{\text{ort}} = C^+ C^+_{\text{ort}} = C^+_{\text{ort}}. \tag{A.40}$$

Note that C^+_{ort} is *uniquely* determined from $N^{\frac{1}{2}}_+$. Still, in the compact case where invertibility of P is not assumed, Z_0 can be $\neq 0$ and then there are other Poisson operators $\widetilde{K}^+$ than K^+_{ort} that map $H^{s-\frac{1}{2}}(E'_1)$ into Z^s_+ and satisfy $\varrho^+ \widetilde{K}^+ = C^+_{\text{ort}}$.

In much of the preceding analysis, C^+, C^- and K^+ can be replaced by C^+_{ort}, $I - C^+_{\text{ort}} = C^{+-}_{\text{ort}}$ and K^+_{ort}. For example, departing from (A.15), we can replace C^+ and C^- by C^+_{ort} and C^{+-}_{ort} in (A.16) and the subsequent discussion. However, the formulas generalizing those in Theorem A.4 will be somewhat more complicated.

We shall call C^+_{ort} the *orthogonal Calderón projector* (recall that $d = 1$). One can argue that it is more natural to consider C^+_{ort} than C^+ — at least when the norm in $L_2(E'_1)$ is in some sense canonically given — on the other hand, C^+ contains more information from P; it is *not* determined from $N^{\frac{1}{2}}_+$ alone but from this together with the complement $N^{\frac{1}{2}}_-$ representing essentially the Cauchy data of exterior null-solutions. (We underline that the *complete symbol* of C^+ is determined from the symbol of P and its derivatives at X', independently of a choice of extension outside X.) At any rate, C^+ is defined regardless of a choice of norm in $L_2(E'_1)$ and gives fairly simple formulas in the application to boundary value problems.

As noted in [S66], the construction of the Calderón projectors $C^{\pm}$ generalizes the construction of projection operators onto the boundary value spaces for holomorphic functions inside resp. outside the unit disk; here the Cauchy-Riemann operator plays the role of P. In fact, $N^{\frac{1}{2}}_+$ then corresponds to the L_2 Hardy space. Here C^+ is orthogonal, but for more general domains in $\mathbb{C} = \mathbb{R}^2$ it need not be so. Then C^+_{ort} corresponds to the Szegö projection operator, whose kernel has been considered with great interest. In higher dimensions, Dirac operators and Clifford analysis provide a tool to generalize the 2-dimensional function theoretic phenomena; see e.g., Calderbank [Ca96] for an account linking this with the ideas around the Calderón projector for Dirac operators.

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