NONHOMOGENEOUS DIRICHLET NAVIER-STOKES PROBLEMS IN LOW REGULARITY L_p SOBOLEV SPACES

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Abstract. The time-dependent Navier-Stokes problem on an interior or exterior smooth domain, with nonhomogeneous Dirichlet boundary condition, is treated in anisotropic L_p Sobolev spaces ($1 < p < \infty$) of Bessel-potential type $H_p^{s+2,s/2+1}$ or Besov type $B_p^{s+2,s/2+1}$ by use of a reformulation of the linearized problem to a parabolic pseudodifferential boundary value problem. Earlier studies required $s > \frac{1}{p} - 1$; the present work extends the solvability to spaces with $s > \frac{1}{p} - 2$ for zero initial data ($s > -2$ if $f = 0$), $s > \frac{2}{p} - 2$ for nonzero initial data, with s, p subject to other conditions stemming from the nonlinearity.

Introduction.

The Navier-Stokes problem with nonhomogeneous Dirichlet or Neumann boundary conditions has been studied in anisotropic L_2 Sobolev spaces in Grubb-Solonnikov [4] and in L_p Sobolev spaces (Bessel-potential spaces $H_p^{s+2,s/2+1}$, Besov spaces $B_p^{s+2,s/2+1}$, $1 < p < \infty$) in [7], extended to exterior domains in [8]. In these papers, solutions were found for $s > \frac{1}{p} - 1$ (with $s + 3 \geq \frac{n+2}{p}$), since the strategy was to transform the linearized (Stokes) problem considered in solenoidal (divergence free) spaces to a parabolic pseudodifferential problem in full Sobolev spaces; the parabolic system thus obtained is necessarily *of class* 2 and lacks a certain continuity for $s \leq \frac{1}{p} - 1$. However, in the Dirichlet case, the original Stokes problem has only class 1, so one could expect results for $s \in]\frac{1}{p} - 2, \frac{1}{p} - 1]$ also.

In the present paper we show how one can use the general parabolic pseudodifferential results in a more efficient way, extending the solvability of the Dirichlet Stokes problem to $s > \frac{1}{p} - 2$ for nonzero boundary values and forces, zero initial values (all $s \in \mathbb{R}$ when $f = 0$). Nonzero initial values are included when $s > \frac{2}{p} - 2$.

For the Dirichlet Navier-Stokes problem, we then obtain extensions of the results in [7], [8] down to $s > \frac{1}{p} - 2$ too ($s > -2$ if $f = 0$, $s > \frac{2}{p} - 2$ for nonzero initial values), with s, p subject to other conditions stemming from the nonlinearity.

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1. Presentation of the problem and the function spaces.

Consider the nonhomogeneous Navier-Stokes problem with Dirichlet boundary condition

$$\begin{aligned} \partial_t u - \Delta u + \kappa \textstyle\sum_{j=1}^n u_j \partial_j u + \operatorname{grad} q &= f \quad \text{on } Q_{I_b} = \Omega \times I_b, \\ \operatorname{div} u &= 0 \quad \text{on } Q_{I_b}, \\ \gamma_0 u &= \varphi \quad \text{on } S_{I_b} = \Gamma \times I_b, \\ r_0 u &= u_0 \quad \text{on } \Omega; \end{aligned} \tag{1.1}$$

for an interior or exterior domain $\Omega \subset \mathbb{R}^n$ with smooth boundary Γ, $I_b =]0, b[$, $b > 0$. The constant κ equals 1; if instead we take $\kappa = 0$, we have the Stokes problem.

Here $u(x,t)$ is the velocity vector $u = \{u_1, \dots, u_n\}$, $q(x,t)$ is the (scalar) pressure. Let $\vec{n} = (n_1, \dots, n_n)$ be the (interior) normal at Γ, and denote by u_ν resp. u_τ the normal resp. tangential component of an n-vector field u defined near Γ:

$$u_\nu = \vec{n} \cdot u = \operatorname{pr}_\nu u, \qquad u_\tau = u - (\vec{n} \cdot u)\, \vec{n} = \operatorname{pr}_\tau u.$$

As usual, $\gamma_k u = (\partial_\nu^k u)|_\Gamma$ with $\partial_\nu = \sum_{j=1}^n n_j \partial_j$, and we write $\gamma_0 u_\nu = \gamma_\nu u$.

We denote by pr_J and pr_{J_0} the usual projection operators in $L_p(\Omega)^n$ (orthogonal for $p = 2$) onto the solenoidal spaces J_p and $J_{0,p}$:

$$\begin{aligned} J_p = J_p(\Omega) &= \{ u \in L_p(\Omega)^n \mid \operatorname{div} u = 0 \}, \\ J_{0,p} = J_{0,p}(\Omega) &= \{ u \in L_p(\Omega)^n \mid \operatorname{div} u = 0,\ \gamma_\nu u = 0 \}; \end{aligned} \tag{1.2}$$

the projections satisfy

$$\operatorname{pr}_J = I + \operatorname{grad} R_D \operatorname{div}, \quad \operatorname{pr}_{J_0} = (I - \operatorname{grad} K_N \gamma_\nu) \operatorname{pr}_J, \tag{1.3}$$

cf. e.g. [4, Th. 2.5]; cf. also [3, Ex. 3.14]. Here $(\, R_D \quad K_D \,) : \{f, \varphi\} \mapsto u$ and $(\, R_N \quad K_N \,) : \{f, \psi\} \mapsto v$ are solution operators for the Dirichlet, resp. Neumann problem for $-\Delta$ on Ω:

$$\begin{cases} -\Delta u = f, \\ \quad \gamma_0 u = \varphi; \end{cases} \qquad \text{resp.} \qquad \begin{cases} -\Delta v = f, \\ \quad \gamma_1 v = \psi. \end{cases} \tag{1.4}$$

For interior domains, the Neumann solution operator is chosen such that it maps data $\{f, \psi\}$ with $\int_\Omega f\, dx - \int_\Gamma \psi\, dx' = 0$ into functions v with $\int_\Omega v\, dx = 0$. For exterior domains, the Dirichlet solution operator is chosen as explained e.g. in [8, Th. 4] (in particular, $\operatorname{grad} K_D$ maps into functions that are $O(|x|^{-n})$ for $|x| \to \infty$). When $\Omega = \mathbb{R}^n$, $\operatorname{pr}_{J_0} = \operatorname{pr}_J = I + \operatorname{grad} R \operatorname{div}$ and is denoted $\operatorname{pr}_{J,\mathbb{R}^n}$; there are no boundary terms.

The data are assumed to satisfy

$$(1-\operatorname{pr}_{J_0})f=0,\quad \operatorname{pr}_\nu\varphi=0,\quad (1-\operatorname{pr}_{J_0})u_0=0. \tag{1.5}$$

When f or u_0 is in a space of distributions in $x\in\mathbb{R}^n$, the condition just means that $\operatorname{div} f=0$ resp. $\operatorname{div} u_0=0$.

For the nonlinear term in (1.1) we observe that $\sum_{j=1}^n u_j\partial_j v=\operatorname{div}(u\otimes v)$ when $\operatorname{div} u=0$, and we write

$$\begin{aligned}&\mathcal{K}(u,v)=\textstyle\sum_{j=1}^n u_j\partial_j v,\ \text{ equal to }\ \operatorname{div}(u\otimes v)\ \text{ when }\ \operatorname{div} u=0,\\ &\mathcal{Q}(u,v)=\operatorname{pr}_{J_0}\mathcal{K}(u,v),\quad \mathcal{K}(u,u)=\mathcal{K}(u),\quad \mathcal{Q}(u,u)=\mathcal{Q}(u).\end{aligned} \tag{1.6}$$

As shown in [4], the problem (1.1) may by application of div and γ_ν in the first line be replaced by the two problems

$$\begin{aligned}\partial_t u-\Delta u+\kappa\mathcal{Q}(u)+G_0u&=f\quad\text{on } Q_{I_b},\\ \gamma_0u&=\varphi\quad\text{on } S_{I_b},\\ r_0u&=u_0\quad\text{on } \Omega;\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}-\Delta q&=\kappa\operatorname{div}\mathcal{K}(u)\qquad\text{on } Q_{I_b},\\ \gamma_1q&=Tu-\kappa\gamma_\nu\mathcal{K}(u)\quad\text{on } S_{I_b},\end{aligned} \tag{1.8}$$

when the ingredients are sufficiently smooth. Here, using the fact that

$$\gamma_\nu\Delta u=-\operatorname{div}'_\Gamma\gamma_1u_\tau+A'_\Gamma\gamma_0u_\tau\ \text{ when }\ \operatorname{div} u=0,\gamma_\nu u=0, \tag{1.9}$$

where $\operatorname{div}'_\Gamma$ and A'_Γ are first-order tangential differential operators (cf. [4, Lemma A.1]), we have set

$$T=(-\operatorname{div}'_\Gamma\gamma_1+A'_\Gamma\gamma_0)\operatorname{pr}_\tau,\quad G_0=\operatorname{grad}K_NT, \tag{1.10}$$

they are both of class 2.

The "class" terminology comes from the theory of pseudodifferential boundary problems of Boutet de Monvel [2]; an operator A is of class $r\ge 0$ when it is of the form $A=B+\sum_{0\le j\le r-1}K_j\gamma_j$ with B well-defined on $L_p(\Omega)$. Negative class was included in [3]; for $r=-m<0$ we say that A is of class $-m$ if $A\partial_\nu^m$ is of class 0.

The projection operators pr_J and pr_{J_0} are of class 0 but not of any negative class; this is important for the discussion of sharpness of estimates.

The procedure used in the mentioned papers was to solve (1.7) first and then use (1.8) to determine q as

$$q=\kappa(R_N\operatorname{div}-K_N\gamma_\nu)\mathcal{K}(u)+K_NTu=\kappa\widetilde{G}\mathcal{K}(u)+K_NTu. \tag{1.12}$$

By [4, Th. 2.6], $\widetilde{G} = R_N \operatorname{div} - K_N\gamma_\nu$ is of class 0 even though the two terms separately are of class 1, and $\operatorname{grad}\widetilde{G}$ equals $\operatorname{pr}_{J_0} - I$, likewise of class 0.

As in [7], we shall treat the problems in anisotropic Bessel-potential spaces $H_p^{s,\,s/2}(\overline{Q}_{I_b})^n$ and Besov spaces $B_p^{s,\,s/2}(\overline{Q}_{I_b})^n$, $1 < p < \infty$. (In the present paper, we drop the parentheses from $(s, s/2)$ since there is no danger of confusion with other spaces.) We briefly recall the main features, referring to [6] or [7] for further details and references to the literature.

The $H_p^{s,\,s/2}$ spaces, $s \in \mathbb{R}$, are generalizations of the positive integer case

$$H_p^{2m,m}(\overline{Q}_{I_b}) = \{\, u(x,t) \in L_p(Q_{I_b}) \mid D_x^\alpha D_t^j u \in L_p(Q_{I_b}) \text{ for } |\alpha| + 2j \le 2m \,\};$$

they are defined by restriction from the spaces

$$H_p^{s,\,s/2}(\mathbb{R}^n{\times}\mathbb{R}) = \{\, u \in \mathcal{S}' \mid \mathcal{F}^{-1}_{(\xi,\tau)\to(x,t)}(|\xi|^4{+}\tau^2{+}1)^{s/4}\hat{u}(\xi,\tau) \in L_p(\mathbb{R}^{n+1}) \,\} \quad (1.13)$$

with norm $\|\mathcal{F}^{-1}_{(\xi,\tau)\to(x,t)}(|\xi|^4{+}\tau^2{+}1)^{s/4}\hat{u}\|_{L_p}$; this is a scale preserved under complex interpolation. The spaces $H_p^{s,\,s/2}(\overline{Q}_{I_b})$ are Banach spaces provided with the norm

$$\|u\|_{H_p^{s,\,s/2}(\overline{Q}_{I_b})} = \inf\{\, \|U\|_{H_p^{s,\,s/2}(\mathbb{R}^{n+1})} \mid u = U \text{ on } Q_{I_b} \,\},$$

where the U run through the extensions of u to $\mathbb{R}^{n+1}$ (they are spaces of *extendible* distributions).

The Besov scale $B_p^{s,\,s/2}$ is defined slightly differently; it arises from the $H_p^{s,\,s/2}$ scale by suitable real interpolation. The B-spaces must be included even if one is mainly interested in finding solutions in H-spaces, because they are the correct boundary value spaces; in fact, γ_j maps $H_p^{s,\,s/2}(\overline{Q}_{I_b})$ as well as $B_p^{s,\,s/2}(\overline{Q}_{I_b})$ continuously onto $B_p^{s-j-\frac1p,(s-j-\frac1p)/2}(\overline{S}_{I_b})$, for $s > j + \frac1p$.

For the problems (1.1) and (1.7) with zero initial data, the appropriate setting is obtained by using spaces of *supported* distributions, namely distributions defined for $t \in]-\infty, b[= I_{-\infty,b}$ and supported for $t \ge 0$:

$$\begin{aligned} H_{p\,(0)}^{s,s/2}(\overline{Q}_{\mathbb{R}_+}) &= \{\, u \in H_p^{s,s/2}(\overline{Q}_{\mathbb{R}}) \mid u = 0 \text{ for } t < 0 \,\},\\ H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b}) &= r_{Q_{I_{-\infty,b}}} H_{p\,(0)}^{s,s/2}(\overline{Q}_{\mathbb{R}_+}); \end{aligned} \quad (1.14)$$

r_M indicates restriction to M. There are corresponding B-spaces, and the spaces are defined also with Q replaced by S.

Functions belonging to $H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b})$ are usually identified with their restriction to $\overline{Q}_{I_b}$ (an extension by 0 for $t < 0$ is tacitly understood), and the space is regarded as a space "over $\overline{Q}_{I_b}$". The elements belonging to $H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b})$ for negative s are in this way a generalization of the functions in $L_p(Q_{I_b})$ that is different from the generalization defined by $H_p^{s,s/2}(\overline{Q}_{I_b})$ (except when $s > \frac{2}{p} - 2$). Smooth functions vanishing near $t = 0$ are dense in $H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b})$.

The trace operator γ_j maps $H_{p\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})$ and $B_{p\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})$ continuously onto $B_{p\,(0)}^{s-j-\frac{1}{p},(s-j-\frac{1}{p})/2}(\overline{S}_{I_b})$, for $s > j + \frac{1}{p}$. We shall denote

$$\mathcal{B}_{p,b}^{s+2} = B_p^{s+2-\frac{1}{p},(s+2-\frac{1}{p})/2}(\overline{S}_{I_b})^n, \quad \mathcal{B}_{p,b\,(0)}^{s+2} = B_{p\,(0)}^{s+2-\frac{1}{p},(s+2-\frac{1}{p})/2}(\overline{S}_{I_b})^n. \tag{1.15}$$

The restriction to a fixed time, $r_{t_0}u = u|_{t=t_0}$, is well-defined for $s > \frac{2}{p}$, in fact r_{t_0} then maps $H_p^{s,\,s/2}(\overline{\Omega} \times \mathbb{R})$ and $B_p^{s,\,s/2}(\overline{\Omega} \times \mathbb{R})$ continuously onto $B^{s-\frac{2}{p}}(\overline{\Omega})$.

We shall also need the spaces of distributions defined for $x \in \mathbb{R}^n$ and supported for $x \in \overline{\Omega}$:

$$\begin{aligned} &H_{p;0}^{s,s/2}(\overline{\Omega}\times\mathbb{R}) = \{\, u \in H_p^{s,s/d}(\mathbb{R}^n \times\mathbb{R}) \mid u = 0 \text{ on } \complement\overline{\Omega}\times\mathbb{R}\,\}, \text{ and e.g.} \\ &H_{p;0\,(0)}^{s,s/2}(\overline{Q}_{I_b}) = r_{Q_{I_{-\infty,b}}}\{\, u \in H_{p;0}^{s,s/2}(\overline{\Omega}\times\mathbb{R}) \mid u = 0 \text{ for } t < 0\,\}, \end{aligned} \tag{1.16}$$

and the corresponding B-spaces. Here smooth functions vanishing near $S_{\mathbb{R}}$, resp. vanishing near $S_{\mathbb{R}}$ and near $t = 0$, are dense. (Also here, functions of x are identified with their restriction to $x \in \overline{\Omega}$ — the extension by 0 for $x \notin \overline{\Omega}$ being tacitly understood.) There are dualities between spaces with opposite exponents, such that a space of extendible distributions is dual to a space of supported distributions (with respect to x and t separately), e.g.,

$$H_{p\,(0)}^{-s,-s/2}(\overline{Q}_{\mathbb{R}_+}) \simeq (H_{p';0}^{s,s/2}(\overline{Q}_{\mathbb{R}_+}))', \text{ with } \tfrac{1}{p'} = 1 - \tfrac{1}{p}. \tag{1.17}$$

For s close to zero, there are identifications between the spaces of supported distributions and extendible distributions, e.g.,

$$\begin{aligned} H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b}) &\simeq H_p^{s,s/2}(\overline{Q}_{I_b}) \text{ for } \tfrac{2}{p} - 2 < s < \tfrac{2}{p}, \\ H_{p;0}^{s,s/2}(\overline{Q}_{I_b}) &\simeq H_p^{s,s/2}(\overline{Q}_{I_b}) \text{ for } \tfrac{1}{p} - 1 < s < \tfrac{1}{p}. \end{aligned} \tag{1.18}$$

For some special considerations we shall need the slightly more general spaces defined in a similar way for two real numbers σ and ϱ, both lying in $\overline{\mathbb{R}}_+$ or in $\overline{\mathbb{R}}_-$, departing from

$$H_p^{\sigma,\varrho}(\mathbb{R}^n \times\mathbb{R}) = \{u \in \mathcal{S}' \mid \mathcal{F}^{-1}_{(\xi,\tau)\to(x,t)}(|\xi|^{|\sigma|} + |\tau|^{|\varrho|} + 1)^{\pm 1}\hat{u}(\xi,\tau) \in L_p(\mathbb{R}^{n+1})\},$$

with ± 1 chosen when $\sigma, \varrho \in \overline{\mathbb{R}}_\pm$. It is useful to know that

$$\begin{aligned} &\text{(i)} \quad H_p^{\sigma,\varrho}(\mathbb{R}^n \times\mathbb{R}) = L_p\big(\mathbb{R}; H_p^{\sigma}(\mathbb{R}^n)\big) \cap H_p^{\varrho}\big(\mathbb{R}; L_p(\mathbb{R}^n)\big) \text{ for } \sigma, \varrho \geq 0; \\ &\text{(ii)} \quad B_p^{\sigma,\varrho}(\mathbb{R}^n \times\mathbb{R}) = L_p\big(\mathbb{R}; B_p^{\sigma}(\mathbb{R}^n)\big) \cap B_p^{\varrho}\big(\mathbb{R}; L_p(\mathbb{R}^n)\big) \text{ for } \sigma, \varrho > 0. \end{aligned} \tag{1.19}$$

We recall moreover that in all the scales,

$$\begin{aligned} B_p^{\sigma,\varrho} \subset H_p^{\sigma,\varrho} \subset B_p^{\sigma-\varepsilon,\varrho-\varepsilon\varrho/\sigma} \text{ if } p \leq 2; \\ H_p^{\sigma,\varrho} \subset B_p^{\sigma,\varrho} \subset H_p^{\sigma-\varepsilon,\varrho-\varepsilon\varrho/\sigma} \text{ if } p \geq 2; \end{aligned} \tag{1.20}$$

with equality of $B_p^{\sigma,\varrho}$ and $H_p^{\sigma,\varrho}$ if and only if $p = 2$. (Here ε is arbitrary > 0.)

2. Linear results.

For the results in this section, $\kappa = 0$ in (1.1). We shall show the following generalization of [7, Th. 1.7], the new feature being that it allows $s \in]\frac{1}{p}-2, \frac{1}{p}-1]$, whereas the earlier result required $s > \frac{1}{p}-1$.

Theorem 2.1. *Let $b \in \mathbb{R}_+$.*

For any $s > \frac{1}{p}-2$ and any

$$\{f,\varphi\} \in H^{s,\,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n \times \mathcal{B}^{s+2}_{p,b\,(0)} \tag{2.1}$$

satisfying (1.5), there is a solution $\{u,q\}$ of the Stokes problem (1.1) with $\kappa = 0$, $u_0 = 0$, such that

$$u \in H^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n, \quad \operatorname{grad} q \in H^{s,s/2}_{p\,(0)}(\overline{Q}_{I_b})^n, \quad q \in H^{s,s/2}_{p\,(0)}(\overline{Q}_{I_b}). \tag{2.2}$$

Here u and $\operatorname{grad} q$ are uniquely determined, and q is unique when, in case of an interior domain, it is chosen in the closure in $H^{s,s/2}_{p\,(0)}(\overline{Q}_{I_b})$ of the smooth functions satisfying $\int_\Omega q(x,t)\,dx = 0$.

The following estimate holds with C_b nondecreasing in b:

$$\begin{aligned}\big(\|u\|^p_{H^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n} + \|\operatorname{grad} q\|^p_{H^{s,\,s/2}_{p\,(0)}(\overline{Q}_{I_b})^n} + \|q\|^p_{H^{s,\,s/2}_{p\,(0)}(\overline{Q}_{I_b})}\big)^{\frac{1}{p}}&\\ \le C_b\big(\|f\|^p_{H^{s,\,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n} + \|\varphi\|^p_{\mathcal{B}^{s+2}_{p,b\,(0)}}\big)^{\frac{1}{p}}&.\end{aligned} \tag{2.3}$$

In case $f = 0$, the solvability and estimates extend to all $s \in \mathbb{R}$.

The analogous result holds with H replaced by B throughout.

Proof.

We first treat the case where $f = 0$. Here we have to find $\{v,q\}$ solving a problem (1.1) of the form

$$\begin{gathered}\partial_t v - \Delta v + \operatorname{grad} q = 0 \text{ on } Q_{I_b},\\ \operatorname{div} v = 0 \text{ on } Q_{I_b}, \quad \gamma_0 v = \psi \text{ on } S_{I_b}, \quad r_0 v = 0 \text{ on } \Omega,\end{gathered} \tag{2.4}$$

with ψ given in $\mathcal{B}^{s+2}_{p,b\,(0)}$. Consider the two associated problems as in (1.7) and (1.8):

$$\partial_t v - \Delta v + G_0 v = 0,\ \gamma_0 v = \psi,\ r_0 v = 0; \tag{2.5}$$

$$-\Delta q = 0,\ \gamma_1 q = Tv. \tag{2.6}$$

Here (2.5) is (in view of the parabolicity shown in [4] and extended to exterior domains in [8]) covered by [6, Th. 3.4], applied as in Cor. 4.5 there, which shows that it is uniquely solvable, by a Poisson solution operator $\mathbf{K}_b$. In fact, this holds not only for "sufficiently large s", for $\mathbf{K}_b$ is continuous from $\mathcal{B}^{s+2}_{p,b\,(0)}$ to $H^{s+2,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})$ *for all* $s\in\mathbb{R}$ (cf. [6, (3.25)]), *regardless of the class of* G_0, and solves (2.5) for all $s\in\mathbb{R}$. Once (2.5) is solved, we can solve (2.6) by use of the Neumann Poisson operator recalled around (1.4), cf. also (1.10)ff., obtaining altogether the solutions

$$v=\mathbf{K}_b\psi,\quad q=K_NTv=K_NT\mathbf{K}_b\psi, \tag{2.7}$$

that solve (2.5) and (2.6) for any s.

Application of K_N to Tv, and the resulting uniqueness of q modulo a side condition, requires a justification that was given for more smooth v in [4, (5.19)ff., Ex. 2.3]; this extends to the present situation by an approximation of ψ by smooth functions, carried out below. We shall first investigate the spaces where $K_NT\mathbf{K}_b$ acts.

With Λ_- denoting the pseudodifferential homeomorphism

$$\Lambda_-\colon H^r_p(\overline{\Omega})\overset{\sim}{\to}H^{r-1}_p(\overline{\Omega}),\ \text{all } r\in\mathbb{R}, \tag{2.8}$$

defined in [3, (5.2)], we can write

$$\begin{aligned} q=K_NT\mathbf{K}_b\psi&=-K_N\operatorname{div}'_\Gamma\gamma_1\operatorname{pr}_\tau\mathbf{K}_b\psi+K_NA'_\Gamma\gamma_0\operatorname{pr}_\tau\mathbf{K}_b\psi\\ &=\Lambda_-^{-1}(-\Lambda_-K_N\operatorname{div}'_\Gamma\gamma_1\operatorname{pr}_\tau\mathbf{K}_b\psi+\Lambda_-K_NA'_\Gamma\gamma_0\operatorname{pr}_\tau\mathbf{K}_b\psi),\end{aligned} \tag{2.9}$$

where $\gamma_1\operatorname{pr}_\tau\mathbf{K}_b$ and $\gamma_0\operatorname{pr}_\tau\mathbf{K}_b$ are continuous from $\mathcal{B}^{s+2}_{p,b\,(0)}$ to $\mathcal{B}^{s+1}_{p,b\,(0)}$, and $-\Lambda_-K_N\operatorname{div}'_\Gamma$ and $\Lambda_-K_NA'_\Gamma$ are Poisson operators *independent of* t of order 1, hence continuous from $\mathcal{B}^{s+1}_{p,b\,(0)}$ to $H^{s,s/2}_{p\,(0)}(\overline{S}_{I_b})^n$ by [7, Lemma 1.5 (iii)]. Thus

$$K_NT\mathbf{K}_b\psi\colon\mathcal{B}^{s+2}_{p,b\,(0)}\to\Lambda_-^{-1}H^{s,s/2}_{p\,(0)}(\overline{S}_{I_b})^n\subset H^{s,s/2}_{p\,(0)}(\overline{S}_{I_b})^n$$

continuously for all $s\in\mathbb{R}$. (In Theorem 2.2 below, we show further estimates of q, where in particular the regularity in t is improved.)

Let $\psi_k\in C^\infty(\overline{S}_{I_b})$, supported in $\Gamma\times\,]0,b]$ and converging to ψ in $\mathcal{B}^{s+2}_{p,b\,(0)}$ for $k\to\infty$; then $\{v,q\}$ is the limit in $H^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n\times H^{s,\,s/2}_{p\,(0)}(\overline{Q}_{I_b})$ of the solutions $\{v_k,q_k\}$ of the problems (2.5), (2.6) with ψ replaced by ψ_k. By [4, Sect. 5.1], the $\{v_k,q_k\}$ solve (2.4) with data $\{0,\psi_k,0\}$, and hence $\{v,q\}$ solves it with data $\{0,\psi,0\}$. It follows in particular that $\operatorname{grad}q\in H^{s,s/2}_{p\,(0)}(\overline{S}_{I_b})^n$.

We get as in [6, Cor. 4.5] (using the method from [4, Th. 6.3]) that the solution operators $\mathbf{K}_b$, $K_NT\mathbf{K}_b$ and $\operatorname{grad}K_NT\mathbf{K}_b$ have norm-estimates with constants nondecreasing in b, showing the relevant version of (2.3).

Next, let $f\neq 0$. Recall that it equals a distribution in $H^{s,s/2}_p(\mathbb{R}^n\times I_{-\infty,b})^n$ vanishing for $t<0$ and for $x\notin\overline{\Omega}$, and that the condition in (1.5) just means that $\operatorname{div}f=0$. By application of [6, Cor. 4.5] to the heat problem

$$\partial_tU-\Delta U=f\text{ on }\mathbb{R}^n\times I_{-\infty,b},\quad U=0\text{ for }t<0, \tag{2.10}$$

we find a unique solution $U = \mathbf{W}_{\mathbb{R}^n,b}f \in H_p^{s+2,s/2+1}(\mathbb{R}^n \times I_{-\infty,b})^n$ (cf. also [6, Th. 3.4]). Moreover, $\operatorname{div} U$ is the unique solution of (2.10) with f replaced by $\operatorname{div} f$, so $\operatorname{div} f = 0$ implies $\operatorname{div} U = 0$. Let $w = r_{Q_{I_{-\infty,b}}}U$; it is in $H_{p\,(0)}^{s+2,\,s/2+1}(\overline{Q}_{I_b})^n$, and $\gamma_0 w$ is defined as an element of $\mathcal{B}_{p,b\,(0)}^{s+2}$ when $s > \frac{1}{p} - 2$. Then u and q solve the problem (1.1) with $\kappa = 0$, $u_0 = 0$, if and only if $v = u - w$ and q solve problem (2.4) with $\psi = \varphi - \gamma_0 w$; here $\psi \in \mathcal{B}_{p,b\,(0)}^{s+2}$. This has been solved above, so we now find the general solution

$$
\begin{aligned}
u &= \mathbf{K}_b(\varphi - \gamma_0 w) + w = (I - \mathbf{K}_b\gamma_0) r_{Q_{I_{-\infty,b}}} \mathbf{W}_{\mathbb{R}^n,b} f + \mathbf{K}_b\varphi, \\
q &= K_N T \mathbf{K}_b(\varphi - \gamma_0 w) = K_N T \mathbf{K}_b(\varphi - \gamma_0 r_{Q_{I_{-\infty,b}}} \mathbf{W}_{\mathbb{R}^n,b} f).
\end{aligned} \tag{2.11}
$$

Since $\mathbf{K}_b\gamma_0$ maps $H_{p\,(0)}^{s+2,\,s/2+1}(\overline{Q}_{I_b})^n$ into itself for $s > \frac{1}{p} - 2$, we find the continuity asserted in (2.3).

This ends the proof for H-spaces, and the proof for B-spaces is similar. □

For $s > \frac{1}{p} - 1$, $s - \frac{1}{p} \notin \mathbb{Z}$, the result is contained in [7, Th. 1.7] and [8], since $H_{p;0\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})^n \subset H_{p\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})^n$ and $B_{p;0\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})^n \subset B_{p\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})^n$ as closed subspaces then.

The estimates of q can be improved as follows:

Theorem 2.2. *When $f = 0$, the pressure q determined in Theorem 2.1 has the following additional properties:*

$$
q \in \Lambda_-^{-1}\big(H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b}) \cap B_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b})\big) \text{ for } s \in \mathbb{R}, \tag{2.12}
$$

$$
q \in H_{p\,(0)}^{s+1,(s_>+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \cap B_{p\,(0)}^{s+1,(s+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \quad \text{for } s > \tfrac{1}{p} - 1 \text{ or } s < -1, \tag{2.13}
$$

$$
q \in H_{p\,(0)}^{s+1,(s_>+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \cap B_{p\,(0)}^{s_<+1,(s_\gtrless+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \quad \text{for } s = -1, \tag{2.14}
$$

$$
q \in H_{p\,(0)}^{s+1-\frac{1}{p},(s_>+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \cap B_{p\,(0)}^{s+1-\frac{1}{p},(s+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \quad \text{for } -1 < s < \tfrac{1}{p} - 1, \tag{2.15}
$$

$$
q \in H_{p\,(0)}^{s+1-\frac{1}{p},(s_>+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \cap B_{p\,(0)}^{s_<+1-\frac{1}{p},(s_\gtrless+1-\frac{1}{p})/2}(\overline{Q}_{I_b}) \quad \text{for } s = \tfrac{1}{p} - 1, \tag{2.16}
$$

where $s_>$ stands for $s - \varepsilon$ if $p > 2$ and s otherwise, $s_<$ stands for $s - \varepsilon$ if $p < 2$ and s otherwise, $s_\gtrless$ stands for $s - \varepsilon$ if $p \gtrless 2$ and s if $p = 2$, ε arbitrary > 0.

When $f \neq 0$, these properties hold for $s > \frac{1}{p} - 2$. They are valid whether the data space for f is taken as $H_{p;0\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})^n$ or $B_{p;0\,(0)}^{s,\,s/2}(\overline{Q}_{I_b})^n$.

Proof. First let $f=0$. Consider q described in (2.9). Since $\gamma_1 \operatorname{pr}_\tau \mathbf{K}_b$ and $\gamma_0 \operatorname{pr}_\tau \mathbf{K}_b$ are continuous from $\mathcal{B}^{s+2}_{p,b\,(0)}$ to $\mathcal{B}^{s+1}_{p,b\,(0)}$, we need to show that $K_N \operatorname{div}'_\Gamma$ and $K_N A'_\Gamma$ map $\varphi_1 \in \mathcal{B}^{s+1}_{p,b\,(0)}$ into the space listed in each line.

(2.12) was shown in the proof of Theorem 2.1.

For (2.13), let first $s > \frac{1}{p} - 1$, so that $\varphi_1 \in B^{\sigma,\sigma/2}_{p\,(0)}(\overline{S}_{I_b})^n$ with $\sigma = s + 1 - \frac{1}{p} > 0$. By (1.19),

$$B^{\sigma,\sigma/2}_{p\,(0)}(S_{\mathbb{R}})^n = L_p\big(\mathbb{R}; B^\sigma_p(\Gamma)\big)^n \cap B^{\sigma/2}_p\big(\mathbb{R}; L_p(\Gamma)\big)^n,$$

and $K_N \operatorname{div}'_\Gamma$, being a Poisson operator of order 0 independent of t, maps the former space into $L_p\big(\mathbb{R}; B^{\sigma+\frac{1}{p}}_p(\overline{\Omega}) \cap H^{\sigma+\frac{1}{p}}_p(\overline{\Omega})\big)$ and the latter into $B^{\sigma/2}_p\big(\mathbb{R}; B^{\frac{1}{p}}_p(\overline{\Omega}) \cap H^{\frac{1}{p}}_p(\overline{\Omega})\big)$. Their intersection is contained in $B^{\sigma+\frac{1}{p},\sigma/2}_p(\overline{Q}_{\mathbb{R}}) \cap H^{\sigma+\frac{1}{p},\sigma/2}_p(\overline{Q}_{\mathbb{R}})$ if $p \le 2$, and in $B^{\sigma+\frac{1}{p},\sigma/2}_p(\overline{Q}_{\mathbb{R}}) \cap H^{\sigma+\frac{1}{p},(\sigma-\varepsilon)/2}_p(\overline{Q}_{\mathbb{R}})$ if $p > 2$ (we have used (1.20)), so we find the first part of (2.13) by specialization to the spaces of functions supported for $t \ge 0$ and restricted to $t < b$. The proof for $K_N A'_\Gamma$ is similar.

The second part of (2.13) is obtained by using that the adjoint of $K_N \operatorname{div}'_\Gamma$ is a trace operator T' of order -1 and class 0. For $\sigma, \varrho \ge 0$, it maps

$$\begin{aligned} T' \colon H^{\sigma,\varrho}_{p';0}(\overline{Q}_{\mathbb{R}}) &= L_{p'}\big(\mathbb{R}; H^\sigma_{p';0}(\overline{\Omega})\big) \cap H^\varrho_{p'}(\mathbb{R}; L_{p'}(\Omega)) \\ &\to L_{p'}\big(\mathbb{R}; B^{\sigma+1-\frac{1}{p'}}_{p'}(\Gamma)\big)^n \cap H^\varrho_{p'}(\mathbb{R}; B^{1-\frac{1}{p'}}_{p'}(\Gamma)\big)^n \subset B^{\sigma+\frac{1}{p},\varrho(-\varepsilon)}_{p'}(S_{\mathbb{R}})^n, \end{aligned} \tag{2.17}$$

with ε subtracted if $p' < 2$, i.e., $p > 2$. Then by duality,

$$K_N \operatorname{div}'_\Gamma \colon B^{-\sigma-\frac{1}{p},-\varrho(+\varepsilon)}_p(S_{\mathbb{R}})^n \to H^{-\sigma,-\varrho}_p(\overline{Q}_{\mathbb{R}}) \text{ for } \sigma \ge 0, \varrho \ge 0.$$

For $s \le -1$, we apply this with $-\sigma = s+1$, $-\varrho(+\varepsilon) = (s+1-\frac{1}{p})/2$, finding that

$$K_N \operatorname{div}'_\Gamma \colon B^{s+1-\frac{1}{p},(s+1-\frac{1}{p})/2}_p(S_{\mathbb{R}})^n \to H^{s+1,(s(-\varepsilon)+1-\frac{1}{p})/2}_p(\overline{Q}_{\mathbb{R}}),$$

as was to be shown. The same argument treats $K_N A'_\Gamma$. There is a similar calculation with H replaced by B, $s < -1$, and no precautions concerning ε. For $s = -1$, the conclusion for B-spaces follows from the result for H-spaces in view of (1.20).

For the remaining values of s, namely $-1 < s \le \frac{1}{p} - 1$, we argue a little differently in order to avoid spaces with opposite sign for the smoothness in x and t. The calculation in (2.17) gives in particular for $\sigma \ge 0$:

$$T' \colon H^{\sigma,(\sigma(+\varepsilon))/2}_{p';0}(\overline{Q}_{\mathbb{R}}) \to B^{\sigma+\frac{1}{p},\sigma/2}_{p'}(S_{\mathbb{R}})^n \subset B^{\sigma,\sigma/2}_{p'}(S_{\mathbb{R}})^n.$$

When $s \le \frac{1}{p} - 1$, we use this with $\sigma = -s - 1 + \frac{1}{p}$ to get by duality:

$$K_N \operatorname{div}'_\Gamma \colon B^{s+1-\frac{1}{p},(s+1-\frac{1}{p})/2}_p(S_{\mathbb{R}})^n \to H^{s+1-\frac{1}{p},(s(-\varepsilon)+1-\frac{1}{p})/2}_p(\overline{Q}_{\mathbb{R}}),$$

obtaining (2.15) and (2.16) for H-spaces. For $s < \frac{1}{p} - 1$, there is a similar proof for B-spaces without precautions concerning ε. When $s = \frac{1}{p} - 1$, we get the result for B-spaces from the H-case with a loss of ε if $p < 2$.

When $f \neq 0$, we need to assume $s > \frac{1}{p} - 2$ in order for $\gamma_0 w$ to be defined, cf. (2.11). Here $w \in H^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n$ resp. $B^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n$ when $f \in H^{s,\,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n$ resp. $B^{s,\,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n$, so in any case, $\gamma_0 w \in \mathcal{B}^{s+2}_{p,b\,(0)}$, entering in (2.11) like φ, and the conclusions are as before. $\square$

The result for $s > \frac{1}{p} - 1$ was essentially given in [7, (1.50)], however the reservation concerning an ε was overlooked there. For (2.12) one could remark that when $s \geq 0$, $\Lambda_-^{-1} H^{s,s/2} \subset H^{s+1,s/2}$, but here the results from (2.13) are stronger. Note that in all cases, the regularity in t is lifted by at least $(1 - \frac{1}{p} - \varepsilon)/2$. (The regularity in x in (2.15) may possibly be improved by working with spaces with different sign for the x- and t-regularity.)

Theorem 1.7 in [7] and its generalization to exterior domains in [8] allow nonzero initial values when $s > \frac{1}{p} - 1$, describing the necessary compatibility conditions at $\Gamma \times \{0\}$ in full. We shall now also allow nonzero initial values for lower values of s:

Corollary 2.3. *Let $\frac{2}{p} - 2 < s < \frac{2}{p} - 1$, and let $\{f, \varphi, u_0\}$ be given in $H^{s,\,s/2}_{p;0}(\overline{Q}_{I_b})^n \times \mathcal{B}^{s+2}_{p,b\,(0)} \times B^{s+2-\frac{2}{p}}_{p;0}(\overline{\Omega})^n$, satisfying* (1.5). *Then the problem* (1.1) *with $\kappa = 0$ and the given data has a solution $\{u, q\}$ in $H^{s+2,\,s/2+1}_p(\overline{Q}_{I_b})^n \times H^{s,\,s/2}_p(\overline{Q}_{I_b})$, where $\{u, \operatorname{grad} q\}$ is uniquely determined, and q is so under a side condition as in Theorem* 2.1, *with estimates*

$$\begin{aligned}\big(\|u\|^p_{H^{s+2,\,s/2+1}_p(\overline{Q}_{I_b})^n} + \|\operatorname{grad} q\|^p_{H^{s,\,s/2}_p(\overline{Q}_{I_b})} + \|q\|^p_{H^{s,\,s/2}_p(\overline{Q}_{I_b})}\big)^{\frac{1}{p}} \\ \leq C_b\big(\|f\|^p_{H^{s,\,s/2}_{p;0}(\overline{Q}_{I_b})^n} + \|\varphi\|^p_{\mathcal{B}^{s+2}_{p,b\,(0)}} + \|u_0\|^p_{B^{s+2-\frac{1}{p}}_{p;0}(\overline{\Omega})^n}\big)^{\frac{1}{p}};\end{aligned} \tag{2.18}$$

C_b being nondecreasing in b. There are similar results with H replaced by B throughout.

The statements on q in Theorem 2.2 *hold in this case with the index* (0) *removed.*

Proof. We recall that f and u_0 identify with a distribution in $H^{s,s/2}_p(\mathbb{R}^n \times I_b)^n$ resp. a function in $B^{s+2-\frac{2}{p}}_p(\mathbb{R}^n)$, supported for $x \in \overline{\Omega}$. By application of [6, Cor. 4.5] to the heat problem

$$\partial_t U - \Delta U = f \text{ on } \mathbb{R}^n \times I_b, \quad U|_{t=0} = u_0, \tag{2.19}$$

we find a unique solution $U \in H^{s+2,s/2+1}_p(\mathbb{R}^n \times I_b)^n$. Since $\operatorname{div} U$ is the unique solution of (2.19) with f and u_0 replaced by $\operatorname{div} f$ and $\operatorname{div} u_0$, $\operatorname{div} U = 0$. Let $w = r_{Q_{I_b}} U$; it is in $H^{s+2,\,s/2+1}_p(\overline{Q}_{I_b})^n$ with $r_0 w = u_0$, and $\gamma_0 w \in B^{s+2-\frac{1}{p},(s+2-\frac{1}{p})/2}_p(\overline{S}_{I_b})^n$. Here

$\gamma_0 u_0 = r_0\gamma_0 w$ when $s \geq \frac{1}{p} - 1$, by [6, Sect. 4.1] (when $s = \frac{1}{p} - 1$, it holds in the sense of coincidence explained there). So since $u_0 \in B_{p;0}^{s+2-\frac{2}{p}}(\overline{\Omega})^n$, we have in fact that $\gamma_0 w \in \mathcal{B}^{s+2}_{p,b\,(0)}$. Then u and q solve the problem (1.1) with $\kappa = 0$, if and only if $v = u - w$ and q solve the problem with f replaced by 0, u_0 replaced by 0 and φ replaced by $\varphi - \gamma_0 w \in \mathcal{B}^{s+2}_{p,b\,(0)}$. This is solved in Theorem 2.1, from which we draw the desired conclusions. □

The initial value space $B_{p;0}^{s+2-\frac{2}{p}}(\overline{\Omega})^n$ equals $B_{p}^{s+2-\frac{2}{p}}(\overline{\Omega})^n$ when $s \in\,]\frac{2}{p} - 2, \frac{3}{p} - 2]$, and comes arbitrarily close to $L_p(\Omega)^n$ when $s \searrow \frac{2}{p} - 2$. When $\varphi = 0$ in (1.1), there are other methods that allow larger initial spaces (including $L_p(\Omega)^n$), e.g. $u_0 \in H_p^r(\overline{\Omega})^n$ for $r > \frac{1}{p} - 1$ in [5, Cor. 1.5, Rem. 1.6]. But the main efforts in the present paper are directed towards the case $\varphi \neq 0$. See also Remark 3.10 below.

Let us also include a version of Theorem 2.1 and its corollary that allows force distributions that are restrictions to Q_{I_b} of solenoidal distributions on $\mathbb{R}^{n+1}$. For this purpose, define

$$\begin{aligned} H^{s,s/2}_{p,\mathrm{div}}(\overline{Q}_{I_b}) &= \{\, f \in \mathcal{D}'(Q_{I_b}) \mid f = r_{Q_{I_b}} F \text{ for some} \\ &\qquad F \in H_p^{s,s/2}(\mathbb{R}^{n+1}) \text{ with } \operatorname{div} F = 0\,\}, \\ H^{s,s/2}_{p,\mathrm{div}\,(0)}(\overline{Q}_{I_b}) &= \{\, f \in \mathcal{D}'(Q_{I_{-\infty,b}}) \mid f = r_{Q_{I_{-\infty,b}}} F \text{ for some} \\ &\qquad F \in H_{p\,(0)}^{s,s/2}(\mathbb{R}^{n} \times I_{-\infty,b}) \text{ with } \operatorname{div} F = 0\,\}, \end{aligned} \tag{2.20}$$

and analogous B-spaces; the first space is provided with the infimum norm (infimum of the norms of the divergence free extensions F), the second is a closed subspace of $H^{s,s/2}_{p,\mathrm{div}}(\overline{Q}_{I_{-\infty,b}})$.

Then we can show:

Corollary 2.4.

1° *Theorems* 2.1 *and* 2.2 *hold with the data space* $H^{s,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n$ *for* f *replaced by* $H^{s,s/2}_{p,\mathrm{div}\,(0)}(\overline{Q}_{I_b})^n$.

2° *Corollary* 2.3 *holds with the data space* $H^{s,s/2}_{p;0}(\overline{Q}_{I_b})^n$ *for* f *replaced by* $H^{s,s/2}_{p,\mathrm{div}}(\overline{Q}_{I_b})^n$.

(There are similar result for B-spaces.)

Proof. 1° is shown by reduction to the result of Theorem 2.1 for $f = 0$. Now, instead of having a distribution f defined for $x \in \mathbb{R}^n$, we use an extension F to $x \in \mathbb{R}^n$ with, say, at most twice as large norm, and proceed as in (2.10)ff. Similarly, for 2° we replace f used in Corollary 2.3 by F. □

3. Nonlinear results.

For the results in this section, $\kappa = 1$ in (1.1). One has the following estimates of the nonlinear term:

Theorem 3.1. *Let $1 < p < \infty$. The constants in this theorem are independent of b. Assume that* $\operatorname{div} u = \operatorname{div} v = 0$.

$1°$ *Let $b \le \infty$. For λ, μ and $\omega \in \mathbb{R}$ such that $\mu \ge 0$, $\omega \ge 0$ and $2\lambda + \mu + \omega >$* $\max\{0, (n+2)(\frac{2}{p} - 1)\}$,

$$\begin{aligned} \|f \cdot g\|_{H_p^{\lambda,\lambda/2}(\overline{Q}_{I_b})} &\le C_1 \|f\|_{H_p^{\lambda+\mu,(\lambda+\mu)/2}(\overline{Q}_{I_b})} \|g\|_{H_p^{\lambda+\omega,(\lambda+\omega)/2}(\overline{Q}_{I_b})}, \\ \|\mathcal{K}(u,v)\|_{H_p^{\lambda-1,(\lambda-1)/2}(\overline{Q}_{I_b})^n} &\le C_1' \|u\|_{H_p^{\lambda+\mu,(\lambda+\mu)/2}(\overline{Q}_{I_b})^n} \|v\|_{H_p^{\lambda+\omega,(\lambda+\omega)/2}(\overline{Q}_{I_b})^n}, \end{aligned} \tag{3.1}$$

when $\lambda + \mu + \omega \ge \frac{n+2}{p}$; except that $\lambda + \mu + \omega > \frac{n+2}{p}$ is assumed if $\mu = 0$ or $\omega = 0$.

$2°$ *Let $s \in \mathbb{R}$ be such that*

$$\begin{aligned} &\text{(i)} \quad s + 3 \ge \tfrac{n+2}{p}, \\ &\text{(ii)} \quad s + 2 > \max\{0, (n+2)(\tfrac{1}{p} - \tfrac{1}{2})\}. \end{aligned} \tag{3.2}$$

Let $\sigma \in [0,1]$ satisfying $\sigma \le s + 3 - \frac{n+2}{p}$, with $\sigma < 1$ if $s + 2 = \frac{n+2}{p}$. Then

$$\|\mathcal{K}(u,v)\|_{H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n} \le C_2 \|u\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n} \|v\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n}. \tag{3.3}$$

$3°$ *Moreover, if $s + 3 > \frac{n+2}{p}$, one has for any $\varepsilon > 0$, when $0 \le \sigma < s + 3 - \frac{n+2}{p}$,*

$$\begin{aligned} &\|\mathcal{K}(u,v)\|_{H_p^{s+\sigma,\,(s+\sigma)/2}(\overline{Q}_{I_b})^n} \\ &\qquad \le \big(\varepsilon \|u\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n} + C_\varepsilon' \|u\|_{L_p(Q_{I_b})^n}\big) \|v\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n}, \end{aligned} \tag{3.4}$$

and, if $s + 2 > \frac{2}{p}$,

$$\begin{aligned} &\|\mathcal{K}(u,v)\|_{H_p^{s+\sigma,\,(s+\sigma)/2}(\overline{Q}_{I_b})^n} \\ &\le \big(\varepsilon \|u\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n} + C_\varepsilon \int_0^b \|u\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_t})^n}\, dt\big) \|v\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n}. \end{aligned} \tag{3.5}$$

$4°$ *The estimates in $2°$ and $3°$ are likewise valid with $\mathcal{K}$ replaced by $\mathcal{Q} = \operatorname{pr}_{J_0} \mathcal{K}$, when $s + 2 > \frac{2(n+2)}{p(n+3)}$; we use the same notation for the constants. Similar results hold with H_p replaced by B_p throughout.*

Proof. The first estimate in $1°$ was shown in Yamazaki [10, Th. 6.1] ([7] includes references to earlier results), and the second estimate follows when we use the second formulation in (1.6). $2°$ is a specialization to $\lambda + \mu = \lambda + \omega = s + 2$, with λ chosen as large as possible under the given side conditions. $3°$ is a variant of [7, Th. 2.1 $4°$]: We first note that as a consequence of (2.1),

$$\|\mathcal{K}(u,v)\|_{H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n} \le C \|u\|_{H_p^{s+2-\delta,(s+2-\delta)/2}(\overline{Q}_{I_b})^n} \|v\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n},$$

when $0 < \sigma < s + 3 - \frac{n+2}{p} - \delta$, $\delta > 0$. Then the elementary inequality, valid for $0 < \delta < s + 2$,

$$\|u\|_{H_p^{s+2-\delta,(s+2-\delta)/2}(\mathbb{R}^n\times\mathbb{R})} \le \varepsilon\|u\|_{H_p^{s+2,s/2+1}(\mathbb{R}^n\times\mathbb{R})} + C_1(\varepsilon)\|u\|_{H_p^{0,0}(\mathbb{R}^n\times\mathbb{R})},$$

and similar versions over subsets and restrictions, lead to (3.4).

For (3.5) we observe that

$$\begin{aligned}\|u\|_{L_p(Q_{I_b})^n} &= \Big(\int_0^b \|r_t u\|^p_{L_p(\Omega)^n}\,dt\Big)^{\frac1p}\\ &\le \sup_{t\in I_b}\|r_t u\|^{(p-1)/p}_{L_p(\Omega)^n}\Big(\int_0^b \|r_t u\|_{L_p(\Omega)^n}\,dt\Big)^{\frac1p}\\ &\le \delta\sup_{t\in I_b}\|r_t u\|_{L_p(\Omega)^n} + C_2(\delta)\int_0^b \|r_t u\|_{L_p(\Omega)^n}\,dt,\end{aligned}$$

for any $\delta > 0$. Since $s+2 > \frac{2}{p}$, we have $B_p^{s+2-2/p}(\overline{\Omega}) \subset L_p(\Omega)$, and

$$\|r_t f\|_{L_p(\Omega)} \le C_0'\|r_t f\|_{B_p^{s+2-2/p}(\overline{\Omega})} \le C_0\|f\|_{H_p^{s+2,s/2+1}(\overline{\Omega}\times\mathbb{R})}$$

for any $t \in \mathbb{R}$, with constants independent of t; this holds also with $H_p^{s+2,s/2+1}$ replaced by $B_p^{s+2,s/2+1}$. We apply this fact to u in the preceding formula, and insert it with $\delta = \varepsilon/(C_0C'_\varepsilon)$ in (3.4); then we get (3.5) (with 2ε instead of ε). The H_p spaces can be exchanged by B_p spaces in the resulting expressions.

Finally, let us show the statements on $\mathcal{Q} = \operatorname{pr}_{J_0}\mathcal{K}$ in 4°. Here, if $s + \sigma > \frac1p - 1$, they follow simply by application of the projection pr_{J_0} as a continuous operator on $H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n$. The best possible σ is $\min\{1, s+3-\frac{n+2}{p}\}(-\varepsilon)$, where ε should be subtracted when $s+2 = \frac{n+2}{p}$. With this σ, $s+\sigma > \frac1p - 1$ as long as

$$2s + 3 - \tfrac{n+2}{p} > \tfrac1p - 1, \text{ i.e., } s+2 > \tfrac{n+3}{2p}. \tag{3.6}$$

When these inequalities do not hold, pr_{J_0} is not directly defined on $H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n$ since it is not of negative class, but then we can use an investigation of Johnsen [9] to pass into other spaces where the projection makes sense. Note that $s+\sigma \le \frac1p - 1$ can only happen when $s \le \frac1p - 1$, and that

$$\tfrac1p - 1 \ge s \ge \tfrac{n+2}{p} - 3 \implies p \ge \tfrac{n+1}{2}. \tag{3.7}$$

By [9, Th. 6.1 and 7.2], applied with $M = (1,\dots,1,2)$, $|M| = n+2$, $s_0 = s_1 = s+2$, $p_0 = p_1 = p$, $q_0 = q_1 = 2$, the mapping $\mathcal{K}\colon (u,v) \mapsto \operatorname{div}(u\otimes v)$

is, when (3.2 ii) holds, continuous from $H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n \times H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n$ to $H_r^{s+1,(s+1)/2}(\overline{Q}_{I_b})^n$, where

$$\tfrac{n+2}{r} = 2\tfrac{n+2}{p} - (s+2), \text{ if } s+2 < \tfrac{n+2}{p}. \tag{3.8}$$

When we consider an s with $s+2 \le \frac{n+3}{2p}$ (in contrast to (3.6)) and satisfying (3.2), the hypotheses for (3.8) are satisfied. Here r is a positive index lower than p. For our application of pr_{J_0} to $H_r^{s+1,(s+1)/2}(\overline{Q}_{I_b})^n$, we need that $r > 1$ (for $r \le 1$, the pseudodifferential boundary operators in anistropic spaces have not been fully investigated). In fact, $r > 1$ in our case, for when s and p are such that (3.2) and the conclusion in (3.7) hold, then

$$\tfrac{n+2}{r} = 2\tfrac{n+2}{p} - (s+2) \le 2\tfrac{n+2}{p} - \tfrac{n+2}{p} + 1 = \tfrac{n+2}{p} + 1 \le \tfrac{2(n+2)}{n+1} + 1 = \tfrac{3n+5}{n+1},$$

and hence

$$r \ge \tfrac{(n+1)(n+2)}{3n+5}, \text{ which is } > 1 \text{ for } n \ge 2.$$

We can then apply pr_{J_0} to $H_r^{s+1,(s+1)/2}(\overline{Q}_{I_b})^n$ when $s+1 > \frac{1}{r} - 1$, i.e., when $s+2 > \frac{1}{r}$. Here we have that

$$s+2 > \tfrac{1}{r} \iff (n+2)(s+2) > 2\tfrac{n+2}{p} - (s+2) \iff s+2 > \tfrac{2(n+2)}{p(n+3)}. \tag{3.9}$$

In the affirmative case,

$$\begin{gathered}\mathcal{Q} = \mathrm{pr}_{J_0}\mathcal{K} \text{ and } \mathcal{K} \text{ are continuous:}\\ H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n \times H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n \to H_r^{s+1,(s+1)/2}(\overline{Q}_{I_b})^n.\end{gathered} \tag{3.10}$$

Finally, by an anisotropic Sobolev imbedding theorem from [10],

$$H_r^{s+1,(s+1)/2}(\overline{Q}_{I_b})^n \hookrightarrow H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n,$$

where

$$s+\sigma - \tfrac{n+2}{p} = s+1-\tfrac{n+2}{r}, \text{ i.e., } \sigma = s+3-\tfrac{n+2}{p}. \tag{3.11}$$

The operators of course also map into the spaces $H_p^{s+\sigma,(s+\sigma)/2}Q_{I_b})^n$ for $\sigma < \min\{1, s+3-\frac{n+2}{p}\}$. The statements in 3° now generalize straightforwardly to $\mathcal{Q}$.

This shows that the results for $\mathcal{K}$ carry over to $\mathcal{Q}$ if in addition $s+2 > \frac{2(n+2)}{p(n+3)}$. □

When $b < \infty$, (3.5) implies that (3.3) holds with C_2 replaced by

$$C_{\varepsilon,b} = \varepsilon + C_\varepsilon b; \tag{3.12}$$

here $C_{\varepsilon,b}$ can be made as small as we want by taking first ε and then $b = b(\varepsilon)$ small enough.

We shall also need the elementary observation that is often used in these matters (cf. e.g. [7, Lemma 3.1]):

Lemma 3.2. *Let $\alpha > 0$, $0 \le \beta < 1$, $\gamma > 0$ and $4\alpha\gamma \le (1-\beta)^2$. Then the smallest root λ_- of the polynomial $\alpha\lambda^2 + (\beta-1)\lambda + \gamma$, $\lambda_- = 2\gamma(1-\beta+\sqrt{(1-\beta)^2 - 4\alpha\gamma})^{-1}$, is positive, and*

$$\lambda_1 \le \alpha\lambda_0^2 + \beta\lambda_0 + \gamma,\ \lambda_0 \le \lambda_- \implies \lambda_1 \le \lambda_-. \tag{3.13}$$

Solvability properties were thoroughly investigated in [7] and [8] for the Navier-Stokes problem in $H_p^{s+2,s/2+1}$-spaces with $s > \frac{1}{p} - 1$. The really new contributions that are now made possible by the linear results in Section 1 are for low values of s, namely $s \in \,]\frac{1}{p} - 2, \frac{1}{p} - 1]$, so let us restrict the attention to this interval. When nonzero initial data enter, we moreover assume $s > \frac{2}{p} - 2$.

Here are some further remarks on the nonlinear estimates: First note that in (3.2), condition (ii) follows from (i) when $s > -2$, as we assume. Secondly, in order to allow spaces of 'supported distributions', we shall elaborate the considerations in the proof of Theorem 3.1 4° as follows, when $\frac{2(n+2)}{p(n+3)} < s + 2 \le \frac{1}{p} + 1$, using the second identification in (1.18):

Let σ be as in 2° or 3°, with $\sigma < 1$ if $s = \frac{1}{p} - 1$. Then if $s + \sigma > \frac{1}{p} - 1$, we have (since $s + \sigma < s + 1 \le \frac{1}{p}$)

$$\begin{aligned} &\mathcal{K}(u,v) \text{ and } \mathcal{Q}(u,v) \in H_p^{s+\sigma,\,(s+\sigma)/2}(\overline{Q}_{I_b}) \\ &\qquad = H_{p;0}^{s+\sigma,\,(s+\sigma)/2}(\overline{Q}_{I_b}) \subset H_{p;0}^{s,\,s/2}(\overline{Q}_{I_b}). \end{aligned} \tag{3.14}$$

For lower values of $s + \sigma$ we can get a similar result by invoking the mapping properties (3.10). In fact, when $\frac{2(n+2)}{p(n+3)} < s + 2 \le \frac{n+3}{2p}$ (cf. (3.6) and (3.9)), and σ is chosen best possible according to (3.11), we have with r as in (3.8),

$$\begin{aligned} &\mathcal{K}(u,v) \text{ and } \mathcal{Q}(u,v) \in H_r^{s+1,\,(s+1)/2}(\overline{Q}_{I_b}) \\ &\qquad = H_{r;0}^{s+1,\,(s+1)/2}(\overline{Q}_{I_b}) \subset H_{p;0}^{s+\sigma,\,(s+\sigma)/2}(\overline{Q}_{I_b}) \subset H_{p;0}^{s,\,s/2}(\overline{Q}_{I_b}). \end{aligned} \tag{3.15}$$

For the first equality it is used not only that $s + 1 > \frac{1}{r} - 1$ (cf. (3.9)) but also that $s + 1 < \frac{1}{r}$. This holds since $s + 2 \le \frac{n+3}{2p}$:

$$\begin{aligned} \tfrac{1}{r} - (s+1) &= \tfrac{2}{p} - \tfrac{s+2}{n+2} - s - 1 = \tfrac{2}{p} + 1 - \tfrac{(s+2)(n+3)}{n+2} \\ &\ge \tfrac{p+2}{p} - \tfrac{(n+3)^2}{2p(n+2)} \ge \tfrac{(n+5)(n+2)-(n+3)^2}{2p(n+2)} = \tfrac{n+1}{2p(n+2)} > 0; \end{aligned}$$

in the last line we used that $2p \ge n + 1$, cf. (3.7).

This shows:

Corollary 3.3. *When $\frac{2(n+2)}{p(n+3)} < s+2 \le \frac{1}{p}+1$, and σ is chosen according to Theorem 3.1 2° or 3°, with $\sigma < 1$ if $s = \frac{1}{p} - 1$, then the estimates (3.3)–(3.5) likewise hold for*

$\mathcal{K}$ and $\mathcal{Q}$ with $H_p^{s+\sigma,\,(s+\sigma)/2}$-norms replaced by $H_{p;0}^{s+\sigma,\,(s+\sigma)/2}$-norms (and likewise for B-spaces).

Consider data

$$\Phi = \{f, \varphi, u_0\} \in H_{p;0}^{s,\,s/2}(\overline{Q}_{I_b})^n \times \mathcal{B}_{p,b\,(0)}^{s+2} \times B_{p;0}^{s+2-\frac{2}{p}}(\overline{\Omega})^n, \tag{3.16}$$

satisfying (1.5) and provided with the data norm $\mathcal{N}_{s,p,b}$:

$$\mathcal{N}_{s,p,b}(\Phi) = (\|f\|^p_{H_{p;0}^{s,\,s/2}(\overline{Q}_{I_b})^n} + \|\psi\|^p_{\mathcal{B}_{p,b\,(0)}^{s+2}} + \|u_0\|^p_{B_{p;0}^{s+2-2/p}(\overline{\Omega})^n}\big)^{\frac{1}{p}}. \tag{3.17}$$

Theorem 3.4. *Let $s \in]\frac{2}{p} - 2, \frac{1}{p} - 1]$ with $s \geq \frac{n+2}{p} - 3$. Let $b \in \mathbb{R}_+$.*

$1°$ *There is at most one solution $\{u, q\}$ with*

$$\{u, \operatorname{grad} q\} \in H_p^{s+2,\,(s+1)/2}(\overline{Q}_{I_b})^n \times H_p^{s,\,s/2}(\overline{Q}_{I_b})^n \tag{3.18}$$

of the Navier-Stokes problem (1.1) *for each set of data Φ satisfying* (1.5) *(where q in the case of interior domains is subject to the side condition mentioned in Theorem* 2.1*).*

$2°$ *There is a constant $N_{s,p,b}$ such that for data Φ with data norm $\mathcal{N}_{s,p,b}(\Phi) < N_{s,p,b}$ there exists a solution $\{u, q\} \in H_p^{s+2,\,s/2+1}(\overline{Q}_{I_b})^n \times H_p^{s,s/2}(\overline{Q}_{I_b})$ of* (1.1) *with* (3.18)*, the norm depending continuously on Φ. When $s \geq s_0$ for some $s_0 > \frac{n+2}{2p} - 2$, the norm condition for existence can be replaced by the condition $\mathcal{N}_{s_0,p,b}(\Phi) < N_{s_0,p,b}$.*

$3°$ *Assume that $s > \frac{n+2}{p} - 3$. One can for each $N > 0$ choose a $b' \leq b$ such that there exists a solution $\{u, q\} \in H_p^{s+2,\,s/2+1}(\overline{Q}_{I_{b'}})^n \times H_p^{s,s/2}(\overline{Q}_{I_{b'}})$ of* (1.1) *(satisfying also* (3.18) *with b replaced by b', and with norm depending continuously on Φ) for any set of data Φ with norm $\mathcal{N}_{s,p,b'}(\Phi) < N$. For $s \geq s_0$, s_0 as above, the solution can be obtained with b' defined relative to s_0.*

The statements hold with H_p replaced by B_p throughout.

Proof. We denote

$$\begin{aligned}
&\|f\|_{H_p^{r,r/2}(\overline{Q}_{I_b})^n} = |||f|||_{r,b}, \quad \|f\|_{H_{p;0}^{r,r/2}(\overline{Q}_{I_b})^n} = |||f|||_{r,b;0},\\
&(\|f\|^p_{H_p^{r+2,r/2+1}(\overline{Q}_{I_b})^n} + \|g\|^p_{H_p^{r,r/2}(\overline{Q}_{I_b})})^{\frac{1}{p}} = |||f, g|||'_{r+2,b}.
\end{aligned} \tag{3.19}$$

Note that since $\frac{2}{p} > \frac{2(n+2)}{p(n+3)}$, the condition $s + 2 > \frac{2(n+2)}{p(n+3)}$ is satisfied for the s we consider. According to Theorem 3.1 and Corollary 3.3, we have for $\sigma < 1$ with $\sigma \leq s + 3 - \frac{n+2}{p}$,

$$\begin{aligned}
|||\mathcal{K}(u, v)|||_{s,b;0} &\leq C_3|||\mathcal{K}(u, v)|||_{s+\sigma,b;0} \leq C_3C_2|||u|||_{s+2,b}|||v|||_{s+2,b},\\
|||\mathcal{Q}(u, v)|||_{s,b;0} &\leq C_3|||\mathcal{Q}(u, v)|||_{s+\sigma,b;0} \leq C_3C_2|||u|||_{s+2,b}|||v|||_{s+2,b}.
\end{aligned} \tag{3.20}$$

First some generalities on the strategy for solving (1.1). We cannot directly use the reduction to (1.7) and (1.8), since u is sought in $H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n$ with $s+2 \le \frac{1}{p}+1$, where G_0 is not in general defined. But thanks to (3.14), (3.15), we can use a splitting of $\mathcal{K}(u)$,

$$\mathcal{K}(u) = \operatorname{pr}_{J_0} \mathcal{K}(u) + (I - \operatorname{pr}_{J_0})\mathcal{K}(u) = \mathcal{Q}(u) + (I - \operatorname{pr}_{J_0})\mathcal{K}(u), \tag{3.21}$$

and write

$$\{u, q\} = \{v, q_1\} + \{w, q_2\}, \tag{3.22}$$

where $\{v, q_1\}$ is the solution according to Corollary 2.3 of the linear problem with the same data:

$$\begin{aligned} \partial_t v - \Delta v + \operatorname{grad} q_1 &= f \quad \text{in } Q_{I_b}, \\ \operatorname{div} v &= 0 \quad \text{in } Q_{I_b}, \\ \gamma_0 v &= \varphi \quad \text{on } S_{I_b}, \\ r_0 v &= u_0 \quad \text{on } \Omega. \end{aligned} \tag{3.23}$$

and $\{w, q_2\}$ is to be constructed so that

$$\begin{aligned} \partial_t w - \Delta w &= -\mathcal{Q}(v + w) \quad \text{in } Q_{I_b}, \\ \operatorname{div} w &= 0 \quad \text{in } Q_{I_b}, \\ \gamma_0 w &= 0 \quad \text{on } S_{I_b}, \\ r_0 w &= 0 \quad \text{on } \Omega; \end{aligned} \tag{3.24}$$

and

$$\operatorname{grad} q_2 = -(I - \operatorname{pr}_{J_0})\mathcal{K}(v + w). \tag{3.25}$$

Then $\{u, q\}$ solves the original problem if and only if $\{w, q_2\}$ solves (3.24)–(3.25). Here we first discuss (3.24); next if w solves (3.24), then q_2 is determined from (3.25) (and the side condition when it applies), since (3.25) implies

$$\begin{aligned} -\Delta q_2 &= -\operatorname{div} \operatorname{grad} q_2 = \operatorname{div}(1 - \operatorname{pr}_{J_0})\mathcal{K}(v + w) = \operatorname{div} \mathcal{K}(v + w), \\ \gamma_1 q_2 &= \gamma_\nu \operatorname{grad} q_2 = \gamma_\nu(-(I - \operatorname{pr}_{J_0})\mathcal{K}(v + w)) = -\gamma_\nu \mathcal{K}(v + w), \end{aligned} \tag{3.26}$$

so that

$$q_2 = \widetilde{G}\mathcal{K}(v + w) \tag{3.27}$$

according to (1.12).

Let us first show the uniqueness. Let $\{u, q\}$ and $\{u', q'\}$ be two solutions of (1.1) on I_b. Define $\{v, q_1\}$ from the data as above, then $\{u, q\} = \{v + w, q_1 + q_2\}$ and $\{u', q'\} = \{v + w', q_1 + q_2'\}$ with $\{w, q_2\}$ and $\{w', q_2'\}$ solving the respective versions of (3.24)–(3.25), and we have to show that $\{w'', q_2''\} = \{w - w', q_2 - q_2'\}$ is zero. Since

$$-\mathcal{Q}(v + w) + \mathcal{Q}(v + w') = \mathcal{Q}(w' - w, v + w) + \mathcal{Q}(v + w', w' - w), \tag{3.28}$$

w'' satisfies

$$\begin{aligned} \partial_t w'' - \Delta w'' &= -\mathcal{Q}(w'', v + w) - \mathcal{Q}(v + w', w''), \\ \operatorname{div} w'' = 0, &\quad \gamma_0 u'' = 0, \quad r_0 u'' = 0. \end{aligned} \tag{3.29}$$

Denote by $\mathbf{H}_b \colon g \mapsto w$ the operator solving the heat problem

$$\begin{aligned} \partial_t w - \Delta w = g \quad \text{in } Q_{I_b}, \\ \gamma_0 w = 0 \quad \text{on } S_{I_b}, \quad r_0 w = 0 \quad \text{on } \Omega; \end{aligned} \tag{3.30}$$

by [6, Cor. 4.5] it satisfies

$$|||w|||_{t+2,b} = |||\mathbf{H}_b g|||_{t+2,b} \le C'_b |||g|||_{t,b} \tag{3.31}$$

for $t \in]\frac{2}{p} - 2, \frac{2}{p}[$, since the values on $\overline{S}_{I_b}$ and $\overline{\Omega} \times \{0\}$ satisfy the relevant compatibility condition. C'_b can be obtained to be nondecreasing in b, and if $\operatorname{div} g = 0$ then $\operatorname{div} w = 0$ in view of the uniqueness of solutions. By (3.20) we have:

$$\begin{aligned} |||w''|||_{s+2,b'} &\le C'_b(|||\mathcal{Q}(w'', v + w) + \mathcal{Q}(v + w', w'')|||_{s,b';0} \\ &\le C'_b C_2 C_3 (2|||v|||_{s+2,b'} + |||w|||_{s+2,b'} + |||w'|||_{s+2,b'})|||w''|||_{s+2,b'} \quad \text{for all } b' \le b. \end{aligned} \tag{3.32}$$

This implies that $w'' = 0$ on $Q_{I_{b'}}$ when

$$C'_b C_2 C_3 (2|||v|||_{s+2,b'} + |||w|||_{s+2,b'} + |||w'|||_{s+2,b'}) < 1,$$

which holds for sufficiently small $b' > 0$ (depending on v, w and w'), so $w = w'$ on $[0, b']$. By (3.27), also $q_2 = q'_2$ on b'.

Replacing 0 by arbitrary points in I_b, we see that if $u = u'$ on $\overline{I}_{b_0} = [0, b_0] \subset [0, b[$, then $u = u'$ on $[0, b'_0]$ for some $b'_0 \in]b_0, b[$, so there is no *largest* $b_0 < b$ where $u = u'$ on $\overline{I}_{b_0}$. Thus $u' = u$ on I_b, and hence also $q = q'$ on I_b. This shows 1°.

Now let us show the existence, for a given set of data $\Phi = \{f, \varphi, u_0\}$. In view of the above analysis, we define $\{v, q_1\}$ as the solution of (3.23) and have to solve (3.24). By (2.18),

$$|||v, q_1|||'_{s+2,b} \le C_b \mathcal{N}_{s,p,b}(\Phi). \tag{3.33}$$

Since $s + \sigma \in]\frac{2}{p} - 2, \frac{2}{p}[$, we can define the mapping $\mathcal{R}_{b,v}$ on $H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n$ by

$$\mathcal{R}_{b,v} \colon w \mapsto \mathbf{H}_b(-\mathcal{Q}(v + w)); \tag{3.34}$$

then (3.24) holds when w is a fixed point for $\mathcal{R}_{b,v}$. The aim is to show that such a fixed point exists when either the data norm is small enough in relation to a given b, or b is small enough in relation to a given data norm estimate.

For $\mathcal{R}_{b,v}$ we have by (3.31) and (3.33), since (3.20) also holds for spaces without ';0',

$$\begin{aligned} |||\mathcal{R}_{b,v}w|||_{s+2,b} &\le C_b'|||\mathcal{Q}(v,v)+\mathcal{Q}(v,w)+\mathcal{Q}(w,v)+\mathcal{Q}(w,w)|||_{s,b} \\ &\le C_b'C_2C_3\big(|||v|||_{s+2,b}^2+2|||v|||_{s+2,b}|||w|||_{s+2,b}+|||w|||_{s+2,b}^2\big) \\ &\le C_b'C_2C_3(C_b\mathcal{N}_{s,p,b}(\Phi)+|||w|||_{s+2,b})^2. \end{aligned} \tag{3.35}$$

We shall first show 2°, where we take $b'=b$ and adapt the norms. Here we apply Lemma 3.2 with $\lambda_0=|||w|||_{s+2,b}$ and $\lambda_1=|||\mathcal{R}_{b,v}w|||_{s+2,b}$, and

$$\alpha=C_b'C_3C_2,\quad \beta=2C_b'C_3C_2C_b\mathcal{N},\quad \gamma=C_b'C_3C_2C_b^2\mathcal{N}^2, \tag{3.36}$$

$\mathcal{N}=\mathcal{N}_{s,p,b}(\Phi)$. This gives that if, for some $\eta\in\,]0,1[\,$,

$$2C_b'C_3C_2C_b\mathcal{N}\le\eta,\qquad (2C_b'C_3C_2C_b\mathcal{N})^2\le(1-\eta)^2, \tag{3.37}$$

then

$$|||w|||_{s+2,b}\le\lambda_-\implies |||\mathcal{R}_{b,v}w|||_{s+2,b}\le\lambda_-,$$

where

$$\lambda_-=\frac{2\gamma}{1-\beta+\sqrt{(1-\beta)^2-4\alpha\gamma}}\le\frac{2\gamma}{1-\beta}\le\frac{2C_b'C_3C_2C_b^2\mathcal{N}^2}{1-\eta}\le\frac{\eta C_b\mathcal{N}}{1-\eta}. \tag{3.38}$$

So $\mathcal{R}_{b,v}$ maps the closed ball $\overline{B}_b(0,\lambda_-)$ in $H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n$ with radius λ_- into itself.

When (3.37) holds and w and $w'\in\overline{B}_b(0,\lambda_-)$, then

$$\begin{aligned} |||\mathcal{R}_{b,v}w-\mathcal{R}_{b,v}w'|||_{s+2,b} &= |||\mathbf{H}_b[-\mathcal{Q}(v+w)+\mathcal{Q}(v+w')]|||_{s+2,b} \\ &\le C_b'|||\mathcal{Q}(w'-w,v+w)+\mathcal{Q}(v+w',w'-w)|||_{s,b} \\ &\le 2C_b'C_2C_3(C_b\mathcal{N}+\lambda_-)|||w'-w|||_{s+2,b}. \end{aligned} \tag{3.39}$$

Since $C_b\mathcal{N}+\lambda_-\le C_b\mathcal{N}(1+\eta(1-\eta)^{-1})=C_b\mathcal{N}(1-\eta)^{-1}$ by (3.38), $\mathcal{R}_{b,v}$ is a proper contraction on $\overline{B}_b(0,\lambda_-)$ if in addition to (3.37)

$$2C_b'C_2C_3C_b\mathcal{N}(1-\eta)^{-1}<1; \tag{3.40}$$

note that this is just a sharpening of the second inequality in (3.37). Then $\mathcal{R}_{b,v}$ has a unique fixed point $\overline{w}\in\overline{B}_b(0,\lambda_-)$ (determined as $\lim_{m\to\infty}\mathcal{R}_{b,v}^m w_0$, for an arbitrary $w_0\in\overline{B}_b(0,\lambda_-)$). This $\overline{w}$ solves (3.24), and we set $u=v+\overline{w}$. As noted in (3.27), the accompanying q_2 is determined by $q_2=\widetilde{G}\mathcal{K}(u)$, and $q=q_1+q_2$.

This proves the main statement in 2°. The modification with s replaced by s_0 is obvious.

The preceding lines are a close generalization of the proof of [7, Th. 3.2 2°]. In a similar way, the proof of [7, Th. 3.2 3°] is generalized straightforwardly to give 3°. Again the crucial step is to construct w; one uses that (3.35)–(3.38) are likewise valid with b replaced by any smaller b' (and the constants $C_{b'}, C'_{b'}$ can be replaced by C_b, C'_b since they are nondecreasing), now the smallness in (3.37) is obtained by making C_2 small, using (3.12). Moreover, the estimates in Theorem 3.1 3° are used. $\square$

With zero initial data, we can extend the above proof to allow slightly lower s in the uniqueness statement and statement on existence for small data norms:

Corollary 3.5. *Let* $s \in]\frac{2(n+2)}{p(n+3)} - 2, \frac{1}{p} - 1]$ *with* $s \geq \frac{n+2}{p} - 3$. *Replace the data spaces and norm in* (3.16)–(3.17) *by*

$$\begin{aligned} \Psi &= \{f, \varphi\} \in H^{s,\,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n \times \mathcal{B}^{s+2}_{p,b\,(0)}, \\ \mathcal{N}_{s,p,b\,(0)}(\Psi) &= (\|f\|^p_{H^{s,\,s/2}_{p;0\,(0)}(\overline{Q}_{I_b})^n} + \|\psi\|^p_{\mathcal{B}^{s+2}_{p,b\,(0)}})^{\frac{1}{p}}. \end{aligned} \tag{3.41}$$

1° *There is at most one solution* $\{u, q\}$ *with*

$$\{u, \operatorname{grad} q\} \in H^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n \times H^{s,\,s/2}_{p\,(0)}(\overline{Q}_{I_b})^n \tag{3.42}$$

of the Navier-Stokes problem (1.1) *for each set of data* Ψ *satisfying* (1.5) *(with the usual side condition on* q*).*

2° *There is a constant* $N_{s,p,b}$ *such that for data* Ψ *with data norm* $\mathcal{N}_{s,p,b\,(0)}(\Psi) < N_{s,p,b}$ *there exists a solution* $\{u, q\} \in H^{s+2,\,s/2+1}_{p\,(0)}(\overline{Q}_{I_b})^n \times H^{s,s/2}_{p\,(0)}(\overline{Q}_{I_b})$ *of* (1.1) *with* (3.42), *the norm depending continuously on* Ψ.

Proof. Note that $\frac{2(n+2)}{p(n+3)} \in]\frac{1}{p}, \frac{2}{p}[$. We can allow s down to $\frac{2(n+2)}{p(n+3)} - 2$, since there is no need to define restrictions to $t = 0$. The proof goes as in Theorem 3.4 1° and 2°, now based directly on Theorem 2.1, omitting explicit mention of the zero initial condition which is built into the spaces with index (0). $\square$

As noted earlier in (3.7), the new results for $s \leq \frac{1}{p} - 1$ are applicable when $p \geq \frac{n+1}{2}$. To see which lower bound on s that is strongest, we observe:

$$\begin{aligned} \max\{\tfrac{1}{p} - 2, \tfrac{n+2}{p} - 3\} &= \begin{cases} \frac{n+2}{p} - 3 & \text{for } p \in [\frac{n+1}{2}, n+1], \\ \frac{1}{p} - 2 & \text{for } p \geq n+1; \end{cases} \\ \max\{\tfrac{2}{p} - 2, \tfrac{n+2}{p} - 3\} &= \begin{cases} \frac{n+2}{p} - 3 & \text{for } p \in [\frac{n+1}{2}, n], \\ \frac{2}{p} - 2 & \text{for } p \geq n. \end{cases} \end{aligned} \tag{3.43}$$

Note that we can get s arbitrarily close to -2 by taking p large enough.

The estimates of q can be improved as follows:

Theorem 3.6. *When $\{u,q\}$ solve the Navier-Stokes problem according to Theorem 3.4, then $q=q_1+q_2$, where q_1 has the properties listed in Theorem 2.2 with (0) removed, and $q_2\in H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})$ (with b replaced by b' in case 3°) for σ satisfying:*

$$\sigma\in[0,1],\ \sigma\le s+3-\tfrac{n+2}{p},\ \sigma<1 \text{ if } s=\tfrac{n+2}{p}-2 \text{ or } \tfrac1p-1. \tag{3.44}$$

The result extends to the cases treated in Corollary 3.5 with H_p-spaces replaced by $H_{p\,(0)}$-spaces.

Similar reults hold for B-spaces.

Proof. We give details for the solutions of Theorem 3.4. The information on q_1 follows since it is the pressure obtained by solving a linear problem, by Corollary 2.3.

For q_2, we use that it equals $\widetilde{G}\mathcal{K}(u)$ where $\widetilde{G}$ is a singular Green operator of order 0 and class 0. By [7, Lemma 1.5], $\widetilde{G}$ is continuous in $H_q^{t,t/2}(\overline{Q}_{I_b})$ when $t>\frac1q-1$. Here, when $s+\sigma>\frac1p-1$, we use (3.14) to apply $\widetilde{G}$ in $H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})$, and when $s+\sigma$ is lower, we use (3.15) to apply $\widetilde{G}$ in $H_r^{s+1,(s+1)/2}(\overline{Q}_{I_b})$, which is subsequently injected continuously into $H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})$. □

It is also possible to treat the Navier-Stokes problem in a slightly different way building on Corollary 2.4.

For $s+2\ge 0$, let $l\colon H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n\to H_p^{s+2,s/2+1}(\mathbb{R}^n\times\overline{I}_b)^n$ be a continuous linear extension operator (depending on s and p), which maps into functions supported in $(\overline{\Omega}+B_\varepsilon)\times\overline{I}_b$, say, with $B_\varepsilon=\{|x|<\varepsilon\}$. Then we can decompose by use of $\operatorname{pr}_{J,\mathbb{R}^n}$,

$$\begin{aligned}\mathcal{K}(u,v)&=r_{Q_{I_b}}\mathcal{K}(lu,lv)\\&=r_{Q_{I_b}}\operatorname{pr}_{J,\mathbb{R}^n}\mathcal{K}(lu,lv)+r_{Q_{I_b}}((I-\operatorname{pr}_{J,\mathbb{R}^n})\mathcal{K}(lu,lv)\\&=\widetilde{Q}(u,v)-r_{Q_{I_b}}\operatorname{grad}R\operatorname{div}\mathcal{K}(lu,lv).\end{aligned}\tag{3.45}$$

When $u,v\in H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n$ with $s\ge\frac{n+2}{p}-3,\quad s\in\,]\frac1p-2,\frac1p-1]$, then it follows from Theorem 3.1 2° that $\widetilde{\mathcal{Q}}(u,v)$ belongs to $H_{p,\mathrm{div}}^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n$ (recall (2.20)) and satisfies:

$$\|\widetilde{\mathcal{Q}}(u,v)\|_{H_p^{s+\sigma,(s+\sigma)/2}(\overline{Q}_{I_b})^n}\le C_2'\|u\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n}\|v\|_{H_p^{s+2,s/2+1}(\overline{Q}_{I_b})^n},\tag{3.46}$$

for the σ described in Theorem 3.1 2°. The estimates (3.4) and (3.5) likewise generalize to $\widetilde{Q}(u,v)$. The advantage of this point of view is that we can use the pseudodifferential operator $\operatorname{pr}_{J,\mathbb{R}^n}$ freely without class restrictions; on the other hand l is subject to a choice.

Assume moreover that $s > \frac{2}{p} - 2$, and consider (1.1) with data as in Corollary 2.4 2°, now decomposing the nonlinear term by (3.45). We proceed as in (3.22)–(3.25), now taking for $\{v, q_1\}$ the solution of (3.23) according to Corollary 2.4 2° and replacing Q by $\widetilde{Q}$, so that (3.25) is replaced by

$$\operatorname{grad} q_2 = r_{Q_{I_b}} \operatorname{grad} R \operatorname{div} \mathcal{K}(lu). \tag{3.47}$$

It is now found just as in the proof of Theorem 3.4 that there is uniqueness of u in a solution, and that u may be constructed either for small data norms in relation to b, or for data norms estimated by a freely chosen constant but with b replaced by a sufficiently small b'. For q_2 we can then simply take

$$q_2 = r_{Q_{I_b}} R \operatorname{div} \mathcal{K}(lu), \tag{3.48}$$

but we only claim uniqueness for $\operatorname{grad} q_2$ (which follows from uniquenes of u since $\operatorname{grad} q = \operatorname{grad} q_1 + \operatorname{grad} q_2$ is determined from u by (1.1)).

This shows:

Theorem 3.7. *Let $s \in]\frac{2}{p} - 2, \frac{1}{p} - 1]$ with $s \geq \frac{n+2}{p} - 3$ (cf. also (3.43)). Replace in (3.16)–(3.17) the data space $H^{s,s/2}_{p;0}(\overline{Q}_{I_b})^n$ for f by the data space $H^{s,s/2}_{p,\mathrm{div}}(\overline{Q}_{I_b})^n$ defined in (2.20).*

1° There is at most one solution $\{u, \operatorname{grad} q\}$ with

$$\{u, \operatorname{grad} q\} \in H_p^{s+2,\, s/2+1}(\overline{Q}_{I_b})^n \times H_p^{s,\, s/2}(\overline{Q}_{I_b})^n \tag{3.49}$$

of the Navier-Stokes problem (1.1) for each set of data Φ satisfying (1.5).

2° There is a constant $N_{s,p,b}$ such that for data Φ with data norm $\mathcal{N}_{s,p,b}(\Phi) < N_{s,p,b}$ there exists a solution $\{u, q\} \in H_p^{s+2,\, s/2+1}(\overline{Q}_{I_b})^n \times H_p^{s,s/2}(\overline{Q}_{I_b})$ of (1.1) with (3.49), the norm depending continuously on Φ.

3° Assume that $s > \frac{n+2}{p} - 3$. One can for each $N > 0$ choose a $b' \leq b$ such that there exists a solution $\{u, q\} \in H_p^{s+2,\, s/2+1}(\overline{Q}_{I_{b'}})^n \times H_p^{s,s/2}(\overline{Q}_{I_{b'}})$ of (1.1) (satisfying also (3.49) with b replaced by b', and with norm depending continuously on Φ) for any set of data Φ with norm $\mathcal{N}_{s,p,b'}(\Phi) < N$.

The statements hold with H_p replaced by B_p throughout.

With zero initial data, we can allow lower s in the uniqueness statement and the statement on existence for small data norms:

Corollary 3.8. *Let $s \in]\frac{1}{p} - 2, \frac{1}{p} - 1]$ with $s \geq \frac{n+2}{p} - 3$ (cf. also (3.43)). Replace the data spaces and norm in (3.10)–(3.11) by*

$$\begin{aligned} \Psi &= \{f, \varphi\} \in H^{s,\, s/2}_{p,\mathrm{div}\,(0)}(\overline{Q}_{I_b})^n \times \mathcal{B}^{s+2}_{p,b\,(0)}, \\ \mathcal{N}_{s,p,b}(\Psi) &= (\|f\|^p_{H^{s,\, s/2}_{p,\mathrm{div}\,(0)}(\overline{Q}_{I_b})^n} + \|\psi\|^p_{\mathcal{B}^{s+2}_{p,b\,(0)}})^{\frac{1}{p}}. \end{aligned} \tag{3.50}$$

1° *There is at most one solution* $\{u, \operatorname{grad} q\}$ *with*

$$\{u, \operatorname{grad} q\} \in H_{p\,(0)}^{s+2,\, s/2+1}(\overline{Q}_{I_b})^n \times H_{p\,(0)}^{s,\, s/2}(\overline{Q}_{I_b})^n \tag{3.51}$$

of the Navier-Stokes problem (1.1) *for each set of data* Ψ *satisfying* (1.5).

2° *There is a constant* $N_{s,p,b}$ *such that for data* Ψ *with data norm* $\mathcal{N}_{s,p,b}(\Psi) < N_{s,p,b}$ *there exists a solution* $\{u, q\} \in H_{p\,(0)}^{s+2,\, s/2+1}(\overline{Q}_{I_b})^n \times H_{p\,(0)}^{s,s/2}(\overline{Q}_{I_b})$ *of* (1.1) *with* (3.51), *the norm depending continuously on* Ψ.

Proof. The proof goes as in Theorem 3.6 1° and 2°, now based on Corollary 2.4 1°. (We can allow $s > \frac{1}{p} - 2$ since there is no need to define restrictions to $t = 0$.) □

There are also generalizations of Theorem 3.6 to the cases in Theorem 3.7 and Corollary 3.8.

In all the cases, it is seen as in [7, Th. 3.7] when $s > \frac{n+2}{p} - 3$, that if f and φ are C^∞ for $t > 0$, then so are u and q.

Remark 3.9. When $f = 0$ in (3.50), the result of Corollary 3.8 can be further improved for $p > n + 1$. Now f and u_0 are zero in (3.23), so that problem can be solved for all $s \in \mathbb{R}$ by Theorem 2.1. Let us see which $s \le \frac{1}{p} - 2$ will be allowed: For (3.24) (with Q replaced by $\widetilde{Q}$), the conditions $s > -2$ and $s \ge \frac{n+2}{p} - 3$ are needed; then the right hand side is in $H_p^{s+\sigma,(s+\sigma)/2}$ with $\sigma = s - \frac{n+2}{p} + 3$. For the application of $\mathbf{H}_b$, cf. (3.30), it suffices that $s + \sigma > \frac{1}{p} - 2$, i.e., $s > \frac{n+3}{2p} - \frac{1}{2} - 2$. This condition is weaker than the condition $s > \frac{1}{p} - 2$ and replaces it, when $p > n + 1$. In particular, when $p \ge n + 3$, *only the hypothesis* $s > -2$ *is needed.*

Remark 3.10. Much of the literature on the Navier-Stokes problem (1.1) deals with the case $\varphi = 0$ (homogeneous boundary condition), see e.g. the survey of H. Amann in [1], where he develops new general results for this case by use of semigroup techniques and interpolation/extrapolation of solenoidal distribution spaces. In a personal communication, Amann has sketched how his results may be extended to allow nonhomogeneous boundary conditions too, by a weak formulation where the boundary data are incorporated in the force distribution f. However, the present results are not readily compared with those of Amann. One fundamental difference is that the nonlinear term in [1] is taken of the form $Q(u)$ (or some extension by continuity in his family of spaces) where the projection pr_{J_0} has already taken place and the pressure is already eliminated; there is no attempt to retrieve the unknown pressure q as we do. The *class* problems that we deal with do not occur then. Another difference is that in the results of [1], the regularities in x and t are separated so that one can have more smoothness in t (and less in x) than in our results, where the regularities are linked by the anisotropic space definitions, giving fractional differentiability in t. It may possibly be of interest to try to combine the strong points of each method.

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References

[1] H. Amann, *On the strong solvability of the Navier-Stokes equations*, J. math. fluid mech. **2** (2000), 16–98.

[2] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Mat. **126** (1971), 11–51.

[3] G. Grubb, *Pseudo-differential boundary problems in L_p spaces*, Comm. Part. Diff. Eq. **15** (1990), 289–340.

[4] G. Grubb and V. A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand. **69** (1991), 217–290.

[5] G. Grubb, *Initial value problems for the Navier-Stokes equations with Neumann conditions*, The Navier-Stokes equations II — Theory and numerical methods, Proceedings Oberwolfach 1991, Lecture Notes in Math. vol. 1530, Springer Verlag, 1992, pp. 262–283.

[6] ———, *Parameter-elliptic and parabolic pseudodifferential boundary problems in global L_p Sobolev spaces*, Math. Zeitschr. **218** (1995), 43–90.

[7] ———, *Nonhomogeneous time-dependent Navier-Stokes problems in L_p Sobolev spaces*, Diff. Int. Equ. **8** (1995), 1013–1046.

[8] ———, *Nonhomogeneous Navier-Stokes problems in L_p Sobolev spaces over interior and exterior domains*, Theory of the Navier-Stokes Equations, Ser. Adv. Math. Appl. Sci. 47 (J. G. Heywood, K. Masuda, R. Rautmann, V. Solonnikov, eds.), World Scientific, Singapore, 1998, pp. 46–63.

[9] J. Johnsen, *Pointwise multiplication of Besov and Triebel-Lizorkin spaces*, Math. Nachr. **175** (1995), 85–133.

[10] M. Yamazaki, *A quasi-homogeneous version of paradifferential operators, I. Boundedness on spaces of Besov type*, J. Fac. Sci. Tokyo **33** (1986), 131–174.

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