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On the Logarithm Component in Trace Defect Formulas

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In asymptotic expansions of resolvent traces $\mathrm{Tr}(A(P-\lambda)^{-1})$ *for classical pseudodifferential operators on closed manifolds, the coefficient* $C_0(A,P)$ *of* $(-\lambda)^{-1}$ *is of special interest, since it is the first coefficient containing nonlocal elements from A; moreover, it enters in index formulas.* $C_0(A,P)$ *also equals the zeta function value at zero when P is invertible.* $C_0(A,P)$ *is a trace modulo local terms, since* $C_0(A,P)-C_0(A,P')$ *and* $C_0([A,A'],P)$ *are local. By use of complex powers* P^s *(or similar holomorphic families of order s), Okikiolu, Kontsevich and Vishik, Melrose and Nistor showed formulas for these trace defects in terms of residues of operators defined from A, A′,* $\log P$ *and* $\log P'$.

The present paper has two purposes. One is to show how the trace defect formulas can be obtained from the resolvents in a simple way without use of the complex powers of P as in the original proofs. We also give here a simple direct proof of a recent residue formula of Scott for $C_0(I,P)$*. The other purpose is to establish trace defect residue formulas for operators on manifolds with boundary, where complex powers are not easily accessible; we do this using only resolvents. We also generalize Scott's formula to boundary problems.*

Keywords Noncommutative residue; Pseudodifferential boundary operators; Residue of logarithm; Resolvent method; Trace defect formula; Zeta function.

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Introduction

Consider a classical pseudodifferential operator (ψdo) A of order σ on an n-dimensional smooth compact boundaryless manifold X. When P denotes an auxiliary elliptic ψdo of order $m>0$ and, say, positive, one can study the generalized zeta function $\zeta(A,P,s)$ defined as the meromorphic extension of $\mathrm{Tr}(AP^{-s})$ to the complex plane, where the complex powers P^{-s} are defined from the resolvent $(P-\lambda)^{-1}$ as in Seeley (1967). It is well known that $\zeta(A,P,s)$ has a Laurent

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expansion at $s = 0$,

$$\zeta(A, P, s) \sim C_{-1}(A, P)s^{-1} + C_0(A, P) + \sum_{l \geq 1} C_l(A, P)s^l, \tag{0.1}$$

where $mC_{-1}(A, P)$ equals the noncommutative residue $\operatorname{res} A$ (Wodzicki, 1984; Guillemin, 1985), and $C_0(A, P)$ equals the canonical trace $\operatorname{TR} A$ in particular cases (Kontsevich and Vishik, 1995; Lesch, 1999; recent extension in Grubb, 2005).

The coefficient $C_0(A, P)$ is not in general independent of P, but then it is viewed as a "regularized trace" (Melrose and Nistor, 1996) or a "weighted trace" (Cardona et al., 2002, 2003). In general it satisfies the *trace defect formulas*

$$C_0(A, P) - C_0(A, P') = -\frac{1}{m}\operatorname{res}(A(\log P - \log P')), \tag{0.2}$$

$$C_0([A, A'], P) = -\frac{1}{m}\operatorname{res}(A[A', \log P]), \tag{0.3}$$

shown by Okikiolu (1995) and Kontsevich and Vishik (1995), respectively Melrose and Nistor (1996), by use essentially of the holomorphic family P^{-s} and the fact that its derivative at zero is $-\log P$.

For a compact manifold X with boundary $\partial X = X'$, the situation is somewhat different. A pseudodifferential calculus that contains differential elliptic boundary value problems and their solution operators and is closed under composition and elliptic inversion is the calculus of Boutet de Monvel (1971); we consider an operator $A = P_+ + G$ lying there. Here P is a ψdo defined on a larger boundaryless manifold $\widetilde{X}$ in which X is imbedded, such that P satisfies the transmission condition at X' (in particular, it is of integer order), $P_+ = r^+Pe^+$ is the truncation to X (e^+ extends by zero, r^+ restricts to X), and G is a singular Green operator (smoothing in the interior, but important near the boundary).

Even the simplest auxiliary operator $P_{1,\mathrm{D}}$ with P_1 equal to the Laplace operator, the D indicating Dirichlet condition, does *not* have its complex powers in the Boutet de Monvel calculus, so the ingredients in the zeta function $\operatorname{Tr}(AP_{1,\mathrm{D}}^{-s})$ are not easily accessible. Nevertheless, by relying on the resolvent family $A(P_{1,\mathrm{D}} - \lambda)^{-1}$, we managed to show in a joint work with Schrohe (Grubb and Schrohe, 2004), that $C_0(A, P_{1,\mathrm{D}}) - C_0(A, P_{2,D})$ and $C_0([A, A'], P_{1,\mathrm{D}})$ are *local*, and to pinpoint the nonlocal content of $C_0(A, P_{1,\mathrm{D}})$ modulo local terms. The question of possible generalizations of the formulas (0.2)–(0.3) remained open then.

In the present paper we show for the boundaryless case how the formulas (0.2)–(0.3) can be derived directly from the knowledge of the resolvent (Section 2). The crucial fact is that the constant comes from a strictly homogeneous term in the symbol of $A((P - \lambda)^{-1} - (P' - \lambda)^{-1})$, respectively $A[A', (P - \lambda)^{-1}]$ which is integrable at $\xi = 0$ (and is $O(\lambda^{-2})$ for $|\lambda| \to \infty$ when $\xi \neq 0$). The operator $\log P$ appears simply because $\log \lambda$ has a jump of $2\pi i$ at the negative real axis; there is no need to construct the P^{-s}.

Before this, we give (in Section 1) a similarly simple proof of the formula shown recently by Scott (2005),

$$C_0(I, P) = -\frac{1}{m}\operatorname{res}(\log P), \tag{0.4}$$

from which he draws consequences on multiplicative properties; here $C_0(I, P)$ equals $\zeta(P, 0)$ + the nullity of P. Scott's proof of (0.4) is based on calculations inspired from Okikiolu (1995), going via results for P^{-s}. In fact, finding the direct proof of (0.4) in terms of the resolvent was the starting point for our present paper.

Next, we discuss possible generalizations of the formulas (0.2)–(0.3) to the situation with boundary. Here we replace the auxiliary family $(P_{1,\mathrm{D}} - \lambda)^{-1}$ used in Grubb and Schrohe (2001, 2004) by its ψdo part $(P_1 - \lambda)_+^{-1}$ (which corresponds to replacing $P_{1,\mathrm{D}}^{-s}$ by $(P_1^{-s})_+$, another family which equals the identity for $s = 0$); this spares us of the technicalities involved in working with a boundary condition for P_1. On the other hand, we allow general higher order choices of the differential operator P_1, where Grubb and Schrohe (2001, 2004) considered the second-order principally scalar case (which provides simple roots in the detailed construction of the resolvent symbol). To handle general choices of P_1, we base the study on the relatively crude methods in the book by Grubb (1996).

In Section 3, we show that (0.2) does generalize in a natural way, since $(\log P_1 - \log P_2)_+$ is a zero order ψdo having the transmission property

$$C_0(A, P_{1,+}) - C_0(A, P_{2,+}) = -\frac{1}{m}\operatorname{res}(A(\log P_1 - \log P_2)_+). \tag{0.5}$$

Here we use the residue definition of Fedosov et al. (1996).

In Section 4, we consider generalizations of (0.3), for two operators $A = P_+ + G$, $A' = P'_+ + G'$ of orders σ and σ', and normal order zero. The leftover terms (singular Green type terms) coming from commutators $[A', (\log P_1)_+]$ are not in the calculus and have not (yet) been covered by residue formulas, so we cannot extend (0.3) directly. However, considering $A[A', (P_1 - \lambda)_+^{-1}]$, we show that the normal trace $\mathcal{S}_\lambda$ of its singular Green operator part $\mathcal{G}_\lambda$ is a ψdo on X' with sufficiently good symbol estimates to allow integration against $\log \lambda$, leading to a classical ψdo S on X' such that

$$C_0([A, A'], P_{1,+}) = -\frac{1}{m}\operatorname{res}_X((P[P', \log P_1])_+) - \frac{1}{m}\operatorname{res}_{X'}(S); \tag{0.6}$$

the right-hand side can be regarded as an interpretation of "$-\frac{1}{m}\operatorname{res}(A[A', (\log P_1)_+])$".

Finally, in Section 5, we show a certain generalization of (0.4) to normal elliptic pseudodifferential boundary problems $(P_+ + G)_T$ as considered in Grubb (1996), and include a remark on Atiyah-Patodi-Singer problems.

1. On the Residue of Logarithm Formula

Let P be an elliptic pseudodifferential operator of order $m \in \mathbb{R}_+$ acting on the sections of a hermitian vector bundle E over a closed (i.e., compact boundaryless) manifold X of dimension n, such that the principal symbol has no eigenvalues on $\mathbb{R}_-$. We can assume that P has no eigenvalues on $\mathbb{R}_-$ (by a small rotation if needed). Then we can define the resolvent $Q_\lambda = (P - \lambda)^{-1}$ in a sector V around $\mathbb{R}_-$. The complex powers and the logarithm are defined by functional calculus as

$$\begin{gathered} P^{-s} = \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-s}(P - \lambda)^{-1}\, d\lambda \qquad \text{for } \operatorname{Re} s > 0, \quad P^{k-s} = P^k P^{-s}; \\ \log P = \lim_{s\to 0} \frac{i}{2\pi}\int_{\mathscr{C}} \lambda^{-s}\log\lambda (P - \lambda)^{-1} d\lambda, \end{gathered} \tag{1.1}$$

with integrations on a curve $\mathscr{C}$ in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ going around the nonzero spectrum of P in the positive direction; hereby P^{-s} and $\log P$ are taken to be zero on $\ker P$.

It is well known that $\operatorname{Tr} P^{-s}$ extends meromorphically to $\mathbb{C}$ as the zeta function $\zeta(P, s)$ (Seeley, 1967); it is regular at $s = 0$. It is also known that the noncommutative residue can be defined for $\log P$ (Lesch, 1999; Okikiolu, 1995). The value at $s = 0$ was recently identified by Scott (2005) with a residue

$$\zeta(P, 0) = -\frac{1}{m}\operatorname{res}(\log P) \tag{1.2}$$

(this is the formula if $\ker P = 0$; also nonzero cases are considered). His method is based on an analysis of the symbol of P^{-s} inspired from Okikiolu (1995). For strongly elliptic *differential* operators, a formal version of the formula was established via heat operator and complex power considerations in Loya (2001b). We shall show below how the formula can be proven directly from the knowledge of the resolvent in a straightforward way.

We assume $m > n$ for convenience. (Otherwise, one can consider $(P - \lambda)^{-N}$ for large N, where the local formulas however boil down to the same calculation, as indicated in a general situation in Remark 3.12 below.) Then Q_λ is trace-class, and its kernel (calculated in local trivializations) has an asymptotic expansion for $\lambda \to \infty$ in V, leading to a trace expansion by integration of the fiber trace in x

$$\begin{aligned} K(Q_\lambda, x, x) &\sim \sum_{j\geq 0} c_j(x)(-\lambda)^{\frac{n-j}{m}-1} + \sum_{k\geq 1}(c'_k(x)\log(-\lambda) + c''_k(x))(-\lambda)^{-k-1}, \\ \operatorname{Tr} Q_\lambda &\sim \sum_{j\geq 0} c_j(-\lambda)^{\frac{n-j}{m}-1} + \sum_{k\geq 1}(c'_k \log(-\lambda) + c''_k)(-\lambda)^{-k-1}. \end{aligned} \tag{1.3}$$

This was first shown by Agranovich (1987) (with reference to the heat trace formulation of Duistermaat and Guillemin, 1975 and the complex power formulation of Seeley, 1967); proof details can also be found in Grubb and Seeley (1995) for the case where m is integer, and in Loya (2001a), Grubb and Hansen (2002) for the general case. In fact, the meromorphic structure of $\zeta(P, s)$ and the asymptotic expansion of $\operatorname{Tr} Q_\lambda$ can be deduced from one another (as accounted e.g. in Grubb and Seeley, 1996). In particular, we can define

$$C_0(P) = c_n = \int_X \operatorname{tr} c_n(x)dx; \qquad \text{then } C_0(P) = \zeta(P, 0) + \nu_0, \tag{1.4}$$

where ν_0 is the algebraic multiplicity of zero as an eigenvalue of P. For ν_0 equals the rank of the eigenprojection $\Pi_0 = \frac{i}{2\pi}\int_{|\lambda|=\varepsilon}(P - \lambda)^{-1}d\lambda$, cf. Kato (1996, Section III 6.8).

We shall base our study of $C_0(P)$ on the resolvent information, and will now recall an elementary deduction of the kernel expansion down to $O(|\lambda|^{-2+\varepsilon})$. In local trivializations, the symbol $q(x, \xi, \lambda)$ of Q_λ has an expansion in quasi-homogeneous terms $q(x, \xi, \lambda) \sim \sum_{j\geq 0} q_{-m-j}(x, \xi, \lambda)$, where $q_{-m} = (p_m - \lambda)^{-1}$, and q_{-m-j} for each $j \geq 1$ is a finite sum of terms with the structure

$$f(x, \xi, \lambda) = g_1 q_{-m}^{\nu_1} g_2 q_{-m}^{\nu_2} \cdots g_M q_{-m}^{\nu_M} g_{M+1}; \tag{1.5}$$

here the ν_k are integers ≥ 1 and the $g_k(x, \xi)$ are ψdo symbols independent of λ and homogeneous of degree r_k for $|\xi| \geq 1$. The index sums $r = \sum_{1\leq k\leq M+1} r_k$ and

$v = \sum_{1 \le k \le M} v_k$ satisfy

$$2 \le v \le 2j + 1, \qquad r = -j + (v - 1)m. \tag{1.6}$$

This is seen by working out the symbol construction in Seeley (1967) in detail (more information and references in Grubb, 1996, Remark 3.3.7). See also Remark 1.6 below. We indicate strictly homogeneous versions (the extensions by homogeneity into the region $|\xi| \le 1$) by an upper index h; the q^h_{-m-j} satisfy

$$q^h_{-m-j}(x, t\xi, t^m \lambda) = t^{-m-j} q^h_{-m-j}(x, \xi, \lambda) \qquad \text{for } t > 0, \quad \text{all } \xi \ne 0. \tag{1.7}$$

Note that in (1.5), f^h is $O(|\xi|^r)$ at $\xi = 0$ (for $\lambda \ne 0$), hence integrable in ξ at $\xi = 0$ if $r > -n$. Then in view of (1.6), q^h_{-m-j} is integrable at $\xi = 0$, when $j < n + m$ and $\lambda \ne 0$ (this is clear for $j = 0$, and for $j \ge 1$, the least integrable contributions are those with $v = 2$). In particular, q^h_{-m-n} is continuous in ξ.

The diagonal kernel $K(Q_\lambda, x, x)$ defined from q equals $\int_{\mathbb{R}^n} q(x, \xi, \lambda) đ\xi$ (where $đ$ stands for $(2\pi)^{-n} d$).

Lemma 1.1. *q has an expansion in strictly homogeneous terms plus a remainder*

$$q(x, \xi, \lambda) = \sum_{0 \le j < m+n} q^h_{-m-j}(x, \xi, \lambda) + q'_{-2m-n}(x, \xi, \lambda), \tag{1.8}$$

where the q^h_{-m-j} ($j < m + n$) and q'_{-2m-n} are integrable in ξ, and $\int q'_{-2m-n} đ\xi = O(|\lambda|^{-2+\varepsilon})$, any $\varepsilon > 0$. Consequently, $K(Q_\lambda, x, x)$ has the expansion

$$\begin{aligned} K(Q_\lambda, x, x) &= \sum_{0 \le j < m+n} c_j(x)(-\lambda)^{\frac{n-j}{m} - 1} + O(|\lambda|^{-2+\varepsilon}), \qquad \textit{where} \\ c_j(x) &= \int_{\mathbb{R}^n} q^h_{-m-j}(x, \xi, -1) đ\xi, \qquad \textit{for } j < m + n. \end{aligned} \tag{1.9}$$

Proof. For $j = 0$,

$$q_{-m} - q^h_{-m} = (p_m - \lambda)^{-1} - (p^h_m - \lambda)^{-1} = (p_m - \lambda)^{-1}(p^h_m - p_m)(p^h_m - \lambda)^{-1}, \tag{1.10}$$

so it is supported in $|\xi| \le 1$ and $O(|\lambda|^{-2})$ there. This also holds for $q_{-m-j} - q^h_{-m-j}$ for general $j \ge 1$ since $v \ge 2$ in (1.5). For $j < m + n$ the $q_{-m-j} - q^h_{-m-j}$ are integrable in ξ, the integrals being $O(\lambda^{-2})$. For the remainder $q - \sum_{j<m+n} q_{-m-n}$, write $m = m' + \delta$, m' integer and $\delta \in]0, 1]$, and note that $j < m + n$ means $j \le m' + n$. The symbol $q - \sum_{0 \le j < m+n} q_{-m-j}$ is of order $-m - m' - n - 1 = -2m - n + \delta - 1$ and satisfies

$$\begin{aligned} \Big| q - \sum_{j<m+n} q_{-m-j} \Big| &\le c(1 + |\xi|^m + |\lambda|)^{-2}(1 + |\xi|)^{-n+\delta-1} \\ &\le c'(1 + |\lambda|)^{-2+\varepsilon}(1 + |\xi|)^{-n+\delta-1-m\varepsilon}, \end{aligned} \tag{1.11}$$

any $\varepsilon \ge 0$. If $\delta < 1$ (the case where m is noninteger), we can take $\varepsilon = 0$, otherwise we take it small positive; then the integral in ξ is $O(|\lambda|^{-2+\varepsilon})$. This shows the statements on (1.8).

Now (1.9) follows directly by integration in ξ, using the calculations

$$\int_{\mathbb{R}^n} q^h_{-m-j}(x,\xi,\lambda)đ\xi = |\lambda|^{\frac{n-j}{m}-1}\int_{\mathbb{R}^n} q^h_{-m-j}(x,\eta,\lambda/|\lambda|)đ\eta. \tag{1.12}$$

For $\lambda \in \mathbb{R}_-$, they show that $c_j(x) = \int_{\mathbb{R}^n} q^h_{-m-j}(x,\xi,-1)đ\xi$; this remains valid on general rays in V since q^h_{-j-m} is holomorphic in λ (cf. e.g. Grubb and Seeley, 1995, Lemma 2.3). □

In the case $j = n$, we get in particular, when the contributions $c_n(x)$ are carried back to the manifold and collected:

$$C_0(P) = c_n = \int \operatorname{tr} c_n(x)dx, \qquad \text{where } c_n(x) = \int_{\mathbb{R}^n} q^h_{-m-n}(x,\xi,-1)đ\xi. \tag{1.13}$$

Now consider the operator $\log P$, (1.1). It is well known that it has a symbol in local coordinates (cf. e.g. Okikiolu, 1995)

$$\operatorname{symb}(\log P) = m\log[\xi]I + b(x,\xi), \tag{1.14}$$

where b is classical of order zero, and $[\xi]$ stands for a smooth positive function equal to $|\xi|$ for $|\xi| \geq 1$. This symbol is found termwise from the symbol of $Q_\lambda = (P-\lambda)^{-1}$ by Cauchy integral formulas as in (1.1); in particular,

$$b_{-n}(x,\xi) = \frac{i}{2\pi}\int_{\mathscr{C}'} \log\lambda q_{-m-n}(x,\xi,\lambda)d\lambda, \tag{1.15}$$

where $\mathscr{C}'$ is a closed curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ encircling the eigenvalues of p_m. According to the definition of noncommutative residues of operators with log-polyhomogeneous symbols (Lesch, 1999; Okikiolu, 1995),

$$\operatorname{res}(\log P) = \int_X \int_{|\xi|=1} \operatorname{tr} b_{-n}(x,\xi)đS(\xi)dx \tag{1.16}$$

(where the integral is known to have an invariant meaning). We want to show that this number equals $-mC_0(P)$. This will be based on a simple lemma.

Lemma 1.2. *Let $f(\lambda)$ be meromorphic on $\mathbb{C}$ and $O(\lambda^{-1-\varepsilon})$ for $|\lambda| \to \infty$ (some $\varepsilon > 0$), with poles lying in a bounded subset of $\mathbb{C}\setminus\overline{\mathbb{R}}_-$. Let $\mathscr{C}$ be a closed curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ encircling the poles in the positive direction. Then*

$$\frac{1}{2\pi i}\int_{\mathscr{C}} \log\lambda f(\lambda)d\lambda = \int_{-\infty}^0 f(t)dt. \tag{1.17}$$

The identity also holds if $f(\lambda)$ is holomorphic in a keyhole region around $\overline{\mathbb{R}}_-$:

$$V_{r,\theta} = \{\lambda \in \mathbb{C} \mid |\lambda| < r \text{ or } |\arg\lambda - \pi| < \theta\} \tag{1.18}$$

(r and $\theta > 0$), and $f(\lambda)$ is $O(\lambda^{-1-\varepsilon})$ for $\lambda \to \infty$ in $V_{r,\theta}$; then $\mathscr{C}$ should be a curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ going around $\complement V_{r,\theta}$ in the positive direction, e.g. defined as the boundary of $V_{r',\theta'}$ for some $r' \in]0,r[$, $\theta' \in]0,\theta[$.

Proof. We can replace $\mathscr{C}$ by the curve $\mathscr{C}_1 + \mathscr{C}_2 + \mathscr{C}_3 + \mathscr{C}_4$ in the complex plane cut-up along $\overline{\mathbb{R}}_-$, where (for a sufficiently large R)

$$\begin{aligned} \mathscr{C}_1 &= \{Re^{i\omega} \mid -\pi \le \omega \le \pi\}, \qquad \mathscr{C}_2 = \left\{se^{i\pi} \mid R \ge s \ge \frac{1}{R}\right\}, \\ \mathscr{C}_3 &= \left\{\frac{1}{R}e^{i\omega} \mid \pi \ge \omega \ge -\pi\right\}, \qquad \mathscr{C}_4 = \left\{se^{-i\pi} \mid \frac{1}{R} \le s \le R\right\}; \end{aligned} \tag{1.19}$$

we shall let $R \to \infty$. Here

$$\begin{aligned} \left| \int_{\mathscr{C}_1} \log \lambda f(\lambda) d\lambda \right| &= O(RR^{-1-\varepsilon} \log R) \to 0 \qquad \text{for } R \to \infty, \\ \left| \int_{\mathscr{C}_3} \log \lambda f(\lambda) d\lambda \right| &= O(R^{-1} \log R) \to 0 \qquad \text{for } R \to \infty; \end{aligned} \tag{1.20}$$

moreover,

$$\log \lambda = \log(se^{i\pi}) = \log s + i\pi \quad \text{on } \mathscr{C}_2 \quad \log \lambda = \log(se^{-i\pi}) = \log s - i\pi \quad \text{on } \mathscr{C}_4,$$

(the difference of the values of log λ from above and from below on $\mathbb{R}_-$ is $2\pi i$). Then

$$\frac{1}{2\pi i} \int_{\mathscr{C}} \log \lambda f(\lambda) d\lambda = \frac{1}{2\pi i} \int_{-R}^{-\frac{1}{R}} 2\pi i f(t) dt + O(R^{-\varepsilon} \log R) = \int_{-\infty}^{0} f(t) dt.$$

For the second statement, we can instead approximate $\mathscr{C}$ by $\mathscr{C}'_R = \mathscr{C}'_1 + \mathscr{C}_2 + \mathscr{C}_3 + \mathscr{C}_4 + \mathscr{C}'_5$, where $\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_4$ are as above, and

$$\mathscr{C}'_1 = \{Re^{i\omega} \mid \pi - \theta' \le \omega \le \pi\}, \qquad \mathscr{C}'_5 = \{Re^{i\omega} \mid -\pi \ge \omega \ge -\pi - \theta'\};$$

then we use that the integrals over $\mathscr{C}'_1$ and $\mathscr{C}'_5$ go to 0 for $R \to \infty$. □

At each x, we have the formula for $c_n(x)$ in (1.13), and the formula with b_{-n}:

$$\int_{|\xi|=1} b_{-n}(x, \xi) đS(\xi) = \int_{|\xi|=1} \frac{i}{2\pi} \int_{\mathscr{C}'} \log \lambda\, q^h_{-m-n}(x, \xi, \lambda) d\lambda đS(\xi), \tag{1.21}$$

so the identification of $C_0(P)$ and $-\frac{1}{m}\text{res}(\log P)$ will be obtained if we show that for each x,

$$\int_{\mathbb{R}^n} q^h_{-m-n}(x, \xi, -1) đ\xi = -\frac{1}{m} \int_{|\xi|=1} \frac{i}{2\pi} \int_{\mathscr{C}'} \log \lambda q^h_{-m-n}(x, \xi, \lambda) d\lambda đS(\xi). \tag{1.22}$$

We transform the left-hand side by use of the quasi-homogeneity (1.7). For later reference, the calculation will be formulated in

Lemma 1.3. *Let $m > 0$. Let $f(\xi, t)$ be continuous for $(\xi, t) \in (\mathbb{R}^n \setminus \{0\}) \times \overline{\mathbb{R}}_-$ and quasi-homogeneous there in the sense that $f(s\xi, s^m t) = s^{-m-n} f(\xi, t)$ for all $s > 0$, and integrable at $\xi = 0$ for each $t \ne 0$. Then*

$$\int_{\mathbb{R}^n} f(\xi, -1) đ\xi = \frac{1}{m} \int_{|\xi|=1} \int_{-\infty}^{0} f(\xi, t) dt\, đS(\xi). \tag{1.23}$$

Proof. Since $|f(\xi,-1)| = |\xi|^{-m-n}|f(\xi/|\xi|, -|\xi|^{-m})|$ is $O(|\xi|^{-m-n})$ for $|\xi|\to\infty$, the function in the left-hand side is integrable. For $|\xi|=1$, we make a calculation using the coordinate change $t=-r^{-m}$, $dt = mr^{-m-1}dr$

$$\int_0^\infty f(r\xi,-1)r^{n-1}dr = \int_0^\infty f(\xi,-r^{-m})r^{-m-1}dr = \frac{1}{m}\int_{-\infty}^0 f(\xi,t)dt, \tag{1.24}$$

which gives

$$\begin{aligned}\int_{\mathbb{R}^n} f(\xi,-1)đ\xi &= \int_{|\eta|=1}\int_0^\infty f(r\eta,-1)r^{n-1}dr\,đS(\eta)\\ &= \frac{1}{m}\int_{|\eta|=1}\int_{-\infty}^0 f(\eta,t)dt\,đS(\eta),\end{aligned}$$

showing (1.23). (We are using the Fubini theorem; in fact (1.24) is valid almost everywhere with respect to $\xi\in S^{n-1}$.) □

Now (1.22) follows by application of (1.23) to $q^h_{-m-n}(x,\xi,t)$ at each x and application of Lemma 1.2 to $\int_{-\infty}^0 q^h_{-m-n}(x,\xi,t)dt$ (the minus comes from replacing $\frac{1}{2\pi i}$ by $\frac{i}{2\pi}$). Integration in x of the fiber trace then gives the desired identity (0.4).

We have shown

Theorem 1.4. *$C_0(P)$ equals $-\frac{1}{m}\operatorname{res}(\log P)$, and this holds pointwise, in that*

$$C_0(P) = \int_X \operatorname{tr} c_n(x)dx = -\frac{1}{m}\operatorname{res}(\log P), \tag{1.25}$$

where, for each x, in local coordinates,

$$c_n(x) = \int_{\mathbb{R}^n} q^h_{-m-n}(x,\xi,-1)đ\xi = -\frac{1}{m}\int_{|\xi|=1} b_{-n}(x,\xi)đS(\xi). \tag{1.26}$$

Remark 1.5. In this application of Lemma 1.3, $f(\xi,t) = q^h_{-m-n}(x,\xi,t)$ is not only integrable at $\xi=0$ but continuous there, for $t\neq 0$. Then for any $\xi\in S^{n-1}$, $f(\xi,t) = |t|^{-1-n/m}f(|t|^{-1/m}\xi,-1)$, where $f(|t|^{-1/m}\xi,-1)\to f(0,-1)$ for $t\to-\infty$, assuring that the integrals in (1.24) exist. We can then say that the identification of the contributions from $q^h_{-m-n}(x,\xi,-1)$ and $-\frac{1}{m}b_{-n}(x,\xi)$ holds *on each ray* $\{s\xi \mid s\geq 0\}$, $\xi\in S^{n-1}$ (holds microlocally in this sense).

Remark 1.6. The bounds on the q^h_{-m-j} $(j<m+n)$ and the remainder q'_{-2m-n} could also be inferred from the fact that q has 'regularity' m, cf. the general rules for regularity numbers of parameter-dependent symbols introduced in Grubb (1996), instead of from the explicit formulas around (1.5). In fact, writing $-\lambda = \mu^m e^{i\theta}$, $\mu\geq 0$ (on each relevant ray with argument θ), we have that $p(x,\xi)+\mu^m e^{i\theta}$ is parameter-elliptic of order m and regularity m, with parametrix symbol $q(x,\xi,\theta,\mu)$ of order $-m$ and regularity m, cf. Grubb (1996, Section 1.5, (2.1.13), Theorem 2.1.22). This assures the desired estimates, as accounted for in Grubb (1996, Theorem 3.3.5), and its proof. The proofs in Section 2 below could also be phrased in terms of the calculus in Grubb (1996). We use it effectively in Sections 3–5 concerned with boundary operators; some basic facts are recalled in the beginning of Section 3.

2. The Trace Defect Formulas for Closed Manifolds

Let A be a classical pseudodifferential operator of order $\sigma \in \mathbb{R}$, and let P be as in the preceding section; we now assume for convenience that $m > n + \sigma$.

It was shown in Grubb and Seeley (1995, Theorem 2.7) (m integer > 0) and Loya (2001a), and Grubb and Hansen (2002) ($m \in \mathbb{R}_+$), that the kernel of $A(P-\lambda)^{-1}$ calculated in local trivializations has an expansion on the diagonal, implying a trace expansion by integration of the fiber trace in x:

$$\begin{aligned} K(A(P-\lambda)^{-1}, x, x) &\sim \sum_{j\geq 0} c_j(x)(-\lambda)^{\frac{\sigma+n-j}{m}-1} + \sum_{k\geq 0}(c'_k(x)\log(-\lambda) + c''_k(x))(-\lambda)^{-k-1}, \\ \operatorname{Tr}(A(P-\lambda)^{-1}) &\sim \sum_{j\geq 0} c_j(-\lambda)^{\frac{\sigma+n-j}{m}-1} + \sum_{k\geq 0}(c'_k\log(-\lambda) + c''_k)(-\lambda)^{-k-1}. \end{aligned} \tag{2.1}$$

Here $\lambda \to \infty$ on rays in an open subsector V of $\mathbb{C}$ containing $\mathbb{R}_-$. It is convenient to assume that the operators are represented, via local coordinate systems, as a finite sum of pieces acting separately in a system of disjoint open sets in $\mathbb{R}^n$ (as e.g. in Grubb, 2005, Section 1), so that we get the trace simply by integrating over $\mathbb{R}^n$.

The $c'_k(x)$ vanish when $\sigma \notin \mathbb{Z}$. We shall *define*

$$c_{n+\sigma}(x) = 0, \qquad c_{n+\sigma} = 0, \qquad \text{if } n + \sigma \notin \mathbb{N}; \tag{2.2}$$

then $c_{n+\sigma}(x)$ and $c_{n+\sigma}$ have a meaning for any σ. (We denote $\{0, 1, 2, \dots\} = \mathbb{N}$.)

The coefficient of $(-\lambda)^{-1}$ in (2.1) will be denoted $C_0(A, P)$

$$C_0(A, P) = c_{\sigma+n} + c''_0. \tag{2.3}$$

Corresponding to (2.1), the generalized zeta function $\zeta(A, P, s)$, defined as $\operatorname{Tr}(AP^{-s})$ for large $\operatorname{Re} s$, has a meromorphic extension to $\mathbb{C}$ with poles at the points $(j-n)/m$, with Laurent coefficients directly related to the coefficients in the expansion (2.1). In particular, $C_0(A, P)$ equals the coefficient of s^0 plus $\operatorname{Tr}(A\Pi_0)$, cf. (1.4)ff.

It is well known that $C_0(A, P)$ is in general nonlocal in the sense that it depends on the full structure of A, not just its homogeneous symbols. However, when A' and P' are another pair of similar operators, one can show that

$$C_0(A, P) - C_0(A, P') \text{ and } C_0([A, A'], P) \text{ are local} \tag{2.4}$$

(depend on a finite set of strictly homogeneous symbol terms of A, A', P and P'); in this sense, $C_0(A, P)$ is a *quasi-trace* on the classical ψdo's A. $C_0(A, P)$ is called a regularized trace or weighted trace by other authors. Explicit formulas for the trace defects in (2.4) were shown by Okikiolu (1995), Kontsevich and Vishik (1995), and Melrose and Nistor (1996):

$$C_0(A, P) - C_0(A, P') = -\frac{1}{m}\operatorname{res}(A(\log P - \log P')), \tag{2.5}$$

$$C_0([A, A'], P) = -\frac{1}{m}\operatorname{res}(A[A', \log P]). \tag{2.6}$$

Here Okikiolu (1995) proved (2.5) by an exact symbol calculation passing via the symbols of the complex powers P^{-s} and $(P')^{-s}$, and Kontsevich and Vishik (1995) proved it by use of their calculus of weakly holomorphic ψdo families. Melrose and Nistor (1996) showed both (2.5) and (2.6) on the basis of the theorem of Guillemin on holomorphic families (Guillemin, 1985) (we have reconstructed a proof based on this idea in Grubb, 2005, pf. of Proposition 3.1). In all these cases, the logarithm $\log P$ comes up as a result of a differentiation of P^{-s} with respect to s.

Our present aim is to show how the formulas (2.5)–(2.6) can be found directly from the knowledge of the resolvent expression $A(P-\lambda)^{-1}$, without worrying about the construction of P^{-s}. (This is important for generalizations to other types of manifolds.) We show that in fact the full operator $\log P$ plays a very minor role; its symbol comes in only because of the jump across the negative real axis as in Lemma 1.2.

Let P and P' be auxiliary operators of order m with resolvents $Q_\lambda = (P-\lambda)^{-1}$, $Q'_\lambda = (P'-\lambda)^{-1}$ (symbols q, respectively q'), and consider the symbol $s(x, \xi, \lambda)$ of

$$S_\lambda = A(Q_\lambda - Q'_\lambda) \tag{2.7}$$

in local coordinates. Much as in Lemma 1.1, we can show

Proposition 2.1. *The symbol s of $S_\lambda = A(Q_\lambda - Q'_\lambda)$ has an expansion in strictly homogeneous terms plus a remainder*:

$$s(x, \xi, \lambda) = \sum_{0\le j<\sigma+m+n} s^h_{-m-j}(x, \xi, \lambda) + s'_{-2m-n}(x, \xi, \lambda), \tag{2.8}$$

where the $s^h_{\sigma-m-j}$ and s'_{-2m-n} are integrable in ξ for $\lambda \neq 0$, and $\int s'_{-2m-n} đ\xi$ is $O(|\lambda|^{-2+\varepsilon})$, any $\varepsilon > 0$. Consequently, $K(S_\lambda, x, x)$ and the trace $\operatorname{Tr} S_\lambda$ have the expansions

$$\begin{aligned} K(S_\lambda, x, x) &= \sum_{j<\sigma+m+n} \tilde{s}_j(x)(-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(|\lambda|^{-2+\varepsilon}), \\ \operatorname{Tr} S_\lambda &= \sum_{j<\sigma+m+n} \tilde{s}_j(-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(|\lambda|^{-2+\varepsilon}), \quad \text{where} \\ \tilde{s}_j(x) = \int_{\mathbb{R}^n} s^h_{-m-j}&(x, \xi, -1) đ\xi, \quad \tilde{s}_j = \int \operatorname{tr} \tilde{s}_j(x) dx, \quad \text{for } j < \sigma+m+n. \end{aligned} \tag{2.9}$$

In particular, when $n+\sigma \notin \mathbb{N}$, there is no term with $(-\lambda)^{-1}$ in the expansion of $\operatorname{Tr} S_\lambda$, and

$$C_0(A, P) - C_0(A, P') = 0. \tag{2.10}$$

When $n+\sigma \in \mathbb{N}$, the coefficient of $(-\lambda)^{-1}$ in $\operatorname{Tr} S_\lambda$ equals

$$C_0(A, P) - C_0(A, P') = \int \operatorname{tr} \tilde{s}_{n+\sigma}(x) dx. \tag{2.11}$$

Proof. We use again the analysis of the resolvent symbol recalled in Section 1. The composition with A in front leads to terms of the form (1.5), where the λ-independent coefficients now furthermore contain information from the symbol a of A. Consider

$$q - q' \sim q_{-m}(x, \xi, \lambda) - q'_{-m}(x, \xi, \lambda) + \sum_{j\ge1}(q_{-m-j}(x, \xi, \lambda) - q'_{-m-j}(x, \xi, \lambda)).$$

All the terms in the sum over $j \geq 1$ are finite sums of expressions as in (1.5), containing at least two principal resolvent factors q_{-m}, respectively q'_{-m}. Moreover,

$$
\begin{aligned}
q_{-m} - q'_{-m} &= (p_m - \lambda)^{-1} - (p'_m - \lambda)^{-1} = (p_m - \lambda)^{-1}(p'_m - p_m)(p'_m - \lambda)^{-1} \\
&= q_{-m}(p'_m - p_m)q'_{-m},
\end{aligned}
$$

showing that it also contains two principal resolvent factors (q_{-m} and q'_{-m}) together with a λ-independent factor. Then an application of the standard composition rule gives that the homogeneous terms in the symbol $s = a \circ (q - q')$ of S_λ are finite sums of expressions that are a slightly generalized version of (1.5), where some of the factors q_{-m} may be replaced by q'_{-m}. The important observation is that there are at least two such factors in each term. Then, taking the order and homogeneity degrees into account, we see that $s(x, \xi, \lambda) \sim \sum_{j \geq 0} s_{\sigma-m-j}(x, \xi, \lambda)$ satisfies

$$
\begin{aligned}
&|s^h_{\sigma-m-j}| \leq c(|\xi|^m + |\lambda|)^{-2}|\xi|^{\sigma+m-j}, \\
&|s^h_{\sigma-m-j} - s_{\sigma-m-j}| \leq c|\lambda|^{-2}(1 + |\xi|^{\sigma+m-j}), \quad \text{supported in } |\xi| \leq 1, \quad \text{any } j, \\
&\Big| s - \sum_{j<N} s_{\sigma-m-j} \Big| \leq c(1 + |\xi|^m + |\lambda|)^{-2}(1 + |\xi|)^{\sigma+m-N}, \qquad \text{any } N.
\end{aligned} \tag{2.12}
$$

For $j < \sigma + m + n$, the first two expressions are integrable in ξ. The remainder $s - \sum_{j<\sigma+m+n} s_{\sigma-m-j}$ is seen as in the treatment of (1.11) to be $O((1 + |\lambda|)^{-2+\varepsilon}(1 + |\xi|)^{-n-\delta'})$ with $\delta' > 0$ and ε arbitrarily small, here ε can be taken $= 0$ if $\sigma + m \notin \mathbb{Z}$. This shows the first part of the lemma, and the second part follows by integration, first in ξ and then (for the fiber trace) in x.

For the third part, observe that there is no term $c(-\lambda)^{-1}$ in (2.9) when $n + \sigma \notin \mathbb{N}$. When $n + \sigma \in \mathbb{N}$, the coefficient of $(-\lambda)^{-1}$ is found from (2.9) for $j = n + \sigma$. □

Note that all the indicated coefficients are local, and that there is no $(-\lambda)^{-1}\log(-\lambda)$ term as in (2.1).

We can now show (2.5) in a precise form, by a calculation as in Section 1. For this we consider

$$
F = A(\log P - \log P').
$$

Since the logarithmic terms in the symbols of $\log P$ and $\log P'$ cancel out (cf. (1.14)), it is a classical ψdo of order σ; we denote its symbol by $f(x, \xi)$. When we define F by the formula

$$
\begin{aligned}
F = A(\log P - \log P') &= A \lim_{s \to 0} \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} \log \lambda ((P - \lambda)^{-1} - (P' - \lambda)^{-1}) d\lambda \\
&= \frac{i}{2\pi} \int_{\mathscr{C}} \log \lambda S_\lambda \, d\lambda,
\end{aligned} \tag{2.13}
$$

then in local coordinates, its symbol is found termwise from the symbol of S_λ by the formulas

$$
f_{\sigma-j}(x, \xi) = \frac{i}{2\pi} \int_{\mathscr{C}'} \log \lambda s_{\sigma-m-j}(x, \xi, \lambda) d\lambda, \tag{2.14}
$$

where $\mathscr{C}'$ is a closed curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ encircling the eigenvalues of p_m and p'_m. This follows from the calculations of the terms in $\log P$ and $\log P'$ described e.g., in Okikiolu (1995), and the composition rule for ψdo's.

When $n+\sigma \notin \mathbb{N}$, there is no term of degree $-n$, so the noncommutative residue of F is zero. When $n+\sigma \in \mathbb{N}$, it is determined by

$$\operatorname{res} F = \int_X \int_{|\xi|=1} \operatorname{tr} f_{-n}(x,\xi)\,đS(\xi)\,dx. \tag{2.15}$$

Theorem 2.2. *Let P and P' be classical ψdo's of order $m>0$ and such that the principal symbol has no eigenvalues on $\overline{\mathbb{R}}_-$, let A be a classical ψdo of order σ, and let $S_\lambda = A((P-\lambda)^{-1}-(P'-\lambda)^{-1})$ and $F = A(\log P - \log P')$ with symbols s resp. f. Assume that $m > n+\sigma$.*

Consider the case $n+\sigma \in \mathbb{N}$. The formula (2.5) is valid, and it holds pointwise, in that

$$C_0(A,P) - C_0(A,P') = \int_X \operatorname{tr} \tilde{s}_{n+\sigma}(x)\,dx = -\frac{1}{m}\operatorname{res}(A(\log P - \log P')) \tag{2.16}$$

where, for each x, in local coordinates,

$$\tilde{s}_{n+\sigma}(x) = \int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,-1)\,đ\xi = -\frac{1}{m}\int_{|\xi|=1} f_{-n}(x,\xi)\,đS(\xi). \tag{2.17}$$

When $n+\sigma \notin \mathbb{N}$, the identities hold trivially (with zero values everywhere).

Proof. The proof consists of rewriting $\int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,-1)\,đ\xi$ in the same way as we did with the integral of q^h_{-m-n} in Section 1:

$$\begin{aligned}\int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,-1)\,đ\xi &= -\frac{1}{m}\int_{|\xi|=1} \frac{i}{2\pi}\int_{\mathscr{C}'} \log\lambda\, s^h_{-m-n}(x,\xi,\lambda)\,d\lambda\, đS(\xi)\\ &= -\frac{1}{m}\int_{|\xi|=1} f_{-n}(x,\xi)\,đS(\xi),\end{aligned} \tag{2.18}$$

where the first equation follows from Lemmas 1.2 and 1.3, and the second equation follows from (2.14). □

There is a related proof of the other trace defect formula, (2.6). Here we consider A of order σ, A' of order σ' and P as before, now assuming for convenience that $\sigma+\sigma'+m > n$.

Here we first observe that by cyclic permutation,

$$\begin{aligned}&\operatorname{Tr}([A,A']Q_\lambda) = \operatorname{Tr}(AA'Q_\lambda) - \operatorname{Tr}(AQ_\lambda A') = \operatorname{Tr}(A[A',Q_\lambda]), \qquad \text{where}\\ &A[A',Q_\lambda] = A(Q_\lambda(P-\lambda)A'Q_\lambda - Q_\lambda A'(P-\lambda)Q_\lambda) = AQ_\lambda[P,A']Q_\lambda.\end{aligned} \tag{2.19}$$

Let

$$T_\lambda = [A,A']Q_\lambda, \tag{2.20}$$

$$R_\lambda = A[A',Q_\lambda] = AQ_\lambda[P,A']Q_\lambda. \tag{2.21}$$

The traces of T_λ and R_λ are identical, and the operators both have order $\sigma + \sigma' - m$. It is seen from the second formula for R_λ that the homogeneous terms $r_{\sigma+\sigma'-m-j}$ in its symbol r are finite sums of terms of the form (1.5) with at least two factors q_{-m}, so that the strictly homogeneous symbols $r^h_{\sigma+\sigma'-m-j}$ are integrable in ξ at $\xi = 0$ for $j < \sigma + \sigma' + m + n$.

We then find, very similarly to the study of S_λ, that the diagonal kernel of R_λ has an expansion

$$K(R_\lambda, x, x) = \sum_{j<\sigma+\sigma'+m+n} \tilde{r}_j(x)(-\lambda)^{\frac{n+\sigma+\sigma'-j}{m}-1} + O(|\lambda|^{-2+\varepsilon}), \tag{2.22}$$

where

$$\tilde{r}_j(x) = \int_{\mathbb{R}^n} r^h_{\sigma+\sigma'-m-j}(x, \xi, -1)\,đ\xi. \tag{2.23}$$

When $n + \sigma + \sigma' \notin \mathbb{N}$, there is no term $c(-\lambda)^{-1}$ in (2.22), hence no such term in the trace expansion of R_λ. Since this is the same as that of T_λ, the term is also missing from $\operatorname{Tr} T_\lambda$, so $C_0([A, A'], P) = 0$. When $n + \sigma + \sigma' \in \mathbb{N}$, the coefficient of $(-\lambda)^{-1}$ in (2.22) equals (2.23) with $j = n + \sigma + \sigma'$, i.e.,

$$\tilde{r}_{n+\sigma+\sigma'}(x) = \int_{\mathbb{R}^n} r^h_{-m-n}(x, \xi, -1)\,đ\xi. \tag{2.24}$$

Then the coefficient of $(-\lambda)^{-1}$ in the expansion of $\operatorname{Tr} R_\lambda$ equals the integral in x of the fiber trace of this (collecting the contributions from local coordinate systems), and since $\operatorname{Tr} T_\lambda$ has the same expansion, we can conclude that

$$C_0([A, A'], P) = \int_X \operatorname{tr} \tilde{r}_{n+\sigma+\sigma'}(x)\,dx. \tag{2.25}$$

On the other hand, we consider $H = A[A', \log P]$, observing that it is a classical ψdo of order $\sigma + \sigma'$ in view of (1.14). Here,

$$\begin{aligned} H &= A(A' \log P - \log P A') = A \lim_{s\to 0} \frac{i}{2\pi} \int_{\mathscr{C}} \lambda^{-s} \log \lambda (A' Q_\lambda - Q_\lambda A')\,d\lambda \\ &= \frac{i}{2\pi} \int_{\mathscr{C}} \log \lambda R_\lambda\, d\lambda. \end{aligned} \tag{2.26}$$

The symbol $h(x, \xi)$ is found termwise in local coordinates from the symbol of R_λ by the formulas

$$h_{\sigma+\sigma'-j}(x, \xi) = \frac{i}{2\pi} \int_{\mathscr{C}'} \log \lambda\, r_{\sigma+\sigma'-m-j}(x, \xi, \lambda)\,d\lambda, \tag{2.27}$$

where $\mathscr{C}'$ is a closed curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ encircling the eigenvalues of p_m.

When $n + \sigma + \sigma' \notin \mathbb{N}$, there is no term of degree $-n$, so the noncommutative residue of H is zero. When $n + \sigma + \sigma' \in \mathbb{N}$, it is determined by

$$\operatorname{res} H = \int_X \int_{|\xi|=1} \operatorname{tr} h_{-n}(x, \xi)\,đS(\xi)\,dx. \tag{2.28}$$

We then get

Theorem 2.3. *With P and A as in Theorem 2.2, let A' be a classical ψdo of order σ', and let $R_\lambda = A[A', (P-\lambda)^{-1}]$ and $H = A[A', \log P]$ with symbols r resp. h. Assume that $m > n + \sigma + \sigma'$.*

Let $n + \sigma + \sigma' \in \mathbb{N}$. The formula (2.6) *is valid, and it holds pointwise, in that*

$$C_0([A, A'], P) = \int_X \operatorname{tr} \tilde{r}_{n+\sigma+\sigma'}(x)dx = -\frac{1}{m}\operatorname{res}(A[A', \log P]) \tag{2.29}$$

where, for each x, in local coordinates,

$$\tilde{r}_{n+\sigma+\sigma'}(x) = \int_{\mathbb{R}^n} r^h_{-m-n}(x, \xi, -1) đ\xi = -\frac{1}{m}\int_{|\xi|=1} h_{-n}(x, \xi) đS(\xi). \tag{2.30}$$

When $n + \sigma + \sigma' \notin \mathbb{N}$, the identities hold trivially (*with zero values everywhere*).

Proof. The identity follows from (2.27) together with Lemmas 1.2 and 1.3, in the same way as in Theorem 2.2. □

Remark 2.4. The observation in Remark 1.5 on the microlocal identification extends to the formulas in Theorems 2.2 and 2.3.

3. The First Trace Defect Formula for Manifolds with Boundary

We shall now discuss extensions of the above results to pseudodifferential boundary operatos (ψdbo's) of Boutet de Monvel's type in the case of manifolds with boundary.

Consider a compact n-dimensional C^∞ manifold X with boundary $\partial X = X'$, and a hermitian C^∞ vector bundle E over X. Let $A = P_+ + G$ be an operator of order σ belonging to the calculus of Boutet de Monvel (1971), acting on sections of E. Here P is a classical ψdo satisfying the transmission condition at ∂X and G is a *singular Green operator* (*s.g.o.*) of class zero with polyhomogeneous symbol. (More details can be found e.g., in Boutet de Monvel, 1971; Grubb, 1996). When $P \neq 0$, we must assume $\sigma \in \mathbb{Z}$ because of the requirements of the transmission condition; when $P = 0$, it is straightforward to allow $\sigma \in \mathbb{R}$. For the results in Section 4, P is moreover assumed to be of normal order ≤ 0 (its symbol is bounded in ξ_n, the boundary conormal variable).

As auxiliary operator we take an elliptic *differential operator* P_1 of order $m > 0$ whose principal symbol has no eigenvalues on $\mathbb{R}_-$; so $(p_{1,m}(x, \xi) - \lambda)^{-1}$ is defined for λ in a sector V around $\mathbb{R}_-$, for all x, all ξ with $|\xi| + |\lambda| \neq 0$. P_1 can be assumed to be given on a larger boundaryless n-dimensional compact manifold $\widetilde{X}$ in which X is smoothly imbedded, acting in a bundle $\widetilde{E}$ extending E and with the same ellipticity properties there. We set

$$Q_\lambda = (P_1 - \lambda)^{-1} \tag{3.1}$$

on $\widetilde{X}$; it is defined except for a discrete subset of $\mathbb{C}$; in particular, it exists for large λ in the sector V.

For the case where $m = 2$ and P_1 is strongly elliptic with scalar principal symbol, defining the Dirichlet realization $P_{1,\mathrm{D}}$, we showed in a joint work with Schrohe (Grubb and Schrohe, 2001) that there is a resolvent trace expansion, when $N > (\sigma + n)/2$:

$$\operatorname{Tr}(A(P_{1,\mathrm{D}} - \lambda)^{-N}) \sim \sum_{j\geq 0} \tilde{c}_j(-\lambda)^{\frac{n+\sigma-j}{2}-N} + \sum_{k\geq 0}(\tilde{c}'_k \log(-\lambda) + \tilde{c}''_k)(-\lambda)^{-\frac{k}{2}-N}, \qquad (3.2)$$

valid for $\lambda \to \infty$ in V. It was used there to show that the coefficient $\tilde{c}'_0$ is proportional to the noncommutative residue of A, as introduced by Fedosov et al. (1996).

The proofs in Grubb and Schrohe (2001) were formulated only for $\sigma \in \mathbb{Z}$; but for more general $\sigma \in \mathbb{R}$, they carry over without difficulty to the case $A = G$. In particular, if $\sigma \in \mathbb{R}\setminus\mathbb{Z}$, the coefficients $\tilde{c}'_k$ vanish (since the ψdo's on the boundary obtained by reduction of $A(P_{1,\mathrm{D}} - \lambda)^{-N}$ are polyhomogeneous of noninteger order). The identification of $\tilde{c}'_0$ with a noncommutative residue then holds with

$$\operatorname{res}(G) = 0, \qquad \text{when } \sigma \notin \mathbb{Z}. \qquad (3.3)$$

As usual, we define

$$C_0(A, P_{1,\mathrm{D}}) = \tilde{c}_{n+\sigma} + \tilde{c}''_0,$$

where $\tilde{c}_{n+\sigma}$ is defined to be zero if $n + \sigma \notin \mathbb{N}$. By a precise analysis of the terms entering in trace expansions like (3.2), we showed in Grubb and Schrohe (2004) that the functional $C_0(A, P_{1,\mathrm{D}})$ has quasi-trace properties as in (2.4); moreover, we singled out some cases where it has a value independent of the auxiliary operator P_1 and vanishes on commutators, so that it can be regarded as a canonical trace in a similar sense as that of Kontsevich and Vishik (1995).

It is shown in Grubb and Schrohe (2004) that the singular Green part $G^{(N)}_\lambda$ of $(P_{1,\mathrm{D}} - \lambda)^{-N} = (Q^N_\lambda)_+ + G^{(N)}_\lambda$ contributes only locally to $C_0(A, P_{1,\mathrm{D}})$. It has an interest to consider the composition $A(Q^N_\lambda)_+$ alone; it likewise has an expansion

$$\operatorname{Tr}(A(Q^N_\lambda)_+) \sim \sum_{j\geq 0} \tilde{a}_j(-\lambda)^{\frac{n+\sigma-j}{2}-N} + \sum_{k\geq 0}(\tilde{a}'_k \log(-\lambda) + \tilde{a}''_k)(-\lambda)^{-\frac{k}{2}-N}, \qquad (3.4)$$

where $\tilde{a}'_0 = \frac{1}{m}\operatorname{res}(A)$, and the coefficient of $(-\lambda)^{-N}$,

$$C_0(A, P_{1,+}) = \tilde{a}_{n+\sigma} + \tilde{a}''_0 \qquad (3.5)$$

is a quasi-trace on the ψdbo's (by the results of Grubb and Schrohe, 2004).

One may remark that in an associated zeta function formulation, the consideration of $(Q^N_\lambda)_+$ alone corresponds to considering compositions with $(P_1^{-s})_+$ alone, where $(P_1^{-s})_+$ is another family of operators than $(P_{1,\mathrm{D}})^{-s}$; both families have the property that they equal I when $s = 0$.

But actually these complex powers lie outside the Boutet de Monvel calculus (when $s \notin \mathbb{Z}$). There is a description in Grubb (1996, Section 4.4) of negative powers ($\operatorname{Re} s > 0$), showing how the *s.g.o.* part satisfies some but not all the standard estimates. But they have not, to our knowledge, been successfully described as a holomorphic family in some sense where results like that of Guillemin (1985, Theorem 7.1), for closed manifolds could be applied to generalize the trace defect formulas (2.5)–(2.6). (The use of Guillemin's result is explained e.g., in Grubb, 2005, pf. of Proposition 3.1.)

Even if one avoids dealing with complex powers, there is still a problem in generalizing the formulas (2.5)–(2.6) that logarithms of ψdbo's have not been studied, and do not in general belong to the Boutet de Monvel calculus. However, $(\log P_1 - \log P_2)_+$ does belong there when P_1 and P_2 are two choices of the auxiliary elliptic operator (of order m), thanks to the cancellation of logarithms resulting from (1.14). But $[A', (\log P_1)_+]$ does not so, except in trivial cases.

We shall show a generalization of (2.5) in this section, and treat (2.6) in the following section.

The papers Grubb and Schrohe (2001, 2004) used the refined calculus of Grubb and Seeley (1995), which allows obtaining complete trace expansions (with remainders $O(\lambda^{-M})$, any M).

Presently, we shall use the cruder (but more generally applicable) calculus from Grubb (1996) to achieve our result, building also on the insight gained in Sections 1 and 2. Notably, we are avoiding some technical challenges by restricting the attention to the trace of $AQ_{\lambda,+}$, without an *s.g.o.* term AG_λ coming from a boundary condition on P_1.

An advantage is that we can allow rather general auxiliary operators P_1 of higher order, with no conditions on root multiplicities in the principal symbol. (In Grubb and Schrohe, 2001, 2004, the order 2 and scalarity assured well separated roots in ξ_n, one in each complex half-plane.) On the other hand, the theory we presently use gives trace expansions with a finite number of terms only (plus a remainder); but this turns out to be just sufficient for studying the trace defect formulas.

Let us first recall some elements of the theory. As usual, $\langle\xi\rangle$ stands for $(1+|\xi|^2)^{\frac{1}{2}}$; moreover, it is convenient to denote $\langle(\xi,\mu)\rangle = \langle\xi,\mu\rangle$ and use the sign $\dot{\leq}$ as shorthand for "$\leq$ a constant times".

A ψdo symbol $s(x,\xi,\mu)$ on $\mathbb{R}^n$ depending on the parameter $\mu \in \overline{\mathbb{R}}_+$ is said to be *of order d and regularity ν* ($d, \nu \in \mathbb{R}$) with uniform estimates (Grubb, 1996, Definition 2.1.1), when it satifies, for all indices α, β, j

$$|D_x^\beta D_\xi^\alpha D_\mu^j s(x,\xi,\mu)| \leq c(\langle\xi\rangle^{\nu-|\alpha|} + \langle\xi,\mu\rangle^{\nu-|\alpha|})\langle\xi,\mu\rangle^{d-\nu-j} \tag{3.6}$$

for $(x,\xi,\mu) \in \mathbb{R}^{2n} \times \overline{\mathbb{R}}_+$, with constants c depending on the indices. It is then said to be *polyhomogeneous*, when it furthermore has an expansion $s(x,\xi,\mu) \sim \sum_{l\in\mathbb{N}} s_{d-l}(x,\xi,\mu)$ in terms s_{d-l} that are homogeneous in (ξ,μ) of degree $d-l$ for $|\xi| \geq 1$, such that $s - \sum_{l<M} s_{d-l}$ is of order $d-M$ and regularity $\nu - M$, for all $M \in \mathbb{N}$. Note that in (3.6), $\langle\xi\rangle^{\nu-|\alpha|}$ can be left out when $|\alpha| \leq \nu$, and $\langle\xi,\mu\rangle^{\nu-|\alpha|}$ can be left out when $|\alpha| \geq \nu$.

For such symbols we have

Lemma 3.1. *Let $s(x,\xi,\mu)$ be polyhomogeneous of order d and regularity ν, d and $\nu \in \mathbb{R}$. Write $\nu = \nu' + \delta$ with ν' integer and $\delta \in]0,1]$. Then*

$$\begin{aligned} |s_{d-l}(x,\xi,\mu)| &\dot{\leq} \langle\xi,\mu\rangle^{d-l} \quad \text{for } l \leq \nu, \text{ with} \\ |s^h_{d-l}(x,\xi,\mu)| &\dot{\leq} |(\xi,\mu)|^{d-l}, \\ |s_{d-l} - s^h_{d-l}| &\dot{\leq} \langle\xi,\mu\rangle^{d-\nu} \quad \text{for } |(\xi,\mu)| \geq c > 0; \end{aligned} \tag{3.7}$$

and the next terms with $l \le \nu + n$ are estimated by

$$\begin{aligned} &|s_{d-l}(x,\xi,\mu)| \stackrel{.}{\le} \langle\xi\rangle^{\nu-l}\langle\xi,\mu\rangle^{d-\nu}, \quad \text{for } \nu < l < \nu+n, \text{ with} \\ &|s^h_{d-l}(x,\xi,\mu)| \stackrel{.}{\le} |\xi|^{\nu-l}|(\xi,\mu)|^{d-\nu}; \end{aligned} \tag{3.8}$$

so that altogether

$$s(x,\xi,\mu) = s^h_d(x,\xi,\mu) + \cdots + s^h_{d-\nu'-n}(x,\xi,\mu) + s'(x,\xi,\mu), \tag{3.9}$$

where $s' = s'' + s'''$, satisfying for $\mu \ge c_0 > 0$,

$$\begin{aligned} &|s''| \stackrel{.}{\le} |\xi|^{\delta-n}|\mu|^{d-\nu}\chi(\xi), \quad \text{with } \chi \in C_0^\infty(\mathbb{R}^n), \chi(\xi) = 1 \text{ for } |\xi| \le 1, \\ &|s'''| \stackrel{.}{\le} \langle\xi\rangle^{\delta-n-1}\langle\xi,\mu\rangle^{d-\nu}. \end{aligned} \tag{3.10}$$

Proof. The proof, given for particular choices of d and ν in Grubb (1996, (3.3.35)ff. and (3.3.69)ff.), extends to the general situation. The first two lines in (3.7) follow readily from the definitions. The third line is less obvious; it is shown in Grubb (1996, Lemma 2.1.9 2°) (by integration of an estimate of a high enough ξ-derivative). (3.8) follows easily from the definitions (one may consult Grubb, 1996, Lemma 2.1.9 1°). Then (3.9) follows in view of (3.10), where s'' collects the differences between homogeneous and strictly homogeneous symbols and s''' is the remainder $s - \sum_{l \le \nu'+n} s_{d-l}$. □

As in Grubb (1996, Theorem 3.3.5 and 3.3.10), we can use the lemma to get a diagonal kernel expansion with $n + \nu'$ precise terms.

Lemma 3.2. *When $d < -n$ in Lemma 3.1, the kernel $K(S_\mu, x, y)$ of $S_\mu = \mathrm{OP}(s(x,\xi,\mu))$ is continuous and has an expansion on the diagonal*

$$K(S_\mu, x, x) = \sum_{0 \le l < n+\nu} \tilde{s}_l(x)\mu^{n+d-l} + \tilde{s}'(x,\mu); \tag{3.11}$$

here

$$\tilde{s}_l(x) = \int_{\mathbb{R}^n} s^h_{d-l}(x,\xi,1)\,đ\xi, \tag{3.12}$$

and $\tilde{s}'(x,\mu)$ is $O(\mu^{d-\nu+\varepsilon})$ for $\mu \to \infty$, any $\varepsilon > 0$. Here if $\nu \notin \mathbb{Z}$, ε can be left out.

Proof. This follows by integration of (3.9) in ξ. For the terms s^h_{d-l} we use the homogeneity, replacing ξ by $\eta = \mu^{-1}\xi$

$$\int_{\mathbb{R}^n} s^h_{d-l}(x,\xi,\mu)\,đ\xi = \mu^{d-l+n}\int_{\mathbb{R}^n} s^h_{d-l}(x,\eta,1)\,đ\eta,$$

and for s' we use the estimates (3.10) (cf. Grubb, 1996, Lemma 3.3.6). □

Remark 3.3. The symbol spaces $S^{r,a}_{\mathrm{phg}}$ ($a \in \mathbb{Z}$) defined in Grubb and Seeley (1995) are somewhat more refined; they fit into the regularity classes as follows. Let $s(x,\xi,\mu)$ belong to $S^{r,a}_{\mathrm{phg}} \cap S^{r+a,0}_{\mathrm{phg}}$, where $r + a < -n$. Then s is of order $r + a$, and

$f(x, \xi, \mu) = \mu^{-a} s(x, \xi, \mu)$ satisfies the requirements (3.6), except those concerning μ-derivatives, for being of order r and regularity r. The fact that $r + a < -n$ makes the symbol integrable in ξ; the information on f assures that the strictly homogeneous terms f^h_{r-l} are integrable at $\xi = 0$ for $l < r + n$ and the remainder $f - \sum_{l<n+r} f^h_{r-l}$ is integrable at $\xi = 0$ (as in Lemma 3.1). Then we get the diagonal expansion of the kernel of $S_\mu = \mathrm{OP}(s)$ as in Lemma 3.2,

$$K(S_\mu, x, x) = \mu^a \sum_{0 \le l < n+r} \tilde{s}_l(x) \mu^{n+r-l} + \tilde{s}'(x, \mu); \qquad \tilde{s}'(x, \mu) = O(\mu^{a+\varepsilon}), \tag{3.13}$$

for $\mu \to \infty$, with locally determined coefficients $\tilde{s}_l$. What the calculus of Grubb and Seeley (1995) moreover gives for the symbols in $S^{r,a}_{\mathrm{phg}} \cap S^{r+a,0}_{\mathrm{phg}}$ is a full expansion of the remainder,

$$\tilde{s}'(x, \mu) \sim \mu^a \left[\sum_{l \ge n+r} \tilde{s}_l(x) \mu^{n+r-l} + \sum_{k \ge 0} (\tilde{s}'_k(x) \log \mu + \tilde{s}''_k(x)) \mu^{-k} \right],$$

with local coefficients $\tilde{s}_l(x)$, $\tilde{s}'_k(x)$ and global coefficients $\tilde{s}''_k(x)$. Some of the $\tilde{s}''_k(x)$ may belong to the same powers as coefficients $\tilde{s}_l(x)$, so the values of a and r are important in the discussion of which terms are local.

Besides ψdo's, we must now deal with singular Green operators. Singular Green symbol-kernels $\tilde{g}(x', x_n, y_n, \xi', \mu)$ of *order* d (degree $d - 1$), regularity ν and class zero satisfy estimates

$$\| D^\beta_{x'} D^\alpha_{\xi'} D^j_\mu \tilde{g}(x', x_n, y_n, \xi', \mu) \|_{L_2(\mathbb{R}_+ \times \mathbb{R}_+)} \dot{\le} (\langle \xi' \rangle^{\nu - |\alpha|} + \langle \xi', \mu \rangle^{\nu - |\alpha|}) \langle \xi', \mu \rangle^{d - \nu - j}, \tag{3.14}$$

along with further estimates for $x^k_n D^{k'}_{x_n} y^m_n D^{m'}_{y_n} \tilde{g}$ ($k, k', m, m' \in \mathbb{N}$) (cf. Grubb, 1996, Section 2.3), and with a suitable definition of polyhomogeneity.

Here is the following rule for normal traces of *s.g.o.* symbol-kernels.

Lemma 3.4. *When $\tilde{g}(x', x_n, y_n, \xi', \mu)$ is a singular Green symbol-kernel of order d, regularity ν and class zero, then the normal trace*

$$s(x', \xi', \mu) = \mathrm{tr}_n\, \tilde{g} = \int_0^\infty \tilde{g}(x', x_n, x_n, \xi', \mu) dx_n \tag{3.15}$$

is a ψdo symbol on $\mathbb{R}^{n-1}$ of order d and regularity $\nu - \frac{1}{4}$, polyhomogeneous if $\tilde{g}$ is so.

Proof. This is shown in Grubb (1996, pf. of Theorem 3.3.9), for $\nu \in \mathbb{Z} \cup (\mathbb{Z} + \frac{1}{2})$; s is denoted $\tilde{\tilde{g}}$ there. The first part of the proof extends to all real $\nu \ge 1$ or ≤ 0; the loss of $\frac{1}{4}$ stems from the negative cases (which occur in symbol terms of low order). The last part of the proof, showing how $\nu = \frac{1}{2}$ is included by use of a derivative, extends to general $\nu \in]0, 1[$. (The considerations in Grubb (1996) were aimed at negative integer values of d, but all the arguments work with arbitrary $d \in \mathbb{R}$ also.) □

Combining Lemma 3.4 with Lemma 3.2 in dimension $n - 1$, we find for *s.g.o.*'s of order $d < 1 - n$ and regularity ν the following lemma.

Lemma 3.5. *Let G_μ be a μ-dependent polyhomogeneous singular Green operator on $\overline{\mathbb{R}}^n_+$ of order $d < 1 - n$ and regularity ν. The normal trace $S_\mu = \mathrm{tr}_n\, G_\mu$ is a ψdo on*

$\mathbb{R}^{n-1}$ *of order d and regularity* $v-\frac{1}{4}$, *whose kernel on the diagonal has an expansion in powers of* μ:

$$K(S_\mu, x', x') = \sum_{0\le l<n-1+v-\frac{1}{4}} \tilde{s}_l(x')\mu^{n-1+d-l} + O(\mu^{d-v+\frac{1}{4}(+\varepsilon)}); \tag{3.16}$$

with $\varepsilon = 0$ *if* $v-\frac{1}{4}\notin\mathbb{Z}$, *any small* $\varepsilon>0$ *if* $v-\frac{1}{4}\in\mathbb{Z}$. *Here*

$$\tilde{s}_l(x') = \int_{\mathbb{R}^{n-1}} s^h_{d-l}(x', \xi', 1)đ\xi'. \tag{3.17}$$

When the kernel has compact (x', y')*-support,* G_μ *is trace-class and the trace has an expansion with coefficients* $\tilde{s}_l = \int \operatorname{tr}\tilde{s}_l(x')dx'$:

$$\operatorname{Tr} G_\mu = \sum_{0\le l<n-1+v-\frac{1}{4}} \tilde{s}_l\mu^{n-1+d-l} + O(\mu^{d-v+\frac{1}{4}\,(+\varepsilon)}); \tag{3.18}$$

with ε *as above.*

Proof. By Lemma 3.4, the operator family $S_\mu = \operatorname{tr}_n G_\mu$ satisfies the hypotheses of Lemma 3.2 in $n-1$ dimensions with v replaced by $v-\frac{1}{4}$, this implies (3.16) with (3.17). Then (3.18) follows by integration in x'. □

Now let us turn to the specific operators we want to study. Consider $A = P_+ + G$ of order $\sigma\in\mathbb{R}$ together with an auxiliary elliptic differential operator P_1 of order $m>n+\sigma$. Recall that if $\sigma\in\mathbb{R}\backslash\mathbb{Z}$, we have $P=0$.

The resolvent $Q_{1,\lambda} = (P_1-\lambda)^{-1}$ depends on λ running in a sector V around $\mathbb{R}_-$ in $\mathbb{C}$, where it is defined for large λ. We consider λ on each ray there, writing $-\lambda = \mu^m e^{i\omega}$, $\mu\ge 0$. Since P_1 is a differential operator, $Q_{1,\lambda}$ is of regularity $+\infty$. By Grubb (1996, (2.1.13), (2.3.54)), $A = P_+ + G$ enters in the parameter-dependent calculus as an operator of order and regularity σ (since G is of class zero). Then the composed operator $AQ_{1,\lambda,+}$ is of order $\sigma - m$ and regularity σ, in view of Grubb (1996, Theorem 2.7.7, Corollary 2.7.8) (no loss of ε regularity thanks to the mentioned theorem).

In the following, we work in a localized situation, as explained e.g., in Grubb and Schrohe (2004, after (3.11)).

For the ψdo $PQ_{1,\lambda}$ we already have a diagonal kernel expansion (2.1) pointwise for $x\in\widetilde{X}$; integration of the fiber trace over X gives the trace expansion

$$\operatorname{Tr}((PQ_{1,\lambda})_+) = \sum_{l\ge 0} c_{l,+}(-\lambda)^{\frac{\sigma+n-l}{m}-1} + \sum_{k\ge 0}(c'_{k,+}\log(-\lambda) + c''_{k,+})(-\lambda)^{-k-1}. \tag{3.19}$$

Lemma 3.5 applied to the singular Green part $G_\lambda = AQ_{1,\lambda,+} - (PQ_{1,\lambda})_+$ gives the expansion

$$\begin{aligned}\operatorname{Tr} G_\lambda &= \sum_{0\le l<n-1+\sigma-\frac{1}{4}} b_l(-\lambda)^{\frac{n-1+\sigma-l}{m}-1} + O(\lambda^{-1+\frac{1}{4m}(+\varepsilon)})\\ &= \sum_{1\le j<n+\sigma-\frac{1}{4}} b'_j(-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(\lambda^{-1+\frac{1}{4m}(+\varepsilon)}),\end{aligned} \tag{3.20}$$

with $b'_j = b_{j-1}$; ε equals zero when $\sigma - \frac{1}{4} \notin \mathbb{Z}$, and can be any small positive number when $\sigma - \frac{1}{4} \in \mathbb{Z}$. Here one first shows the expansion on each ray, noting that for λ on the ray $\mathbb{R}_-$ we get (3.20); then the holomorphy assures that the expansion is the same on the other rays (as in Grubb and Seeley, 1995, Lemma 2.3). When $\sigma \notin \mathbb{Z}$, $G_\lambda = AQ_{1,\lambda,+}$, so (3.20) shows its trace expansion. When $\sigma \in \mathbb{Z}$, addition of (3.19) and (3.20) gives

$$\operatorname{Tr}(AQ_{1,\lambda,+}) = \sum_{0 \le l < n+\sigma} c_l(-\lambda)^{\frac{n+\sigma-l}{m}-1} + O(\lambda^{-1+\frac{1}{4m}}). \tag{3.21}$$

This expansion does *not* show the appearance of a term $c(-\lambda)^{-1}$. We shall obtain that by proving two things:

1) When P_1 is replaced by another auxiliary operator P_2 of order m, then the difference of the traces $\operatorname{Tr}(AQ_{1,\lambda,+}) - \operatorname{Tr}(AQ_{2,\lambda,+})$ has a better expansion,

$$\operatorname{Tr}(A(Q_{1,\lambda} - Q_{2,\lambda})_+) = \sum_{0 \le j < n+\sigma+\frac{1}{4}} d_j(-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}), \tag{3.22}$$

with $\varepsilon = 0$ if $\sigma - \frac{1}{4} \notin \mathbb{Z}$.

2) There exist particular choices of P_1, where one has a better expansion than (3.21):

$$\operatorname{Tr}(AQ_{1,\lambda,+}) = \sum_{0 \le j < n+\sigma+\frac{1}{4}} c_j(-\lambda)^{\frac{n+\sigma-j}{m}-1} + (c'_0 \log(-\lambda) + c''_0)(-\lambda)^{-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}), \tag{3.23}$$

with $\varepsilon = 0$ if $\sigma - \frac{1}{4} \notin \mathbb{Z}$.

Then an expansion (3.23) is obtained for general choices of P_1 by use of (3.22).
For point 1) in this program, let us denote

$$\mathcal{Q}_\lambda = Q_{1,\lambda} - Q_{2,\lambda}, \qquad \text{with symbol } \mathfrak{q}(x, \xi, \lambda). \tag{3.24}$$

Then we can write, since $A = P_+ + G$,

$$\begin{aligned} A(Q_{1,\lambda} - Q_{2,\lambda})_+ = A\mathcal{Q}_{\lambda,+} = (P\mathcal{Q}_\lambda)_+ + \mathcal{G}_\lambda, \quad \text{with} \\ \mathcal{G}_\lambda = -G^+(P)G^-(\mathcal{Q}_\lambda) + G\mathcal{Q}_{\lambda,+}. \end{aligned} \tag{3.25}$$

The last identity refers to a localized situation. In $\mathbb{R}^n$, $G^+(P) = r^+Pe^-J$ and $G^-(P) = Jr^-Pe^+$, where $e^\pm$ denote extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$, $r^\pm$ denote restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$, and J maps $u(x', x_n)$ to $u(x', -x_n)$, cf. Grubb (1996, p. 252 and (A.32)). For the present operators, $P_+\mathcal{Q}_{\lambda,+} = (P\mathcal{Q}_\lambda)_+ - G^+(P)G^-(\mathcal{Q}_\lambda)$.

The desired formula for the ψdo term can be found pointwise in $x \in \widetilde{X}$, by use of Theorem 2.2, and then integrated over X. It is the singular Green term that requires a new effort.

Theorem 3.6. *Let $A = P_+ + G$, of order $\sigma \in \mathbb{R}$ with G of class zero, assuming $P = 0$ if $\sigma \notin \mathbb{Z}$. Let P_1 and P_2 be auxiliary elliptic differential operators of order m, as described in the beginning of this section, with $m > n + \sigma$.*

The singular Green part $\mathcal{G}_\lambda$ of $A(Q_{1,\lambda} - Q_{2,\lambda})_+$ is of order $\sigma - m$, class zero and regularity $\sigma + \frac{1}{2}$. Consequently, in local coordinates, its normal trace $\mathcal{S}_\lambda = \operatorname{tr}_n \mathcal{G}_\lambda$ is a ψdo on $\mathbb{R}^{n-1}$ of order $\sigma - m$ and regularity $\sigma + \frac{1}{4}$. Denoting its symbol $\mathfrak{s}(x', \xi', \lambda) \sim \sum_{l \geq 0} \mathfrak{s}_{\sigma-m-l}(x', \xi', \lambda)$, we have the trace expansion

$$\operatorname{Tr} \mathcal{G}_\lambda = \sum_{1 \leq j < n+\sigma+\frac{1}{4}} d_j \, (-\lambda)^{\frac{n+\sigma-j}{m} - 1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}), \tag{3.26}$$

with $\varepsilon = 0$ if $\sigma - \frac{1}{4} \notin \mathbb{Z}$, $\varepsilon > 0$ if $\sigma - \frac{1}{4} \in \mathbb{Z}$. Here

$$d_j = \tilde{\mathfrak{s}}_{j-1} = \int \operatorname{tr} \tilde{\mathfrak{s}}_{j-1}(x') dx', \qquad \text{where } \tilde{\mathfrak{s}}_l(x') = \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{\sigma-m-l}(x', \xi', -1) đ\xi'. \tag{3.27}$$

Proof. In a localized situation, $\mathcal{G}_\lambda$ is the sum of the operators $-G^+(P)G^-(\mathcal{Q}_\lambda)$ and $G\mathcal{Q}_{\lambda,+}$. We first study $G^+(P)G^-(\mathcal{Q}_\lambda)$.

We have from Section 2 (where the notation $P, Q_\lambda, P', Q'_\lambda$ was used for what we now call $P_1, Q_{1,\lambda}, P_2, Q_{2,\lambda}$) that the symbol $\mathfrak{q}$ of $\mathcal{Q}_\lambda$ in (3.24) has an expansion $\sum_{j \in \mathbb{N}} \mathfrak{q}_{-m-j}$ in homogeneous symbols

$$\mathfrak{q} \sim \mathfrak{q}_{-m} + \sum_{j \geq 1} \mathfrak{q}_{-m-j}, \qquad \mathfrak{q}_{-m-j}(x, \xi, \lambda) = q_{1,-m-j}(x, \xi, \lambda) - q_{2,-m-j}(x, \xi, \lambda),$$

where all the terms in the sum over $j \geq 1$ are finite sums of expressions as in (1.5), and

$$q_{1,-m} - q_{2,-m} = (p_{1,m} - \lambda)^{-1} - (p_{2,m} - \lambda)^{-1} = q_{1,-m}(p_{2,m} - p_{1,m})q_{2,-m};$$

so all the homogeneous terms $\mathfrak{q}_{-m-j}$ are sums of terms of the form generalizing (1.5) with at least two factors $(p_{1,m} - \lambda)^{-1}$ or $(p_{2,m} - \lambda)^{-1}$. Since P_1 and P_2 are differential operators, we need not smooth out around $\xi = 0$, but can take the exact symbols (for $\lambda \neq 0$).

When $p_{1,m}$ and $p_{2,m}$ are scalar (functions times the identity matrix), the analysis is a little simpler than in the general matrix case, so let us describe this case first. Here the factors $q_{i,-m} = (p_{i,m} - \lambda)^{-1}$ can be collected to the right, so in fact the terms in $\mathfrak{q}$ are of the form

$$f(x, \xi, \lambda) = f_0(x, \xi) q_{1,-m}^{\nu_1} q_{2,-m}^{\nu_2}, \tag{3.28}$$

with $\nu_1 + \nu_2 \geq 2$ and f_0 polynomial in ξ. Then for each $j \geq 0$, $\mathfrak{q}_{-m-j}$ is a sum of terms of the form $r'(x, \xi) q''(x, \xi, \lambda)$, where r' is the symbol of a differential operator of order m independent of λ and q'' is of order $-2m - j$, likewise with structure as in (3.28), smooth in all variables (for $|\xi| + |\lambda| \neq 0$). The operator $\mathrm{OP}(r'q'')$ can be further decomposed into a finite sum of terms $RQ' = \mathrm{OP}(r)\mathrm{OP}(q')$, where r and q' have a similar structure as r' and q'' (we need this modification to get a composition of two operators instead of a product of symbols). Now we treat each term

$$G^+(P)G^-(RQ')$$

separately. We are working in $\mathbb{R}^n$, where the manifold corresponds to $\overline{\mathbb{R}}^n_+$, and can assume that the symbols of P, R and Q' are defined on $\mathbb{R}^n$. Here we write

$$\begin{aligned} G^-(RQ') &= Jr^-RQ'e^+ = Jr^-R(e^+r^+ + e^-r^-)Q'e^+ \\ &= Jr^-Re^-JJr^-Q'e^+ = \overline{R}_+G^-(Q'), \end{aligned} \tag{3.29}$$

where we have used that $r^-Re^+ = 0$ since R is a differential operator, and denoted $JRJ = \overline{R}$, again a differential operator. Thus $G^+(P)G^-(RQ') = G^+(P)\overline{R}_+G^-(Q')$, where $G^+(P)\overline{R}_+$ is a λ-independent *s.g.o.* of order $\sigma + m$ and class m. It enters in the parameter-dependent calculus as an operator of order $\sigma + m$, class m and regularity $\sigma + \frac{1}{2}$, cf. Grubb (1996, (2.3.55)). (It is the presence of the normal derivatives of order $\leq m$ in the differential operator $\overline{R}$ that brings the regularity down to $\sigma + \frac{1}{2}$, not $\sigma + m$ as in the considerations for closed manifolds, but the gain of $\frac{1}{2}$ will be just enough to serve our purposes.) Composing with $G^-(Q')$ of order $-2m - j$, class zero and regularity $+\infty$, we find that

$$\begin{aligned} G^+(P)G^-(RQ') = G^+(P)\overline{R}_+G^-(Q') \quad &\text{is of order } \sigma - m - j, \\ &\text{class zero, and regularity } \sigma + \frac{1}{2}. \end{aligned} \tag{3.30}$$

Collecting the terms (finitely many for each order) we find that the homogeneous terms in $\mathfrak{q}$ contribute to an *s.g.o.* of order $\sigma - m$, class zero and regularity $\sigma + \frac{1}{2}$. Since the remainder of $\mathfrak{q}$ after subtraction of N homogeneous terms is $O(\langle\xi, \mu\rangle^{-m-N-1})$, its contribution will, when N gets large, reach arbitrary low orders and estimates $O(|\lambda|^{-N'})$ for any N', so it complies with the regularity $\sigma + \frac{1}{2}$.

There is a very similar proof for $G\mathcal{Q}_{\lambda,+}$. Again we use that each $\mathrm{OP}(\mathfrak{q}_{-m-j})$ can be written as a finite sum of terms RQ', where R is a differential operator of order m and Q' has symbol structure as in (3.28) and order $-2m - j$. Now for each term, since $G^+(R) = 0$,

$$G(RQ')_+ = GR_+Q'_+,$$

where GR_+ is a parameter-independent *s.g.o.* of order $\sigma + m - j$ and class m, hence has regularity $\sigma + \frac{1}{2}$ when taken into the parameter-dependent theory. Then $GR_+Q'_+$ is of order $\sigma - m - j$, class zero, and regularity $\sigma + \frac{1}{2}$. Collecting the terms and treating remainders as above, we get that $G\mathcal{Q}_{\lambda,+}$ has order $\sigma - m$, class zero and regularity $\sigma + \frac{1}{2}$.

This shows the asserted symbol properties of $\mathcal{G}_\lambda$. Its normal trace $\mathcal{S}_\lambda$ is of order $\sigma - m$ and regularity $\sigma + \frac{1}{4}$ by Lemma 3.4. By Lemma 3.5, its kernel has an expansion on the diagonal

$$\begin{aligned} K(\mathcal{S}_\lambda, x', x') &= \sum_{0 \leq l \leq n-1+\sigma} \tilde{\mathfrak{s}}_l(x')(-\lambda)^{\frac{n-1+\sigma-m-l}{m}} + O\big(\lambda^{\frac{\sigma-m-\sigma-\frac{1}{4}(+\varepsilon'')}{m}}\big) \\ &= \sum_{0 \leq l \leq n-1+\sigma} \tilde{\mathfrak{s}}_l(x')(-\lambda)^{\frac{n-1+\sigma-l}{m}-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}), \end{aligned} \tag{3.31}$$

with $\varepsilon = 0$, unless $\sigma + \frac{1}{4} \in \mathbb{Z}$, and $\tilde{\mathfrak{s}}_l(x')$ defined as in (3.27). In the proof, the lemma is applied for each ray; the ray $\mathbb{R}_-$ gives the value (3.27) for the coefficients, and the holomorphy assures that their values are the same on the other rays (as in Grubb and Seeley, 1995, Lemma 2.3). Finally, integration in x' of the fiber trace then gives

$$\operatorname{Tr}_{\mathbb{R}^n_+} \mathscr{G}_\lambda = \operatorname{Tr}_{\mathbb{R}^{n-1}} \mathscr{S}_\lambda = \sum_{0 \le l < n-1+\sigma+\frac{1}{4}} \tilde{\mathfrak{s}}_l (-\lambda)^{\frac{n-1+\sigma-l}{m}-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)})$$
$$= \sum_{1 \le j < n+\sigma+\frac{1}{4}} d_j (-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}), \tag{3.32}$$

with d_j defined as in (3.27). This ends the proof when $p_{1,m}$, $p_{2,m}$ are scalar.

It remains to explain how the case where the $p_{i,m}$ are general matrices is treated. Consider the entries in the matrix $\mathfrak{q}_{-m-j} = (\mathfrak{q}_{-m-j,k,l})_{k,l=1,\dots,\dim E}$. Since the entries in the matrices $(p_{i,m}(x,\xi) - \lambda)^{-1}$ are rational functions of (ξ, λ) (with C^∞ coefficients in x) that are homogeneous of degree $-m$ in $(\xi, |\lambda|^{1/m})$ on each ray in V, hence $O((|\xi|^m + |\lambda|)^{-1})$, the entries in $\mathfrak{q}_{-m-j}$ have the form of sums of products

$$f(x,\xi,\lambda) = g_1(x,\xi)\varrho_1(x,\xi,\lambda)g_2(x,\xi)\varrho_2(x,\xi,\lambda)\cdots g_M(x,\xi)\varrho_M(x,\xi,\lambda)g_{M+1}(x,\xi),$$

with homogeneous polynomials $g_1, \dots, g_M$ in ξ independent of λ and rational functions $\varrho_1, \dots, \varrho_M$ in (ξ, λ) that are homogeneous of degree $-m$ in $(\xi, |\lambda|^{1/m})$ on each ray in V; here $M \ge 2$, and the total degree in $(\xi, |\lambda|^{1/m})$ is $-m-j$. In each entry, the functions g_i can be moved up front so that we get $f(x,\xi,\lambda) = g_1 \dots g_{M+1} \varrho_1 \dots \varrho_M$, with a structure similar to (3.28). The compositions $G^+(P)G^-(\mathscr{Q}_\lambda)$ and $G\mathscr{Q}_{\lambda,+}$ are now worked out for each vector component as sums of compositions of the form $\operatorname{OPG}(g)G^-(\operatorname{OP}(f))$, respectively $\operatorname{OPG}(g)\operatorname{OP}(f)_+$, which can be analyzed just as described above after (3.28). This gives a large but finite sum of terms from each homogeneous terms in $\mathfrak{q}$, which sum up to give expansions as above. □

Observe a direct consequence.

Corollary 3.7. *Assumptions as in Theorem* 3.6.
The trace of $A(Q_{1,\lambda} - Q_{2,\lambda})_+$ *has an expansion* (3.22).

Proof. If $\sigma \notin \mathbb{Z}$, there is no ψdo part, and the expansion is (3.26). If $\sigma \in \mathbb{Z}$, the ψdo part has an expansion

$$\operatorname{Tr}((P(Q_{1,\lambda} - Q_{2,\lambda}))_+) = \sum_{0 \le j \le n+\sigma} c_{j,+}(-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(\lambda^{-2+\varepsilon'}) \tag{3.33}$$

(any $\varepsilon' > 0$), found from (2.9) by taking fiber traces and integrating over X. When we add this to (3.26), we find (3.22). □

Now we turn to point 2) in the program for showing (3.23) in general.

Lemma 3.8. *Let P_0 be selfadjoint positive of order 2 with scalar principal symbol, and let $A = P_+ + G$ be as above. For k so large that $2k > n + \sigma$, there is a trace expansion for $\lambda \to \infty$ in $\mathbb{C} \setminus \mathbb{R}_+$:*

$$\operatorname{Tr}(A(P_0^k - \lambda)_+^{-1}) \sim \sum_{j \ge 0} c_j (-\lambda)^{\frac{n+\nu-j}{2k}-1} + \sum_{l \ge 0} (c'_l \log(-\lambda) + c''_l)(-\lambda)^{-\frac{l}{2k}-1}. \tag{3.34}$$

Here $c'_0 = \frac{1}{2k} \operatorname{res} A$, and if $n + \sigma \notin \mathbb{N}$, $C_0(A, (P_0^k)_+) = c''_0 = \operatorname{TR} A$ $(= \operatorname{Tr} A$ if $\sigma < -n)$.

Proof. Here we use that (3.4) is known for P_0 from Grubb and Schrohe (2004), and translate it to a statement on the meromorphic structure of the generalized zeta function $\zeta(A, P_{0,+}, s)$, which allows replacing P_0 by P_0^k. This gives the structure of $\zeta(A, (P_0^k)_+, s)$, which translates back to a trace expansion (3.34). Here Grubb and Seeley (1996, Proposition 2.9 and Corollary 3.5), are used. In details, we define $\zeta(A, P_{0,+}, s)$ and $\zeta(A, (P_0^k)_+, s)$ as the meromorphic extensions of $\mathrm{Tr}(A(P_0^{-s})_+)$, respectively $\mathrm{Tr}(A(P_0^{-sk})_+)$, defined à priori for large $\mathrm{Re}\, s$. It is well known that the expansion (3.4) implies the following meromorphic structure of $\zeta(A, P_{0,+}, s)$:

$$\Gamma(s)\zeta(A, P_{0,+}, s) \sim \sum_{j\geq 0} \frac{\tilde{c}_j}{s + \frac{j-n-\sigma}{2}} + \sum_{l\geq 0}\left(\frac{\tilde{c}'_l}{(s+\frac{l}{2})^2} + \frac{\tilde{c}''_l}{s+\frac{l}{2}}\right) \tag{3.35}$$

(by use of e.g., Grubb and Seeley, 1996, Proposition 2.9). Dividing out the Gamma factor, we obtain a meromorphic structure somewhat similar to (3.35),

$$\zeta(A, P_{0,+}, s) \sim \sum_{j\geq 0} \frac{\tilde{a}_j}{s + \frac{j-n-\sigma}{2}} + \sum_{l\geq 0}\left(\frac{\tilde{a}'_l}{(s+\frac{l}{2})^2} + \frac{\tilde{a}''_l}{s+\frac{l}{2}}\right) \tag{3.36}$$

except that the double poles vanish for l even, since they are turned into simple poles by the cancellations from the zeros of $\Gamma(s)^{-1}$ at $0, -1, -2, \ldots$. Since P_0 is self-adjoint positive, the complex powers agree with the definition by spectral theory, so $P_0^{-s} = (P_0^k)^{-s'}$, $s' = s/k$. Then we can replace the formula for P_0^{-s} by the formula for $(P_0^k)^{-s'}$ simply by replacing the variable s by $s'k$, so we get

$$\begin{aligned}\zeta(A, (P_0^k)_+, s') &\sim \sum_{j\geq 0} \frac{\tilde{a}_j}{s'k + \frac{j-n-\sigma}{2}} + \sum_{l\geq 0}\left(\frac{\tilde{a}'_l}{(s'k+\frac{l}{2})^2} + \frac{\tilde{a}''_l}{s'k+\frac{l}{2}}\right)\\ &\sim \sum_{j\geq 0} \frac{\tilde{b}_j}{s' + \frac{j-n-\sigma}{2k}} + \sum_{l\geq 0}\left(\frac{\tilde{b}'_l}{(s'+\frac{l}{2k})^2} + \frac{\tilde{b}''_l}{s'+\frac{l}{2k}}\right),\end{aligned}$$

with the double poles vanishing for l even. Multiplication by $\Gamma(s')$ gives still another expansion

$$\Gamma(s')\zeta(A, (P_0^k)_+, s') \sim \sum_{j\geq 0} \frac{\tilde{d}_j}{s' + \frac{j-n-\sigma}{2k}} + \sum_{l\geq 0}\left(\frac{\tilde{d}'_l}{(s'+\frac{l}{2k})^2} + \frac{\tilde{d}''_l}{s'+\frac{l}{2k}}\right), \tag{3.37}$$

where we get double poles back at the values where l/k is even (a subset of the set where they were removed before).

Finally, we use Grubb and Seeley (1996, Proposition 2.9) in the direction from $\zeta(s)$ to $f(\lambda)$, in the same way as in the proof of Grubb and Seeley (1996, Corollary 3.5). The cited proposition shows how the meromorphic structure of $\Gamma(1-s')\Gamma(s')\zeta(A, (P_0^k)_+, s')$ carries over to an asymptotic expansion of $f(-\lambda) = \mathrm{Tr}(A(P_0^k - \lambda)_+^{-1})$. The needed exponential decrease for $|\mathrm{Im}\, s'| \to \infty$ follows from the similar property of $\Gamma(1-s)\Gamma(s)\zeta(A, P_{0,+}, s)$. That $f(-\lambda)$ satisfies an $O(|\lambda|^{-\alpha})$ estimate (with $\alpha > 0$) for $\lambda \to \infty$ in the considered sector is assured by (3.21) above, with $m = 2k > n + \sigma$. The positivity of P_0 assures that f is regular at zero. The method introduces some possible new integer poles on the positive real axis (coming from $\Gamma(1-s')$), but in the end result they are not present, since we already have the corresponding part of the expansion in powers known from (3.21).

It is known from Grubb and Schrohe (2001) that $\tilde{c}'_0 = \frac{1}{2}\operatorname{res} A$ in (3.35), also equal to the coefficient of $(\lambda)^{-1}\log(-\lambda)$ in the corresponding resolvent trace expansion. Following the reduction, we see that $\tilde{d}'_0$ in (3.37) equals $\frac{1}{k}\tilde{c}'_0 = \frac{1}{2k}\operatorname{res} A$.

If $n+\sigma \notin \mathbb{N}$, $\operatorname{res} A = 0$ and we have from Grubb and Schrohe (2001) that $C_0(A, P_{0,\mathrm{D}}) = \operatorname{TR} A$. Hence also $C_0(A, P_{0,+}) = \operatorname{TR} A$ (since there is no contribution from $G_\lambda^{(N)}$), so $\zeta(A, P_{0,+}, s)$ is regular at zero with value $\operatorname{TR} A$. Then also $\zeta(A, (P_0^k)_+, 0)$ equals $\operatorname{TR} A$. $\square$

Theorem 3.9. *Assumptions as in Theorem* 3.6.

$\operatorname{Tr}(AQ_{1,\lambda,+})$ *has a trace expansion* (3.23); *in particular,* $C_0(A, P_{1,+})$ *is well defined as the coefficient of* $(-\lambda)^{-1}$ (*equal to* $\operatorname{TR} A$ *if* $n+\sigma \notin \mathbb{N}$), *and* $c'_0 = \frac{1}{m}\operatorname{res} A$.

Proof. First let m be even $= 2k$. Then we can compare an arbitrary auxiliary operator P_1 with $P_3 = P_0^k$ from Lemma 3.8 (with resolvent $Q_{3,\lambda} = (P_3-\lambda)^{-1}$). Here (3.22) for the trace difference and (3.34) for $\operatorname{Tr}(AQ_{3,\lambda,+})$ add up to give:

$$\begin{aligned}\operatorname{Tr}(AQ_{1,\lambda,+}) &= \operatorname{Tr}(A(Q_{1,\lambda}-Q_{3,\lambda})_+) + \operatorname{Tr}(AQ_{3,\lambda,+})\\ &= \sum_{0\le j<n+\sigma+\frac{1}{4}} c_j\,(-\lambda)^{\frac{n+\sigma-j}{m}-1} + (c'_0\log(-\lambda) + c''_0)(-\lambda)^{-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}),\end{aligned} \tag{3.38}$$

with $c'_0 = \frac{1}{m}\operatorname{res} A$, $c''_0 = \operatorname{TR} A$ if $n+\sigma \notin \mathbb{N}$. So the assertions hold for m even.

Next, let m be odd. Necessarily, P_1 cannot have its spectrum in a sector with opening $< \pi$, since the principal symbol is odd in ξ, so iterated powers are not easy to use (e.g., for a selfadjoint Dirac operator D, $(D^2)^{\frac{1}{2}} = |D|$ is different from D). Instead, we shall use an idea of doubling up, found in Grubb and Seeley (1995). For a given P_1 of order m, consider

$$\mathscr{P}_1 = \begin{pmatrix} 0 & -P_1^* \\ P_1 & 0 \end{pmatrix}, \qquad \mathscr{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix},$$

acting in the bundle $E \oplus E$. $\mathscr{P}_1$ is skew-self-adjoint, with resolvent

$$(\mathscr{P}_1 - \lambda)^{-1} = \begin{pmatrix} -\lambda(P_1^*P_1+\lambda^2)^{-1} & P_1^*(P_1P_1^*+\lambda^2)^{-1} \\ -P_1(P_1^*P_1+\lambda^2)^{-1} & -\lambda(P_1P_1^*+\lambda^2)^{-1} \end{pmatrix},$$

for $\lambda \in \mathbb{C}\setminus i\mathbb{R}$. Now

$$\operatorname{Tr}(\mathscr{A}(\mathscr{P}_1-\lambda)_+^{-1}) = -\lambda\operatorname{Tr}(A(P_1^*P_1+\lambda^2)_+^{-1}) - \lambda\operatorname{Tr}(A(P_1P_1^*+\lambda^2)_+^{-1}).$$

In the right-hand side, $P_1^*P_1$ and $P_1P_1^*$ are self-adjoint elliptic of even order $2m$ (and ≥ 0), so by the result already shown for even-order auxiliary operators, applied to the two traces, we get

$$\begin{aligned}
&\operatorname{Tr}(\mathscr{A}(\mathscr{P}_1-\lambda)_+^{-1})\\
&= -\lambda\Big(\sum_{0\le j<n+\sigma+\frac14} a_j\lambda^{2(\frac{n+\sigma-j}{2m}-1)} + (a_0'\log(\lambda^2)+a_0'')\lambda^{-2} + O(\lambda^{2(-1-\frac{1}{8m}(+\varepsilon))})\Big)\\
&= \sum_{0\le j<n+\sigma+\frac14} b_j(-\lambda)^{\frac{n+\sigma-j}{m}-1} + (2a_0'\log(-\lambda)+a_0'')(-\lambda)^{-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}),
\end{aligned} \tag{3.39}$$

with coefficients modified because of powers of $-1=e^{i\pi}$; here $a_0' = 2\frac{1}{2m}\operatorname{res} A = \frac{1}{m}\operatorname{res} A$, and $a_0'' = 2\operatorname{TR} A$ if $n+\sigma\notin\mathbb{N}$. Now $\mathscr{P}_1$ can be compared with

$$\mathscr{P}_2 = \begin{pmatrix} P_1 & 0\\ 0 & P_1\end{pmatrix},$$

and a calculation as in (3.38) gives that $\operatorname{Tr}(\mathscr{A}(\mathscr{P}_2-\lambda)_+^{-1})$ likewise has an expansion as in the last line of (3.39), with the same coefficient $2a_0'$ of the logarithmic term, and the same a_0'' if $n+\sigma\notin\mathbb{N}$. Then $\operatorname{Tr}(A(P_1-\lambda)_+^{-1}) = \frac12\operatorname{Tr}(\mathscr{A}(\mathscr{P}_2-\lambda)_+^{-1})$ likewise has an expansion, with log-coefficient a_0', and $C_0(A,P_{1,+}) = \operatorname{TR} A$ if $n+\sigma\notin\mathbb{N}$. □

One can also see from these proofs that the value of $C_0(A,P_{1,+})$ *modulo local terms* is as described in Grubb and Schrohe (2004), namely, in local coordinates, a sum of integrals over X resp. X' of finite part integrals in ξ, respectively ξ' of the symbols of P, respectively $\operatorname{tr}_n G$. (The identity $C_0(G,P_{1,+}) = \operatorname{TR} G$ extends to certain integer order parity cases as in Grubb and Schrohe, 2004.)

Now the coefficient of $(-\lambda)^{-1}$ in $\operatorname{Tr}(A(Q_{1,\lambda}-Q_{2,\lambda})_+)$ will be studied in detail.

Note that when $n+\sigma\in\mathbb{N}$, the sum in (3.22) goes from zero to $n+\sigma$ and the last term is $d_{n+\sigma}(-\lambda)^{-1}$. When $n+\sigma\notin\mathbb{N}$, we see that there is no term with $(-\lambda)^{-1}$ in the expansion, so (as it should be)

$$C_0(A,P_{1,+}) - C_0(A,P_{2,+}) = 0 \quad \text{if } n+\sigma\notin\mathbb{N}. \tag{3.40}$$

We shall finally show

Theorem 3.10. *Assumptions as in Theorem* 3.6. *One has that*

$$C_0(A,P_{1,+}) - C_0(A,P_{2,+}) = -\frac{1}{m}\operatorname{res}(A(\log P_1 - \log P_2)_+). \tag{3.41}$$

Proof. Denote

$$L = \log P_1 - \log P_2, \tag{3.42}$$

with symbol $l(x,\xi)$; in view of (1.14), it is classical of order zero, and (cf. (3.24)) the homogeneous terms $l_{-j}(x,\xi)$ are determined for $|\xi|\ge 1$ by the formulas

$$l_{-j}(x,\xi) = \frac{i}{2\pi}\int_{\mathscr{C}'} \log\lambda\, \mathfrak{q}_{-m-j}(x,\xi,\lambda)\,d\lambda, \tag{3.43}$$

where $\mathscr{C}'$ is a closed curve in $\mathbb{C}\backslash\overline{\mathbb{R}}_-$ encircling the eigenvalues of $p_{1,m}(x,\xi)$ and $p_{2,m}(x,\xi)$. From the fact that P_1 and P_2 are differential operators, it is easily checked that L satisfies the transmission condition at $x_n = 0$.

We have that

$$A(\log P_1 - \log P_2)_+ = AL_+ = (P_+ + G)L_+ = (PL)_+ - G^+(P)G^-(L) + GL_+. \quad (3.44)$$

According to Fedosov et al. (1996) (with the sign of the *s.g.o.*-term corrected in Grubb and Schrohe, 2001), the residue is determined by the formula

$$\begin{aligned}\operatorname{res}(AL_+) &= \int_{\mathbb{R}^n_+}\int_{|\xi|=1} \operatorname{tr}\operatorname{symb}_{-n}(PL)\,đS(\xi)\,dx \\ &\quad + \int_{\mathbb{R}^{n-1}}\int_{|\xi'|=1} \operatorname{tr}\operatorname{symb}_{1-n}(\operatorname{tr}_n(-G^+(P)G^-(L)+GL_+))\,đS(\xi')\,dx', \quad (3.45)\end{aligned}$$

where symb_k stands for "the homogeneous term of degree k in the symbol of the operator".

Consider first the case where $\sigma \notin \mathbb{Z}$, $P = 0$. Then the left-hand side in (3.41) is zero in view of (3.40), and the right-hand side is zero, since $A(\log P_1 - \log P_2)_+$ is an *s.g.o.* of noninteger order. So the formula is verified for $\sigma \notin \mathbb{Z}$, and we can restrict the attention to the case where $\sigma \in \mathbb{Z}$.

The calculations leading to Theorem 2.2 show that

$$-\frac{1}{m}\int_{\mathbb{R}^n_+}\int_{|\xi|=1} \operatorname{tr}\operatorname{symb}_{-n}(PL)\,đS(\xi)\,dx = \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n} \operatorname{tr}\operatorname{symb}^h_{-m-n}(P\mathcal{Q}_\lambda)|_{\lambda=-1}\,đ\xi\,dx; \quad (3.46)$$

this gives the ψdo part of the desired formula.

Now consider the *s.g.o.* part (recall (3.25)). For the operator $\mathscr{G}_\lambda$ and its normal trace $\mathscr{S}_\lambda$, we denote the symbols $\mathfrak{g}(x',\xi',\xi_n,\eta_n,\lambda)$ resp. $\mathfrak{s}(x',\xi',\lambda)$. Moreover, we denote

$$G' = -G^+(P)G^-(L) + GL_+, \qquad S' = \operatorname{tr}_n(-G^+(P)G^-(L) + GL_+), \quad (3.47)$$

with symbols $g'(x',\xi',\xi_n,\eta_n)$, $s'(x',\xi')$.

From (3.27) we have in particular:

$$\tilde{\mathfrak{s}}_{\sigma+n-1}(x) = \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x',\xi',-1)\,đ\xi', \quad (3.48)$$

and the integral of its fiber trace gives the contribution to $C_0(A, P_{1,+}) - C_0(A, P_{2,+})$.

In the following, consider first the case where $\mathfrak{q}$ is independent of x_n.

The term of order $1-n$ in the symbol of S' is constructed for $|\xi'| \geq 1$ as the term of homogeneity degree $1-n$ in the symbol

$$\begin{aligned}s'(x',\xi') &= \int_{\mathbb{R}} (-g^+(p)\circ g^-(l) + g\circ l_+)(x',\xi',\xi_n,\xi_n)\,đ\xi_n \\ &= \int_{\mathbb{R}}\bigg(-g^+(p)\circ g^-\Big(\frac{i}{2\pi}\int_{\mathscr{C}''}\log\lambda\mathfrak{q}\,d\lambda\Big) + g\circ\Big(\frac{i}{2\pi}\int_{\mathscr{C}''}\log\lambda\,\mathfrak{q}\,d\lambda\Big)_+\bigg)đ\xi_n; \\ &\qquad\qquad (3.49)\end{aligned}$$

here $\mathscr{C}''$ is a curve in $\mathbb{C}\backslash\overline{\mathbb{R}}_-$ formed as the boundary of a set $V_{r,\theta}$ (1.18) with θ, r and ε taken so small that the eigenvalues of the principal symbols $p_{1,m}(x,\xi)$ and

$p_{2,m}(x,\xi)$ lie in the complement of $V_{r+\varepsilon,\theta+\varepsilon}$ for all x, all $|\xi'|\geq 1$. Such sets exist since $p_{1,m}$ and $p_{2,m}$ are homogeneous of degree m in ξ for $|\xi|\geq 1$ and the ellipticity condition holds uniformly in x (originally running in the compact manifold X). We have:

$$s'_{1-n}(x',\xi') = \sum_{j+k+|\alpha|=\sigma+n-1} \frac{(-i)^{|\alpha|}}{\alpha!}\bigg(\int_{\mathbb{R}}\bigg(-\partial^{\alpha}_{\xi'}g^{+}(p)_{\sigma-j}\circ_n g^{-}\bigg(\frac{i}{2\pi}\int_{\mathscr{C}''}\log\lambda\,\partial^{\alpha}_{x'}\mathfrak{q}_{-m-k}d\lambda\bigg)đ\xi_n + \int_{\mathbb{R}}\partial^{\alpha}_{\xi'}g_{\sigma-j}\circ_n\bigg(\frac{i}{2\pi}\int_{\mathscr{C}''}\log\lambda\,\partial^{\alpha}_{x'}\mathfrak{q}_{-m-k}d\lambda\bigg)_{+}\bigg)đ\xi_n\bigg), \tag{3.50}$$

where $\circ_n$ stands for symbol composition with respect to the normal variables (cf. Grubb, 1996, Section 2.6), and we denote the homogeneous term of order r in an *s.g.o.* symbol g by g_r (it is of *degree* $r-1$; this index was used in Grubb, 1996). There are finitely many terms. In each term, the integration in λ and the factor $\log\lambda$ can be moved outside $\circ_n$ and g^-, since these operations preserve the holomorphy in $V_{r+\varepsilon,\theta+\varepsilon}$ and preserve sufficient decrease in λ for $|\lambda|\to\infty$ (in view of the detailed rules in (Grubb, 1996, Section 2.6), and the analysis in Theorem 3.6). Furthermore, the integrations in λ and ξ_n can be interchanged. So if we define

$$\varphi(x',\xi',\lambda) = \sum_{j+k+|\alpha|=\sigma+n-1}\frac{(-i)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}}\Big[-\partial^{\alpha}_{\xi'}g^{+}(p)_{\sigma-j}\circ_n g^{-}(\partial^{\alpha}_{x'}\mathfrak{q}_{-m-k}(x',0,\xi,\lambda)) + \partial^{\alpha}_{\xi'}g_{\sigma-j}\circ_n(\partial^{\alpha}_{x'}\mathfrak{q}_{-m-k}(x',0,\xi,\lambda))_{+}\Big]đ\xi_n, \tag{3.51}$$

we have that

$$s'_{1-n}(x',\xi') = \frac{i}{2\pi}\int_{\mathscr{C}''}\log\lambda\,\varphi(x',\xi',\lambda)d\lambda. \tag{3.52}$$

An application of Lemma 1.2 gives:

$$\frac{i}{2\pi}\int_{\mathscr{C}''}\log\lambda\,\varphi(x',\xi',\lambda)d\lambda = -\int_{-\infty}^{0}\varphi(x',\xi',t)dt. \tag{3.53}$$

One checks from (3.51) that φ has the quasi-homogeneity property $\varphi(x',t\xi',t^m\lambda)= t^{-m-n+1}\varphi(x',\xi',\lambda)$ for $|\xi'|\geq 1$, $t\geq 1$. Taking strictly homogeneous symbols everywhere gives φ^h, which is integrable at $\xi'=0$ for $\lambda\neq 0$ in view of the regularity properties shown in Theorem 3.6. Now we can apply Lemma 1.3 with dimension n replaced by $n-1$, finding that

$$\begin{aligned}\int_{|\xi'|=1}s'_{1-n}(x',\xi')đS(\xi') &= -\int_{|\xi'|=1}\int_{-\infty}^{0}\varphi^h(x',r\xi',t)dt\,đS(\xi')\\ &= -m\int_{\mathbb{R}^{n-1}}\varphi^h(x',\xi',-1)đ\xi'\\ &= -m\sum_{j+k+|\alpha|=\sigma+n-1}\frac{(-i)^{|\alpha|}}{\alpha!}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\\ &\quad\times\Big[-\partial^{\alpha}_{\xi'}g^{+}(p)^h_{\sigma-j}\circ_n g^{-}(\partial^{\alpha}_{x'}\mathfrak{q}^h_{-m-k}(x',0,\xi,-1))\\ &\quad+\partial^{\alpha}_{\xi'}g^h_{\sigma-j}\circ_n(\partial^{\alpha}_{x'}\mathfrak{q}^h_{-m-k}(x',0,\xi,-1))_{+}\Big]đ\xi_n\,đ\xi',\end{aligned} \tag{3.54}$$

which we recognize as

$$= -m \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x', \xi', -1) đ\xi'. \tag{3.55}$$

This shows that the contribution from $S' = \operatorname{tr}_n G'$ (cf. (3.47)) matches the coefficient $\tilde{\mathfrak{s}}_{n+\sigma}(x')$ of $(-\lambda)^{-1}$ in the diagonal kernel expansion of $\mathscr{S}_\lambda = \operatorname{tr}_n \mathscr{G}_\lambda$, pointwise in x', cf. (3.48), (3.27). Integration of the fiber trace in x' gives

$$\operatorname{res}(G') = -m\tilde{\mathfrak{s}}_{n-1+\sigma}, \tag{3.56}$$

where $\tilde{\mathfrak{s}}_{n-1+\sigma}$ is the coefficient of $(-\lambda)^{-1}$ in the trace expansion of $\mathscr{S}_\lambda$ (and the trace expansion of $\mathscr{G}_\lambda$), cf. (3.32). Adding this identity to (3.46), we find (3.41).

There remains to include the case where the symbol $\mathfrak{q}$ depends on x_n, but this is easy to do. One takes a Taylor expansion of $\mathfrak{q}$ in x_n at $x_n = 0$; since a factor x_n^k lowers the order in the resulting *s.g.o.*'s by k steps (cf. Grubb, 1996, Lemma 2.4.3), only the first $\sigma + n$ terms can contribute to the constants we are studying. Each of these terms enters by the standard composition rules in a very similar way as above, only now one also has to keep track of the effect of powers x_n^k. Again this leads to (3.41). □

Remark 3.11. The proof shows that the identity (3.41) holds in a partly localized way, namely, the pseudodifferential contributions from each side match pointwise in $x \in X$ (before integration in x), and for the singular Green contributions, the ψdo's on X' obtained after taking tr_n match pointwise in $x' \in X'$ (before integration in x').

Remark 3.12. The identity (3.41) holds also when the P_i are taken of order $m = 2$ as in Grubb and Schrohe (2004), which necessitates a replacement of $Q_{i,\lambda}$ by $Q^N_{i,\lambda}$ for a large enough N. For, writing $Q^N_{i,\lambda} = \frac{\partial_\lambda^{N-1}}{(N-1)!} Q_{i,\lambda}$, we see that the term with $(-\lambda)^{-N}$ in $\operatorname{Tr}(A(Q^N_{1,\lambda} - Q^N_{2,\lambda})_+)$ is found by integration of compositions where the symbol terms $\mathfrak{q}^h_{-m-j}$ are replaced by $\frac{\partial_\lambda^{N-1}}{(N-1)!}\mathfrak{q}^h_{-m-j}$. We just give the argument for the *s.g.o.* part. The analysis in Grubb and Schrohe (2004) shows the needed fall-off in λ and integrability in ξ' in this case. Since (with notation as in the proof of Theorem 3.10)

$$(-\lambda)^{-1} \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x', \xi', -1) đ\xi' = \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x', \xi', \lambda) đ\xi'$$

for $\lambda \in \mathbb{R}_-$, an application of $\frac{\partial_\lambda^{N-1}}{(N-1)!}$ gives for the corresponding function $\mathfrak{s}^{(N)h}_{-Nm+1-n}(x', \xi', \lambda)$ resulting from insertion of the $\frac{\partial_\lambda^{N-1}}{(N-1)!}\mathfrak{q}^h_{-m-j}$

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \mathfrak{s}^{(N)h}_{-Nm+1-n}(x', \xi', \lambda) đ\xi' &= \frac{\partial_\lambda^{N-1}}{(N-1)!} \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x', \xi', \lambda) đ\xi' \\ &= \frac{\partial_\lambda^{N-1}}{(N-1)!} [(-\lambda)^{-1} \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x', \xi', -1) đ\xi'] \\ &= (-\lambda)^{-N} \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m+1-n}(x', \xi', -1) đ\xi', \end{aligned} \tag{3.57}$$

showing that the coefficient of $(-\lambda)^{-N}$ in the expansion of $\mathrm{Tr}(A(Q_{1,\lambda}^N - Q_{2,\lambda}^N)_+)$ obeys the same formulas as the coefficient of $(-\lambda)^{-1}$ in Theorem 3.10. Then it is set in relation to the residue in exactly the same way as we did there.

Remark 3.13. The hypothesis that P_1 and P_2 are differential operators was convenient in the proof of Theorem 3.6, but can probably be removed; this would require keeping track of leftover terms resulting from decompositions as in (3.29) with R replaced by a ψdo (having the transmission property). A similar remark can be made for the analysis in Section 4 below, where allowing P_1 to be a ψdo would create a number of extra terms in Lemma 4.1, which however seem manageable. This can be taken up if necessitated by applications.

Let us also remark that the argumentation in Section 4, based on the identity (4.3), would apply to the problem treated in Section 3, giving the desired trace expansions but not the full information on the symbol structure of $\mathscr{G}_\lambda$ in (3.25).

4. The Second Trace Defect Formula for Manifolds with Boundary

Now some words on possible extensions of the other trace defect formula (2.6) to the situation of ψdbo's. Here we assume $m > \sigma + \sigma' + n$ in order to have a trace-class operator $[A, A']Q_{\lambda,+}$. Clearly,

$$\mathrm{Tr}([A, A']Q_{\lambda,+}) = \mathrm{Tr}(A[A', Q_{\lambda,+}]), \tag{4.1}$$

so one might strive to show that $-m\, C_0([A, A'], P_{1,+})$ should equal

$$\mathrm{res}(A[A', (\log P_1)_+]). \tag{4.2}$$

But there are several problems with such a formula. The ψdo part of $A[A', (\log P_1)_+]$ is $P[P', \log P_1]$, hence classical in view of (1.14). But there will in addition be *s.g.o.*-like elements that are not covered by existing theories. One is $G^+(\log P_1) = r^+ \log P_1 e^- J$, which is *not* a standard *s.g.o.*, for example, $G^+(\log(-\Delta))$ on $\mathbb{R}^n_+$ has symbol-kernel $c(x_n + y_n)^{-1} e^{-|\xi'|(x_n+y_n)}$ (for $|\xi'| \geq 1$) with a singularity at $x_n = y_n = 0$. Furthermore, compositions of $(\log P_1)_+$ with ψdbo's will also contain nonstandard terms.

We shall proceed in a different way. Namely, we show for the singular Green part $\mathscr{G}_\lambda$ of $A[A', Q_{\lambda+}]$ that its normal trace $\mathscr{S}_\lambda$ has sufficiently good symbol estimates to allow a "log-transform" (integration together with $\log \lambda$ over a curve $\mathscr{C}''$ as in Theorem 3.10) resulting in a classical ψdo S over X', such that the contribution from $\mathscr{G}_\lambda$ equals $-\frac{1}{m}\mathrm{res}\, S$.

As in Grubb and Schrohe (2004), we assume that the ψdo's P and P' are of normal order 0. (Normal order k means that the symbol and its derivatives are $O(\langle \xi_n \rangle^k)$ at the boundary, here $k \leq$ the order. In general, when P satisfies the transmission condition, it is the sum of a ψdo of normal order -1, a differential operator, and a ψdo vanishing to a very high order at the boundary.)

There is a delicate argument in Grubb and Schrohe (2001, 2004) for showing that terms containing compositions with $G^\pm(Q_\lambda)$ contribute to C_0 with local coefficients; this relies on the exact structure of the symbol of Q_λ at $x_n = 0$ as a function of the roots of the polynomial $p_{1,2}(x', 0, \xi', \xi_n) - \lambda$ in ξ_n. Here we shall replace this argument with an argument using that

$$Q_\lambda = (P_1 - \lambda)^{-1} = -\frac{1}{\lambda} + \frac{1}{\lambda} P_1 (P_1 - \lambda)^{-1} = -\frac{1}{\lambda} + \frac{1}{\lambda} P_1 Q_\lambda, \tag{4.3}$$

where the contributions from the term $\frac{1}{\lambda}$ cancel out in the calculations of commutators. A difficulty in using this is that $P_1 Q_\lambda$ is only of order zero, not of large negative order.

We work in a localized situation (as in Section 3). The singular Green terms appearing in the treatment of $A[A', Q_{\lambda+}]$ are calculated in the following lemma.

Lemma 4.1. *Let $A = P_+ + G$, $A' = P'_+ + G'$ of orders σ, respectively σ', the ψdo's being of normal order zero and the s.g.o.'s being of class zero; assume that P and P' are zero if σ or σ' is noninteger. The singular Green part $\mathcal{G}_\lambda$ of $A[A', Q_{\lambda+}]$ is the sum of terms*

$$\mathcal{G}_\lambda = G[G', Q_{\lambda,+}] + P_+[G', Q_{\lambda,+}] + G[P'_+, Q_{\lambda,+}] + G_{1,\lambda}, \tag{4.4}$$

with the following properties:

1°. *$G[G', Q_{\lambda,+}]$, $P_+[G', Q_{\lambda,+}]$ and $G[P'_+, Q_{\lambda,+}]$ are singular Green operators satisfying the primary formulas*

$$\begin{aligned} G[G', Q_{\lambda,+}] &= GG'Q_{\lambda,+} - GQ_{\lambda,+}G', \\ P_+[G', Q_{\lambda,+}] &= P_+G'Q_{\lambda,+} - P_+Q_{\lambda,+}G', \\ G[P'_+, Q_{\lambda,+}] &= GP'_+Q_{\lambda,+} - GQ_{\lambda,+}P'_+ \end{aligned} \tag{4.5}$$

and the secondary formulas

$$\begin{aligned} G[G', Q_{\lambda,+}] &= \frac{1}{\lambda} GG'P_{1,+}Q_{\lambda,+} - \frac{1}{\lambda} GP_{1,+}Q_{\lambda,+}G', \\ P_+[G', Q_{\lambda,+}] &= \frac{1}{\lambda} P_+G'P_{1,+}Q_{\lambda,+} - \frac{1}{\lambda} P_+P_{1,+}Q_{\lambda,+}G', \\ G[P'_+, Q_{\lambda,+}] &= \frac{1}{\lambda} GP'_+P_{1,+}Q_{\lambda,+} - \frac{1}{\lambda} GP_{1,+}Q_{\lambda,+}P'_+. \end{aligned} \tag{4.6}$$

2°. *$P_+[P'_+, Q_{\lambda,+}]$ is the sum of the ψdo term $(P[P', Q_\lambda])_+$ and the singular Green term $G_{1,\lambda}$ satisfying primarily*

$$G_{1,\lambda} = -G^+(P)G^-([P', Q_\lambda]) - P_+G^+(P')G^-(Q_\lambda) + P_+G^+(Q_\lambda)G^-(P'), \tag{4.7}$$

with

$$\begin{aligned} G^-([P', Q_\lambda]) &= G^-(P'Q_\lambda) - G^-(Q_\lambda P') \\ &= G^-(P')Q_{\lambda,+} + \overline{P'}_+G^-(Q_\lambda) - G^-(Q_\lambda)P'_+ - (\overline{Q_\lambda})_+G^-(P'), \end{aligned} \tag{4.8}$$

and secondarily

$$\begin{aligned} G_{1,\lambda} = &-\frac{1}{\lambda} G^+(P)G^-([P', P_1Q_\lambda]) \\ &- \frac{1}{\lambda} P_+G^+(P')\overline{P}_{1,+}G^-(Q_\lambda) + \frac{1}{\lambda} P_+P_{1,+}G^+(Q_\lambda)G^-(P'), \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} G^-([P', P_1 Q_\lambda]) &= G^-(P' P_1 Q_\lambda) - G^-(P_1 Q_\lambda P') \\ &= G^-(P' P_1) Q_{\lambda,+} + (\overline{PP_1})_+ G^-(Q_\lambda) - P_{1,+} G^-(Q_\lambda) P'_+ \\ &\quad - P_{1,+} (\overline{Q_\lambda})_+ G^-(P'). \end{aligned} \tag{4.10}$$

Proof. The cases in 1° follow easily by insertion of (4.3), since multiplication by $\frac{1}{\lambda}$ commutes with G' and with P'_+, and (since P_1 is a differential operator)

$$(P_1 Q_\lambda)_+ = r^+ P_1 Q_\lambda e^+ = r^+ P_1 e^+ r^+ Q_\lambda e^+ = P_{1,+} Q_{\lambda,+}. \tag{4.11}$$

For the case 2°, we calculate

$$P_+[P'_+, Q_{\lambda,+}] = P_+[P', Q_\lambda]_+ - P_+ G^+(P') G^-(Q_\lambda) + P_+ G^+(Q_\lambda) G^-(P'), \tag{4.12}$$

where we have used in the last expression that P' has normal order ≤ 0. Here, since $G^{\pm}(\frac{1}{\lambda}) = 0$,

$$P_+ G^+(P') G^-(Q_\lambda) = \frac{1}{\lambda} P_+ G^+(P') G^-(P_1 Q_\lambda) = \frac{1}{\lambda} P_+ G^+(P') \overline{P}_{1,+} G^-(Q_\lambda)$$

as in (3.29); similarly,

$$P_+ G^+(Q_\lambda) G^-(P') = \frac{1}{\lambda} P_+ G^+(P_1 Q_\lambda) G^-(P') = \frac{1}{\lambda} P_+ P_{1,+} G^+(Q_\lambda) G^-(P')$$

in view of (4.11). This explains the last two terms in (4.9).

For the first term in the right-hand side of (4.12) we observe:

$$P_+[P', Q_\lambda]_+ = (P[P', Q_\lambda])_+ - G^+(P) G^-([P', Q_\lambda]).$$

The *s.g.o.* term satisfies:

$$G^+(P) G^-([P', Q_\lambda]) = \frac{1}{\lambda} G^+(P) G^-([P', P_1 Q_\lambda]),$$

in view of (4.3). This shows (4.7) and (4.9), and (4.8) and (4.10) follow by calculations such as:

$$G^-(P'' Q_\lambda) = J r^- P''(e^+ r^+ + e^- J J r^-) Q_\lambda e^+ = G^-(P'') Q_{\lambda,+} + \overline{P''}_+ G^-(Q_\lambda). \tag{4.13}$$

Here $\overline{P''} = J P'' J$ is likewise a ψdo satisfying the transmission condition, and the calculation holds regardless of the normal order of P''. We give details for the formulas in (4.10):

$$\begin{aligned} G^-(P' P_1 Q_\lambda) &= G^-(P' P_1) Q_{\lambda,+} + (\overline{P' P_1})_+ G^-(Q_\lambda), \\ G^-(P_1 Q_\lambda P') &= P_{1,+} G^-(Q_\lambda P') = P_{1,+} G^-(Q_\lambda) P'_+ + P_{1,+} (\overline{Q_\lambda})_+ G^-(P'). \end{aligned}$$

□

Lemma 4.2. *Hypotheses as in Lemma* 4.1.

1°. *The singular Green terms appearing in the primary formulas in Lemma* 4.1 *are all of one of the forms*

$$A''Q_{\lambda,+}, \quad A''G^{\pm}(Q_\lambda), \quad A''Q_{\lambda,+}A''', \quad A''G^{\pm}(Q_\lambda)A''', \tag{4.14}$$

(*or the same expressions with* Q_λ *replaced by* $\overline{Q}_\lambda$), *where* $A'' = P''_+ + G''$ *and* $A''' = P'''_+ + G'''$ *are of normal order zero and class zero.*

2°. *The singular Green terms appearing in the secondary formulas in Lemma* 4.1, *with* $\frac{1}{\lambda}$ *omitted, are all of one of the forms* (4.14) (*or the same expressions with* Q_λ *replaced by* $\overline{Q}_\lambda$), *where* $A'' = P''_+ + G''$ *has* P'' *of normal order* $\leq m$ *and* G'' *of class* $\leq m$. *The right factor* $A''' = P'''_+ + G'''$ *is of normal order zero and class zero. The only resulting terms where* $A'' = P''_+$ *of normal order* $= m$ *can occur, are of the form*

$$(PP_1)_+Q_{\lambda,+}G' \quad \text{or} \quad (PP_1)_+G^+(Q_\lambda)G^-(P'), \tag{4.15}$$

with an s.g.o. to the right.

Proof. For the terms in (4.5), this is clear from the basic rules of calculus, cf. e.g., Grubb (1996). For the terms in (4.6) with $\frac{1}{\lambda}$ omitted,

$$\begin{aligned} &GG'P_{1,+}Q_{\lambda,+}, \quad GP_{1,+}Q_{\lambda,+}G', \quad P_+G'P_{1,+}Q_{\lambda,+}, \\ &P_+P_{1,+}Q_{\lambda,+}G', \quad GP'_+P_{1,+}Q_{\lambda,+}, \quad GP_{1,+}Q_{\lambda,+}P'_+, \end{aligned} \tag{4.16}$$

all the expressions except the fourth one have *s.g.o.*'s of class m to the left of $Q_{\lambda,+}$, since, when $P_{1,+}$ is composed to the left with an *s.g.o.* of class zero, we get an *s.g.o.* of class m. For the fourth expression, we observe that since P_1 is a differential operator of order m,

$$P_+P_{1,+} = (PP_1)_+ + \sum_{0\leq j\leq m-1} K_j\gamma_j, \tag{4.17}$$

where PP_1 is of normal order m and $\sum_{0\leq j\leq m-1} K_j\gamma_j$ is an *s.g.o.* of class m; here the K_j are Poisson operators of order $\sigma + m - j$ and the γ_j are the standard trace operators ($\gamma_j u = (D_n^j u)|_{x_n=0}$). Thus

$$P_+P_{1,+}Q_{\lambda,+}G' = (PP_1)_+Q_{\lambda,+}G' + G''Q_{\lambda,+}G', \tag{4.18}$$

where PP_1 has normal order m and G'' has class m.

Now consider the terms in (4.7). The second and third term are clearly of the asserted form with A'' of normal order and class zero. For the first term we use the decomposition of $G^-([P', Q_\lambda])$ given in (4.8) to reach this conclusion.

Finally, consider the expressions in (4.9) with the additional decomposition of a factor in the first term given in (4.10), and $\frac{1}{\lambda}$ omitted:

$$\begin{gathered} G^+(P)[G^-(P'P_1)Q_{\lambda,+} + (\overline{PP_1})_+G^-(Q_\lambda) - P_{1,+}G^-(Q_\lambda)P'_+ - P_{1,+}(\overline{Q_\lambda})_+G^-(P')], \\ G^+(P')\overline{P}_{1,+}G^-(Q_\lambda), \quad P_+P_{1,+}G^+(Q_\lambda)G^-(P') \end{gathered} \tag{4.19}$$

All the expressions have P_1 or $\overline{P}_1$ entering in compositions to the left of a λ-dependent factor. The first line and the first expression in the second line lead to expressions with *s.g.o.*'s of class m to the left. The last expression leads in view of (4.17) to

$$P_+P_{1,+}G^+(Q_\lambda)G^-(P') = (PP_1)_+G^+(Q_\lambda)G^-(P') + G''G^+(Q_\lambda)G^-(P',)$$

where PP_1 has normal order m and G'' has class m. □

We now investigate the normal traces.

Proposition 4.3. *Let G_λ be a parameter-dependent singular Green operator of a form as in Lemma 4.3, and such that the sum of the orders of A'' and A''' is ϱ; then $S_\lambda = \operatorname{tr}_n G_\lambda$ is a ψdo in the parameter-dependent calculus of order $\varrho - m$ with symbol $s(x', \xi', \lambda) \sim \sum_{j\ge 0} s_{\varrho-m-j}(x', \xi', \lambda)$.*

1°. *When A'' is of normal order and class 0, S_λ is of regularity $\varrho - \frac{1}{4}$, and the symbol satisfies estimates (where $\mu = |\lambda|^{\frac{1}{m}}$):*

$$\left|\partial^{\beta,\alpha}_{x',\xi'}\left[s(x', \xi', \lambda) - \sum_{j<J} s_{\varrho-m-j}(x', \xi', \lambda)\right]\right| \dot{\le} \langle\xi'\rangle^{\varrho-\frac{1}{4}-|\alpha|-J}\langle\xi', \mu\rangle^{-m+\frac{1}{4}}, \tag{4.20}$$

for all indices.

2°. *When A'' is of normal order m and class m, S_λ is of regularity $\varrho - m + \frac{1}{4}$, the symbol satisfying estimates*

$$\left|\partial^{\beta,\alpha}_{x',\xi'}\left[s(x', \xi', \lambda) - \sum_{j<J} s_{\varrho-m-j}(x', \xi', \lambda)\right]\right| \dot{\le} \langle\xi'\rangle^{\varrho-m+\frac{1}{4}-|\alpha|-J}\langle\xi', \mu\rangle^{-\frac{1}{4}}, \tag{4.21}$$

for all indices.

Proof. Note that G_λ is in all cases of class zero, since Q_λ is of order $-m$ and A''' is of normal order and class zero.

Consider first the case where there is no factor A''' (or when $A''' = I$); here we get the results fairly easily. Then A'' is of order ϱ. If the class of G'' is zero, A'' enters in the parameter-dependent calculus as the sum of a ψdo and an *s.g.o.* both of order ϱ and regularity ϱ, so when it is composed with $Q_{\lambda,+}$ or $G^\pm(Q_\lambda)$ of order $-m$ and regularity $+\infty$ we get an operator of order $\varrho - m$ and regularity ϱ, in view of Grubb (1996, Theorem 2.7.7, Corollary 2.7.8). By Lemma 3.4, tr_n of it is a ψdo on X' of order $\varrho - m$ and regularity $\varrho - \frac{1}{4}$.

When $\varrho \le \frac{1}{4}$, the estimates (4.20) hold automatically (i.e., are standard symbol estimates), since the power of $\langle\xi', \mu\rangle$ in the parenthesis as in (3.6) can be left out when $\nu \le 0$. For larger ϱ, we compose to the left with $\Lambda^\varrho\Lambda^{-\varrho}$, where $\Lambda^t = \mathrm{OP}'(\langle\xi'\rangle^t)$; it is accounted for in Grubb (1996, Section 2.8) that this defines operators within the calculus (not only for ψdo's on $\mathbb{R}^{n-1}$ but also for *s.g.o.*'s). The preceding considerations now apply to the expression composed to the left with $\Lambda^{-\varrho}$, which satisfies estimates with an extra factor $\langle\xi'\rangle^{-\varrho}$. The resulting operator will satisfy (4.20) with ϱ replaced by zero, and when we recompose with Λ^ϱ to the left, it is easily checked from the composition rules that we obtain an operator satisfying (4.20).

When $A'' = G''$ of class m, its regularity is only $\varrho - m + \frac{1}{2}$ (by Grubb, 1996, (2.3.55)); then the composed operator has regularity $\varrho - m + \frac{1}{2}$, and tr_n of it has regularity $\varrho - m + \frac{1}{4}$ by Lemma 3.4. (The central fact here is that $G'' = G_1 + \sum_{l<m} K_l \gamma_l$, such that the term with the weakest decrease in λ will be $K_{m-1}\gamma_{m-1}Q_{\lambda,+}$, where $\gamma_{m-1}Q_{\lambda,+}$ is a trace operator whose symbol norm is $O(\langle \xi', \mu \rangle^{-\frac{1}{2}})$.) If $\varrho - m < -\frac{1}{4}$, the estimates (4.21) are automatically satisfied, otherwise we obtain them by pulling out a factor $\Lambda^t \Lambda^{-t}$ for a large t as above.

Next, we consider the case where A''' is nontrivial. Here we have to make some extra efforts, both since regularity numbers in compositions are not in general additive, and since we have to deal with some inconvenient terms (4.15). There are now three factors, with the λ-dependent factor in the middle.

Let A'' and A''' have orders ϱ_1 and ϱ_2, so that $\varrho = \varrho_1 + \varrho_2$. Invoking the trick of composing to the left with $\Lambda^{\varrho_1}\Lambda^{-\varrho_1}$ if $\varrho_1 > 0$ and to the right with $\Lambda^{\varrho_2}\Lambda^{-\varrho_2}$ if $\varrho_2 > 0$, we can assume that $\varrho_1, \varrho_2 \leq 0$. To begin with, assume also that q is independent of x_n.

The normal trace of G_λ is found by applying (3.15) to its symbol-kernel $g(x', \xi', x_n, y_n, \lambda)$. We recall from Grubb (1996) the notation $g(x', \xi', D_n, \lambda)$ (or just $g(D_n)$) for the operator on $\mathbb{R}_+$ defined for each (x', ξ', λ) by applying the *s.g.o.* definition in one variable x_n; we use again the notation $\circ_n$ for the composition of such one-dimensional operators. For operators on $\mathbb{R}_+$ of normal order and class zero, tr_n is the usual trace, so there is a certain commutativity, namely e.g.,

$$\begin{aligned} \operatorname{tr}_n(g(D_n) \circ_n g'(D_n)) &= \operatorname{tr}_n(g'(D_n) \circ_n g(D_n)), \\ \operatorname{tr}_n(g(D_n) \circ_n p(D_n)_+) &= \operatorname{tr}_n(p(D_n)_+ \circ_n g(D_n)); \end{aligned} \tag{4.22}$$

the *s.g.o.*'s are smoothing. We shall use this to reduce the most difficult estimates for three components to cases of two components with better properties. Consider e.g., a composition $A''Q_{\lambda,+}A'''$. Here

$$\operatorname{tr}_n(a'' \circ_n q_+ \circ_n a''') = \operatorname{tr}_n(a''' \circ_n a'' \circ_n q_+), \tag{4.23}$$

since $a'' \circ_n q_+$ and a''' are both of normal order and class zero. In the compositions coming from the primary cases in Lemma 4.1, $a''' \circ_n a''$ will be of normal order and class zero. In the compositions coming from the secondary cases in Lemma 4.1, $a''' \circ_n a''$ will be *a singular Green operator of class* m; this is clear if a'' is such one, and if $a'' = p''_+$ is of normal order m, a''' is necessarily an *s.g.o.* of class zero according to Lemma 4.2, so the composite is an *s.g.o.* of class m. The important fact is that we get rid of contributions of the form $(pp_1)_+ q_+ g'$, where a direct attack need not give estimates with a decrease in λ since

$$\sup_{\xi_n} |p_{1,m}(x, \xi)(p_{1,m}(x, \xi) - \lambda)^{-1}| = 1. \tag{4.24}$$

Now the results from the beginning of the proof for compositions $A''Q_{\lambda,+}$ can be applied. When $a''' \circ_n a''$ is of normal order and class zero, this gives a symbol of order $\varrho - m$ and regularity $\varrho - \frac{1}{4}$, and when $a''' \circ_n a''$ is an *s.g.o.* of class m, we get a symbol of order $\varrho - m$ and regularity $\varrho - m + \frac{1}{4}$; since $\varrho \leq 0$, the estimates in (4.20)–(4.21) are automatic.

The commutation is only allowed on the one-dimensional level. To find the full composition of a'', q_+ and a''', we note that

$$\operatorname{tr}_n(a'' \circ q_+ \circ a''') \sim \sum_{\alpha,\beta\in\mathbb{N}^{n-1}} \frac{(-i)^{\alpha+\beta}}{\alpha!\beta!}\operatorname{tr}_n(\partial^{\alpha}_{\xi'}a'' \circ_n \partial^{\alpha}_{x'}\partial^{\beta}_{\xi'}q_+ \circ_n \partial^{\alpha}_{x'}\partial^{\beta}_{x'}a'''), \tag{4.25}$$

and perform the above commutation idea for each term, to find the desired symbol information.

Concerning remainders, an analysis shows that it is only the part of normal order m of PP_1, giving a term $D'D_n^m Q_{\lambda,+}G'''$, that needs special treatment; for the part P_{m-1} of PP_1 of normal order $\le m-1$ one can appeal to the estimate

$$\sup_{\xi_n} |p_{m-1}(p_{1,m}-\lambda)^{-1}| \dot{\le} \langle\xi'\rangle^{\sigma+1}\langle\xi',\mu\rangle^{-1}. \tag{4.26}$$

In the usual remainder term (as in e.g., Grubb and Schrohe, 2004, pf. of Proposition 3.8) in the calculation of the composition inside $\operatorname{tr}_n((D_n^m Q_\lambda)_+G''')$, one can then perform a commutation (4.22) inside the integral w.r.t. h.

If q depends on x_n, it must be Taylor expanded in x_n and each term treated individually; here one uses that in the terms with kth powers of x_n, $k\ge 1$, the symbols coming from q are $O(\lambda^{-2})$ and the order of the *s.g.o.*'s are lowered by k.

There is a similar analysis when $Q_{\lambda,+}$ is replaced by $\overline{Q}_{\lambda,+}$, $G^{\pm}(Q_\lambda)$ or $G^{\pm}(\overline{Q}_\lambda)$. □

It may be remarked that the fraction $\frac{1}{4}$ comes in because of the general application of Lemma 3.4. Particular efforts applied to the individual compositions may give an improvement to $\frac{1}{2}$ in (4.21)—and an analysis extending that of Grubb and Schrohe (2001) would give further improvements, cf. Remark 3.3. But the gain of $\frac{1}{4}$ is sufficient for the present purposes.

We can now conclude with

Theorem 4.4. *Let $A = P_+ + G$ of order σ and normal order and class zero, let $A' = P'_+ + G'$ of order σ' and normal order and class zero, and let P_1 be an auxiliary elliptic differential operator of order $m > \sigma+\sigma'+n$, with no eigenvalues of the principal symbol on $\overline{\mathbb{R}}_-$ (so that $Q_\lambda = (P_1-\lambda)^{-1}$ is defined for large λ in a sector V around $\mathbb{R}_-$). We assume that P and P' are zero if σ or σ' is noninteger.*

Let $\mathcal{S}_\lambda = \operatorname{tr}_n \mathcal{G}_\lambda$ with symbol $\mathfrak{s}(x',\xi',\lambda)$, where $\mathcal{G}_\lambda$ is the singular Green part of $A[A', Q_{\lambda,+}]$. Then $\mathcal{S}_\lambda$ is a family of ψdo's on X' with the properties:

1°. *$\mathcal{S}_\lambda$ is of order $\sigma+\sigma'-m$ and regularity $\sigma+\sigma'-\frac{1}{4}$, the symbol satisfying:*

$$\left|\partial^{\beta,\alpha}_{x',\xi'}\left[\mathfrak{s}(x',\xi',\lambda) - \sum_{j<J}\mathfrak{s}_{\sigma+\sigma'-m-j}(x',\xi',\lambda)\right]\right| \dot{\le} \langle\xi'\rangle^{\sigma+\sigma'-\frac{1}{4}-|\alpha|-J}\langle\xi',\mu\rangle^{-m+\frac{1}{4}}, \tag{4.27}$$

on the rays in V (with $-\lambda = \mu^m e^{i\theta}$, $\mu > 0$), for all α, β, J.

2°. *$\lambda\mathcal{S}_\lambda$ is of order $\sigma+\sigma'$ and regularity $\sigma+\sigma'+\frac{1}{4}$, and for all α, β, J,*

$$\left|\partial^{\beta,\alpha}_{x',\xi'}\left[\mathfrak{s}(x',\xi',\lambda) - \sum_{j<J}\mathfrak{s}_{\sigma+\sigma'-m-j}(x',\xi',\lambda)\right]\right| \dot{\le} \langle\xi'\rangle^{\sigma+\sigma'+\frac{1}{4}-|\alpha|-J}\langle\xi',\mu\rangle^{-\frac{1}{4}}\mu^{-m}. \tag{4.28}$$

Proof. This follows immediately from Proposition 4.3 in view of the description of $\mathcal{G}_\lambda$ given in Lemmas 4.1 and 4.2. □

We can then establish trace expansions. Here we first consider the case where σ and σ' are integers.

Theorem 4.5. *Assumptions as in Theorem* 4.4, *with σ and $\sigma' \in \mathbb{Z}$.*
There is a trace expansion

$$\operatorname{Tr}([A, A']Q_{\lambda,+}) = \sum_{0\le j\le n+\sigma+\sigma'} c_j\,(-\lambda)^{\frac{n+\sigma+\sigma'-j}{m}-1} + O(\lambda^{-1-\frac{1}{4m}}), \tag{4.29}$$

so that

$$C_0([A, A'], P_{1,+}) = c_{n+\sigma+\sigma'} \tag{4.30}$$

(*taken equal to zero if $n+\sigma+\sigma' < 0$*) *is well defined.*
The symbol $s(x', \xi')$ deduced from the symbol $\mathfrak{s}(x', \xi', \lambda)$ of $\mathcal{S}_\lambda$ by

$$s(x', \xi') = \frac{i}{2\pi}\int_{\mathscr{C}''} \log\lambda\, \mathfrak{s}(x', \xi', \lambda)d\lambda \tag{4.31}$$

(*with $\mathscr{C}''$ a curve in $\mathbb{C}\backslash\overline{\mathbb{R}}_-$ encircling the sectorial set containing the eigenvalues of $p_{1,m}(x, \xi)$ for $x \in X$, $|\xi'| \ge 1$*), *is a classical ψdo symbol of order $\sigma+\sigma'$, defining a ψdo S such that*

$$C_0([A, A'], P_{1,+}) = -\frac{1}{m}\operatorname{res}((P[P', \log P_1])_+) - \frac{1}{m}\operatorname{res}(S). \tag{4.32}$$

Proof. For $\mathscr{R}_\lambda = P[P', Q_\lambda]$ we have a diagonal kernel expansion as in (2.22)ff. with coefficients $\tilde r_j(x)$. Integrating over the coordinate patches intersected with $\overline{\mathbb{R}}^n_+$, we find that

$$\begin{aligned} \operatorname{Tr}\mathscr{R}_{\lambda,+} &= \sum_{j<\sigma+\sigma'+m+n} \tilde r_{j,+}(-\lambda)^{\frac{n+\sigma+\sigma'-j}{m}-1} + O(|\lambda|^{-2+\varepsilon}), \qquad \text{where} \\ \tilde r_{j,+} &= \int_{\mathbb{R}^n_+} \operatorname{tr}\tilde r_j(x), \qquad \tilde r_j(x) = \int_{\mathbb{R}^n} r^h_{-m+\sigma+\sigma'-j}(x, \xi, -1)đ\xi. \end{aligned} \tag{4.33}$$

The calculations around Theorem 2.3 apply to this situation, showing that the coefficient of $(-\lambda)^{-1}$ identifies with the residue

$$\tilde r_{n+\sigma+\sigma',+} = -\frac{1}{m}\operatorname{res}((P[P', \log P_1])_+). \tag{4.34}$$

For $\mathcal{S}_\lambda$, the information that it is of order $\sigma+\sigma'-m$ and regularity $\sigma+\sigma'-\frac14$ leads by Lemma 3.5 to a trace expansion

$$\begin{aligned} \operatorname{Tr}_{\mathbb{R}^{n-1}}\mathcal{S}_\lambda &= \sum_{1\le j<n+\sigma+\sigma'-\frac14} d_j\,(-\lambda)^{\frac{n+\sigma+\sigma'-j}{m}-1} + O(\lambda^{-1+\frac{1}{4m}}), \\ d_j &= \tilde{\mathfrak{s}}_{j-1} = \int \operatorname{tr}\tilde{\mathfrak{s}}_{j-1}(x')dx', \\ \tilde{\mathfrak{s}}_l(x') &= \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{\sigma+\sigma'-m-l}(x', \xi', -1)đ\xi', \end{aligned} \tag{4.35}$$

which just misses having a precise term $c(-\lambda)^{-1}$. But we can improve the expansion by using the additional information we have on the symbol in Theorem 4.4. In fact, for $j = \sigma + \sigma' + n$, $l = \sigma + \sigma' + n - 1$, we have a term (taken equal to 0 if $\sigma + \sigma' \leq -n$) satisfying

$$\begin{aligned} |\mathfrak{s}_{-m-n+1}(x', \xi', \lambda)| &\dot{\leq} \langle \xi' \rangle^{-\frac{1}{4}-n+1} \langle \xi', \mu \rangle^{-m+\frac{1}{4}}, \\ |\mathfrak{s}_{-m-n+1}(x', \xi', \lambda)| &\dot{\leq} \langle \xi' \rangle^{\frac{1}{4}-n+1} \langle \xi', \mu \rangle^{-\frac{1}{4}} \mu^{-m}, \end{aligned} \tag{4.36}$$

and the remainder $\mathfrak{s}' = \mathfrak{s} - \sum_{l<\sigma+\sigma'+n} \mathfrak{s}_{\sigma+\sigma'-m-l+1}$ after this term satisfies

$$\begin{aligned} |\mathfrak{s}'| &\dot{\leq} \langle \xi' \rangle^{-\frac{1}{4}-n} \langle \xi', \mu \rangle^{-m+\frac{1}{4}}, \\ |\mathfrak{s}'| &\dot{\leq} \langle \xi' \rangle^{\frac{1}{4}-n} \langle \xi', \mu \rangle^{-\frac{1}{4}} \mu^{-m}. \end{aligned} \tag{4.37}$$

From (4.36) follows as in Grubb (1996, Lemma 2.1.9) that

$$\begin{aligned} |\mathfrak{s}^h_{-m-n+1}(x', \xi', \lambda)| &\dot{\leq} |\xi'|^{-\frac{1}{4}-n+1} |\xi', \mu|^{-m+\frac{1}{4}}, \\ |\mathfrak{s}^h_{-m-n+1}(x', \xi', \lambda)| &\dot{\leq} |\xi'|^{\frac{1}{4}-n+1} |\xi', \mu|^{-\frac{1}{4}} \mu^{-m}, \end{aligned} \tag{4.38}$$

so $\mathfrak{s}^h_{-m-n+1}$ is integrable at $\xi' = 0$ (besides being so for $|\xi'| \to \infty$) when $\lambda \neq 0$. Then

$$\begin{aligned} &\operatorname{Tr}(\mathrm{OP}'(\mathfrak{s}^h_{-m-n+1})) = d_{\sigma+\sigma'+n} (-\lambda)^{-1}, \quad \text{with} \\ &d_{\sigma+\sigma'+n} = \tilde{\mathfrak{s}}_{\sigma+\sigma'+n-1} = \int \operatorname{tr} \tilde{\mathfrak{s}}_{\sigma+\sigma'+n-1}(x') dx', \\ &\tilde{\mathfrak{s}}_{\sigma+\sigma'+n-1}(x') = \int_{\mathbb{R}^{n-1}} \mathfrak{s}^h_{-m-n+1}(x', \xi', -1) đ\xi', \end{aligned} \tag{4.39}$$

as in (4.35). This gives the needed extra term, but we also have to show that remainders do not interfere. (4.28) shows that $|\mathfrak{s}'| \dot{\leq} \langle \xi' \rangle^{\frac{1}{4}-n} \mu^{-m-\frac{1}{4}}$, which integrates in $(n-1)$-space to give an estimate by $\mu^{-m-\frac{1}{4}}$. The difference $\mathfrak{s}^h_{-m-n+1} - \mathfrak{s}_{-m-n+1}$ is $O(\mu^{-m-\frac{1}{4}})$ on its support contained in $\{|\xi'| \leq 1\}$, so it likewise integrates to an $O(\mu^{-m-\frac{1}{4}})$ term. This also holds for the preceding terms, the differences $\mathfrak{s}^h_{\sigma+\sigma'-m-l} - \mathfrak{s}_{\sigma+\sigma'-m-l}$ with $l < \sigma + \sigma' + n - 1$. Then we can finally conclude (4.29) from this and (4.33).

We shall now show that the integral in (4.31) is well defined so that the symbol properties can be checked directly. Again we use the estimates in Theorem 4.4. Note that (4.27) gives too little decrease in λ to allow the integration (4.31), whereas (4.28) gives enough decrease in λ, but much less in ξ'. Using that $\mathfrak{s}$, its terms and remainders are $O(\lambda^{-1-\frac{1}{4}})$, we can insert $\mathfrak{s}$ in (4.31) in order to obtain $s(x', \xi') \sim \sum_{j \geq 0} s_{\sigma+\sigma'-j}(x', \xi')$. Here $s_{\sigma+\sigma'-j}$ is homogeneous of degree $\sigma + \sigma' - j$ in ξ' for $|\xi'| \geq 1$, in view of the following calculation with $t \geq 1$, $\varrho = t^{-m}\lambda$

$$\begin{aligned} s_{\sigma+\sigma'-j}(x', t\xi') &= \frac{i}{2\pi} \int_{\mathscr{C}''} \mathfrak{s}_{\sigma+\sigma'-m-j}(x', t\xi', \lambda) \log \lambda \, d\lambda \\ &= \frac{i}{2\pi} \int_{\mathscr{C}''} t^{\sigma+\sigma'-m-j} \mathfrak{s}_{\sigma-\sigma'-m-j}(x', \xi', \varrho)(\log \varrho + m \log t) t^m \, d\varrho \\ &= t^{\sigma+\sigma'-j} \frac{i}{2\pi} \int_{\mathscr{C}''} \mathfrak{s}_{\sigma+\sigma'-m-j}(x', \xi', \varrho) \log \varrho \, d\varrho = t^{\sigma+\sigma'-j} s_{\sigma+\sigma'-j}(x', \xi'), \end{aligned} \tag{4.40}$$

where we have used that $\frac{i}{2\pi}\int_{\mathscr{C}''} \tilde{s}_{\sigma+\sigma'-m-j}(x',\xi',\varrho)\,d\varrho = 0$ since the integrand is holomorphic on the region to the left of $\mathscr{C}''$ and $O(\lambda^{-\frac{5}{4}})$ for $\lambda \to \infty$ there.

Remainders satisfy

$$|s(x',\xi') - \sum_{j<J} s_{\sigma+\sigma'-j}(x',\xi')| \dot{\le} \langle\xi'\rangle^{\sigma+\sigma'+\frac{1}{4}-J} \tag{4.41}$$

for all J, in view of (4.28). Using the exact terms for $j < J' = J+1$ and the remainder estimate (4.41) with J replaced by J', we can improve (4.41) to

$$|s(x',\xi') - \sum_{j<J} s_{\sigma+\sigma'-j}(x',\xi')| \dot{\le} \langle\xi'\rangle^{\sigma+\sigma'-J}, \tag{4.42}$$

which is the appropriate estimate for showing that s is polyhomogeneous of order $\sigma+\sigma'$. Estimates of derivatives are included in a similar way.

So now s is well defined as a classical symbol of order $\sigma+\sigma'$; it defines the operator S with the residue

$$\operatorname{res} S = \int_{\mathbb{R}^{n-1}} \int_{|\xi'|=1} \operatorname{tr} s_{1-n}(x',\xi')\,đS(\xi')dx'. \tag{4.43}$$

From the fact that

$$s_{1-n}(x',\xi') = \frac{i}{2\pi}\int_{\mathscr{C}''} \log\lambda\, \tilde{s}^{h}_{-m-n+1}(x',\xi',\lambda)d\lambda$$

for $|\xi'| \ge 1$, it is found by use of Lemma 1.2 and Lemma 1.3 for dimension $n-1$, that

$$-\frac{1}{m}\operatorname{res} S = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \tilde{s}_{-m-n+1}(x',\xi',-1)đ\xi'dx' = \tilde{s}_{\sigma+\sigma'+n}. \tag{4.44}$$

Collecting the residues and contributions to $C_0([A,A'],P_{1,+})$ from (4.34) and (4.44), we find (4.32). $\square$

Noninteger orders are included as follows.

Theorem 4.6. *Assumptions as in Theorem 4.4, with σ and $\sigma' \in \mathbb{R}$ and $P = P' = 0$. There is a trace expansion*

$$\operatorname{Tr}([A,A']Q_{\lambda,+}) = \sum_{0\le j<n+\sigma+\sigma'+\frac{1}{4}} c_j\,(-\lambda)^{\frac{n+\sigma+\sigma'-j}{m}-1} + O(\lambda^{-1-\frac{1}{4m}(+\varepsilon)}), \tag{4.45}$$

where $\varepsilon = 0$ if $\sigma+\sigma'+\frac{1}{4} \notin \mathbb{Z}$. Define $C_0([A,A'],P_{1,+}) = c_{n+\sigma+\sigma'}$ if $n+\sigma+\sigma' \in \mathbb{N}$, $C_0([A,A'],P_{1,+}) = 0$ otherwise. Then defining S as in Theorem 4.5, we have that

$$C_0([A,A'],P_{1,+}) = -\frac{1}{m}\operatorname{res}(S). \tag{4.46}$$

Proof. There is no ψdo term in this case. For the *s.g.o.* term $\mathscr{G}_\lambda$ we proceed as in the preceding proof. It goes over verbatim if $\sigma+\sigma' \in \mathbb{Z}$, whereas one has to modify the indexations when $\sigma+\sigma' \notin \mathbb{Z}$. Actually, that is a case where there will be no nontrivial term $c(-\lambda)^{-1}$, and all one has to check is remainder estimates. Since S is of noninteger order then, $\operatorname{res} S$ is also zero. □

Remark 4.7. The local constant $C_0([A, A'], P_{1,+})$ enters into index formulas as follows (similarly to Melrose and Nistor, 1996).

If $A = P_+ + G$ is elliptic of order and class zero, and B is a parametrix of A (necessarily also of order and class zero), then

$$\operatorname{ind}(A) = C_0([A, B], P_{1,+}), \tag{4.47}$$

for any auxiliary $P_{1,+}$. In fact,

$$\operatorname{ind}(A) = \operatorname{Tr}(AB - I) - \operatorname{Tr}(BA - I) = C_0(AB - I - BA + I, P_{1,+}) = C_0([A, B], P_{1,+}),$$

where the first equality is well known, and the second follows from Theorem 3.9, since $AB - I$ and $BA - I$ are of order $-\infty$. Since the index is invariant under homotopies, we moreover have that

$$\operatorname{ind}(A) = C_0([A^0, B^0], P_{1,+}), \tag{4.48}$$

where A^0 is defined from the principal symbol a^0 of A and B^0 is defined from a parametrix of a^0. By (4.32), the expressions in (4.47) and (4.48) equal residues.

Remark 4.8. When $A = P_+ + G$ goes from a bundle E over X to a bundle F over X, and $A' = P'_+ + G'$ goes from F to E (both of normal order and class zero), then

$$C_0(AA', P_{2,+}) - C_0(A'A, P_{1,+}) \tag{4.49}$$

is a residue, when P_1 and P_2 are auxiliary operators in E resp. F of the same order m. For,

$$\begin{aligned}
&\operatorname{Tr}_F(AA'(P_2 - \lambda)_+^{-1}) - \operatorname{Tr}_E(A'A(P_1 - \lambda)_+^{-1}) \\
&\quad = \operatorname{Tr}_E(A'(P_2 - \lambda)_+^{-1}A - A'A(P_1 - \lambda)_+^{-1}) \\
&\quad = \operatorname{Tr}_E\Big(A'\frac{1}{\lambda}[(P_2(P_2 - \lambda)^{-1})_+A - A(P_1(P_1 - \lambda)^{-1})_+]\Big),
\end{aligned}$$

which can be analyzed in local coordinates just as we did above, to show that the coefficient of $(-\lambda)^{-1}$ is a residue (with ψdo part $(P' \log P_2 P - P'P \log P_1)_+$). When A is elliptic of order zero and B is a parametrix, we conclude that

$$\begin{aligned}
\operatorname{ind}(A) &= C_0(AB - I, P_{2,+}) - C_0(BA - I, P_{1,+}) \\
&= C_0(AB, P_{2,+}) - C_0(BA, P_{1,+}) - C_0(I, P_{2,+}) + C_0(I, P_{1,+})
\end{aligned} \tag{4.50}$$

is a residue, where $C_0(I, P_{i,+}) = -\frac{1}{m}\operatorname{res}(\log P_i)_+$ is obtained by integrating the fiber trace of the pointwise formula (1.26) over X.

5. Extension of the Res of Log Formula to Pseudodifferential Boundary Problems

With these techniques at hand, we shall also investigate possible extensions of the res of log formula (0.4) to realizations of elliptic pseudodifferential boundary problems. Consider a normal elliptic realization $(P_+ + G)_T$, as defined in Grubb (1996, Section 3.3). Here P is a classical ψdo in E of integer order $m > 0$ satisfying the transmission condition at X', G is a singular Green operator in E of order and class m, and $T = \{T_0, \dots, T_{m-1}\}$ is a normal trace operator with entries T_k of order and class k going from E to F_k, all polyhomogeneous. E and the F_k are hermitian C^∞ vector bundles over X resp. X'. We assume that the conditions for uniform parameter-ellipticity in Grubb (1996, Definition 3.3.1) are satisfied on the rays in a sector V around $\mathbb{R}_-$.

The resolvent

$$((P_+ + G)_T - \lambda)^{-1} = R_\lambda = Q_{\lambda,+} + G_\lambda \tag{5.1}$$

was constructed in Grubb (1996, Section 3.3) and shown to belong to the parameter-dependent calculus set up in the book. Complex powers $((P_+ + G)_T)^z$ were described to some extent in Grubb (1996, Section 4.4), just for $\operatorname{Re} z < 0$, where it was shown that their singular Green part has some, but not all of the symbol estimates of standard *s.g.o.*'s. The logarithm of $(P_+ + G)_T$ has not, to our knowledge, been discussed anywhere.

Since the complex powers were only considered for $\operatorname{Re} z < 0$, we cannot draw conclusions about a derivative at $z = 0$, but one can try a formula as in (1.1); it generally leads to an operator outside the Boutet de Monvel calculus. Rather than going into a deeper analysis of such operators and the possibility of defining residues on them, we shall show a generalization of (0.4) where a residue of the logarithm of the ψdo part does enter, and the *s.g.o.* part is reduced to the residue of a classical ψdo on X'; the "nice part" of the log contribution from G_λ.

It is shown in Grubb (1996, Theorem 3.3.5, 3.3.10), that when $m > n$, the resolvent has a trace expansion with at least $n + 1$ exact terms:

$$\operatorname{Tr} R_\lambda = \sum_{0 \le j \le n} c_j (-\lambda)^{\frac{n-j}{m} - 1} + O(\lambda^{-1 - \frac{1}{4m}}), \tag{5.2}$$

valid for $\lambda \to \infty$ in the sector of parameter-ellipticity. (If the regularity is greater than 1, there will be more terms in the expansion.) The coefficients c_j are defined by integration of the strictly homogeneous terms in the symbols of $Q_{\lambda,+}$ and G_λ; in particular, the coefficient of $(-\lambda)^{-1}$,

$$C_0(I, (P_+ + G)_T) = c_n \tag{5.3}$$

is defined from the term of order $-m - n$ in the symbol of $Q_{\lambda,+}$ and the term of order $-m + 1 - n$ in the symbol of G_λ (in local coordinates). As usual, Q_λ is the inverse of $P - \lambda$, defined on a larger compact n-dimensional manifold $\widetilde{X}$ in which X is smoothly imbedded.

In the following, we work in a localization to $\mathbb{R}^n$ (with X carried over to subsets of $\overline{\mathbb{R}}^n_+$), as in the preceding sections. Let Q_λ, G_λ and $S_\lambda = \operatorname{tr}_n G_\lambda$ have symbols q, g

and $s = \operatorname{tr}_n g$, respectively, with expansions, e.g.,

$$q(x, \xi, \lambda) \sim \sum_{j\geq 0} q_{-m-j}(x, \xi, \lambda),$$
$$s(x', \xi', \lambda) \sim \sum_{j\geq 0} s_{-m-j}(x', \xi', \lambda). \tag{5.4}$$

Then

$$c_n = c^P_{n,+} + c^G_n, \qquad \text{with}$$
$$c^P_{n,+} = \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n} \operatorname{tr} q^h_{-m-n}(x, \xi, -1)đ\xi\, dx, \tag{5.5}$$
$$c^G_n = \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \operatorname{tr} s^h_{-m+1-n}(x', \xi', -1)đ\xi'\, dx'.$$

Consider the elliptic system $\{P_+ + G, T\}$ defining the operator $(P_+ + G)_T$ we are interested in. The order is m, the regularity of P is m, and the regularity ν of the full system is an integer or half-integer lying in the interval $[\frac{1}{2}, m]$ (cf. Grubb, 1996, (3.3.11))—unless the operators are purely differential, in which case the regularity is $+\infty$ (any $\nu \in \mathbb{R}$ works then). As shown in Grubb (1996, Theorem 3.3.2), Q_λ is of order $-m$ and regularity m, and G_λ is of order $-m$, class zero and regularity ν (the regularities being replaced by $+\infty$ in the differential operator case).

With reference to the lemmas in Section 3 here, the proof of (5.2) in Grubb (1996, Section 3.3) consists of applying Lemma 3.2 to the pseudodifferential part $Q_{\lambda,+}$ to get pointwise expansions of the diagonal kernel of Q_λ and integrate these over $\mathbb{R}^n_+$, applying Lemma 3.5 to the normal trace of the *s.g.o.* part G_λ to get pointwise expansions of the diagonal kernel and integrate these over $\mathbb{R}^{n-1}$ (contributions from interior patches are smoothing and $O(\lambda^{-1-\frac{1}{4m}})$), and adding the expansions.

Now we want to relate the coefficients $c^P_{n,+}$ and c^G_n to residues. $c^P_{n,+}$ is immediately understood on the basis of Theorem 1.4 (integrating the pointwise version over $\mathbb{R}^n_+$). For c^G_n, we have the following lemma.

To explain the curve $\mathscr{C}''$ used there, we recall from Grubb (1996) that the ellipticity hypothesis assures that the strictly homogeneous principal symbol $p^h_m(x, \xi) - \lambda$ and principal boundary symbol operator $\{p^h(x', 0, \xi', D_n) + g^h(x', \xi', D_n) - \lambda, t^h(x', \xi', D_n)\}$ are invertible for λ in a sector around $\mathbb{R}_-$, $(\xi', \lambda) \neq 0$, such that the resolvent exists in a keyhole region $V_{r,\varepsilon}$ (1.18) except at finitely many points. By a small rotation, we can assure that no eigenvalues are on $\mathbb{R}_-$. As $\mathscr{C}''$ we take a curve in $\mathbb{C}\setminus\overline{\mathbb{R}}_-$ around $\complement V_{r,\varepsilon}$ and the spectrum except possibly zero; it can be the boundary of $V_{r',\varepsilon'}$ with suitably small r' and ε'.

Lemma 5.1. *Define from s and S_λ the reduced symbol s' and the corresponding operator S'_λ*

$$s'(x', \xi', \lambda) = s(x', \xi', \lambda) - s_{-m}(x', \xi', \lambda),$$
$$S'_\lambda = \mathrm{OP}'(s'(x', \xi', \lambda)), \tag{5.6}$$

and set

$$B = \frac{i}{2\pi}\int_{\mathscr{C}''} S'_\lambda \log\lambda\, d\lambda,$$
$$b_{-j}(x', \xi') = \frac{i}{2\pi}\int_{\mathscr{C}''} s_{-m-j}(x', \xi', \lambda)\log\lambda\, d\lambda \quad \textit{for } j \geq 1, \quad |\xi'| \geq 1. \tag{5.7}$$

Then B is a classical ψdo on $\mathbb{R}^{n-1}$ *of order* -1 *with symbol* $b \sim \sum_{j\geq 1} b_{-j}$.

Proof. Since s' is of order $-m-1$ and regularity $\nu - \frac{5}{4}$, we have that

$$s' \text{ is } O\Big(\Big(\langle\xi'\rangle^{\nu-\frac{5}{4}} + \langle\xi', \mu\rangle^{\nu-\frac{5}{4}}\Big)\langle\xi', \mu\rangle^{-m-1-\nu+\frac{5}{4}}\Big),$$

hence falls off like λ to the power $\max\{-1-\frac{1}{m}, -1-\frac{\nu-\frac{1}{4}}{m}\}$, so the symbol multiplied by $\log\lambda$ is $O(\lambda^{-1-\delta})$ with a $\delta > 0$. There are similar estimates for derivatives. Then B is defined as a bounded operator in L_2, and its symbol terms b_{-j} are found by integration of the terms in s' as stated. To see that b_{-j} is homogeneous of degree $-j$ in ξ' for $|\xi'| \geq 1$, we write for $t \geq 1$, with $\varrho = t^{-m}\lambda$

$$\begin{aligned} b_{-j}(x', t\xi') &= \frac{i}{2\pi}\int_{\mathscr{C}''} s_{-m-j}(x', t\xi', \lambda)\log\lambda\, d\lambda \\ &= \frac{i}{2\pi}\int_{\mathscr{C}''} t^{-m-j}s_{-m-j}(x', \xi', \varrho)(\log\varrho + m\log t)t^m\, d\varrho \\ &= t^{-j}\frac{i}{2\pi}\int_{\mathscr{C}''} s_{-m-j}(x', \xi', \varrho)\log\varrho\, d\varrho = t^{-j}b_{-j}(x', \xi'), \end{aligned} \tag{5.8}$$

where the term with $m\log t$ drops out as in (4.40). Derivatives in x' and ξ' and remainders are easily checked. □

B can in a sense be considered as the "nice ψdo part" of the logarithmic contribution from the normal trace of the singular Green term G_λ in the resolvent; we have only left out the principal symbol of G_λ. (It is not clear what kind of operator comes out of applying the log Cauchy formula to this term in general.)

Theorem 5.2. *Consider a normal elliptic realization* $(P_+ + G)_T$, *where P is of integer order* $m > 0$, *G is of order and class m, and* $T = \{T_0, \ldots, T_{m-1}\}$ *is normal, with entries* T_k *of order and class k. Assume that* $m > n$.

With B defined in Lemma 5.1, *we have that*

$$C_0(I, (P_+ + G)_T) = -\frac{1}{m}\mathrm{res}((\log P)_+) - \frac{1}{m}\mathrm{res}(B). \tag{5.9}$$

Here

$$c^P_{n,+} = -\frac{1}{m}\mathrm{res}((\log P)_+), \qquad c^G_n = -\frac{1}{m}\mathrm{res}(B). \tag{5.10}$$

Proof. This goes as in Theorems 3.10 and 4.5. The necessary symbol information has been provided above, so we just have to identify the contributions from the specific homogeneous terms. □

In some cases one can get a more informative formula, as the following example (similar to Grubb and Schrohe, 2004, Remark 4.2) shows.

Example 5.3. Consider a second-order strongly elliptic differential operator P, of the form

$$P = -\partial_{x_n}^2 I + P' \tag{5.11}$$

in a collar neighborhood of X', where P' is a positive self-adjoint second-order elliptic operator on X'. Let $T = \gamma_0$, restriction to X'; then P_{γ_0} is the Dirichlet realization of P. The resolvent R_λ does not have high enough order to be trace-class, but we can iterate it, considering

$$R_\lambda^N = \frac{\partial_\lambda^{N-1}}{(N-1)!} R_\lambda = \frac{\partial_\lambda^{N-1}}{(N-1)!} Q_{\lambda,+} + \frac{\partial_\lambda^{N-1}}{(N-1)!} G_\lambda = (Q_\lambda)_+^N + G_\lambda^{(N)} \tag{5.12}$$

for $N > n/2$ instead. It is easily verified (further details in Grubb and Schrohe, 2004, Remark 4.2) that $\operatorname{tr}_n G_\lambda = -\frac{1}{4}(P' - \lambda)^{-1}$, a resolvent on X' (times a constant). The interior contribution to the coefficient of $(-\lambda)^{-N}$ is

$$-\frac{1}{2}\operatorname{res}_X((\log P)_+), \tag{5.13}$$

in view of the considerations in Remark 3.12. The same considerations plus the information from Section 1 for closed manifolds, applied to $-\frac{1}{4}(P' - \lambda)^{-1}$, gives that the *s.g.o.* contribution is

$$\frac{1}{8}\operatorname{res}_{X'}(\log P'). \tag{5.14}$$

So here

$$C_0(I, (P_+)_{\gamma_0}) = -\frac{1}{2}\operatorname{res}_X((\log P)_+) + \frac{1}{8}\operatorname{res}_{X'}(\log P'), \tag{5.15}$$

where we have logarithmic operators in both terms.

It may be remarked as in Grubb and Schrohe (2004) that the interior term vanishes when n is odd, the boundary term vanishes when n is even.

Remark 5.4. Realizations defined by spectral boundary conditions are not covered by the above theorem, since the boundary condition is not normal. Let us however make some remarks on what can be said for them. Let P be a second-order strongly elliptic differential operator on X which is as in (5.11) on X'. Consider the realization P_T defined as in Grubb (2003) by a boundary condition

$$\Pi_1\gamma_0 u = 0, \qquad \Pi_2(\gamma_1 u + B\gamma_0 u) = 0, \tag{5.16}$$

where Π_1 is a pseudodifferential projection, $\Pi_2 = I - \Pi_1$, and B is a first-order ψdo on X' (all classical). Under the assumption that Π_1 commutes principally with P' and a suitable parameter-ellipticity condition is satisfied (cf. Grubb, 2003, Theorem 2.10), the resolvent $R_\lambda = (P_T - \lambda)^{-1} = Q_{\lambda,+} + G_\lambda$ exists in a sector of $\mathbb{C}$; this includes the case where $P_T = (D_\Pi)^* D_\Pi$ for a Dirac-type operator with a well-posed boundary condition $\Pi\gamma_0 u = 0$, Π an orthogonal ψdo projection (playing the role of Π_1). A particular case is where D_Π represents the Atiyah-Patodi-Singer problem. For N so large that R_λ^N $(= \frac{\partial_\lambda^{N-1}}{(N-1)!} R_\lambda)$ is trace-class (i.e., $2N > n$), there is an expansion

$$\operatorname{Tr} R_\lambda^N = \sum_{0\le j\le n} c_j(-\lambda)^{\frac{n-j}{2}-N} + (c_0' \log(-\lambda) + c_0'')(-\lambda)^{-N} + O(\lambda^{-N-\frac{1}{2}+\varepsilon}) \tag{5.17}$$

(with more terms that are unimportant here). The coefficient of $(-\lambda)^{-N}$, $c_n + c_0''$ is generally nonlocal, so there is no generalization of (5.9). However, one can show that the *difference between two such coefficients* with different P and B but *the same projection* Π_1, is a residue.

In fact, for two such realizations P_T and $\overline{P}_{\overline{T}}$, consider the resolvent difference and its iterated versions:

$$\begin{aligned} R_\lambda - \overline{R}_\lambda &= (Q_\lambda - \overline{Q}_\lambda)_+ + G_\lambda - \overline{G}_\lambda, \\ R_\lambda^N - \overline{R}_\lambda^N &= (Q_\lambda^N - \overline{Q}_\lambda^N)_+ + G_\lambda^{(N)} - \overline{G}_\lambda^{(N)}; \end{aligned} \tag{5.18}$$

with $G_\lambda^{(N)} = \frac{\partial_\lambda^{N-1}}{(N-1)!} G_\lambda$. The interior part $(Q_\lambda^N - \overline{Q}_\lambda^N)_+$ has for $2N > n$ a complete expansion in pure powers with local coefficients, where the coefficient of $(-\lambda)^{-N}$ is identified with $-\frac{1}{2}\operatorname{res}((\log P - \log \overline{P})_+)$, by use of the local formulas in Theorem 2.2 (cf. also Remark 3.12). For the *s.g.o.* part, we have according to Grubb (2003, Theorem 4.1, (4.7)).

$$\begin{aligned} \operatorname{tr}_n(G_\lambda - \overline{G}_\lambda) &= \mathscr{S}_{1,\lambda} + \mathscr{S}_{2,\lambda} + \mathscr{S}_{3,\lambda}, \quad \text{with} \\ \mathscr{S}_{1,\lambda} &= -\frac{1}{2}\Pi_2((P' - \lambda)^{-1} - (\overline{P}' - \lambda)^{-1}), \end{aligned} \tag{5.19}$$

$\frac{\partial_\lambda^{N-1}}{(N-1)!}\mathscr{S}_{2,\lambda}$ strongly polyhomogeneous of degree $-2N$, and $\frac{\partial_\lambda^{N-1}}{(N-1)!}\mathscr{S}_{3,\lambda}$ having its symbol in $S^{1,-2N-1} \cap S^{-2N,0}$, in the notation of Grubb and Seeley (1995), for any $N \ge 1$. All three terms have trace expansions when $2N > n$:

$$\operatorname{Tr} \frac{\partial_\lambda^{N-1}}{(N-1)!}\mathscr{S}_{i,\lambda} = \sum_{0\le j\le n-1} d_{i,j}^{(N)}(-\lambda)^{\frac{n-1-j}{2}-N} + O(\lambda^{-N-\frac{1}{2}+\varepsilon}) \tag{5.20}$$

with coefficients determined from strictly homogeneous symbols. For $\mathscr{S}_{1,\lambda}$ this is seen as in Section 2, for $\mathscr{S}_{2,\lambda}$ it is straightforward, and for $\mathscr{S}_{3,\lambda}$ it is seen from Grubb (2003) or Remark 3.3. Here the $d_{i,n-1}^{(N)}$ are independent of N. Defining

$$S = \frac{i}{2\pi}\int_{\mathscr{C}''} \operatorname{tr}_n(G_\lambda - \overline{G}_\lambda)\log\lambda\, d\lambda = \frac{i}{2\pi}\int_{\mathscr{C}''}\sum_{i=1}^{3}\mathscr{S}_{i,\lambda}\log\lambda\, d\lambda,$$

we can identify $\sum_{i=1}^3 d_{i,n-1}^{(N)}$ with $-\frac{1}{2}\operatorname{res} S$, as in the earlier proofs. Then we conclude that

$$C_0(I, P_T) - C_0(I, \overline{P}_{\overline{T}}) = -\frac{1}{2}\operatorname{res}((\log P - \log \overline{P})_+) - \frac{1}{2}\operatorname{res} S, \tag{5.21}$$

when P_T and $\overline{P}_{\overline{T}}$ have the same projection Π_1.

References

Agranovich, M. S. (1987). Some asymptotic formulas for elliptic pseudodifferential operators. *Functional Analysis Appl.* 21:53–56.

Boutet de Monvel, L. (1971). Boundary problems for pseudodifferential operators. *Acta Math.* 126:11–51.

Cardona, A., Ducourtioux, C., Paycha, S. (2003). From tracial anomalies to anomalies in quantum field theory. *Comm. Math. Phys.* 242:31–65.

Cardona, A., Ducourtioux, C., Magnot, J. P., Paycha, S. (2002). Weighted traces on algebras of pseudodifferential operators. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 5:503–540.

Duistermaat, J. J., Guillemin, V. W. (1975). The spectrum of positive elliptic operators and periodic bicharacteristics. *Inventiones Math.* 39–79.

Fedosov, B. V., Golse, F., Leichtnam, E., Schrohe, E. (1996). The noncommutative residue for manifolds with boundary. *J. Funct. Anal.* 142:1–31.

Grubb, G. (1996). *Functional Calculus of Pseudodifferential Boundary Problems. Progress in Math. vol. 65, Second Edition*. Boston: Birkhäuser (first edition: 1986).

Grubb, G. (2003). Spectral boundary conditions for generalizations of Laplace and Dirac operators. *Comm. Math. Phys.* 240:243–280.

Grubb, G. (2005). *A Resolvent Approach to Traces and Zeta Laurent Expansions.* Am. Math. Soc. Contemp. Math. Proc., vol. 366, pp. 67–93. arXiv: math.AP/0311081.

Grubb, G., Hansen, L. (2002). Complex powers of resolvents of pseudodifferential operators. *Comm. Part. Diff. Eq.* 27:2333–2361.

Grubb, G., Schrohe, E. (2001). Trace expansions and the noncommutative residue for manifolds with boundary. *J. Reine Angew. Math.* 536:167–207.

Grubb, G., Schrohe, E. (2004). Traces and quasi-traces on the Boutet de Monvel algebra. *Ann. Inst. Fourier* 54:1641–1696.

Grubb, G., Seeley, R. (1995). Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems. *Invent. Math.* 121:481–529.

Grubb, G., Seeley, R. (1996). Zeta and eta functions for Atiyah-Patodi-Singer operators. *J. Geom. An.* 6:31–77.

Guillemin, V. (1985). A new proof of Weyl's formula on the asymptotic distribution of eigenvalues. *Adv. Math.* 102:184–201.

Kato, T. (1996). *Perturbation Theory for Linear Operators, Grundlehren vol. 132*. Berlin, Heidelberg: Springer-Verlag.

Kontsevich, M., Vishik, S. (1995). Geometry of determinants of elliptic operators. In: Gindikin, S., et al., eds., *Functional Analysis on the Eve of the 21'st Century (Rutgers Conference in honor of I. M. Gelfand 1993)*. Vol. I, Progr. Math. 131. Boston: Birkhäuser, pp. 173–197.

Lesch, M. (1999). On the noncommutative residue for pseudodifferential operators with log-polyhomogeneous symbols. *Ann. Global Anal. Geom.* 17:151–187.

Loya, P. (2001a). The structure of the resolvent of elliptic pseudodifferential operators. *J. Funct. Anal.* 184:77–134.

Loya, P. (2001b). Tempered operators and the heat kernel and complex powers of elliptic pseudodifferential operators. *Comm. Part. Diff. Equ.* 26:1253–1321.

Melrose, R., Nistor, V. (1996). *Homology of pseudodifferential operators I. Manifolds with boundary*. arXiv: funct-an/9606005. Manuscript.

Okikiolu, K. (1995). The multiplicative anomaly for determinants of elliptic operators. *Duke Math. J.* 79:723–750.

Scott, S. (2005). The residue determinant. *Comm. Part. Diff. Equ.* 30:483–507.

Seeley, R. T. (1967). *Complex Powers of an Elliptic Operator*. Am. Math. Soc. Proc. Symp. Pure Math. X, 1966 Providence: Amer. Math. Soc., pp. 288–307.

Wodzicki, M. (1984). Local invariants of spectral asymmetry. *Invent. Math.* 75:143–178.