Trace defect formulas and zeta values for boundary problems

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Abstract. The aim of this lecture is to describe a circle of results for classical pseudodifferential operators (ψdo's) on closed manifolds, and to report on their extension to pseudodifferential boundary operators (ψdbo's) on compact manifolds with boundary.

1. Recollection of results for manifolds without boundary

We first consider a compact n-dimensional C^∞ manifold X without boundary, provided with a smooth hermitian vector bundle E. A classical ψdo A in E of order $\sigma \in \mathbb{R}$ is defined in local charts by formulas

$$Au(x) = \mathrm{OP}(a(x,\xi))u(x) = \int e^{i(x-y)\cdot\xi} a(x,\xi)u(y)\, dy đ\xi,$$

where $đ\xi = (2\pi)^{-n} d\xi$; here the symbol $a(x,\xi)$ is a C^∞ function with an expansion in homogeneous terms

$$\begin{aligned} &a(x,\xi) \sim \sum_{j\in\mathbb{N}} a_{\sigma-j}(x,\xi), \quad \mathbb{N} = \{0,1,2,\dots\}; \\ &a_{\sigma-j}(x,t\xi) = t^{\sigma-j} a_{\sigma-j}(x,\xi) \text{ for } t \geq 1,\ |\xi| \geq 1,\ \text{all } j; \\ &\partial_x^\beta \partial_\xi^\alpha [a(x,\xi) - \sum_{j<J} a_{\sigma-j}(x,\xi)] \text{ is } O([\xi]^{\sigma-J}),\ \text{all } \alpha,\beta \in \mathbb{N}^n, J \in \mathbb{N}. \end{aligned} \tag{1.1}$$

The symbol $[\xi]$ stands for a smooth positive function of ξ coinciding with $|\xi|$ for $|\xi| \geq 1$.

Besides the usual trace $\mathrm{Tr}\, A$ that exists for $\sigma < -n$ (where A is trace-class), two other trace functionals have in recent years been introduced. By a trace functional on a class of operators $\mathcal{M}$ we mean a linear functional $\ell(A)$ such that $\ell([A,A']) = 0$ when A, A' and $[A,A']$ belong to $\mathcal{M}$; here $[A,A']$ is the commutator $AA' - A'A$. The two new traces are (i) the noncommutative residue, (ii) the canonical trace.

(i) *The noncommutative residue* $\mathrm{res}(A)$.

This is a functional defined on classical ψdo's, introduced ca. 1984 by Wodzicki [**W1**] and independently Guillemin [**Gu**], see also Kassel [**K**] for a nice overview. It has the properties:

- It is defined for all classical ψdo's in E, uniquely (up to a factor) if X is connected.

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- A formula for it is

$$\operatorname{res}(A) = \int_X \int_{|\xi|=1} \operatorname{tr} a_{-n}(x,\xi)\, đS(\xi) dx, \tag{1.2}$$

where part of the information is that the integrand w.r.t. x has a coordinate invariant meaning; here a_{-n} is set equal to 0 if $\sigma - j$ does not take the value $-n$. (The symbol tr indicates fiber trace.)
- It is tracial: $\operatorname{res}([A, A']) = 0$ for all classical ψdo's A, A'.
- It is *local*, also called *symbolic* (in the sense that it depends only on the homogeneous terms in the symbol down to a certain order, here $-n$, in any localization).
- It is 0 if $\sigma \notin \mathbb{Z}$ (for then $a_{-n} = 0$).
- It is 0 if $\sigma < -n$ (for then $a_{-n} = 0$). Hence it does not extend the functional $\operatorname{Tr} A$!

The noncommutative residue plays an important role in the study of geometric invariants; more on this later. The fact that it does not extend the standard trace makes its role a little different from the role of Tr. There is another trace functional, defined only on a subset of the classical ψdo's, and tracial only under suitable circumstances, which however does extend the standard trace:

(ii) *The canonical trace* $\operatorname{TR} A$.

This was introduced ca. 1994 by Kontsevich and Vishik [**KV**]; further information and extensions are found in Lesch [**L**] (1999) and [**G3**] (2005). It has the properties:

- It is *global* (depends on the full structure of A as an operator on X).
- It is defined only for some A. It is so in the cases:

  (1) $\sigma < -n$, then $\operatorname{TR} A = \operatorname{Tr} A$;
  (2) $\sigma \notin \mathbb{Z}$;
  (3) $\sigma \in \mathbb{Z}$, n odd, A is even-even;
  (4) $\sigma \in \mathbb{Z}$, n even, A is even-odd.

  Here A is said to have *even-even* alternating parity (in short: be even-even), when

$$a_{\sigma-j}(x,-\xi) = (-1)^{\sigma-j} a_{\sigma-j}(x,\xi) \text{ for } |\xi| \geq 1, \text{ all } j; \tag{1.3}$$

this holds e.g. for differential operators and their solution operators.

A is said to have *even-odd* alternating parity (in short: be even-odd), when

$$a_{\sigma-j}(x,-\xi) = (-1)^{\sigma-j-1} a_{\sigma-j}(x,\xi) \text{ for } |\xi| \geq 1, \text{ all } j; \tag{1.4}$$

this hold e.g. for $|D|$ and for $D|D|^{-1}$ when D is a Dirac operator.
- In the cases (1)–(4),

$$\operatorname{TR}(A) = \int_X ⨍ \operatorname{tr} a(x,\xi)\, đ\xi dx, \tag{1.5}$$

where the integrand w.r.t. x has a coordinate invariant meaning. Here $⨍ f(x,\xi)\, đ\xi$ is a finite-part integral (*partie finie* in the sense of Hadamard), defined as follows: When $f(x,\xi)$ is a classical symbol of order σ, then the integral over $\{|\xi| \le R\}$ has an asymptotic expansion in R:

$$\int_{|\xi|\le R} f(x,\xi)\,d\!\!{}^-\!\xi \sim \sum_{j\in\mathbb{N}, j\ne n+\sigma} a_j(x)R^{\sigma+n-j} + a_0'(x)\log R + a_0''(x) \text{ for } R\to\infty, \tag{1.6}$$

and one sets $⨍ f(x,\xi)\,d\!\!{}^-\!\xi = a_0''(x)$, the R-independent term.

- The trace property holds in the sense that when A and A' are of order σ resp. σ', $\mathrm{TR}([A,A']) = 0$ when
  - (1′) $\sigma + \sigma' < -n$,
  - (2′) $\sigma + \sigma' \in \mathbb{R}\setminus\mathbb{Z}$,
  - (3′) σ and $\sigma' \in \mathbb{Z}$, n is odd, A and A' are both even-even or both even-odd,
  - (4′) σ and $\sigma' \in \mathbb{Z}$, n is even, A is even-odd and A' is even-even.

[**KV**] gave a definition of $\mathrm{TR}(A)$ in the cases (1)–(3) based on homogeneous distributions; it is also used by Connes and Moscovici in [**CM**]. [**L**] showed the equivalence of the definition of [**KV**] with the above one based on the finite-part integral $⨍ a(x,\xi)\,d\!\!{}^-\!\xi$, in case (2). [**G3**] extended the equivalence to the case (3), adding the case (4), by a calculatory proof that avoids comparison of meromorphic extensions. [**L**] generalized the definition in case (2) to log-polyhomogeneous symbols; this was followed up for the cases (3) and (4) in [**G3**].

The *log-polyhomogeneous symbols* of order σ and log-degree k are introduced in [**L**] as the functions of the form

$$r(x,\xi) \sim \sum_{j\in\mathbb{N},\, l=0,\dots,k} r_{\sigma-j,l}(x,\xi)(\log[\xi])^l, \tag{1.7}$$

where the $r_{\sigma-j,l}(x,\xi)$ are homogeneous in ξ of order $\sigma - j$ for $|\xi| \ge 1$, and $r(x,\xi) - \sum_{j<J,\, l\le k} r_{\sigma-j,l}(x,\xi)(\log[\xi])^l$ is $O([\xi]^{\sigma-J+\varepsilon})$, all J, with similar estimates of derivatives (as in (1.1).

The even-even operators are called 'odd-class' in [**KV**] and subsequent literature (possibly because $\mathrm{TR}\,A$ makes sense for these operators when $\dim X$ *is odd*).

Now let us describe the role of these functionals in the study of geometric invariants. We shall focus on the generalized zeta function, and the associated resolvent trace expansion. For this, let A be a classical ψdo in E of order $\sigma\in\mathbb{R}$ and consider, along with it, an elliptic operator P_1 of order $m\in\mathbb{R}_+$. Assuming that the resolvent set of P_1 contains a ray — which we for simplicity take to be the negative half-axis $\mathbb{R}_-$ — we can define the complex powers P_1^z by an integral

$$P_1^z = \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^z (P_1-\lambda)^{-1}\,d\lambda \tag{1.8}$$

when $\operatorname{Re} z < 0$; here λ^z is continuous on $\mathbb{C}\setminus\mathbb{R}_-$, and $\mathcal{C}$ is a Laurent loop

$$\mathcal{C} = \{\lambda = re^{i\pi} \mid \infty > r > r_0\} \cup \{\lambda = r_0e^{i\theta} \mid \pi\ge\theta\ge-\pi\} \cup \{\lambda = re^{-i\pi} \mid r_0 < r < \infty\} \tag{1.9}$$

going around the nonzero spectrum of P_1 in the positive direction. The definition extends to general z by composition with integer powers of P_1, and it is known from Seeley [**S**] that the P_1^z are classical ψdo's (and how their symbols are found): the complex powers.

The generalized zeta function $\zeta(A,P_1,s)$ is defined as the meromorphic extension of $\mathrm{Tr}(AP_1^{-s})$ (trace-class for $\operatorname{Re} s > (\sigma+n)/m$) to the complex plane. It has simple poles at the real numbers $(\sigma+n-l)/m$, $l \in \mathbb{N}$. This is known from the works of Wodzicki, and can also be deduced from the trace expansion of the resolvent, shown by use of local coordinates in Grubb and Seeley [**GS1**] (1995):

$$\operatorname{Tr}(A(P_1-\lambda)^{-N}) \sim \sum_{j\ge 0} \tilde c_j^{(N)}(-\lambda)^{\frac{\sigma+n-j}{m}-N} + \sum_{k\ge 0}\big(\tilde c_k^{(N)\prime}\log(-\lambda) + \tilde c_k^{(N)\prime\prime}\big)(-\lambda)^{-k-N}, \tag{1.10}$$

for $\lambda \to \infty$ in a sector around $\mathbb{R}_+$ ($N > (\sigma+n)/m$), which implies

$$\Gamma(s)\zeta(A,P_1,s) \sim \sum_{j\ge 0}\frac{c_j}{s+\frac{j-\sigma-n}{m}} - \frac{\operatorname{Tr}(A\Pi_0(P_1))}{s} + \sum_{k\ge 0}\Big(\frac{c'_k}{(s+k)^2}+\frac{c''_k}{s+k}\Big), \tag{1.11}$$

where the right-hand side indicates the pole structure of the meromorphic extension. Division by $\Gamma(s)$ gives the simple poles. (More precisely, [**GS1**] treats the case where m is integer; general $m \in \mathbb{R}_+$ are treated in Loya [**Lo**], Grubb and Hansen [**GH**].)

The term with $\Pi_0(P_1)$ (the generalized eigenprojection for the zero eigenvalue of P_1) appears because P_1^z is defined as 0 on the generalized zero eigenspace of P_1 for all z.

It is seen in particular that $\zeta(A,P_1,s)$ has the Laurent expansion at $s=0$:

$$\zeta(A,P_1,s) \sim c'_0 s^{-1} + \Big(c_{\sigma+n} + c''_0 - \operatorname{Tr}(A\Pi_0(P_1))\Big)s^0 + \sum_{j\ge 1} C_j s^j, \tag{1.12}$$

where we set $c_{\sigma+n}=0$ when $\sigma+n \notin \mathbb{N}$. We call the coefficient of s^0 the "regular value at $s=0$".

The term $\operatorname{Tr}(A\Pi_0(P_1))$ does not occur in resolvent expansions but only in statements concerning the zeta function $\zeta(A,P_1,s)$, where it can be considered an artificial nuisance; it is $c_{\sigma+n}+c''_0$ that has the nice analytic properties.

The constants $c_{\sigma+n}+c''_0$ and $c_{\sigma+n}+c''_0-\operatorname{Tr}(A\Pi_0(P_1))$ are both independent of the choice of local coordinates, whereas the splitting in $c_{\sigma+n}$ and c''_0 (the latter containing a global contribution in general) depends on the coordinates, see e.g. the detailed account in [**G3**]. We define *the basic zeta coefficient* as

$$C_0(A,P_1) = c_{\sigma+n} + c''_0; \tag{1.13}$$

it equals the regular value of $\zeta(A,P_1,s)$ at $s=0$ plus $\operatorname{Tr}(A\Pi_0(P_1))$. One also has, in reference to (1.10), that $\tilde c_0^{(N)\prime} = c'_0$ for all N, and

$$\begin{aligned} &C_0(A,P_1) = \tilde c_{\sigma+n}^{(1)} + \tilde c_0^{(1)\prime\prime} \text{ if } N=1,\\ &C_0(A,P_1) = \tilde c_{\sigma+n}^{(N)} + \tilde c_0^{(N)\prime\prime} + \alpha_N c'_0 \text{ in general, with } \alpha_N = \sum_{1\le j<N}\tfrac{1}{j}. \end{aligned} \tag{1.14}$$

The latter formula is explained in detail in [**G5**]; it is consistent with comparison of expansions for different powers N, in view of the fact that

$$\begin{aligned} (P_1-\lambda)^{-N} &= \tfrac{\partial_\lambda^{N-1}}{(N-1)!}(P_1-\lambda)^{-1};\\ \tfrac{\partial_\lambda^{N-1}}{(N-1)!}[(-\lambda)^{-1}\log(-\lambda)] &= (-\lambda)^{-N}\log(-\lambda) - \alpha_N(-\lambda)^{-N}. \end{aligned} \tag{1.15}$$

The Laurent expansion (1.12) is connected with the two new traces as follows: (i) *The coefficient* c'_0 *equals* $\frac{1}{m}\operatorname{res} A$. In particular, it is independent of P_1. (ii) *In the cases* (1), (2), $C_0(A,P_1) = \operatorname{TR} A$. *In the cases* (3), (4), $C_0(A,P_1) = \operatorname{TR} A$ *holds when m is even and P_1 is even-even.* The first result stems from [**W1**], [**W2**], the second result from [**KV**], [**L**] and [**G3**]. One has in the cases (1)–(4) that $\operatorname{res} A = 0$.

The next question we shall address is: What can be said about $C_0(A, P_1)$ when we are *not* in one of the cases (1)–(4). Then it depends on P_1, and need not vanish on commutators. However, the independence of P_1 and the vanishing on commutators hold in general *modulo local terms*. More precisely, one has the so-called *trace defect* formulas (when P_2 is another auxiliary elliptic operator, of order m'):

$$\begin{aligned} C_0(A, P_1) - C_0(A, P_2) &= -\operatorname{res}(A(\tfrac{1}{m}\log P_1 - \tfrac{1}{m'}\log P_2)), \\ C_0([A, A'], P_1) &= -\tfrac{1}{m}\operatorname{res}(A[A', \log P_1]), \end{aligned} \tag{1.16}$$

shown by Okikiolu [**O**], [**KV**], and Melrose and Nistor [**MN**]. Here $\log P_1$ is defined by

$$\log P_1 = \lim_{s\searrow 0} \tfrac{i}{2\pi}\int_{\mathcal{C}} \lambda^{-s}\log\lambda\,(P_1-\lambda)^{-1}\,d\lambda, \tag{1.17}$$

and its symbol is of the form, in local coordinates,

$$\operatorname{symb}(\log P_1) = m\log[\xi] + l(x,\xi), \tag{1.18}$$

where $l(x,\xi)$ is a classical symbol of order 0. Thus $\frac{1}{m}\log P_1 - \frac{1}{m'}\log P_2$ and $[A, \log P_1]$ are classical.

Very recently, Paycha and Scott [**PS**] have given a considerable improvement of the above informations on $C_0(A, P_1)$. They show a formula for $C_0(A, P_1)$ in general, with ingredients both of the canonical trace-type and of the noncommutative residue-type:

$$C_0(A, P_1) = \int_X \big(\operatorname{TR}_x(A) - \tfrac{1}{m}\operatorname{res}_{x,0}(A\log P_1)\big)\,dx. \tag{1.19}$$

The integrand is defined in a local coordinate system by:

$$\operatorname{TR}_x(A) = ⨍ \operatorname{tr} a(x,\xi)\,đ\xi, \quad \operatorname{res}_{x,0}(A\log P_1) = \int_{|\xi|=1}\operatorname{tr} r_{-n,0}(x,\xi)đS(\xi); \tag{1.20}$$

here $⨍ a(x,\xi)\,đ\xi$ is as in (1.6)ff., and r is the symbol of $R = A\log P_1$, log-polyhomogeneous of order σ and log-degree 1 (cf. (1.7)). Moreover, the expression $\big(\operatorname{TR}_x(A) - \operatorname{res}_{x,0}(A\log P_1)\big)\,dx$ has an invariant meaning as a density on X, although its two terms individually do not so in general.

Having this more general formula (1.19), one can verify the previous formulas (1.16) and the identifications of $C_0(A, P_1)$ with $\operatorname{TR} A$ in the cases (1)–(4).

The methods of [**W1**], [**Gu**], [**KV**], [**O**], [**MN**], [**PS**] rely in an essential way on the concept of holomorphic operator families of complex orders. The typical operator family is P_1^z, $z \in \mathbb{C}$, which is classical of order mz (with complex homogeneities of the symbol terms), but also AP_1^z (classical of order $\alpha(z) = \sigma + mz$ holomorphic in z) and more general operator families are of interest. There are various definitions of the associated symbol calculus; here we find in [**PS**] a clear presentation of the fact that the z-derivative of such an operator family is no longer a family of classical (polyhomogeneous) ψdo's, but of log-polyhomogeneous ψdo's. In fact, each differentiation with respect to z increases the log-degree by 1.

REMARK 1.1. It may be useful to observe that in local coordinates,

$$\operatorname{res}_{x,0}(A \log P_1) = \operatorname{res}_x(A(\log P_1)^0), \tag{1.21}$$

where $(\log P_1)^0$ is the classical ψdo $\mathrm{OP}(l(x,\xi)) = \log P_1 - \mathrm{OP}(m \log[\xi])$ (cf. (1.18)), and res_x stands for the integral over $\{|\xi| = 1\}$ of the fiber trace of the symbol of order $-n$.

2. Similar questions for manifolds with boundary

We next consider extensions of the above concepts to manifolds with boundary. So now X denotes a compact n-dimensional C^∞ manifold with boundary $\partial X = X'$; here X' is a smooth boundaryless manifold of dimension $n-1$. We can assume that $X \subset \widetilde{X}$ for a smooth n-dimensional manifold $\widetilde{X}$ without boundary, so that ∂X is its boundary there.

For given smooth vector bundles E and E' over X (extending to $\widetilde{E}$ and $\widetilde{E}'$ over $\widetilde{X}$), F and F' over X', we consider pseudodifferential boundary operators (ψdbo's) of order σ as defined by Boutet de Monvel [**B**]:

$$\begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X,E) \\ \times \\ C^\infty(X',F) \end{matrix} \to \begin{matrix} C^\infty(X,E') \\ \times \\ C^\infty(X',F') \end{matrix} ; \tag{2.1}$$

here
- P is a classical ψdo on $\widetilde{X}$, $\quad P_+ = r^+ P e^+$,
- G is a singular Green operator (s.g.o.) from X to X,
- T is a trace operator from X to X',
- K is a Poisson operator from X' to X,
- S is a classical ψdo on X'.

The truncated operator $P_+ = r^+ P e^+$ applies to $u \in C^\infty(X,E)$; here e^+ indicates extension by zero on $\widetilde{X} \setminus X$ and r^+ indicates restriction from $\widetilde{X}$ to X. In order for this “brutal” truncation to introduce no new singularities, one assumes that P is of integer order and satisfies the *transmission condition* at X', namely, in local coordinates at the boundary (with tangential variable x', normal variable x_n, the boundary represented by $x_n = 0$):

$$\partial_x^\beta \partial_\xi^\alpha p_{\sigma-j}(x',0,0,-\xi_n) = (-1)^{\sigma-j-|\alpha|} \partial_x^\beta \partial_\xi^\alpha p_{\sigma-j}(x',0,0,\xi_n) \text{ for } |\xi_n| \geq 1. \tag{2.2}$$

When $\sigma \in \mathbb{Z}$, (2.2) implies that P_+ maps $C^\infty(X,E)$ into $C^\infty(X,E')$.

Taking the trace of (2.1) is relevant when $E = E'$ and $F = F'$, here the new object to study the trace of is the operator $B = P_+ + G$. For the definitions of T and K we refer to [**B**] or [**G1**].

A singular Green operator G of class 0 (i.e., well-defined on $L_2(X)$, this assures that the operators of low order are trace-class) is defined in local coordinates near X' from a symbol $g(x',\xi',\xi_n,\eta_n)$ or the associated symbol-kernel $\tilde{g}$ obtained by inverse Fourier transformation and co-Fourier transformation in ξ_n and η_n:

$$\tilde{g}(x',x_n,y_n,\xi') = \mathcal{F}^{-1}_{\xi_n \to x_n} \overline{\mathcal{F}}^{-1}_{\eta_n \to y_n} g(x',\xi',\xi_n,\eta_n) \text{ for } x_n, y_n \geq 0, \tag{2.3}$$

by formulas

$$\begin{aligned}Gu(x) &= \int_{\mathbb{R}^{2n+1}} e^{i(x'-y')\cdot\xi'+ix_n\xi_n-iy_n\eta_n} g(x',\xi',\xi_n,\eta_n)u(y',y_n)\, dy đ\xi' đ\xi_n đ\eta_n \\ &= \int_{\mathbb{R}^n_+} e^{ix'\cdot\xi'} \tilde{g}(x',x_n,y_n,\xi')\acute{u}(\xi',y_n)\, dy_n đ\xi',\end{aligned}$$

where $\acute{u}(\xi',y_n) = \mathcal{F}_{y'\to\xi'}u(y)$. The symbol g has an expansion in smooth terms that are homogeneous in (ξ',ξ_n,η_n) for $|\xi'|\ge 1$:

$$\begin{aligned}&g(x',\xi',\xi_n,\eta_n) \sim \sum_{j\in\mathbb{N}} g_{\sigma-j}(x',\xi',\xi_n,\eta_n), \\ &g_{\sigma-j}(x',t\xi',t\xi_n,t\eta_n) = t^{\sigma-j-1} g_{\sigma-j}(x',\xi',\xi_n,\eta_n) \text{ for } t\ge 1,\ |\xi'|\ge 1,\ \text{all } j.\end{aligned} \tag{2.4}$$

(The enumeration of the homogeneous symbols differs by 1 from the enumeration in [**G1**], to fit with the convention for the normal trace, see (2.6)ff. below.)

Moreover, g satifies estimates that are most easily explained for the symbol-kernel $\tilde{g}$, namely

$$\begin{aligned}&\sup_{x_n,y_n\ge 0} |x_n^k \partial_{x_n}^{k'} y_n^l \partial_{y_n}^{l'} \partial_{x'}^{\beta}\partial_{\xi'}^{\alpha}[\tilde{g}(x',x_n,y_n,\xi') - \sum_{j<J}\tilde{g}_{\sigma-j}(x',x_n,y_n,\xi')]| \\ &\qquad \le C[\xi']^{\sigma-J-|\alpha|-k+k'-l+l'},\ \text{all } \alpha,\beta\in\mathbb{N}^{n-1}, J,k,k',l,l'\in\mathbb{N}.\end{aligned} \tag{2.5}$$

In the definition of G, $\sigma\in\mathbb{R}$ is allowed.

In trace calculations, an important step is to "reduce the s.g.o.s to the boundary": In local coordinates, one defines *the normal trace* $\operatorname{tr}_n G$ as the ψdo on the boundary with symbol

$$(\operatorname{tr}_n g)(x',\xi') = (\operatorname{tr}_n \tilde{g})(x',\xi') = \int_0^\infty \tilde{g}(x',x_n,x_n,\xi')\, dx_n; \tag{2.6}$$

then $s(x',\xi') = \operatorname{tr}_n g$ is polyhomogeneous of order σ when g is of order σ, the j'th term $s_{\sigma-j}(x',\xi')$ (of homogeneity degree $\sigma-j$ in ξ') being derived by (2.3), (2.6) from $g_{\sigma-j}(x',\xi',\xi_n,\eta_n)$.

Example 2.1. Let $P = 1-\Delta$ on $\mathbb{R}^n$, then P_+ is $1-\Delta$ on $\mathbb{R}^n_+$ (the plus-index may be omitted for differential operators since they act locally). The two semihomogeneous Dirichlet problems on $\mathbb{R}^n_+$,

$$\begin{cases}(1-\Delta)u &= f,\\ \gamma_0 u &= 0;\end{cases} \qquad \begin{cases}(1-\Delta)u &= 0,\\ \gamma_0 u &= \varphi;\end{cases}$$

are uniquely solvable, with solution operators R resp. K. An easy calculation shows that $R = Q_+ + G$, where Q is the ψdo $(1-\Delta)^{-1}$ with symbol $(|\xi|^2+1)^{-1}$, and G is the singular Green operator with symbol-kernel $\frac{1}{2\langle\xi'\rangle}e^{-\langle\xi'\rangle(x_n+y_n)}$; $\langle\xi'\rangle = (|\xi'|^2+1)^{\frac12}$. Moreover, K is the Poisson operator with symbol-kernel $e^{-\langle\xi'\rangle x_n}$. Thus we may write, denoting $\gamma_0 = T$,

$$\begin{pmatrix}P_+\\ T\end{pmatrix}^{-1} = \begin{pmatrix}Q_+ + G & K\end{pmatrix},$$

providing examples of matrices (2.1) with $\dim F = 0$, resp. $\dim F' = 0$. Note that

$$\operatorname{tr}_n\big(\tfrac{1}{2\langle\xi'\rangle}e^{-\langle\xi'\rangle(x_n+y_n)}\big) = \tfrac{1}{4\langle\xi'\rangle^2} = \tfrac14(|\xi'|^2+1)^{-1},$$

the symbol of the ψdo $\operatorname{tr}_n G = \frac{1}{4}(1-\Delta_{x'})^{-1}$ on $\mathbb{R}^{n-1}$.

A *noncommutative residue* was assigned to the operators $B = P_+ + G$ by Fedosov, Golse, Leichtnam and Schrohe [**FGLS**] (1996):
(2.7)

$$\operatorname{res}(B) = \int_X \int_{|\xi|=1} \operatorname{tr} p_{-n}(x,\xi)\, đS(\xi)dx + \int_{X'} \int_{|\xi'|=1} \operatorname{tr}(\operatorname{tr}_n g)_{1-n}(x',\xi')\, đS(\xi')dx',$$

with similar properties as in the case of closed manifolds. When $B = G$ and $\sigma \notin \mathbb{Z}$, $\operatorname{res} G = 0$.

A *canonical trace* was introduced in our joint work with Schrohe [**GSc2**] (2004). For P_+,

$$\operatorname{TR}(P_+) = \int_X ⨍ \operatorname{tr} p(x,\xi)\, đ\xi dx, \tag{2.8}$$

has an invariant meaning when P is as in one of the cases (1), (3), (4) (relative to $\widetilde{X}$). For G,

$$\operatorname{TR}(G) = \operatorname{TR}_{X'}(\operatorname{tr}_n G) = \int_{X'} ⨍ \operatorname{tr}(\operatorname{tr}_n g)(x',\xi')\, đ\xi' dx' \tag{2.9}$$

(defined on X' when a normal coordinate has been chosen and G is supported near X'), has an invariant meaning when the ψdo $\operatorname{tr}_n G$ is as in one of the cases (1)–(4) *pertaining to* X' *of dimension* $n-1$. They are then called canonical traces.

Note that for the joint expression $B = P_+ + G$, the two terms have a canonical trace when their parities are *opposite*, since $\widetilde{X}$ has dimension n and X' has dimension $n-1$. This will rarely happen in geometrically interesting cases. For example, the inverse of the Dirichlet realization of a strongly elliptic second-order differential operator is of the form $Q_+ + G$ with Q even-even and $\operatorname{tr}_n G$ even-even, both of order -2 (cf. Example 2.1 for a special, localized case). For n odd, Q_+ but not G will have a canonical trace, and the roles are exchanged when n is even.

Now consider the relation to geometric invariants. For definiteness, take the order σ of B to be integer (we refer to [**G4**] for the results when $\sigma \notin \mathbb{Z}$ and $P = 0$). In [**FGLS**], $\operatorname{res} B$ was not identified with a residue of a generalized zeta function as in the closed manifold case, simply because the relevant zeta function expansions or resolvent expansions had not been developed yet. We did so in a joint work with Schrohe [**GSc1**] (2001). Taking as an auxiliary operator the Dirichlet realization $P_{1,\mathrm{D}}$ of a second-order strongly elliptic differential operator P_1 in E, principally scalar near X', we showed the expansion (when $N > (\sigma+n)/2$)
(2.10)

$$\operatorname{Tr}(B(P_{1,\mathrm{D}}-\lambda)^{-N}) \sim \sum_{j\ge 0} \tilde{c}_j^{(N)}(-\lambda)^{\frac{\sigma+n-j}{2}-N} + \sum_{k\ge 0} \big(\tilde{c}_k^{(N)\prime} \log(-\lambda) + \tilde{c}_k^{(N)\prime\prime}\big)(-\lambda)^{-\frac{k}{2}-N},$$

for $\lambda \to \infty$ in a sector around $\mathbb{R}_-$, which implies
(2.11)

$$\Gamma(s)\zeta(B, P_{1,\mathrm{D}}, s) \sim \sum_{j\ge 0} \frac{c_j}{s + \frac{j-\sigma-n}{2}} - \frac{\operatorname{Tr}(A\Pi_0(P_{1,\mathrm{D}}))}{s} + \sum_{k\ge 0} \Big(\frac{c_k'}{(s+\frac{k}{2})^2} + \frac{c_k''}{s+\frac{k}{2}} \Big),$$

as a description of the poles of the meromorphic extension $\zeta(B, P_{1,\mathrm{D}}, s)$ of $\operatorname{Tr}(BP_{1,\mathrm{D}}^{-s})$ (extended to $s \in \mathbb{C}$ from $\operatorname{Re} s > (\sigma+n)/2$). We then showed moreover that $\operatorname{res} B$

equals 2 times the first log-coefficient, hence in the transition to (2.11),

$$\operatorname{res} B = 2c_0'; \tag{2.12}$$

it is indeed proportional to the residue of $\zeta(B, P_{1,\mathrm{D}}, s)$ at $s = 0$.

A particular point here was that when the resolvent $(P_{1,\mathrm{D}}-\lambda)^{-N}$ is decomposed into its ψdo part and its s.g.o. part:

$$(P_{1,\mathrm{D}} - \lambda)^{-N} = (P_1 - \lambda)_+^{-N} + G_\lambda^{(N)}, \tag{2.13}$$

then $\operatorname{Tr}(BG_\lambda^{(N)})$ has an expansion as in (2.10) *but with k running over $k \geq 1$*; this trace does not contribute to the residue, and its contribution to the coefficient of $(-\lambda)^{-N}$ is local.

Define, similarly to the closed manifold case, the basic zeta coefficient by

$$C_0(B, P_{1,\mathrm{D}}) = c_{\sigma+n} + c_0''. \tag{2.14}$$

Then we showed in [**GSc2**] that

$$C_0(B, P_{1,\mathrm{D}}) = \operatorname{TR} B \tag{2.15}$$

holds essentially in the cases where $\operatorname{TR} B$ could be defined.

Since these cases far from cover all interesting operators, it is important to get information on $C_0(B, P_{1,\mathrm{D}})$ in general. We showed in [**GSc2**] that the "global content" of $C_0(B, P_{1,\mathrm{D}})$ (the value modulo local contributions) is expressed by integrals such as (2.8), (2.9) in local coordinates, and that

$$C_0(B, P_{1,\mathrm{D}}) - C_0(B, P_{2,\mathrm{D}}) \text{ and } C_0([B, B'], P_{1,\mathrm{D}}) \text{ are local.} \tag{2.16}$$

(For the statement on $C_0([B, B'], P_{1,\mathrm{D}})$, we moreover needed to assume that the ψdo's P and P' entering in B and B' have normal order ≤ 0.)

All this was based on a painstaking analysis of the resolvent composition $B(P_{1,\mathrm{D}} - \lambda)^{-N}$ and its symbol, using in particular that the principal interior resolvent symbol $(p_{1,2} - \lambda)^{-1}$ has just two simple poles, one in the upper and one in the lower halfplane of $\mathbb{C}$.

We underline that all the work was carried out in the resolvent framework. Here we used a combination of the Boutet de Monvel calculus (for each λ) and the calculus of parameter-dependent ψdo's developed in [**GS1**], applicable after the considerations had been reduced to the boundary. The occurrence of complex powers in the generalized zeta function is justified by a transition from expansions such as (2.10) to pole structure information such as (2.11), as accounted for e.g. in [**GS2**], but direct calculations on the powers $P_{1,\mathrm{D}}^{-s}$ were not performed.

In fact, the powers $P_{1,\mathrm{D}}^{-s}$ do not belong to the Boutet de Monvel calculus except when s is integer, and there is, to our knowledge, not a sufficiently refined holomorphic calculus available for such powers in order to get results on expansion coefficients.

For further precision on the constant $C_0(B, P_{1,\mathrm{D}})$, a first question to deal with would be to search for generalizations of the trace defect formulas (1.16). The proofs of (1.16) in [**O**], [**KV**], [**MN**] all rely on studies of the holomorphic family P^{-s} (and certain other families), where the fact that causes $\log P$ to appear is that

$$\tfrac{d}{ds} P^{-s}|_{s=0} = -\log P. \tag{2.17}$$

Since we do not have a nice theory of holomorphic families of ψdbo's, we have to understand from resolvent calculations how $\log P$ comes into the picture.

3. Trace defect formulas proved by resolvent methods

For a short while we shall now go back to the closed manifold situation, giving a brief explanation of how the trace defect formulas can be derived purely from resolvent considerations; then this will be generalized to manifolds with boundary.

There are several ingredients in this. The first observation is that when we define a trace expansion of one of our λ-dependent families of ψdo's S_λ from its symbol $s(x,\xi,\lambda) \sim \sum_{j\in\mathbb{N}} s_{\sigma-j}(x,\xi,\lambda)$,

$$\operatorname{Tr} S_\lambda = \int \operatorname{tr} s(x,\xi,\lambda)\, d\!\!{}^-\xi dx,$$

we get exact local contributions $c_j(-\lambda)^{m_j}$ from those terms for which the *strictly homogeneous version* $s^h_{\sigma-j}(x,\xi,\lambda)$ (extended from $s_{\sigma-j}$ by homogeneity to the full region $\xi \neq 0$) are integrable in ξ at $\xi = 0$. For in these terms we can use the joint (quasi-)homogeneity in ξ and λ to reduce the contribution to an integral of $s^h_{\sigma-j}(x,\xi,-1)$ over the whole space times a power of $-\lambda$, and the difference between $s_{\sigma-j}$ and $s^h_{\sigma-j}$ produces something of lower order in λ. So whether $(-\lambda)^{-1}$ gets a local coefficient defined as an integral over the full space depends on whether the term in $s(x,\xi,\lambda)$ with the appropriate homogeneity has an integrable strictly homogeneous version.

The second observation builds on the following simple fact: When $\log\lambda$ is integrated along a Laurent loop (1.9) together with a function $f(\lambda)$ that is holomorphic on a neighborhood of $\mathbb{R}_-$ and is $O(\lambda^{-1-\varepsilon})$ for $|\lambda| \to \infty$ there, then

$$\int_{\mathcal{C}} \log\lambda\, f(\lambda)\, d\lambda = 2\pi i \int_{-\infty}^{0} f(t)\, dt. \tag{3.1}$$

For, $\log\lambda$ just produces a jump of $2\pi i$ to be multiplied with f in the integration along the ray $\mathbb{R}_-$; the contributions from $\log|\lambda|$ cancel out. In other words, we can replace the integration over the Laurent loop with an integration over a ray, whereby the "mysterious" $\log\lambda$ disappears. Or, reading (3.1) from the right to the left, we can turn the integral over a ray into a log-integral.

The observations apply to $C_0(A,P_1) - C_0(A,P_2)$ as follows: Let, for simplicity, P_1 and P_2 be of order $m > \sigma + n$. Let $S_\lambda = A((P_1-\lambda)^{-1} - (P_2-\lambda)^{-1})$. An inspection of the symbol $s(x,\xi,\lambda) \sim \sum_{j\in\mathbb{N}} s_{\sigma-m-j}(x,\xi,\lambda)$ shows that the strictly homogeneous symbol terms $s^h_{\sigma-m-j}(x,\xi,\lambda)$ are integrable in ξ at $\xi = 0$ for $j < \sigma + n + m$, with a convenient estimate of the remainder. (The homogeneity here is a joint homogeneity in (ξ,μ) for $\lambda = -\mu^m e^{i\theta}$, each θ.) The term that produces the coefficient $C_0(A,P_1) - C_0(A,P_2)$ is s^h_{-m-n} (taken equal to 0 when $\sigma + n \notin \mathbb{N}$), with

$$\int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,\lambda)\, d\!\!{}^-\xi = (-\lambda)^{-1} \int_{\mathbb{R}^n} s^h_{-m-n}(x,\xi,-1)\, d\!\!{}^-\xi \equiv (-\lambda)^{-1} b_0(x), \tag{3.2}$$

in local coordinates. Then $\operatorname{tr} b_0(x)$ can be rewritten as the asserted residue integral involving $\log P_1$ and $\log P_2$ (cf. (1.16), (1.17)) by use of (3.1) to get $\log\lambda$ into the picture, combined with the homogeneity and polar coordinates (details in [**G4**]). More general orders of P_1 and P_2 are included as indicated in Remark 3.12 there, details for this are given in [**G5**]. The result follows by integration in x.

In [**G4**] we used these principles to deduce the formulas (1.16), as well as the formula

$$C_0(I, P_1) = -\tfrac{1}{m}\operatorname{res}(\log P_1) \tag{3.3}$$

by Scott [**Sco**]. Next, in the heavier part of [**G4**], we generalized the formulas to the situation of manifolds with boundary:

The calculus of parameter-dependent ψdbo's presented in the book [**G1**] is more crude than the parameter-dependent calculi of [**GS1**] and [**G2**], but it has just what it takes to discuss how far the integrability of the strictly homogeneous symbols at 0 stays valid when j increases in the symbol sequence; notably, this works also for the singular Green terms that are far more complicated than the ψdo terms. The central idea is the concept of "regularity ν", measuring to what extent the symbols satisfy the estimates required for strongly polyhomogeneous symbols (such as those arising from purely differential problems). When a μ-dependent ψdo symbol in the calculus of [**G1**] has regularity $\nu > 0$, and is of order $d < -n$, it has a trace expansion in powers μ^{d+n-l} for $0 \le l < n+\nu$, with a remainder that is $O(\mu^{d-\nu+\varepsilon})$ (any $\varepsilon > 0$). There is a similar rule for s.g.o.s, applied to their normal trace.

When using this calculus, we shall make do with *finite* expansions of the λ-dependent traces, aiming for just enough terms to capture the coefficient of $(-\lambda)^{-1}$. On the other hand, the calculus allows the use of general auxiliary elliptic differential operators P_1, without delicate conditions on separation of the roots of the characteristic polynomial, or scalarity. In [**G4**], they are taken of a sufficiently high order m, to avoid having to deal with N'th powers of the resolvent (but there are means to get around this).

Since the singular Green part of the resolvent, cf. (2.13)ff., is known to contribute locally (in the cases studied in [**GSc1**], [**GSc2**]), we focus on the contribution from $(P_1-\lambda)_+^{-1}$ alone. This corresponds to studying the zeta function

$$\zeta(B, P_{1,+}, s), \text{ meromorphic extension of } \operatorname{Tr}(B(P_1^{-s})_+), \tag{3.4}$$

where $(P_1^{-s})_+$ is an s-dependent family that equals the identity for $s = 0$ (just as a family of powers of an elliptic realization of P_1 on X would do).

The expansion of $\operatorname{Tr}(B(P_1-\lambda)_+^{-1})$ available directly from the theory of [**G1**] does not extend far enough down to include the power $(-\lambda)^{-1}$. However, we showed in [**G4**] that $B(P_1-\lambda)_+^{-1} - B(P_2-\lambda)_+^{-1}$ (with P_2 of order m) has better regularity than $B(P_1-\lambda)_+^{-1}$ alone, with a symbol leading to the expansion formula

$$\operatorname{Tr}(B(P_1-\lambda)_+^{-1} - B(P_2-\lambda)_+^{-1}) = \sum_{0\le j\le \sigma+n} d_j\,(-\lambda)^{\frac{n+\sigma-j}{m}-1} + O(\lambda^{-1-\frac{1}{4m}}), \tag{3.5}$$

where all the explicit coefficients are determined from integrals of strictly homogeneous functions. Combining (3.5) with the expansions (2.10) known for particular choices of P_1, we could deduce that

$$\operatorname{Tr}(B(P_1-\lambda)_+^{-1}) = \sum_{0\le j\le \sigma+n} c_j\,(-\lambda)^{\frac{n+\sigma-j}{m}-1} + (c_0'\log(-\lambda) + c_0'')(-\lambda)^{-1} + O(\lambda^{-1-\frac{1}{4m}}) \tag{3.6}$$

holds for general elliptic differential operators P_1 of order $m > \sigma + n$ having $\mathbb{R}_-$ as a spectral cut. Then the basic zeta coefficient

$$C_0(B, P_{1,+}) = c_{\sigma+n} + c_0'' \tag{3.7}$$

is defined also in these cases.

A further analysis of the local coefficient of $(-\lambda)^{-1}$ in (3.5), using the principles described above for the case of closed manifolds, shows that the coefficient can be identified with a sum of two integrals over $S^*(X)$ resp. $S^*(X')$ involving an integration along with $\log\lambda$ over a Laurent loop (1.9). In general we find:

$$C_0(B,P_{1,+})-C_0(B,P_{2,+})=-\tfrac{1}{m}\operatorname{res}_+(P(\log P_1-\log P_2))-\tfrac{1}{m}\operatorname{res}_{X'}(S'), \tag{3.8}$$

where the two terms are worked out from localizations: $\operatorname{res}_+(P(\log P_1-\log P_2))$ is the integral over X, carried back from integrals over $\mathbb{R}^n_+$ of residue-type integrals

$$\operatorname{res}_x(R)=\int_{|\xi|=1}\operatorname{tr} r_{-n}(x,\xi)dS(\xi); \tag{3.9}$$

with $R=P(\log P_1-\log P_2)$, its symbol denoted r in local coordinates. $\operatorname{res}_{X'}(S')$ is defined from S', a classical ψdo obtained by integration of the normal trace of the s.g.o. part $\mathcal{G}_\lambda$ of $B((P_1-\lambda)^{-1}-(P_2-\lambda)^{-1})_+$ along with $\log\lambda$ on a Laurent loop (1.9), in local coordinates. Both terms in (3.8) have an invariant meaning as the coefficient of $(-\lambda)^{-1}$ in the trace expansion of the ψdo part, resp. s.g.o. part, of $B(P_1-\lambda)^{-1}_+-B(P_2-\lambda)^{-1}_+$.

When m is even, the formula can be written as

$$C_0(B,P_{1,+})-C_0(B,P_{2,+})=-\tfrac{1}{m}\operatorname{res}(B(\log P_1-\log P_2)_+), \tag{3.10}$$

using that the classical ψdo $\log P_1-\log P_2$ satisfies the transmission condition (since m is even), so that $B(\log P_1-\log P_2)_+$ is in the ψdbo calculus and the residue is as defined by [**FGLS**]. (The need for evenness of m, in order for the classical parts of $\log P_1$ and $\log P_2$ to satisfy the transmission condition, was pointed out in [**G5**].)

For the second trace defect formula, for the commutator of $B=P_++G$ and $B'=P'_++G'$, it is shown in [**G4**] that when $m>\sigma+\sigma'+n$ and P and P' have normal order ≤ 0, there is a trace expansion with local coefficients

$$\operatorname{Tr}([B,B'](P_1-\lambda)^{-1}_+)=\sum_{0\le j\le n+\sigma+\sigma'}c_j\,(-\lambda)^{\frac{n+\sigma+\sigma'-j}{m}-1}+O(\lambda^{-1-\frac{1}{4m}}), \tag{3.11}$$

so that $C_0([B,B'],P_{1,+})=c_{n+\sigma+\sigma'}$ (taken equal to 0 if $n+\sigma+\sigma'<0$) is local. Here it is found to have the form

$$C_0([B,B'],P_{1,+})=-\tfrac{1}{m}\operatorname{res}_+(P[P',\log P_1])-\tfrac{1}{m}\operatorname{res}_{X'}(S), \tag{3.12}$$

where S is a classical ψdo on X' constructed in local coordinates by integrating the normal trace of the s.g.o. part of $B[B',(P_1-\lambda)^{-1}_+]$ together with $\log\lambda$ on a Laurent loop (1.9).

When m is even, $P[P',\log P_1]$ is a classical ψdo having the transmission property, so the first term in the right-hand side of (3.12) can be written as $-\frac{1}{m}\operatorname{res}_X((P[P',\log P_1])_+)$, as defined in [**FGLS**]. The whole right-hand side in the formula can be regarded as an interpretation of $-\frac{1}{m}\operatorname{res}(B[B',(\log P_1)_+])$, although $[B',(\log P_1)_+]$ is generally not in the Boutet de Monvel calculus.

4. Formulas for the basic zeta coefficient

Finally we shall report on our efforts, worked out in detail in [**G6**], to find a generalization of the formula (1.19) of Paycha and Scott to manifolds with boundary. We are looking for a description of $C_0(B,P_{1,+})$ when B is given in the Boutet de

Monvel calulus as described above and P_1 is a suitably elliptic differential operator. Our strategy is:

- *Find* $C_0(B, P_{1,+})$ *for one particularly manageable choice of* P_1.
- *Extend to more general* P_2 *by combination with the trace defect formula* (3.10) *(or* (3.8)*).*

We showed in [**G5**] how this can be done for $C_0(A, P_1)$ in the boundaryless case, by placing the calculatory part in a localized framework where the operator P_1 has constant coefficients and is of a simple form. The explicit calculation takes place in $\mathbb{R}^n$ for the case of a compactly supported A, and $P_1 = \mathrm{OP}(|\xi|^m + 1)$ with m large even. It was found that in this special case, since $\log P_1$ has symbol $\log(|\xi|^m + 1) = m \log |\xi| + O(|\xi|^{-m})$ (with no classical term of order 0), the contribution to $C_0(A, P_1)$ comes entirely from $\mathrm{TR}_x A$, whereas $\mathrm{res}_{x,0}(A \log P_1)$ vanishes. Pulling the information back to the manifold, we could justify (1.19) by combining this with the resolvent proof of the first formula in (1.16).

The situation is different in the case with boundary. To keep the calculations as simple as possible, we use $P_1 = -\Delta + 1$ as auxiliary operator, and then have to work with iterated resolvents

$$Q_\lambda^N = (P_1 - \lambda)^{-N} = \tfrac{\partial_\lambda^{N-1}}{(N-1)!}(P_1 - \lambda)^{-1}, \tag{4.1}$$

with N so large that $BQ_{\lambda,+}^N$ is trace-class. ($Q_{\lambda,+}^N$ is short for $(Q_\lambda^N)_+$.)

Consider first the case $B = P_+$. By use of local coordinates and a suitable partition of unity (explained in detail in [**G5**], [**G6**]), we reduce to the situation where the distribution kernel of P has compact support in $\mathbb{R}^n$, taking P_1 as $1 - \Delta$ there. The operator on $\mathbb{R}^n_+$ may be written as

$$\begin{aligned} P_+ Q_{\lambda,+}^N &= r^+ P e^+ r^+ Q_\lambda^N e^+ = r^+ P Q_\lambda^N e^+ - r^+ P e^- J J r^- Q_\lambda^N e^+ \\ &= (PQ_\lambda^N)_+ - G^+(P) G^-(Q_\lambda^N), \end{aligned} \tag{4.2}$$

where r^- and e^- denote the restriction and extension-by-zero operators for $\mathbb{R}^n_- \subset \mathbb{R}^n$, and J is the reflection operator $J : u(x', x_n) \mapsto u(x', -x_n)$; it is used here that $e^+ r^+ + e^- r^- = I$ on functions. The last term is a composition of the s.g.o.s

$$G^+(P) = r^+ P e^- J, \quad G^-(Q_\lambda^N) = J r^- Q_\lambda^N e^+. \tag{4.3}$$

The term $(PQ_\lambda^N)_+$ has a kernel expansion which is simply the restriction to $\mathbb{R}^n_+$ of the kernel expansion for the operators considered on $\mathbb{R}^n$, so the formulas from the case without boundary can be used (with elementary modifications resulting from going via N'th powers of the resolvent). The only contribution to $C_0(P_+, P_{1,+})$ from this term is the integral of $\mathrm{TR}_x P$.

The new term $G^+(P)G^-(Q_\lambda^N)$, however, contributes in a different way. As noted in [**G4**], Sect. 4, it has a better regularity than PQ_λ^N in the sense of [**G1**], for one can use the formula

$$(P_1 - \lambda)^{-1} = (-\lambda)^{-1} + \lambda^{-1} P_1 (P_1 - \lambda)^{-1} \tag{4.4}$$

to write $G^-(P_1 - \lambda)^{-1} = \lambda^{-1} G^-(P_1(P_1 - \lambda)^{-1})$, with similar formulas for N'th powers. The extra factor P_1 improves the regularity, and the new expression decreases better in λ (but is, on the other hand, of higher order). Then all the informations taken together allow the conclusion that the relevant strictly homogeneous symbol is integrable at $\xi' = 0$; no term with $(-\lambda)^{-N} \log(-\lambda)$ appears in the trace expansion. It is possible to reformulate the contribution to be expressed as an integral of

$\operatorname{tr}_n[G^+(P)G^-(Q_\lambda^N)]$ together with $\log\lambda$ over a Laurent loop; this gives a classical ψdo S on X' such that the relevant coefficient is a multiple of the residue of S. In a formal sense, the contribution is

$$\tfrac12 \operatorname{res}_{X'} S = \tfrac12 \operatorname{res}_{X'}(\operatorname{tr}_n(G^+(P)G^-(\log P_1))) = \tfrac12 \operatorname{res}(G^+(P)G^-(\log P_1)); \tag{4.5}$$

but here $G^-(\log P_1)$ is not a standard singular Green operator, so the last expression is not covered by [**FGLS**]. More precisely, $G^-(\log P_1)$ is an operator with symbol-kernel $\frac{-1}{x_n+y_n}e^{-[\xi'](x_n+y_n)}$ (cf. (1.1)ff.) plus more smooth lower order terms, showing a singularity at $x_n = y_n = 0$ that is mild enough to allow taking tr_n of the composition with a singular Green operator.

So in contrast with the boundaryless case, we do not just get a TR_x-type contribution from P when $P_1 = 1-\Delta$.

We can choose P_1 on the manifold so that it obeys these formulas in the specially selected local coordinates, and we get $C_0(P_+, P_{1,+})$ as a sum of contributions of the form

$$\varphi\Big(\int [\operatorname{TR}_x P - \tfrac12 \operatorname{res}_{x,0}(P\log(1-\Delta))]\,dx\psi \\ + \varphi \int \tfrac12 \operatorname{res}_{x'} \operatorname{tr}_n(G^+(P)G^-(\log(1-\Delta)))\,dx'\psi, \tag{4.6}$$

with cutoff functions φ,ψ; here $\operatorname{res}_{x,0}(P\log(1-\Delta))$ is 0, and the integral in x' vanishes if the local coordinate patch is inside $\mathbb{R}^n_+$.

Now we can combine the special choice of P_1 with general choices of P_2 and use (3.10) to conclude:

THEOREM 4.1. *For a ψdo P of order $\sigma\in\mathbb{Z}$ satisfying the transmission condition at X', together with a general elliptic differential operator P_2 of order 2 having $\mathbb{R}_-$ as a spectral cut, $C_0(P_+, P_{2,+})$ is found in local coordinates to be of the form*

$$\begin{aligned} C_0(P_+, P_{2,+}) &= \int_X [\operatorname{TR}_x P - \tfrac12 \operatorname{res}_{x,0}(P\log P_2)]\,dx \\ &\quad + \tfrac12 \int_{X'} \operatorname{res}_{x'} \operatorname{tr}_n(G^+(P)G^-(\log P_2))\,dx'. \end{aligned} \tag{4.7}$$

Next, consider the case $B=G$. When $\sigma\notin\mathbb{Z}$, $C_0(G,P_{2,+}) = \operatorname{TR} G$ by a slight extension of [**GSc2**]; we are presently considering integer σ. The main difficulty in handling $GQ^N_{\lambda,+}$ is that the homogeneities of the s.g.o. symbol terms play together with the homogeneity of the ψdo symbol in a bad way; the singular Green symbol terms are homogeneous in (ξ',ξ_n,η_n) but only for $|\xi'|\ge 1$, so it is not fruitful to use polar coordinates with respect to (ξ',ξ_n,η_n). One can easily see that there will be a global contribution $\operatorname{TR}_{x'}(\operatorname{tr}_n G)$, which in the term of order $-n+1$ stems from an integration over $|\xi'|\le 1$, but the remaining integral over $|\xi'|\ge 1$ is difficult to pin down. Here we take recourse to one more trace concept, namely that of an infinite dimensional matrix (indexed by $l,m\in\mathbb{N}$).

We use that the symbol of G has a Laguerre expansion

$$g(x',\xi',\xi_n,\eta_n) = \sum_{l,m\in\mathbb{N}} c_{lm}(x',\xi')\hat\varphi_l([\xi'],\xi_n)\bar{\hat\varphi}_m([\xi'],\eta_n), \tag{4.8}$$

where the $\hat\varphi_l$ form an orthonormal basis of $\mathcal{F}_{x_n\to\xi_n}(e^+L_2(\mathbb{R}_+))$ for each ξ'. Writing g as a sum of its diagonal and off-diagonal parts,

$$g = g_{\mathrm{diag}} + g_{\mathrm{off}}, \quad g_{\mathrm{diag}} = \sum_{l\in\mathbb{N}} c_{ll}(x',\xi')\hat\varphi_l([\xi'],\xi_n)\bar{\hat\varphi}_l([\xi'],\eta_n),$$

we find that the off-diagonal part contributes with a homogeneous term that is integrable at $\xi'=0$ (and can be handled somewhat like the last term in (4.2), giving a local contribution), whereas in the contribution from the diagonal part, taking tr_n for each l has an effect independent of l (leading to simpler formulas than when $l\neq m$).

After separate calculations for the two parts of $GQ^N_{\lambda,+}$ arising from this decomposition, one can put the pieces together and write $C_0(G,P_{1,+})$ as a sum of contributions, when P_1 has the form $1-\Delta$ in special local coordinates:

$$\varphi(\int [\mathrm{TR}_{x'}\,\mathrm{tr}_n\, G - \tfrac12 \mathrm{res}_{x',0}\,\mathrm{tr}_n(G\,(\log(1-\Delta))_+)]\,dx'\psi;$$

the normal trace of $G\log(1-\Delta)_+$ is log-polyhomogeneous as in (1.7) with $k=1$.

Finally, one can combine the special choice of P_1 with general choices of P_2 and use (3.10) to conclude:

THEOREM 4.2. *For a singular Green operator G of order $\sigma\in\mathbb{Z}$ and class 0, together with a general elliptic differential operator P_2 of order 2 having $\mathbb{R}_-$ as a spectral cut, $C_0(G,P_{2,+})$ is found in local coordinates to be of the form*

$$C_0(G,P_{2,+}) = \int_{X'} [\mathrm{TR}_{x'}\,\mathrm{tr}_n\, G - \tfrac12 \mathrm{res}_{x',0}\,\mathrm{tr}_n(G\,(\log P_2)_+)]\,dx'. \tag{4.9}$$

Details are in [**G6**].

We shall end this survey by reporting on the basic zeta coefficient in another interesting case, namely for the zeta function without the “smearing factor” B, for an elliptic operator in the Boutet de Monvel calculus. Here a generalization of the formula (3.3) holds; this was worked out in [**G4**]. Consider a system $\{P_+ + G, T\}$, where P is of integer order $m>0$, G is of order and class m and $T=\{T_0,\dots,T_{m-1}\}$ is a system of normal trace operators of order and class $0,\dots,m-1$, such that $\{P_+ + G-\lambda, T\}$ satisfies the conditions of parameter-ellipticity in [**G1**], Def. 3.3.1, for λ on the rays in a sector V around $\mathbb{R}_-$. This system defines the realization $(P+G)_T$ acting as $P_+ + G$ and with domain consisting of the sections $u\in H^m(X,E)$ (Sobolev space) such that $Tu=0$. The resolvent $R_\lambda = ((P+G)_T-\lambda)^{-1}$ exists on each ray in V for sufficiently large $|\lambda|$. As noted in [**G1**], Remark 3.3.11, even though R_λ is only trace-class itself when $m>n$, the operator constructed from the symbol minus its homogeneous terms of orders $\geq -n$ is always trace-class, and its trace has an expansion in a number of terms with powers of $-\lambda$. The terms match corresponding terms in trace expansions of higher powers of the resolvent $R^N_\lambda = \frac{\partial_\lambda^{N-1}}{(N-1)!}R_\lambda$ and in the zeta function $\zeta(I,(P+G)_T,s)=\mathrm{Tr}(P+G)_T^{-s}$. In fact,

$$\mathrm{Tr}(R_\lambda - \sum_{0\leq l\leq n-m} R_{\lambda,-m-l}) = \sum_{n-m<l\leq n} a_{-m-l}(-\lambda)^{(n-m-l)/m} + O(\lambda^{-1-\frac{1}{4m}}), \tag{4.10}$$

and here the coefficient of the term with $l=n$ is equal to $C_0(I,(P+G)_T)$.

This is a situation where the strictly homogeneous symbol of order $-m-n$ for R_λ is integrable for $\xi\to 0$ in the ψdo part, resp. for $\xi'\to 0$ in the s.g.o. part, which makes it possible to verify a formula generalizing (3.3):

The resolvent has the structure

$$R_\lambda = Q_{\lambda,+} + G_\lambda, \tag{4.11}$$

where $Q_\lambda = (P-\lambda)^{-1}$ on $\widetilde{X}$ and G_λ is the singular Green part. The operator $\log P$ is defined on $\widetilde{X}$ by the usual Cauchy integral (1.17), and we can, in local coordinates, define the residue integral $\operatorname{res}_x(\log P)$ by integration for $|\xi|=1$ of the fiber trace of the homogeneous symbol of order $-n$ (recall that it is classical, cf. (1.18)). It will contribute to $C_0(I,(P+G)_T)$ by integrals over the coordinate patches intersected with $\mathbb{R}^n_+$ (coming from X) multiplied by $-\frac{1}{m}$. We denote the contribution $\operatorname{res}_+(\log P)$, similarly to the notation used above.

When m is even, the classical part of $\log P$ has the transmission property at X' and the contribution can be regarded as $\operatorname{res}_X((\log P)_+)$ in a sense of residues as defined in [**FGLS**], except that we subtract the principal symbol (or just its log-term) in order to have an operator in the Boutet de Monvel calculus. (Here we are in fact dealing with $(\log P)^0$, cf. Remark 1.1.)

For the contribution from G_λ, we define $S_\lambda = \operatorname{tr}_n G_\lambda$ in local coordinates near the boundary (noting that G_λ on sets with positive distance from the boundary is of order $-\infty$ and $O(\lambda^{-1-\frac{1}{4m}})$). Let S'_λ be defined from the symbol of S_λ minus its principal part; integrating this along with $\log\lambda$ on a Laurent loop gives a classical ψdo S' on the boundary. The contribution to $C_0(I,(P+G)_T)$ is the integral over X' (carried back from the localized pieces) of the residue integrals $\operatorname{res}_{x'}(S')$ as in (3.9) in dimension $n-1$ (multiplied by $-\frac{1}{m}$).

In this sense, we can write the formula

$$C_0(I,(P+G)_T) = -\tfrac{1}{m}\operatorname{res}_+(\log P) - \tfrac{1}{m}\operatorname{res}_{X'}(S'), \tag{4.12}$$

generalizing (3.3), the terms being defined from local calculations. (The above account extends the formula from large m in [**G4**] to general $m>0$.)

The right-hand side in (4.12) (times $-m$) can be viewed as an interpretation of the residue of $\log((P+G)_T)$ in a generalized sense. The operator $\log((P+G)_T) = (\log P)_+ + G^{\log}$, is studied in more detail in a forthcoming paper, where we show that $G^{\log}$, although certainly not belonging to the Boutet de Monvel calculus, does satisfy some of the symbol-kernel estimates, as in the study of generalized s.g.o.s arising from complex powers in [**G1**], Sect. 4.4.

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