LOGARITHMIC TERMS IN TRACE EXPANSIONS OF ATIYAH-PATODI-SINGER PROBLEMS

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Abstract. For a Dirac-type operator D on a manifold X with a spectral boundary condition (defined by a pseudodifferential projection), the associated heat operator trace has an expansion in integer and half-integer powers and log-powers of t; the interest in the expansion coefficients goes back to the work of Atiyah, Patodi and Singer. In the product case considered by APS, it is known that all the log-coefficients vanish when $\dim X$ is odd, whereas the log-coefficients at integer powers vanish when $\dim X$ is even. We here investigate whether this partial vanishing of logarithms holds more generally. One type of result, shown for general D with well-posed boundary conditions, is that a perturbation of D by a tangential differential operator vanishing to order k on the boundary leaves the first k log-power terms invariant (and the non-local power terms of the same degree are only locally perturbed). Another type of result is that for perturbations of the APS product case by tangential operators commuting with the tangential part of D, all the logarithmic terms vanish when $\dim X$ is odd (whereas they can all be expected to be nonzero when $\dim X$ is even). The treatment is based on earlier joint work with R. Seeley and a recent systematic parameter-dependent pseudodifferential boundary operator calculus, applied to the resolvent.

[1]Introduction.

Let D be a first-order differential operator of Dirac-type from $C^\infty(X,E_1)$ to $C^\infty(X,E_2)$ (E_1 and E_2 Hermitian N-dimensional vector bundles over a compact n-dimensional C^∞ manifold X with boundary $\partial X = X'$), and let $D_\geq$ be the L_2-realization defined by the boundary condition $\Pi_\geq(u|_{X'}) = 0$; here $\Pi_\geq$ is the orthogonal projection onto the nonnegative eigenspace for a certain selfadjoint operator A over X' entering in D. For $\Delta_B = D_\geq^* D_\geq$ (and likewise for $D_\geq D_\geq^*$), the following heat trace expansion was shown in a joint work with Seeley [GS95]:

$$\operatorname{Tr}(\varphi e^{-t\Delta_B}) \sim \sum_{-n\leq k<0} a_k t^{\frac{k}{2}} + \sum_{k\geq 0}\big(a'_k \log t + a''_k\big) t^{\frac{k}{2}} \text{ for } t\to 0+\,. \tag{0.1}$$

Here φ is a smooth morphism in E_1; the coefficient a'_0 vanishes when $\varphi = 1$ near X'. The coefficient a''_0 enters in the index of $D_\geq$; the geometric content of the first four a_k (with $k<0$) has been investigated by Dowker, Gilkey and Kirsten [DGK99], [GK02].

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For the case with product structure near X', as studied originally by Atiyah, Patodi and Singer in [APS75], the coefficients were described in [GS96] in terms of the expansion coefficients of zeta and eta functions of A. In particular, it was found that the coefficients a'_k vanish for k even > 0; moreover, if n is odd, they vanish for all $k \geq 0$. The remaining coefficients with $k \neq 0$ are nonzero in general even when $\varphi = 1$, cf. Gilkey and Grubb [GG98].

We shall here investigate to what extent this "partial vanishing of logarithms" may hold in non-product cases. Our principal results are:

1) Consider two choices D_1 and D_2 of D, provided with the same well-posed boundary condition. If they differ by a first-order tangential differential operator $x_n^l P$ (where x_n is the normal coordinate), then the expansions (0.1) for D_1 and D_2 have, for $0 \leq k \leq l$, the same log-coefficients a'_k, and the coefficients a''_k differ only by local terms. In particular, the coefficient a'_1 is preserved under perturbations of an operator with product structure D^0 near X' by terms vanishing at X'. (Section 3.)

2) If D is a perturbation of the product case D^0 (near X') with $\Pi = \Pi_{\geq}$, by a tangential first-order differential operator *commuting with* A, then all log-coefficients are zero if n is odd. When n is even, nontrivial log-terms can in general be expected for both even and odd k. (Section 5.)

We also derive the related expansions for resolvent traces and zeta functions, and we allow φ to be replaced by a differential operator F (tangential or acting in all variables). Similar results are shown for the operator families associated with the eta function.

In preparation for these results, Section 2 gives a review of the underlying parameter-dependent pseudodifferential boundary operator calculus, and Section 4 shows the structure of the resolvent in the commuting case.

Throughout this paper, D_1 and D_2 are provided with the same boundary condition. Perturbations of the boundary condition are considered e.g. in [G01′] and in [G02].

1. Representation formulas.

E_1 and E_2 have Hermitian metrics, and X has a smooth volume element, defining Hilbert space structures on the sections, $L_2(E_1)$, $L_2(E_2)$. The restrictions of the E_i to the boundary X' are denoted E'_i. A neighborhood of X' in X has the form $X_c = X' \times [0, c[$, and there the E_i are isomorphic to the pull-backs of the E'_i. We let x_n denote the coordinate in $[0, c[$. $L_2(E'_i))$ is defined with respect to the volume element $v(x', 0)dx'$ on X' induced by the element $v(x', x_n)dx'dx_n$ on X.

When D is a first-order elliptic differential operator from $C^\infty(E_1)$ to $C^\infty(E_2)$, it may always be written in the following form over X_c:

$$D = \sigma(\partial_{x_n} + A_1), \tag{1.1}$$

where σ is a homeomorphism from $E_1|_{X_c}$ to $E_2|_{X_c}$ and A_1 for each x_n is an elliptic operator in the x'-variable. We say that D is of *product type* when σ is independent of x_n and is unitary from E'_1 to E'_2, and $A_1 = A$ independent of x_n and selfadjoint in $L_2(E'_1)$; here the product measure $dx'dx_n$ is used on X_c. We say that D is of *non-product type* when σ is still independent of x_n and unitary, but the condition on A_1 is relaxed to:

$$A_1 = A + x_n P_1 + P_0, \tag{1.2}$$

where A is as above and the P_j are smooth x_n-dependent differential operators in x' (in short: tangential differential operators) of order $\leq j$. Since $P_0 = P_0(0) + x_n P'_0(x_n)$ with P'_0

of order 0, we may absorb $x_nP'(0)$ in the term x_nP_1, so *we can assume that P_0 is constant in x_n on X_c*. (In [GS95], [G99], these operators of product type and of non-product type were said to be "of Dirac-type". Some other authors restrict that notation to operators that moreover satisfy

$$\sigma^2=-I,\quad \sigma A=-A\sigma,\quad D \text{ is selfadjoint on } X \text{ and } D^2 \text{ principally scalar},$$

which we do not assume here.)

Integration by parts shows that the formal adjoint D^* equals

$$D^*=(-\partial_{x_n}+A_1')\sigma^*,\quad A_1'=A+x_nP_1^*+P_0',\ \text{on } X_c,$$

where $P_0'=P_0^*-v^{-1}\partial_{x_n}v$. When $\partial_{x_n}v(x',0)=0$, D^* may also be written in the form $D^*=(-\partial_{x_n}+A+x_nP_1'+P_0^*)\sigma^*$, with $P_1'-P_1^*$ of order 0. If $P_0=0$ and $P_1=x_n^{l-1}P_l$ for some $l\ge 1$, so that $D=\sigma(\partial_{x_n}+A+x_n^lP_l)$, then D^* can be written in the form $D^*=(-\partial_{x_n}+A+x_n^lP_l')\sigma^*$ if $\partial_{x_n}^jv(x',0)=0$ for $1\le j\le l$.

In the product case we often use the notation

$$D^0=\sigma(\partial_{x_n}+A),\quad D^{0\prime}=(-\partial_{x_n}+A)\sigma^*; \tag{1.3}$$

these operators have a meaning on $X^0=X'\times\overline{\mathbb{R}}_+$; and $D^{0\prime}$ is the formal adjoint of the operator D^0 going from $L_2(E_1^0)$ to $L_2(E_2^0)$, where the E_i^0 are the liftings of the E_i' to X^0, and the product measure is used.

By $V_>$, $V_\ge$, $V_<$ or $V_\le$ we denote the subspaces of $L^2(E_1')$ spanned by the eigenvectors of A corresponding to eigenvalues which are >0, ≥ 0, <0, or ≤ 0. (For precision one can write $V_>(A)$, etc.) V_0 is the nullspace of A. The corresponding projections are denoted $\Pi_>$, $\Pi_\ge$, etc. (note that $\Pi_\ge=\Pi_>+\Pi_0$ and $\Pi_<=I-\Pi_\ge$). They are pseudodifferential operators (ψdo's) of order 0; Π_0 has finite rank and is a ψdo of order $-\infty$. We also define

$$A_\lambda=(A^2-\lambda)^{\frac12},\ \text{for } \lambda\in\mathbb{C}\setminus\operatorname{spec}A^2\supset\mathbb{C}\setminus\overline{\mathbb{R}}_+. \tag{1.4}$$

Moreover, we set

$$\begin{aligned} &|A|=(A^2)^{\frac12},\quad A'=A+\Pi_0,\qquad \text{so that } |A'|=|A|+\Pi_0 \text{ and}\\ &\Pi_>=\tfrac12\tfrac{|A|+A}{|A'|}=\tfrac12\tfrac{A}{|A'|}+\tfrac12-\tfrac12\Pi_0,\quad \Pi_\ge=\Pi_>+\Pi_0=\tfrac12\tfrac{A}{|A'|}+\tfrac12+\tfrac12\Pi_0.\end{aligned} \tag{1.5}$$

In Sections 4–5 of this paper, we consider the product and non-product cases with the boundary condition

$$\Pi_\ge(A)\gamma_0u=0, \tag{1.6}$$

where $\gamma_0u=u|_{X'}$, defining the realizations $D_\ge$ and $D_\ge^0$, and we denote $D_\ge{}^*D_\ge=\Delta_B$, $D_\ge^0{}'D_\ge^0=\Delta_B^0$. However, the more qualitative results in Section 3 allow the consideration of a general first-order elliptic operator D with a boundary condition

$$\Pi\gamma_0u=0, \tag{1.7}$$

where Π is an orthogonal pseudodifferential projection that is *well-posed* with respect to D (cf. [S69] or [G99]). This means that when we at each (x',ξ') in the cotangent sphere bundle of X' denote by $N^+(x',\xi')\subset\mathbb{C}^N$ the space of boundary values of null-solutions of the model operator (defined from the principal symbol d^0 of D),

$$N^+(x',\xi')=\{\,z(0)\in\mathbb{C}^N\mid d^0(x',0,\xi',D_{x_n})z(x_n)=0,\ z(x_n)\in L_2(\mathbb{R}_+)^N\,\},$$

then the principal symbol $\pi^0(x',\xi')$ of Π maps $N^+(x',\xi')$ bijectively onto the range of $\pi^0(x',\xi')$ in $\mathbb{C}^N$. We denote the realization of D defined by (1.7) by D_Π and again denote ${D_\Pi}^*D_\Pi=\Delta_B$; it likewise has a trace expansion (0.1), cf. [G99].

As shown in [G01′], the coefficients a_k and a'_k with $k\le J-n$ in the trace expansion (0.1) are unaffected by a replacement of Π by a closed range operator $\Pi+S$, where S is a pseudodifferential operator of order $\le -J$ for some $J\ge 1$.

It is explained e.g. in [GS95], [GS96] how the heat trace expansion (0.1) is equivalent with the derived resolvent expansion

$$\text{(1.8)}\quad \operatorname{Tr}(\varphi\partial_\lambda^r(\Delta_B-\lambda)^{-1})\sim\sum_{-n\le k<0}\tilde a_k(-\lambda)^{-\frac k2-r-1}+\sum_{k\ge0}\big(\tilde a'_k\log(-\lambda)+\tilde a''_k\big)(-\lambda)^{-\frac k2-r-1},$$

where $r+1>\frac n2$ and $\lambda\to\infty$ on rays in $\mathbb{C}\setminus\mathbb{R}_+$. The coefficients $\tilde a_k$, $\tilde a'_k$ and $\tilde a''_k$ are proportional to the coefficients a_k, a'_k, a''_k in (0.1), respectively, by universal nonzero proportionality factors (depending on r). We shall henceforth work with the resolvent. It is well-known that (0.1) is likewise equivalent with the zeta function expansion

$$\text{(1.9)}\qquad \Gamma(s)\operatorname{Tr}(\varphi\Delta_B^{-s})\sim\sum_{-n\le k<0}\frac{a_k}{s+\frac k2}-\frac{\operatorname{Tr}(\varphi\Pi_0(\Delta_B))}{s}+\sum_{k\ge0}\Big(-\frac{a'_k}{(s+\frac k2)^2}+\frac{a''_k}{s+\frac k2}\Big),$$

describing the pole structure of the meromorphic extension of $\Gamma(s)\operatorname{Tr}(\varphi\Delta_B^{-s})$ from $\operatorname{Re}s>\frac n2$ to $s\in\mathbb{C}$. Here Δ_B^{-s} is defined by functional calculus on $V_0(\Delta_B)^\perp$ and is taken to be zero on $V_0(\Delta_B)$. $\operatorname{Tr}(\varphi\Delta_B^{-s})$ is also denoted $\zeta(\varphi,\Delta_B,s)$, the zeta function.

The coefficients $\tilde a_k,\tilde a'_k,a_k,a'_k$ are locally determined. The first sum in (0.1), (1.8), (1.9), is sometimes written as a summation over all $k\ge -n$; we presently use a convention where such local contributions for $k\ge0$ are absorbed in the generally nonlocal coefficients $\tilde a''_k,a''_k$.

There exist several ways of representing the resolvent. A direct way is described in [G92]. Another way, introduced in [GS95] (see also [G99] for the case (1.7)), is to identify the resolvent as a block in the resolvent of an enlarged first-order system, acting in the bundle $E=E_1\oplus E_2$ over X: Let

$$\text{(1.10)}\qquad \mathcal{D}=\begin{pmatrix}0&-D^*\\D&0\end{pmatrix},\quad \mathcal{D}_{\mathcal{B}}=\begin{pmatrix}0&-{D_\Pi}^*\\D_\Pi&0\end{pmatrix};$$

here $\mathcal{D}_{\mathcal{B}}$ is the realization of $\mathcal{D}$ defined by the boundary condition

$$\text{(1.11)}\qquad \mathcal{B}\gamma_0u=0,\ \text{with } \mathcal{B}=(\,\Pi\quad\Pi^\perp\sigma^*\,),\quad u=\begin{pmatrix}u_1\\u_2\end{pmatrix}.$$

The operator $\mathcal{D}$ in (1.10) is formally skew-selfadjoint on X. When $\widetilde{D}$ is an extension of D to an open n-dimensional C^∞ manifold $\widetilde{X}$ in which X is smoothly imbedded, we define $\widetilde{\mathcal{D}}$ from $\widetilde{D}$ as in (1.10) and set, for $\mu \in \mathbb{C} \setminus i\mathbb{R}$,

$$\mathcal{Q}_\mu = \begin{pmatrix} \mu(\widetilde{D}^*\widetilde{D} + \mu^2)^{-1} & \widetilde{D}^*(\widetilde{D}\widetilde{D}^* + \mu^2)^{-1} \\ -\widetilde{D}(\widetilde{D}^*\widetilde{D} + \mu^2)^{-1} & \mu(\widetilde{D}\widetilde{D}^* + \mu^2)^{-1} \end{pmatrix}, \tag{1.12}$$

where $(\widetilde{D}^*\widetilde{D}+\mu^2)^{-1}$ (resp. $(\widetilde{D}\widetilde{D}^*+\mu^2)^{-1}$) is a parametrix of $\widetilde{D}^*\widetilde{D}+\mu^2$ (resp. of $\widetilde{D}\widetilde{D}^*+\mu^2$); it can be taken as an inverse when $\widetilde{X}$ is compact. Then $\mathcal{Q}_\mu$ is a parametrix — an inverse if $\widetilde{X}$ is compact — of $\widetilde{\mathcal{D}} + \mu$, as is easily checked.

The operator $\mathcal{D}_\mathcal{B}$ is skew-selfadjoint as an unbounded operator in $L_2(E)$, so it has a resolvent $\mathcal{R}_\mu = (\mathcal{D}_\mathcal{B} + \mu)^{-1}$ for $\mu \in \mathbb{C} \setminus i\mathbb{R}$, equal to

$$\mathcal{R}_\mu = (\mathcal{D}_\mathcal{B} + \mu)^{-1} = \begin{pmatrix} \mu(D_\Pi^* D_\Pi + \mu^2)^{-1} & D_\Pi^*(D_\Pi D_\Pi^* + \mu^2)^{-1} \\ -D_\Pi(D_\Pi^* D_\Pi + \mu^2)^{-1} & \mu(D_\Pi D_\Pi^* + \mu^2)^{-1} \end{pmatrix}. \tag{1.13}$$

Thus the resolvent $R_\lambda = (D_\Pi^* D_\Pi - \lambda)^{-1} = (\Delta_B - \lambda)^{-1}$ which we want to analyze, can be retrieved as

$$R_{-\mu^2} = \mu^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathcal{R}_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mu^{-1}\mathcal{R}_{\mu,11}, \quad \lambda = -\mu^2. \tag{1.14}$$

The resolvent has the structure

$$R_\lambda = Q_{\lambda,+} + G_\lambda, \text{ where } Q_{-\mu^2} = \mu^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \mathcal{Q}_\mu \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.15}$$

and G_λ is a singular Green operator (more about pseudodifferential boundary operators in Section 2).

When D is of non-product type and $\Pi = \Pi_\geq$, we consider along with $\mathcal{D}$ and $\mathcal{D}_\mathcal{B}$ the associated operators of product type

$$\mathcal{D}^0 = \begin{pmatrix} 0 & -D^{0\prime} \\ D^0 & 0 \end{pmatrix}, \quad \mathcal{D}^0_\mathcal{B} = \begin{pmatrix} 0 & -D^{0\,\prime}_\geq \\ D^0_\geq & 0 \end{pmatrix}; \tag{1.16}$$

here $\mathcal{D}^0$ and $\mathcal{D}^0_\mathcal{B}$ act in $E^0 = E^0_1 \oplus E^0_2$, and $\mathcal{D}^0_\mathcal{B}$ is determined by the same boundary condition (1.11) as $\mathcal{D}_\mathcal{B}$. $\mathcal{D}^0$ extends to the bundle $\widetilde{E}$ over $\widetilde{X}^0 = X' \times \mathbb{R}$ obtained by lifting $E' = E'_1 \oplus E'_2$.

In the product situation, we can define the ingredients by functional calculus from A, using the Fourier transform in the x_n-variable only. We can write $Q^0_\lambda = (D^{0\prime} D^0 - \lambda)^{-1}$ as follows:

$$Q^0_\lambda = (D^2_{x_n} + A^2 - \lambda)^{-1} = \mathcal{F}^{-1}_{\xi_n \to x_n}(\xi_n^2 + A^2 - \lambda)^{-1}\mathcal{F}_{x_n \to \xi_n}. \tag{1.17}$$

Moreover, we can describe the boundary operators using the following notation for the elementary Poisson operator K_{A_λ}, trace operator T_{A_λ} of class 0, and singular Green operator

G_{A_λ} of class 0:

$$\begin{aligned} K_{A_\lambda} &= \mathrm{OPK}_n(e^{-x_nA_\lambda}) = \mathrm{OPK}_n\Big(\frac{1}{A_\lambda + i\xi_n}\Big),\\ T_{A_\lambda} &= \mathrm{OPT}_n(e^{-x_nA_\lambda}) = \mathrm{OPT}_n\Big(\frac{1}{A_\lambda - i\xi_n}\Big),\\ G_{A_\lambda} &= \mathrm{OPG}_n(e^{-(x_n+y_n)A_\lambda}) = \mathrm{OPG}_n\Big(\frac{1}{(A_\lambda + i\xi_n)(A_\lambda - i\eta_n)}\Big), \end{aligned} \tag{1.18}$$

here we have used both the symbol-kernel and the symbol notation, with respect to the x_n-coordinate. (One can write $(A_\lambda + i\xi_n)^{-1}$ etc. as fractions in these formulas, since they are all commuting functions of the selfadjoint operator A.) Explicitly, for $v \in C^\infty(E_1')$, $u \in C^\infty(E_1^0)$ with compact support in X_c,

$$[K_{A_\lambda}v](x) = e^{-x_nA_\lambda}v(x'), \quad [T_{A_\lambda}u](x') = \int_0^\infty e^{-x_nA_\lambda}u(x',x_n)\,dx_n, \tag{1.19}$$

in the sense of functional calculus, and

$$G_{A_\lambda}u = K_{A_\lambda}T_{A_\lambda}u. \tag{1.20}$$

The use of formulas based on functional calculus will be pursued in Section 4.

Remark 1.2. For a resolvent $(T-\lambda)^{-1}$, one has that

$$\partial_\lambda^r (T-\lambda)^{-1} = r!(T-\lambda)^{-r-1}, \tag{1.21}$$

so it makes no difference whether one refers to powers or to λ-derivatives when describing trace expansions for iterated resolvents. However, in [GS95], the variable $-\lambda$ was replaced by μ^2 and the primary results were expressed for μ-derivatives, since this looked less complicated than a description of the many terms resulting from raising the resolvent to a power. There were not given many details on how one gets back to the desired expansions of λ-derivatives. In fact, a direct consideration of powers (still departing from a reformulation in the variable μ) would have been more adequate; this road was followed in subsequent treatments (partly in [G99], fully in [G01]), and is also followed below. The difference lies in the fact that, with notation as (1.14), the power formula $R_\lambda^2 = \mu^{-2}\mathcal{R}_{\mu,11}^2$ shows a decrease in the order (for fixed μ) which is harder to see from the differentiation formula

$$\partial_\lambda R_\lambda = \partial_\mu R_{-\mu^2}\partial_\lambda\mu = \partial_\mu(\mu^{-1}\mathcal{R}_{\mu,11})\tfrac{-1}{2}\mu^{-1} = c_1\mu^{-3}\mathcal{R}_{\mu,11} + c_2\mu^{-1}\partial_\mu\mathcal{R}_{\mu,11}. \tag{1.22}$$

2. The parameter-dependent symbol calculus.

Let us briefly recall the symbol spaces for pseudodifferential boundary operators (ψdbo's) introduced in [G01]. It is simplest to explain for operators of class 0, which is essentially all we need here (we refer to [G01] for the full calculus).

One considers systems of μ-dependent operators

$$\mathcal{A}(\mu) = \begin{pmatrix} P(\mu)_+ + G(\mu) & K(\mu) \\ T(\mu) & Q(\mu) \end{pmatrix} : \begin{matrix} C^\infty(\overline{\mathbb{R}}^n_+)^N \\ \times \\ C^\infty(\mathbb{R}^{n-1})^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\mathbb{R}}^n_+)^{N'} \\ \times \\ C^\infty(\mathbb{R}^{n-1})^{M'} \end{matrix}, \tag{2.1}$$

which for each fixed μ belong to the calculus of Boutet de Monvel [BM71]: $P(\mu)$ is a ψdo on $\mathbb{R}^n$ satisfying the transmission condition at $x_n = 0$, $G(\mu)$ is a singular Green operator (s.g.o.), $T(\mu)$ is a trace operator, $K(\mu)$ is a Poisson operator and $Q(\mu)$ is a ψdo on $\mathbb{R}^{n-1}$. For the reader who is not familiar with this calculus, we refer to e.g. [G96, Ch. 1] or [G01, Sect. 1].

The starting point in the present parameter-dependent case is the ψdo symbol spaces from [GS95], based on $x' \in \mathbb{R}^{n-1}$ and now allowed to take values in Banach spaces B such as $L_p(\mathbb{R}_+)$, $L_p(\mathbb{R}^2_{++})$ (we write $\mathbb{R}^2_{++} = \mathbb{R}_+ \times \mathbb{R}_+$). Moreover, we now take powers of $|(\xi', \mu)|$ into the definition. To smooth out the behavior of $|(\xi', \mu)|$ near 0, it is convenient to replace it by $[(\xi', \mu)]$; here $[x]$ denotes a C^∞ function of $x \in \mathbb{R}^N$ satisfying $[x] = |x|$ for $|x| \geq 1$, $[x] \in [\frac{1}{2}, 1]$ for $|x| \leq 1$. We also use the notation $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Note that $[(\xi', 1/z)] = |(\xi', 1/z)|$ for $|z| \leq 1$, and that

$$|(\xi', 1/z)| = |(\xi', \mu)| = |z|^{-1}\langle z\xi' \rangle, \text{ when } \mu = 1/z. \tag{2.2}$$

$[(\xi', \mu)]$ is more briefly written $[\xi', \mu]$; it will in the following often be denoted κ (as in [G96]), so from now on,

$$\kappa = [\xi', 1/z] = [\xi', \mu], \text{ with } \mu = 1/z. \tag{2.3}$$

We denote $\{0, 1, 2, \dots\} = \mathbb{N}$.

Definition 2.1. *Let $m \in \mathbb{R}$, d and $s \in \mathbb{Z}$. Then $S^{m,0,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma, B)$ consists of the C^∞ functions $p(x', \xi', \mu)$ valued in B which satisfy, with $1/\mu = z$,*

$$\begin{aligned} &\partial^j_{|z|} p(\cdot, \cdot, 1/z) \in S^{m+j}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, B) \text{ for } 1/z \in \Gamma, \\ &\qquad \text{with uniform estimates for } |z| \leq 1, 1/z \text{ in closed subsectors of } \Gamma, \end{aligned} \tag{2.4}$$

for all $j \in \mathbb{N}$. Moreover, we define

$$S^{m,d,s}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma, B) = \mu^d [\xi', \mu]^s S^{m,0,0}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma, B). \tag{2.5}$$

The indication $(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}, \Gamma, B)$ is often abbreviated to (Γ, B), or just (Γ) if $B = \mathbb{C}$. Keeping the identification of μ with $1/z$ in mind, we shall also say that $p(x', \xi', 1/z)$ lies in $S^{m,d,s}(\Gamma, B)$.

We leave the requirement (from [GS95]) of being holomorphic in $\mu \in \Gamma^\circ$ out of the definition since $\kappa = [\xi', \mu]$ is not so (one could instead work with a variant of κ that is holomorphic on suitable sectors, as in [G96], (A.2$''$)–(A.2$'''$)). The symbol is just assumed to be C^∞ in $\mu \in \Gamma$ considered as a subset of $\mathbb{R}^2$. Accordingly, we write $\partial_{|z|}$ (instead of ∂_z), since it is a control of the radial derivative that is needed (uniformly when the argument of z runs in a compact interval), and $|z|$ in the following enters as a real parameter.

To define symbol-kernels for the boundary operators, we use the cases $B = L_\infty(\mathbb{R}_+)$ and $B = L_\infty(\mathbb{R}^2_{++})$, with variables x_n or u_n, resp. (x_n, y_n) or (u_n, v_n); these variables will then be mentioned in the detailed description of the function.

We denote by $r^\pm$ the restriction from distributions on $\{x_n \in \mathbb{R}\}$ to distributions on $\{x_n \gtrless 0\}$, and by $e^\pm$ the extension from functions on $\{x_n \gtrless 0\}$ to functions on $\{x_n \in \mathbb{R}\}$

by assigning zero values on $\{x_n \lessgtr 0\}$, respectively. With $\mathcal{S}(\mathbb{R}^n)$ denoting the Schwartz space, we denote $r^\pm\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\overline{\mathbb{R}}^n_\pm)$, $\mathcal{S}(\overline{\mathbb{R}}_\pm) = \mathcal{S}_\pm$. We denote $r^+_{x_n} r^+_{y_n}\mathcal{S}(\mathbb{R}^2) = \mathcal{S}_{++}$.

Let us also recall the notation developed from [BM71]: For $n = 1$, the Fourier transformed spaces are denoted $\mathcal{H}^+ = \mathcal{F}e^+\mathcal{S}(\overline{\mathbb{R}}_+)$, $\mathcal{H}^-_{-1} = \mathcal{F}e^-\mathcal{S}(\overline{\mathbb{R}}_-)$; they consist of C^∞ functions that extend holomorphically to $t \in \mathbb{C}_-$ resp. $\mathbb{C}_+$ ($\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \operatorname{Im} z \gtrless 0\}$) and are $O(t^{-1})$ there. Adding to $\mathcal{H}^-_{-1}$ the space $\mathbb{C}[t]$ of polynomials in t, we get the space $\mathcal{H}^-$. We denote

$$\mathcal{H} = \mathcal{H}^+ \dot{+} \mathcal{H}^-, \text{ with projections } h^\pm\colon \mathcal{H} \to \mathcal{H}^\pm. \tag{2.6}$$

The space $\mathcal{F}_{x_n\to\xi_n}\overline{\mathcal{F}}_{y_n\to\eta_n} e^+_{x_n} e^+_{y_n}\mathcal{S}_{++}$ identifies with $\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1}$. (Our notation for the Fourier transform and the conjugate Fourier transform is:

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx, \quad (\overline{\mathcal{F}}f)(\xi) = \int_{\mathbb{R}^n} e^{+ix\cdot\xi} f(x)\,dx, \tag{2.7}$$

so that $\mathcal{F}^{-1} = (2\pi)^{-n}\overline{\mathcal{F}}$; they are sometimes just applied in the x'-variable or the x_n-variable alone.)

The appropriate definition of parameter-dependent symbol-kernels for boundary operators involves a scaling in the x_n-variable:

Definition 2.2. *Let $m \in \mathbb{R}$, d and $s \in \mathbb{Z}$.*

(i) *The space $\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \Gamma, \mathcal{S}_+)$ (briefly denoted $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_+)$) consists of the complex functions $\tilde{f}(x', x_n, \xi', \mu)$ in $C^\infty(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}_+\times\mathbb{R}^{n-1}\times\Gamma)$ satisfying, for all $l, l' \in \mathbb{N}$,*

$$\langle z\xi'\rangle^{l-l'} u_n^l \partial_{u_n}^{l'} \tilde{f}(x', |z|u_n, \xi', 1/z) \in S^{m,d,s+1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \Gamma, L_{\infty,u_n}(\mathbb{R}_+)) \tag{2.8}$$

(equivalently, $u_n^l \partial_{u_n}^{l'} \tilde{f}(x', |z|u_n, \xi', 1/z)$ belongs to $\in S^{m,d+l-l',s+1-l+l'}(\Gamma, L_{\infty,u_n}(\mathbb{R}_+))$).

(ii) *The space $\mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \Gamma, \mathcal{S}_{++})$ (briefly denoted $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_{++})$) consists of the complex functions $\tilde{f}(x', x_n, y_n, \xi', \mu)$ in $C^\infty(\mathbb{R}^{n-1}\times\overline{\mathbb{R}}^2_{++}\times\mathbb{R}^{n-1}\times\Gamma)$ satisfying, for all $l, l', k, k' \in \mathbb{N}$,*

$$\begin{aligned}\langle z\xi'\rangle^{l-l'+k-k'} u_n^l \partial_{u_n}^{l'} v_n^k \partial_{v_n}^{k'} \tilde{f}(x', |z|u_n, |z|v_n, \xi', 1/z)&\\ \in S^{m,d,s+2}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}, \Gamma, L_{\infty,u_n,v_n}(\mathbb{R}^2_{++})).&\end{aligned} \tag{2.9}$$

In details, the statement in (2.8) means that for all j,

$$\|\partial^j_{|z|}(z^d\kappa^{-s-1}\langle z\xi'\rangle^{l-l'} u_n^l \partial_{u_n}^{l'} \tilde{f}(x', |z|u_n, \xi', 1/z))\|_{L_{\infty,u_n}} \mathrel{\dot{\leq}} \langle\xi'\rangle^{m+j}, \tag{2.10}$$

with similar estimates for derivatives $\partial^\alpha_{\xi'}\partial^\beta_{x'}$ with m replaced by $m - |\alpha|$. There is a related explanation of (2.9). We here use $\dot{\leq}$ to indicate "$\leq$ a constant times"; also $\dot{\geq}$ will be used, and $\doteq$ indicates that both $\dot{\leq}$ and $\dot{\geq}$ hold.

The third upper index s is included to keep track of factors $\kappa = [\xi', 1/z]$ in a manageable way. When $s = 0$, we may lave it out of the notation, consistently with [GS95]:

$$S^{m,d,0} = S^{m,d}. \tag{2.11}$$

For the trace formulas later on, it is important to know that we always have inclusions (that follow from [GS95, Lemma 1.13]):

$$
\begin{aligned}
S^{m,d,s} &\subset S^{m+s,d,0} \cap S^{m,d+s,0} \text{ if } s \le 0,\\
S^{m,d,s} &\subset S^{m+s,d,0} + S^{m,d+s,0} \text{ if } s \ge 0,
\end{aligned} \tag{2.12}
$$

for S- as well as for $\mathcal{S}$-spaces. We denote

$$
\bigcap_{m\in\mathbb{R}} S^{m,d,s} = S^{-\infty,d,s}, \quad \bigcup_{m\in\mathbb{R}} S^{m,d,s} = S^{\infty,d,s}, \qquad \text{etc.}; \tag{2.13}
$$

observe that by (2.12), $S^{m,d,-\infty} = S^{-\infty,-\infty,-\infty}$.

The following rule follows from the definition (proof details are given in [G01, Lemma 2.10]):

Lemma 2.3.

(i) *When* $\tilde f \in \mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_+)$, *then* $x_n^j \partial_{x_n}^{j'} \tilde f \in \mathcal{S}^{m,d,s-j+j'}(\Gamma,\mathcal{S}_+)$ *for* $j, j' \in \mathbb{N}$.

(ii) *When* $\tilde f \in \mathcal{S}^{m,d,s}(\Gamma, \mathcal{S}_{++})$, *then* $x_n^i \partial_{x_n}^{i'} y_n^j \partial_{y_n}^{j'} \tilde f \in \mathcal{S}^{m,d,s-i+i'-j+j'}(\Gamma,\mathcal{S}_{++})$ *for* i, i', j, $j' \in \mathbb{N}$.

In the applications to trace formulas, the symbols moreover have to be holomorphic in μ for $\mu \in \Gamma^\circ$ with $|(\xi',\mu)| \ge \varepsilon$ (some $\varepsilon > 0$); we call such symbols holomorphic in μ, and this property is preserved in compositions.

Definition 2.4.

1° *The functions in* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_+)$ *are the* **Poisson symbol-kernels** *and* **trace symbol-kernels of class 0**, *of degree* $m+d+s$, *in the parametrized calculus.*

2° *The functions in* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_{++})$ *are the* **singular Green symbol-kernels of class** 0 *and degree* $m+d+s$ *in the parametrized calculus.*

Operators are defined from these symbol-kernels as follows:

$$
\begin{aligned}
T &= \mathrm{OPT}(\tilde f)\colon u(x) \mapsto \int_{\mathbb{R}^{2(n-1)}} \int_0^\infty e^{i(x'-y')\cdot\xi'} \tilde f(x',x_n,\xi',\mu) u(y',x_n)\, dx_n dy' đ\xi',\\
K &= \mathrm{OPK}(\tilde f)\colon v(x') \mapsto \int_{\mathbb{R}^{2(n-1)}} e^{i(x'-y')\cdot\xi'} \tilde f(x',x_n,\xi',\mu) v(y')\, dy' đ\xi',\\
G &= \mathrm{OPG}(\tilde f)\colon u(x) \mapsto \int_{\mathbb{R}^{2(n-1)}} \int_0^\infty e^{i(x'-y')\cdot\xi'} \tilde f(x',x_n,y_n,\xi',\mu) u(y)\, dy đ\xi';
\end{aligned} \tag{2.14}
$$

note that the usual ψdo definition is used with respect to the x'-variable. Here $đ\xi'$ stands for $(2\pi)^{1-n} d\xi'$. When these definitions are applied with respect to the x_n-variable alone, we write OPT_n, OPK_n, OPG_n.

Let us also mention the definition of the associated symbols, where a Fourier transformation has been performed in the x_n-variable.

Definition 2.5.

(i) *By Fourier transformation in* x_n, $e^+\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_+)$ *is carried over to the space* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^+)$ *of* **Poisson symbols** *of degree* $m+s+d$, *and conjugate Fourier transformation in* x_n *gives the space* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^-_{-1})$ *of* **trace symbols of class** *0 and degree* $m+s+d$.

(ii) *By Fourier transformation in* x_n *and conjugate Fourier transformation in* y_n, $e^+_{x_n}e^+_{y_n}\mathcal{S}^{m,d,s}(\Gamma,\mathcal{S}_{++})$ *is carried over to the space* $\mathcal{S}^{m,d,s}(\Gamma,\mathcal{H}^+\hat{\otimes}\mathcal{H}^-_{-1})$ *of* **singular Green symbols of class** 0 *and degree* $m+d+s$.

The μ-dependent ψdo's P given on $\mathbb{R}^n$ should in addition to the conditions in Definition 2.1 satisfy an appropriate transmission condition at $x_n=0$, which assures that the truncated ψdo $P_+=r^+Pe^+$ enters in the calculus in a good way. The general condition is explained in [G01, Sect. 6], where the class of such symbols, of degree s, is denoted $S^{0,0,s}_{\mathrm{ut}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma)$ (here "ut" stands for "uniform transmission condition", cf. also [G96]).

When $\tilde f_{m-j}\in S^{m-j,d,s}(\Gamma,\mathcal{S}_+)$ for $j\in\mathbb{N}$ and $\tilde f\in S^{m,d,s}(\Gamma,\mathcal{S}_+)$, we say that $\tilde f\sim\sum_j\tilde f_{m-j}$ in $S^{m,d,s}(\Gamma,\mathcal{S}_+)$ if $\tilde f-\sum_{j<J}\tilde f_{m-j}\in S^{m-J,d,s}(\Gamma,\mathcal{S}_+)$ for any $J\in\mathbb{N}$. For any given sequence $\tilde f_{m-j}\in S^{m-j,d,s}(\Gamma,\mathcal{S}_+)$, one can construct an $\tilde f$ such that $\tilde f\sim\sum_j\tilde f_{m-j}$ in $S^{m,d,s}(\Gamma,\mathcal{S}_+)$. (Similar statements hold with $\mathcal{S}_{++}$.)

Of particular interest are the subspaces of the above symbol-kernel spaces consisting of the functions $\tilde f\in S^{m,d,s}$ that are asymptotic series of terms $\tilde f_{m-j}\in S^{m-j,d,s}$ with a specific quasi-homogeneity in (x_n,ξ',μ) or (x_n,y_n,ξ',μ), the corresponding Fourier transformed terms being ordinarily homogeneous in (ξ_n,ξ',μ) resp. (ξ_n,η_n,ξ',μ) for $|\xi'|\ge c>0$, of degree $m-j+d+s$. Such symbol-kernels and symbols are called (weakly) polyhomogeneous.

The explanation for the $+1$ resp. $+2$ in the third upper index in (2.8) resp. (2.9) is, that with this choice, $m+d+s$ is consistent with the top degree of homogeneity in the Fourier transformed situation for polyhomogeneous symbols, where the scalings in x_n and y_n lead to shifts in the indices. Further details in [G01].

The label *strongly polyhomogeneous* is reserved for those symbol-kernels and symbols for which the terms have the homogeneity property on the larger set where $|(\xi',\mu)|\ge c$ ($\xi'\in\mathbb{R}^{n-1}$, $\mu\in\Gamma\cup\{0\}$), and standard estimates for symbols-kernels in one more cotangent variable hold when the extra variable is identified with $|\mu|$ on each ray in Γ. Such symbol-kernels and symbols form a subset of the weakly polyhomogeneous symbol-kernel and symbol spaces by [GS95, Th. 1.16] and [G01, Th. 3.2]:

Theorem 2.6. *(On strongly polyhomogeneous symbol-kernels and symbols.)*

(i) *When* p *is a standard polyhomogeneous* ψ*do symbol of degree* m *with respect to* n *variables (with global estimates), then the symbol obtained by fixing* x_n *and replacing* ξ_n *by* $\mu\in\mathbb{R}_+$ *is in* $S^{0,0,m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathbb{R}_+)$.

(ii) *When* p *is as in* (i) *with* n *replaced by* $n+1$ *and satisfies the uniform transmission condition at* $x_n=0$*, with respect to the* x_n*-variable, then the symbol obtained by fixing* x_{n+1} *and replacing* ξ_{n+1} *by* $\mu\in\mathbb{R}_+$ *is in* $S^{0,0,m}_{\mathrm{ut}}(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}_+)$.

(iii) *When* $\tilde t$, $\tilde k$ *or* $\tilde g$ *is a polyhomogeneous trace, Poisson or singular Green symbol-kernel with respect to* $n+1$ *variables* $(x_1,\dots,x_{n+1})$ *in the standard* ψ*dbo calculus of degree* m *(with global estimates),* x_n *denoting the normal variable, then the symbol-kernels obtained by fixing* x_{n+1} *and replacing* ξ_{n+1} *by* $\mu\in\mathbb{R}_+$ *are in* $\mathcal{S}^{0,0,m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathbb{R}_+,\mathcal{S}_+)$ *resp.* $\mathcal{S}^{0,0,m}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\mathbb{R}_+,\mathcal{S}_{++})$.

Proof. (i) follows by applying [GS95, Th. 1.16] to the symbol $[\xi]^{-m}p$ of degree 0.

As for (ii), the uniform transmission condition is defined for parameter-dependent symbols in [G01, Sect. 6] precisely so that it holds in this situation.

(iii) is shown in [G01, Th. 3.2]. $\square$

The class of strongly polyhomogeneous ψdo symbols satisfing the uniform transmission condition as in this theorem, on each ray in Γ, is denoted $\mathcal{S}^{0,0,m}_{\mathrm{sphg,ut}}(\Gamma)$.

For a simple example of a Poisson operator as in the theorem, see [G01, Ex. 3.4].

The symbol-kernel spaces can of course also be defined for x' runnning in open subsets U' of $\mathbb{R}^{n-1}$; then $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ is replaced by $U'\times\mathbb{R}^{n-1}$ in the formulas in Definitions 2.1 and 2.2. Likewise, $\mathbb{R}^n\times\mathbb{R}^n$ can be replaced by $U\times\mathbb{R}^n$ in the definition of ψdo symbols. The cotangent variable ξ' need only run in a conical subset of $\mathbb{R}^{n-1}$. The operators and symbols behave in a standard way under coordinate transformations (one just has to keep check of the uniformity in z of the relevant estimates); we shall not give any details here but just mention that this allows the definition of operators acting in vector bundles over manifolds, by use of local coordinates and local trivializations.

The following composition rules are proved in [G01, Ths. 6.7–6.9] (recalled here primarily for operators of class zero):

Theorem 2.7. *Let P (ψdo on $\mathbb{R}^n$), G , T and K (class 0 singular Green, trace resp. Poisson operator for $\mathbb{R}^n_+$), and Q (ψdo on $\mathbb{R}^{n-1}$) be parameter-dependent with symbol(-kernels) $p,\tilde g,\tilde t,\tilde k,q$ satisfying (for some $m,d,s\in\mathbb{Z}$):*

$$\begin{aligned} p(x,\xi,\mu)&\in S^{0,0,s}_{\mathrm{ut}}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma),\\ \tilde g(x',x_n,y_n,\xi',\mu)&\in \mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++}),\\ \tilde t(x',x_n,\xi',\mu),\tilde k(x',x_n,\xi',\mu)&\in \mathcal{S}^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{+}),\\ q(x',\xi',\mu)&\in S^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma),\end{aligned}\tag{2.15}$$

and let P', G', T', K' and Q' be given similarly with m, d, s replaced by m', d', and s'. Define

$$m''=m+m',\quad d''=d+d',\quad s''=s+s'.\tag{2.16}$$

Assume that s resp. s' is ≤ 0 in the formulas where P resp. P' enter. Then (omitting the indication $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$):

(i) *TP'_+ (trace operator) has symbol-kernel in $\mathcal{S}^{m,d,s''}(\Gamma,\mathcal{S}_+)$,*
(ii) *$\gamma_0P'_+$ (trace op.) has symbol-kernel in $\mathcal{S}^{0,0,s'}(\Gamma,\mathcal{S}_+)$,*
(iii) *P_+K' (Poisson op.) has symbol-kernel in $\mathcal{S}^{m',d',s''}(\Gamma,\mathcal{S}_+)$,*
(iv) *P_+G' (s.g.o.) has symbol-kernel in $\mathcal{S}^{m',d',s''}(\Gamma,\mathcal{S}_{++})$,*
(v) *GP'_+ (s.g.o.) has symbol-kernel in $\mathcal{S}^{m,d,s''}(\Gamma,\mathcal{S}_{++})$,*
(vi) *GG' (s.g.o.) has symbol-kernel in $\mathcal{S}^{m'',d'',s''+1}(\Gamma,\mathcal{S}_{++})$,*
(vii) *KT' (ψdo) has symbol-kernel in $\mathcal{S}^{m'',d'',s''}(\Gamma,\mathcal{S}_{++})$,*
(viii) *TG' (trace op.) and GK' (Poisson op.) have symbol-kernels in $\mathcal{S}^{m'',d'',s''+1}(\Gamma,\mathcal{S}_+)$,*
(ix) *γ_0G' (trace op.) has symbol-kernel in $\mathcal{S}^{m',d',s'+1}(\Gamma,\mathcal{S}_+)$,*
(x) *TK' (ψdo) has symbol in $S^{m'',d'',s''+1}(\Gamma)$,*
(xi) *γ_0K' (ψdo) has symbol in $S^{m,d,s''+1}(\Gamma)$,*
(xii) *QT' (trace op.) and KQ' (Poisson op.) have symbol-kernels in $\mathcal{S}^{m'',d'',s''}(\Gamma,\mathcal{S}_+)$,*
(xiii) *QQ' (ψdo) has symbol in $S^{m'',d'',s''}(\Gamma)$,*

(xiv) $P_+P'_+ = (PP')_+ - G^+(P)G^-(P')$, *where* PP' *(ψdo) has symbol in* $S^{0,0,s''}_{\rm ut}(\mathbb{R}^n\times\mathbb{R}^n,\Gamma)$, $G^+(P)$ *(s.g.o.) has symbol-kernel in* $\mathcal{S}^{0,0,s-1}(\Gamma,\mathcal{S}_{++})$, *and* $G^-(P')$ *(s.g.o.) has symbol-kernel in* $\mathcal{S}^{0,0,s'-1}(\Gamma,\mathcal{S}_{++})$.

Observe the general principle that the third upper index is lifted by 1 in the cases where the composition involves an integration in x_n.

When $\tilde g$ is a singular Green symbol-kernel, we define the *normal trace* by

$$(\operatorname{tr}_n\tilde g)(x',\xi',\mu)=\int_0^\infty \tilde g(x',x_n,x_n,\xi',\mu)\,dx_n \tag{2.17}$$

This a ψdo symbol in the calculus:

Proposition 2.8. *When* $\tilde g(x',x_n,y_n,\xi',\mu)\in\mathcal{S}^{m,d,s-1}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_{++})$, *then the normal trace of* $\tilde g$ *is a ψdo symbol in* $S^{m,d,s}(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1},\Gamma,\mathbb{C}.)$

When $G=\mathrm{OPG}(\tilde g)$, we denote the ψdo with symbol $\operatorname{tr}_n\tilde g$ by $\operatorname{tr}_n G$. Then in fact, when the traces exist,

$$\operatorname{Tr}_{\mathbb{R}^n_+}G=\operatorname{Tr}_{\mathbb{R}^{n-1}}\operatorname{tr}_n G, \tag{2.18}$$

and there is a similar rule for the operators carried over to the manifold situation, when the symbol-kernel of G is supported in X_c and the product measure is used on X_c:

$$\operatorname{Tr}_X G=\operatorname{Tr}_{X'}\operatorname{tr}_n G. \tag{2.19}$$

In this way, the calculation of traces of s.g.o.s is reduced to the calculation of traces of ψdo's on X', for which we have the results of [GS95] for operators with symbols in the spaces $S^{m,d,0}(\Gamma)$. (When G is given as a finite sum of compositions of Poisson and trace operators, $G=\sum_{j\le J}K_jT_j$, one has by linearity and circular perturbation that $\operatorname{Tr}_X G=\operatorname{Tr}_{X'}(\sum_{j\le J}T_jK_j)$, which is a closely related "reduction to the boundary" that avoids explicit mention of normal and tangential variables.)

Let $\zeta(x_n)$ be a C^∞ function on $\mathbb{R}$ such that

$$\zeta(x_n)=1 \text{ for } |x_n|\le\tfrac13,\ \zeta(x_n)\in[0,1] \text{ for } |x_n|\in[\tfrac13,\tfrac23],\ \zeta(x_n)=0 \text{ for } |x_n|\ge\tfrac23; \tag{2.20}$$

denote $\zeta(x_n/\varepsilon)$ by ζ_ε, $\varepsilon>0$. Recall from [G01, Lemma 7.1ff.]:

Lemma 2.9. *For a singular Green operator* G *of class* 0 *in the calculus,* $(1-\zeta_\varepsilon)G$ *and* $G(1-\zeta_\varepsilon)$ *have symbol-kernels in* $\mathcal{S}^{-\infty,-\infty,-\infty}(\Gamma,\mathcal{S}_{++})$*; hence they are trace-class with traces that are* $O(|\lambda|^{-N})$ *for* $|\lambda|\to\infty$ *in* Γ*, any* N.

This relies on the fact that such operators can be written with a factor x_n^k resp. y_n^k for any k, where Lemma 2.3 applies.

Theorem 2.10. *Let* m, d *and* $s\in\mathbb{Z}$, *with* $s\le 0$.

(i) *Let* G *be a* μ*-dependent singular Green operator of class* 0, *with polyhomogeneous symbol-kernel in* $\mathcal{S}^{m,d,s-1}(\Gamma,\mathcal{S}_{++})$ *in local coordinates, holomorphic in* μ. *If* $m+s>-n$, *assume furthermore that the homogeneous terms in the symbol of* $S=\operatorname{tr}_n G$ *of degree*

$m+d+s-j$ with $m+s-j>-n$ are integrable i ξ'. Then G is trace-class and its trace has an asymptotic expansion in μ for $|\mu|\to\infty$ in Γ:

$$\operatorname{Tr} G \sim \sum_{j\in\mathbb{N}} c_j \mu^{m+d+s+n-1-j} + \sum_{k\in\mathbb{N}} (c'_k \log\mu + c''_k)\mu^{d+s-k}; \tag{2.21}$$

here the coefficients c_j and c'_k with $k=-m+j-n+1$ are determined from the j'th homogeneous term in the symbol of $\operatorname{tr}_n G$ (are "local"), whereas the c''_k depend on the full operator (are "global"). Such an expansion likewise hold for $\operatorname{Tr}_{X'} S$, when S is a ψdo on X' with polyhomogeneous symbol in $S^{m,d,s}(\Gamma)$, under the same additional assumption as above when $m+s>-n$.

(ii) *In particular, if the symbol of G is strongly polyhomogeneous, then*

$$\operatorname{Tr} G \sim \sum_{j\in\mathbb{N}} d_j \mu^{m+d+s+n-1-j}, \tag{2.22}$$

where d_j is determined from the j'th homogeneous term in the symbol of G. A similar statement holds for $\operatorname{Tr}_{X'} S$ when the symbol of S is strongly polyhomogeneous.

Proof. This is shown in [G01, Th. 7.3], but since we need to refer to specific coefficients, we recall some ingredients of the proof here.

By Lemma 2.9, the expansion (2.21) (or (2.22)) is unaffected by replacing the given s.g.o. G by an operator $G'=\zeta_\varepsilon G\zeta_\varepsilon$ with symbol-kernel supported in X_c, where (2.19) can be used.

We have from (2.12) that the symbol-kernel $\tilde g$ of G' is in

$$\mathcal{S}^{m,d,s-1}(\Gamma,\mathcal{S}_{++}) \subset \mathcal{S}^{m+s,d,-1}(\Gamma,\mathcal{S}_{++}) \cap \mathcal{S}^{m,d+s,-1}(\Gamma,\mathcal{S}_{++}),$$

in local coordinates. Hence, by Proposition 2.8, $\operatorname{tr}_n \tilde g \in S^{m+s,d,0}(\Gamma)\cap S^{m,d+s,0}(\Gamma)$. Denote $\operatorname{tr}_n G' = S$, then it is a ψdo on X' with symbol

$$s(x',\xi',\mu) \in S^{m+s,d,0}(\Gamma)\cap S^{m,d+s,0}(\Gamma), \tag{2.23}$$

in local coordinates. Now one simply applies [GS95, Th. 2.1] to S, and from here on, the considerations apply to any ψdo S on X' satisfying the stated assumptions:

If $m+s\le -n$, the inclusion in the first space in (2.23) assures trace-class and integrability in ξ' of all terms in the symbol. An application of [GS95, Th. 2.1] then gives after integration in $x'\in X'$ that $\operatorname{Tr}_{X'} S$ has an expansion as in (2.21) but with $d+s$ replaced by d in the second series. The inclusion in the second space in (2.23) allows us to replace d by the lower integer $d+s$ in the second series. To explain the central part of the proof, let $s(x',\xi',\mu)\in S^{m',d',0}(\Gamma)$ and let $s'=\mu^{-d'}s$ (in a localized situation). Here the j'th homogeneous term $s'_j(x',\xi',\mu)$ in the symbol $s'(x',\xi',\mu)$ has homogeneity degree $m'-j$, denoted m_j in [GS95, Th. 2.1]. Its contribution to the diagonal value of the kernel $K(x',y',\mu)$ of S is

$$K_{s'_j}(x',x',\mu) = \int_{\mathbb{R}^{n-1}} s'_j(x',\xi',\mu)\, đ\xi' = \Big(\int_{|\xi'|\ge|\mu|} + \int_{|\xi'|\le 1} + \int_{1\le|\xi'|\le|\mu|}\Big) s'_j\, đ\xi'; \tag{2.24}$$

here $\int_{|\xi'|\ge|\mu|} s'_j \,đ\xi'$ gives a power term $c(x')\mu^{m'-j+n-1}$ (by homogeneity), $\int_{|\xi'|\le 1} s'_j \,đ\xi'$ gives a series of power terms, and $\int_{1\le|\xi'|\le|\mu|} s'_j \,đ\xi'$ gives a log-power term $c'(x')\mu^{m'-j+n-1}\log\mu$ if $m'-j+n-1$ is a nonpositive integer and besides this some power terms. Thus in the original symbol, $s_j = \mu^{d'} s'_j$ gives a log-power term $c'(x')\mu^{m'+d'-j+n-1}\log\mu$ when $j\ge m'+n-1$. We note that the log-power terms begin with $c\mu^{d'}\log\mu$; also the nonlocal terms begin at this power.

If $m+s>-n$, the supplementary integrability assumption for the terms with $m+s-j>-n$ assures trace-class, and all terms are treated as above.

In (ii), one gets the refined expansion (2.22) since in the strongly polyhomogeneous case, the homogeneous terms in the symbol are strictly homogeneous in $(\xi',\mu)\in\mathbb{R}^{n-1}\times\{\mu\in\Gamma \mid |\mu|\ge\varepsilon\}$ and integrable in $\xi'\in\mathbb{R}^{n-1}$, so that the terms $d_j\mu^{m+d+s+n-1-j}$ are produced directly by integration of the j'th symbol in ξ' and x' using the homogeneity. (Here one does not need to decompose the integral into three regions as in (2.24).)

One can also derive (2.22) from the fact that the strongly polyhomogeneous symbols are as in [G96] with regularity number $\nu=+\infty$, so that full trace expansions with purely power terms hold as shown there (and recalled in [G01, Prop. 7.2]). $\square$

When the operator acts on the sections of a vector bundle over X, one takes the fiber trace in (2.24). There is a similar result for $\operatorname{Tr}_X P_+$ when P has symbol in $S^{m,d,s}(\Gamma)$, only with $n-1$ replaced by n.

Let us now show how the elementary operators introduced in (1.18)–(1.20) fit into the calculus. In the statements below, it is tacitly understood that the symbols are $N\times N$-matrix valued. This could be indicated by adding $\otimes\mathcal{L}(\mathbb{C}^N)$ to all the mentioned symbol spaces, but that would make the reading unnecessarily heavy.

Proposition 2.11. *Consider the operators from Section* 1 *as parametrized by*

$$\mu=(-\lambda)^{\frac12},\ \text{in } \Gamma=\{\mu\in\mathbb{C}\mid \operatorname{Re}\mu>0\}. \tag{2.25}$$

(i) *The ψdo $Q^0_\lambda=(A^2+D^2_{x_n}+\mu^2)^{-1}$ has symbol in $S^{0,0,-2}_{\mathrm{sphg,ut}}(U\times\mathbb{R}^n,\Gamma)$, in local trivializations.*

(ii) *The ψdo's*

$$\begin{aligned} A_\lambda&=(A^2-\lambda)^{\frac12}=(A^2+\mu^2)^{\frac12}\\ A_\lambda^{-1}&=(A^2-\lambda)^{-\frac12}=(A^2+\mu^2)^{-\frac12}\end{aligned} \tag{2.26}$$

have strongly polyhomogeneous symbols in $S^{0,0,1}(U'\times\mathbb{R}^{n-1},\Gamma)$ resp. $S^{0,0,-1}(U'\times\mathbb{R}^{n-1},\Gamma)$, in local trivializations. Moreover, for $m\in\mathbb{N}$,

$$\begin{aligned} &\partial_\lambda^r A_\lambda \text{ has symbol in } S^{0,0,1-2r}(U'\times\mathbb{R}^{n-1},\Gamma),\\ &\partial_\lambda^r A_\lambda^{-1} \text{ has symbol in } S^{0,0,-1-2r}(U'\times\mathbb{R}^{n-1},\Gamma).\end{aligned} \tag{2.27}$$

(iii) *The ψdo $(A_\lambda+|A|)^{-1}$ has symbol in $S^{0,0,-1}(U'\times\mathbb{R}^{n-1},\Gamma)$, in local trivializations, and its ∂_λ^r-derivatives have symbols in $S^{0,0,-1-2r}(U'\times\mathbb{R}^{n-1},\Gamma)$.*

(iv) *The Poisson operator K_{A_λ} and the trace operator T_{A_λ} have strongly polyhomogeneous symbol-kernels in $\mathcal{S}^{0,0,-1}(U'\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+)$, in local trivializations. Moreover,*

$$\partial_\lambda^r K_{A_\lambda} \text{ and } \partial_\lambda^r T_{A_\lambda} \text{ have symbol-kernels in } \mathcal{S}^{0,0,-1-2r}(U'\times\mathbb{R}^{n-1},\Gamma,\mathcal{S}_+). \tag{2.28}$$

Proof. (i), essentially known from [GS95], follows from the fact that $A^2+D^2_{x_n}+e^{2i\theta}D^2_{x_{n+1}}$ is an elliptic differential operator on $X'\times\mathbb{R}^2$, for any $|\theta|<\frac{\pi}{2}$; then we can for each ray in Γ use Theorem 2.6 (ii) and the fact that elliptic differential operators and their parametrices satisfy the transmission condition.

The results of (ii) and (iii) are also essentially known from [GS95]. In (ii), we can compare A_λ and A_λ^{-1} with $(A^2+e^{2i\theta}D^2_{x_n})^{\frac12}$ resp. $(A^2+e^{2i\theta}D^2_{x_n})^{-\frac12}$ defined according to Seeley [S69]; the latter are elliptic of degree 1 resp. -1. Then Theorem 2.6 (i) gives the statement for $r=0$. The cases $r>0$ are included by use of the formula $\partial_\lambda A_\lambda=-\frac12 A_\lambda^{-1}$ and the composition rules.

Concerning (iii), it is shown in the proof of [GS95, Prop. 3.5] that $A_\lambda(|A|+A_\lambda)^{-1}$ has symbol in $S^{0,0}(\Gamma)$, equal to $S^{0,0,0}(\Gamma)$ by (2.11), so the result follows for $r=0$ by composition with A_λ^{-1}, using (ii) and the composition rule (xiii) in Theorem 2.7. For $r>0$, we use that

$$\partial_\lambda^r(|A|+A_\lambda)^{-1}=\sum_{j+k=2m,j,k\ge1}c_{jk}(|A|+A_\lambda)^{-1-j}A_\lambda^{-k}, \tag{2.29}$$

so that (iii) follows from the previous results by use of rule (xiii).

For (iv), we apply Theorem 2.6 (iii). In fact, when $\mu=e^{i\theta}\varrho$ is replaced by $e^{i\theta}\xi_{n+1}$, the symbol-kernel of K_{A_λ} is replaced by the symbol-kernel of the solution operator (in a parametrix sense) $\widetilde K\colon\varphi\mapsto u$ of the Dirichlet problem

$$\begin{aligned}(A+D^2_{x_n}+e^{i2\theta}D^2_{x_{n+1}})u(x',x_n,x_{n+1})&=0 \text{ on } \mathbb{R}^{n-1}\times\mathbb{R}_+\times\mathbb{R},\\ u(x',0,x_{n+1})&=\varphi(x',x_{n+1}) \text{ on } \mathbb{R}^{n-1}\times\mathbb{R},\end{aligned} \tag{2.30}$$

which is a standard Poisson operator relative to $\mathbb{R}^{n-1}\times\mathbb{R}_+\times\mathbb{R}$. Then Theorem 2.6 (iii) implies that the symbol-kernel of K_{A_λ} is in $\mathcal{S}^{0,0,-1}(\Gamma,\mathcal{S}_+)$. The λ-derivatives are included by functional calculus (cf. (1.19)) and composition rules. There is a similar proof for T_{A_λ}. □

3. Preservation of log-terms under general perturbations.

We shall here study the resolvent $R_\lambda=(\Delta_B-\lambda)^{-1}$ by use of the representation (1.14). It is shown in [GS95, Th. 3.9] for the non-product case (with $\Pi=\Pi_>+B_0$), in [G99, Cor. 8.3] for the general case, that $\mathcal{R}_\mu$ has the structure

$$\mathcal{R}_\mu=\mathcal{Q}_{\mu,+}+\mathcal{G}_\mu,\quad \mathcal{G}_\mu=\mathcal{K}_\mu\mathcal{S}_\mu\mathcal{T}_\mu, \tag{3.1}$$

where $\mathcal{S}_\mu$ is a weakly polyhomogeneous ψdo on X' with symbol in $S^{0,0}(\Gamma)$, $\mathcal{K}_\mu$ is a strongly polyhomogeneous Poisson operator of degree -1, and $\mathcal{T}_\mu$ is a strongly polyhomogeneous trace operator of class 0 and degree -1. $\mathcal{Q}_\mu$ is the parametrix described in (1.12).

From the point of view of the more recent calculus in recalled in Section 2, $\mathcal{S}_\mu$ has symbol in $S^{0,0,0}(\Gamma)$ (cf. (2.11)), and $\mathcal{K}_\mu$ and $\mathcal{T}_\mu$ have symbol-kernels in $\mathcal{S}^{0,0,-1}(\Gamma,\mathcal{S}_+)$ since they are strongly polyhomogeneous (cf. Theorem 2.6). So by the elementary composition rules (xii) and (vii) in Theorem 2.7, we find that $\mathcal{G}_\mu$ has symbol-kernel in $\mathcal{S}^{0,0,-2}(\Gamma,\mathcal{S}_{++})$, in local trivializations.

We use this to see from (1.14), (1.15) that

$$R_{-\mu^2}=Q_{-\mu^2,+}+G_{-\mu^2},\quad G_{-\mu^2}=\mu^{-1}\begin{pmatrix}1&0\end{pmatrix}\mathcal{G}_\mu\begin{pmatrix}1\\0\end{pmatrix}, \tag{3.2}$$

where $G_{-\mu^2}$ has symbol-kernel in $\mathcal{S}^{0,-1,-2}(\Gamma,\mathcal{S}_{++})$, in local trivializations. $Q_{-\mu^2}$ is strongly polyhomogeneous of degree -2 and has symbol in $S^{0,0,-2}_{\mathrm{spgh,ut}}(\Gamma)$, by Theorem 2.6 (ii).

Lemma 3.1. *For any* $r \in \mathbb{N}$,

$$R^r_{-\mu^2} = (Q^r_{-\mu^2})_+ + G^{(r)}_{-\mu^2}, \tag{3.3}$$

where $Q^r_{-\mu^2}$ *is strongly polyhomogeneous of degree* $-2r$*, with symbol in* $S^{0,0,-2r}_{\rm spgh,ut}(\Gamma)$*, and* $G^{(r)}_{-\mu^2}$ *has symbol in* $\mathcal{S}^{0,-r,-1-r}(\Gamma,\mathcal{S}_{++})$*, in local trivializations.*

Proof. The statement for the case $r=1$ is shown above. The iterated expressions:

$$R^r_{-\mu^2} = (Q_{-\mu^2,+} + G_{-\mu^2})\circ\cdots\circ(Q_{-\mu^2,+} + G_{-\mu^2}). \tag{3.4}$$

are included by use of rules (iv)–(vi) and (xiv) in Theorem 2.7. ☐

Remark 3.2. By a direct study of the resolvent of $D^*_\Pi D_\Pi$, as carried out for the non-product case with $\Pi = \Pi_\geq$ in [G92], and for more general cases in [G02], one can show that $G_{-\mu^2} = K_\mu S_\mu T_\mu$, where K_μ and T_μ are strongly polyhomogeneous of degree -1 and S_μ has symbol is in $S^{0,0,-1}(\Gamma)$; hence $G_{-\mu^2}$ has symbol-kernel in $\mathcal{S}^{0,0,-3}(\Gamma,\mathcal{S}_{++})$, which leads to the conclusion that $G^{(r)}_{-\mu^2}$ in fact has symbol-kernel in $\mathcal{S}^{0,0,-1-2r}(\Gamma,\mathcal{S}_{++})$. However, the above information suffices for the results on perturbations that we pursue here.

It is well-known how $\operatorname{Tr} Q^r_{-\mu^2,+}$ has an asymptotic development in pure powers of μ (when $2r > n$). When we in addition apply Theorem 2.10 to $G^{(r)}_{-\mu^2}$ we get (1.8), reconfirming the result of [GS95]. Let us also describe the trace expansion in cases where $R^r_{-\mu^2}$ is composed with a differential operator:

Theorem 3.3. *Let* F *be a differential operator of order* m'*. Then for* $r > \frac{n+m'}{2}$*,* $FR^r_{-\mu^2}$ *is trace-class and has an expansion*

$$\operatorname{Tr} FR^r_{-\mu^2} \sim \sum_{-n\le k<0} \tilde a_k \mu^{m'-k-2r} + \sum_{k\ge 0}(\tilde a'_k \log\mu + \tilde a''_k)\mu^{m'-k-2r}. \tag{3.5}$$

If F *is tangential on* X_c*, then*

$$\operatorname{Tr} FR^r_{-\mu^2} \sim \sum_{-n\le k<m'} \tilde a_k \mu^{m'-k-2r} + \sum_{k\ge m'}(\tilde a'_k \log\mu + \tilde a''_k)\mu^{m'-k-2r}. \tag{3.6}$$

The coefficients $\tilde a_k$ *and* $\tilde a'_k$ *are locally determined. If* m' *is odd,* $\tilde a_{-n} = 0$.

Proof. Since the operator is of order $m' - 2r$ for fixed μ, it is trace-class when $m' - 2r < -n$, i.e., $r > \frac{n+m'}{2}$. It is well-known that the ψdo part $FQ^r_{-\mu^2,+}$ has an expansion $\sum_{k\ge -n} c_k\mu^{m'-2r-k}$ without logarithmic terms. Here the terms with k even/odd vanish when m' is odd/even, respectively, since they are defined by integration in $\xi \in \mathbb{R}^n$ of symbols that are odd in ξ. One finds using Lemma 2.3 that the s.g.o. part $FG^{(r)}_{-\mu^2}$ has symbol-kernel in $\mathcal{S}^{0,-r,m'-r-1}(\Gamma,\mathcal{S}_{++})$ in local trivializations, so it follows from Theorem 2.10, applied with $(m,d,s) = (0,-r,m'-r)$, that this term contributes a trace expansion as in (3.5) but starting with $k = 1-n$. Then (3.5) follows by summation; in particular, the term with $k=-n$ vanishes if m' is odd.

If F is tangential, the symbol-kernel of $FG^{(r)}_{-\mu^2}$ is instead in $\mathcal{S}^{m',-r,-r-1}(\Gamma,\mathcal{S}_{++})$, so we can use Theorem 2.10 with $(m,d,s)=(m',-r,-r)$, which lowers the starting power $d+s$ in the series with logarithms to $-2r$; this results in (3.6). $\square$

Now we consider two choices of D as in (1.1), with the same well-posed choice of boundary condition:

$$D_i=\sigma(\partial_{x_n}+A_{1i}),\quad \Pi\gamma_0 u=0,\quad i=1,2. \tag{3.7}$$

The associated other operators in the direct and the doubled-up situation will be marked by index 1 or 2. The realizations of $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined by the same boundary condition (1.11), so the full systems in the doubled-up situations are:

$$\begin{pmatrix}\mathcal{D}_1+\mu\\ \mathcal{B}\gamma_0\end{pmatrix},\quad \begin{pmatrix}\mathcal{D}_2+\mu\\ \mathcal{B}\gamma_0\end{pmatrix}, \tag{3.8}$$

with inverses

$$\begin{pmatrix}\mathcal{D}_1+\mu\\ \mathcal{B}\gamma_0\end{pmatrix}^{-1}=(\,\mathcal{R}_{1,\mu}\quad \mathcal{K}_{1,\mu}\,),\quad \begin{pmatrix}\mathcal{D}_2+\mu\\ \mathcal{B}\gamma_0\end{pmatrix}=(\,\mathcal{R}_{2,\mu}\quad \mathcal{K}_{2,\mu}\,). \tag{3.9}$$

We can write (on X_c)

$$D_1-D_2=x_n^l\overline{P}_l,\quad D_2^*-D_1^*=x_n^l\overline{P}_l^* \tag{3.10}$$

for some $l\ge 0$, some tangential x_n-dependent first-order differential operator $\overline{P}_l$. Then

$$\mathcal{D}_2-\mathcal{D}_1=x_n^l\overline{\mathcal{P}}_l,\ \text{where } \overline{\mathcal{P}}_l=\begin{pmatrix}0&-\overline{P}_l^*\\ \overline{P}_l&0\end{pmatrix}. \tag{3.11}$$

Let us (somewhat abusively) apply the notation $x_n^l\overline{\mathcal{P}}_l$ to $\mathcal{D}_2-\mathcal{D}_1$ on all of X. Then

$$\begin{aligned}(\,\mathcal{R}_{2,\mu}\quad \mathcal{K}_{2,\mu}\,)&=(\,\mathcal{R}_{1,\mu}\quad \mathcal{K}_{1,\mu}\,)\begin{pmatrix}\mathcal{D}_1+\mu\\ \mathcal{B}\gamma_0\end{pmatrix}(\,\mathcal{R}_{2,\mu}\quad \mathcal{K}_{2,\mu}\,)\\ &=(\,\mathcal{R}_{1,\mu}\quad \mathcal{K}_{1,\mu}\,)\begin{pmatrix}\mathcal{D}_2-x_n^l\overline{\mathcal{P}}_l+\mu\\ \mathcal{B}\gamma_0\end{pmatrix}(\,\mathcal{R}_{2,\mu}\quad \mathcal{K}_{2,\mu}\,)\\ &=(\,\mathcal{R}_{1,\mu}\quad \mathcal{K}_{1,\mu}\,)-(\,\mathcal{R}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{R}_{2,\mu}\quad \mathcal{R}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{K}_{2,\mu}\,).\end{aligned} \tag{3.12}$$

In particular,

$$\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu}=-\mathcal{R}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{R}_{2,\mu}. \tag{3.13}$$

Theorem 3.4. *Consider $D_{1,\Pi}$ and $D_{2,\Pi}$ defined from* (3.7)*. For any $r\in\mathbb{N}$, write*

$$R^r_{i,-\mu^2}=(Q^r_{i,-\mu^2})_++G^{(r)}_{i,-\mu^2},\ i=1,2; \tag{3.14}$$

according to Lemma 3.1, $Q^r_{i,-\mu^2}$ has symbol in $S^{0,0,-2r}_{\mathrm{spgh,ut}}(\Gamma)$ and $G^{(r)}_{i,-\mu^2}$ has symbol-kernel in $\mathcal{S}^{0,-r,-1-r}(\Gamma,\mathcal{S}_{++})$ in local trivializations. Write

$$R^r_{2,-\mu^2}-R^r_{1,-\mu^2}=(Q^r_{2,-\mu^2}-Q^r_{1,-\mu^2})_+ +\overline{G}^{(r)}_\mu . \tag{3.15}$$

When (3.10) *holds for some some $l\ge 0$ (with a first-order tangential differential operator $\overline{P}_l$), then $\overline{G}^{(r)}_\mu$ has symbol-kernel in $\mathcal{S}^{1,-r,-2-r-l}(\Gamma,\mathcal{S}_{++})$, in local trivializations.*

Proof. We begin with the case $r=1$. By (3.13) and (3.2),

$$\begin{aligned}\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu}&=-\mathcal{R}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{R}_{2,\mu}=-(\mathcal{Q}_{1,\mu,+}+\mathcal{G}_{1,\mu})x_n^l\overline{\mathcal{P}}_l(\mathcal{Q}_{2,\mu,+}+\mathcal{G}_{2,\mu})\\&=-\mathcal{Q}_{1,\mu,+}x_n^l\overline{\mathcal{P}}_l\mathcal{Q}_{2,\mu,+}-\mathcal{Q}_{1,\mu,+}x_n^l\overline{\mathcal{P}}_l\mathcal{G}_{2,\mu}-\mathcal{G}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{Q}_{2,\mu,+}-\mathcal{G}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{G}_{2,\mu}.\end{aligned} \tag{3.16}$$

We first show how the three last terms are treated by Theorem 2.7 and Lemma 2.3. The lemma shows that multiplication of a singular Green symbol-kernel by x_n^l or y_n^l lowers the third upper index by l steps. Thus $x_n^l\mathcal{G}_{2,\mu}$ has symbol-kernel in $\mathcal{S}^{0,0,-2-l}(\Gamma,\mathcal{S}_{++})$, in local trivializations. A similar result holds for $\mathcal{G}_{1,\mu}x_n^l$, where the composition with x_n^l to the right has the effect of multiplying the symbol-kernel with y_n^l. The composition with the μ-independent first-order tangential differential operator $\mathcal{P}_l$ has the effect of lifting the first upper index by 1 step. When we take these effects into account and use the composition rules, we find that the three last terms in (3.16) are s.g.o.s with symbol-kernels in $\mathcal{S}^{1,0,-3-l}(\Gamma,\mathcal{S}_{++})$, in local trivializations. The remaining term equals

$$\mathcal{Q}_{1,\mu,+}x_n^l\overline{\mathcal{P}}_l\mathcal{Q}_{2,\mu,+}=(\mathcal{Q}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{Q}_{2,\mu})_+-G^+(\mathcal{Q}_{1,\mu})x_n^l\overline{\mathcal{P}}_lG^-(\mathcal{Q}_{2,\mu}), \tag{3.17}$$

cf. (xiv) of Theorem 2.7. Here the ψdo part is strongly polyhomogeneous of order -1 and the $G^\pm(\mathcal{Q}_{i,\mu})$ have symbol-kernels in $\mathcal{S}^{0,0,-2}(\Gamma,\mathcal{S}_{++})$, in local trivializations. By Lemma 2.3, $G^+(\mathcal{Q}_{i,\mu})x_n^l$ has symbol-kernel in $\mathcal{S}^{0,0,-2-l}(\Gamma,\mathcal{S}_{++})$. $\overline{\mathcal{P}}_l$ lifts the first upper index by 1. Then by Theorem 2.7 (vi), the last term in (3.17) has symbol-kernel in $\mathcal{S}^{1,0,-3-l}(\Gamma,\mathcal{S}_{++})$, in local trivializations.

Now by (1.14),

$$R_{2,-\mu^2}-R_{1,-\mu^2}=\mu^{-1}\begin{pmatrix}1&0\end{pmatrix}(\mathcal{R}_{2,\mu}-\mathcal{R}_{1,\mu})\begin{pmatrix}1\\0\end{pmatrix}=-\mu^{-1}\begin{pmatrix}1&0\end{pmatrix}\mathcal{R}_{1,\mu}x_n^l\overline{\mathcal{P}}_l\mathcal{R}_{2,\mu}\begin{pmatrix}1\\0\end{pmatrix}, \tag{3.18}$$

so from what we showed for (3.16) follows in view of (3.2) that $R_{2,-\mu^2}-R_{1,-\mu^2}$ is the sum of a truncated ψdo with symbol in $S^{0,0,-2}_{\mathrm{sphg,ut}}(\Gamma)$ and an s.g.o. with symbol-kernel in $\mathcal{S}^{1,-1,-3-l}(\Gamma,\mathcal{S}_{++})$, in local trivializations. This shows the assertion for $r=1$.

For higher r, we use the formula

$$R^r_{2,-\mu^2}-R^r_{1,-\mu^2}=(R_{2,-\mu^2}-R_{1,-\mu^2})(R^{r-1}_{2,-\mu^2}+R^{r-2}_{2,-\mu^2}R_{1,-\mu^2}+\cdots+R^{r-1}_{1,-\mu^2}). \tag{3.19}$$

The first factor is described above. For the terms in the second factor we use the information in (3.14)ff. An application of the composition rules gives an operator whose ψdo part has symbol in $S^{0,0,-2r}_{\mathrm{sphg,ut}}(\Gamma)$ and whose s.g.o. part has symbol-kernel in

$$\mathcal{S}^{1,-1,-3-l}(\Gamma,\mathcal{S}_{++})\circ\mathcal{S}^{0,-(r-1),-1-(r-1)}(\Gamma,\mathcal{S}_{++})\subset\mathcal{S}^{1,-r,-2-r-l}(\Gamma,\mathcal{S}_{++}). \quad\square$$

It is remarkable in this result (and important for the applications below) that both factors x_n^l (for $l>0$) and $\mathcal{R}_{2,\mu}$ lower the third upper index, whereas $\overline{\mathcal{P}}_l$ only lifts the first upper index. In the case $l=0$, the proof in fact allows D_2-D_1 to be an arbitrary first-order tangential differential operator (as long as ellipticity is respected).

Pursuing the indications in Remark 3.2, one could get a still better result, placing the decrease by $2r$ fully in the third upper index. However, the above result suffices to conclude:

Theorem 3.5. *Let D_1 and D_2 be two first-order elliptic operators on X as in* (1.1), (3.7), *provided with the same well-posed boundary condition $\Pi\gamma_0 u=0$ (with Π being an orthogonal pseudodifferential projection); then* (3.10) *holds for some $l\ge 0$, and we denote the largest such integer by l. Let $D_{1,\Pi}$ and $D_{2,\Pi}$ be the realizations defined by the boundary condition $\Pi\gamma_0 u=0$, and let $\Delta_{i,B}=D^*_{i,\Pi}D_{i,\Pi}$. Let F be a differential operator in E_1 of order m' and let $r>\frac{n+m'}{2}$. Then*

$$F(R^r_{2,\lambda}-R^r_{1,\lambda})=(F(Q^r_{2,\lambda}-Q^r_{1,\lambda}))_+ + F\overline{G}^{(r)}_\mu, \tag{3.20}$$

where $F(Q^r_{2,-\mu^2}-Q^r_{1,-\mu^2})$ has symbol in $S^{0,0,m'-2r}_{\mathrm{spgh,ut}}(\Gamma)$ and $F\overline{G}^{(r)}_\mu$ has symbol-kernel in $\mathcal{S}^{1,-r,m'-2-r-l}(\Gamma,\mathcal{S}_{++})$ (in $\mathcal{S}^{m'+1,-r,-2-r-l}(\Gamma,\mathcal{S}_{++})$ if F is tangential), in local trivializations.

The ψdo part has an asymptotic trace expansion

$$\operatorname{Tr}[(F(Q^r_{2,\lambda}-Q^r_{1,\lambda}))_+]\sim\sum_{-n\le k<\infty}\tilde p_k(-\lambda)^{\frac{m'-k}{2}-r}, \tag{3.21}$$

where $\tilde p_k=0$ for $k-m'+n$ odd.

The s.g.o. part has an asymptotic trace expansion

$$\operatorname{Tr}[F\overline{G}^{(r)}_\mu)]\sim\sum_{-n+1+l\le k<k_0}\tilde g_k(-\lambda)^{\frac{m'-k}{2}-r}+\sum_{k\ge k_0}\big(\tilde g'_k\log(-\lambda)+\tilde g''_k\big)(-\lambda)^{\frac{m'-k}{2}-r}, \tag{3.22}$$

where

$$k_0=l+1 \text{ when } F \text{ is general}, k_0=m'+l+1 \text{ when } F \text{ is tangential on } X_c. \tag{3.23}$$

It follows that

$$\operatorname{Tr}[F(R^r_{2,\lambda}-R^r_{1,\lambda})]\sim\sum_{-n\le k<k_0}\tilde c_k(-\lambda)^{\frac{m'-k}{2}-r}+\sum_{k\ge k_0}\big(\tilde c'_k\log(-\lambda)+\tilde c''_k\big)(-\lambda)^{\frac{m'-k}{2}-r}, \tag{3.24}$$

with k_0 as above. For $k\le l-n$, the $\tilde c_k$ vanish when $k-m'+n$ is odd.

The coefficients $\tilde c_k$ and $\tilde c'_k$ are locally determined.

Proof. Recall that $\lambda=-\mu^2$. The statement on the decomposition (3.20) follows from Theorem 3.4 and Lemma 3.1. The trace expansion of the ψdo part is well-known. The trace expansion of the s.g.o. part is obtained in general by application of Theorem 2.10, with

$$m=1,\quad d=-r,\quad s=m'-1-r-l.$$

Here $m+s=m'-r-l$, so if $r+l\ge n+m'$, there is no need to check integrability conditions. We only assume $r>\frac{n+m'}{2}$; this is allowed because the singular Green part of the resolvent power is in fact of order $-2r$ (degree $-2r-1$) for each fixed λ (since it comes from the resolvent of an elliptic problem of order $2r$), hence tr_n of its composition with F is a ψdo on X' of order $m'-2r$. So all the homogeneous terms in the considered ψdo symbol are integrable in ξ', when $r>\frac{n+m'}{2}$. Then Theorem 2.10 gives an expansion of the trace of the s.g.o. part of the form (2.21), with $m+d+s=m'-2r-l$, $d+s=m'-1-2r-l$ (resp. $-1-2r-l$ if F is tangential). Here the expansion starts with the power $m'-l+n-1-2r$, and the log-terms start with the power $m'-1-l-2r$ (resp. $-1-l-2r$ if F is tangential), so (3.22) is obtained after some relabelling.

When we add the contributions, we find (3.24). □

Remark 3.6. Note in particular that the terms with "global" coefficients $\tilde c''_k$ begin with the power $(-\lambda)^{\frac{m'-l-1}{2}-r}$ for general F, $(-\lambda)^{\frac{-l-1}{2}-r}$ when F is tangential.

There is a similar result for $D_{i,\Pi}D_{i,\Pi}{}^*$. We have furthermore:

Corollary 3.7. *Hypotheses and definitions as in Theorem 3.5. There are expansions*

(3.25)
$$\operatorname{Tr}[F(e^{-t\Delta_{2,B}}-e^{-t\Delta_{1,B}})]\sim\sum_{-n\le k<k_0}c_kt^{\frac{k-m'}{2}}+\sum_{k\ge k_0}\big(c'_k\log t+c''_k\big)t^{\frac{k-m'}{2}},$$
$$\Gamma(s)\operatorname{Tr}[F(\Delta_{2,B}^{-s}-\Delta_{1,B}^{-s})]\sim\sum_{-n\le k<k_0}\frac{c_k}{s+\frac{k-m'}{2}}+\sum_{k\ge k_0}\Big(\frac{-c'_k}{(s+\frac{k-m'}{2})^2}+\frac{c''_k}{s+\frac{k-m'}{2}}\Big)$$
$$-\frac{\operatorname{Tr}[F(\Pi_0(\Delta_{2,B})-\Pi_0(\Delta_{1,B}))]}{s}.$$

The coefficients c_k and c'_k are locally determined; the c_k, c'_k, c''_k are proportional to $\tilde c_k$, $\tilde c'_k$, $\tilde c''_k$ in (3.24) by universal factors. For $k\le l-n$, the c_k vanish when $k-m'+n$ is odd.

Proof. Here one uses the transition formulas explained e.g. in [GS96]. The passage from (3.24) to the zeta function expansion in the second formula of (3.25) is based on Corollary 2.10 there, and the passage to the heat trace expansion is based on Section 5 there. The subtracted term in the second line is explained by the fact that $\Delta_{i,B}^{-s}$ is defined to be zero on $V_0(\Delta_{i,B})$. □

We can also formulate the result as follows:

Corollary 3.8. *Hypotheses and definitions as in Theorem 3.5. For the trace expansion*

$$\operatorname{Tr}FR^r_{1,\lambda}\sim\sum_{-n\le k<0}\tilde a_k(-\lambda)^{\frac{m'-k}{2}-2}+\sum_{k\ge0}\big(\tilde a'_k\log(-\lambda)+\tilde a''_k\big)(-\lambda)^{\frac{m'-k}{2}-r}\tag{3.26}$$

(the summation limit 0 replaced by m' if F is tangential), the replacement of D_1 by D_2 leaves the log-coefficients $\tilde a'_k$ invariant for $k<k_0$. The other coefficients with $k<k_0$ are modified only by local terms; those with $k\le l-n$ and $k-m'+n$ odd are invariant.

There are similar results for the associated heat trace and zeta function.

Proof. (3.26) is a reformulation of (3.5)–(3.6). Since $FR^r_{2,\lambda}=FR^r_{1,\lambda}+F(R^r_{2,\lambda}-R^r_{1,\lambda})$, the result follows for $FR^r_{2,\lambda}$ by addition of (3.24) to (3.26). □

Recall from [GS96] that when $F = \varphi^0$ (a morphism independent of x_n on X_c), the coefficient a_1' in (0.1) and (1.9) in the product case, with Π equal to $\Pi_>$ plus a projection in the nullspace $V_0(A)$, satisfies:

$$a_1' = -\pi^{-1} e_1(\varphi^0, A^2), \tag{3.27}$$

where e_1 is the coefficient of $t^{\frac{1}{2}}$ in the heat trace expansion for A^2 on X':

$$\operatorname{Tr}(\varphi^0 e^{-tA^2}) \sim \sum_{k=1-n}^{\infty} e_k(\varphi^0, A^2) t^{\frac{k}{2}}.$$

Here $e_k(\varphi^0, A^2) = 0$ for $k - n + 1$ odd; in particular, $e_1 = 0$ if n is odd. By [G01′], the value a_1' is the same also for projections $\Pi = \Pi_> + \mathcal{S}$ with $\mathcal{S}$ of order $\le -n-1$.

The above methods moreover allow us to conclude:

Theorem 3.9. *The coefficient a_1' in* (0.1) *and* (1.9) *(as well as the coefficient $\tilde{a}_1'$ in* (1.8)*), is the same for a non-product type operator D* (1.1), (1.2) *over X_c with volume form $v(x)\,dx$ and the associated product type operator D^0* (1.3) *with volume form $v(x',0)\,dx$, when $P_0 = 0$ and $\partial_{x_n} v(x',0) = 0$. Moreover, the coefficient a_1'' (as well as $\tilde{a}_1''$) differs in the cases of D and D^0 by a local contribution only. Here, when $\varphi = \varphi^0$ (independent of x_n on X_c) and $\Pi = \Pi_> + \mathcal{S}$, $\mathcal{S}$ of order $\le -n-1$, a_1' satisfies* (3.27)*.*

Generally, when $D = D^0 + x_n^l \overline{P}_l$ on X_c for some l, and $\partial_{x_n}^j v(x',0) = 0$ for $1 \le j \le l$, then the terms a_k' for $0 \le k \le l$ are the same in the expansions for D and for D^0, and the nonlocal terms a_k'' differ by local contributions only.

Proof. Consider the first mentioned case, where $l = 1$; here $D = D^0 + x_n \overline{P}_1$ with $\overline{P}_1 = \sigma P_1$. Note that the adjoints D^* of D and $D^{0\prime}$ of D^0 are defined differently because of the different volume forms. However, as noted after (1.2), the hypothesis $\partial_{x_n} v(x',0)$ assures that D^* has the form $(-\partial_{x_n} + A + x_n P_1')\sigma^* = D^{0\prime} + x_n \overline{P}_1'$ with $\overline{P}_1' = P_1'\sigma^*$. Then on X_c,

$$\mathcal{D} - \mathcal{D}^0 = x_n \overline{\mathcal{P}}_1, \text{ where } \overline{\mathcal{P}}_1 = \begin{pmatrix} 0 & -\overline{P}_1' \\ \overline{P}_1 & 0 \end{pmatrix}.$$

The proofs of Theorems 3.4 and 3.5 extend immediately to this situation. In the expansion corresponding to (3.22) in this case, the second sum begins at the index $k_0 = 2$. This shows the stability, and the statement on the value follows from the remarks before the theorem.

In the case with general l, the hypotheses assure (in view of the remarks after (1.2)) that $\mathcal{D} - \mathcal{D}^0 = x_n^l \overline{\mathcal{P}}_l$ for a suitable first-order tangential operator $\overline{\mathcal{P}}_l$, and the proof goes as in Theorems 3.4 and 3.5. □

We recall (e.g. from [GG98]) that e_1 is generically nonzero when n is even (cf. also (3.29) below).

Note also the general result in the case $l = 0$, where P_0 can be nonzero: Here a_0' is preserved when D is replaced by D^0, and a_0'' is perturbed only by local contributions. This stability was established for the case with $\Pi = \Pi_\ge(A)$ in [G92], and in [GS95] for $\Pi = \Pi_>(A)$ plus certain finite rank projections.

But a perturbation with $P_0 \ne 0$ will in general change a_1':

Remark 3.10. Consider the simple case where $P_0 = \alpha I$ (on X_c), $\alpha \in \mathbb{R}$, and $F = \varphi^0$ as above. If we also change the boundary projection to $\Pi_{\geq}(A + \alpha I)$, we have a new product case, where the operator is

$$D^0_\alpha = \sigma(\partial_{x_n} + A + \alpha I) \text{ (on } X_c) \text{ with boundary condition } \Pi_{\geq}(A + \alpha I)\gamma_0 u = 0, \tag{3.28}$$

and the heat trace has an expansion as in (0.1) with log-coefficients that vanish according to the description after (0.1). Here the coefficient a'_1 equals $-\pi^{-1}e_1(\varphi^0, (A + \alpha I)^2)$, by the preceding explanation. Now since $\Pi_{\geq}(A + \alpha) - \Pi(A)$ is a finite linear combination of eigenprojections of A, it is a ψdo of order $-\infty$, so a replacement of $\Pi_{\geq}(A)$ by $\Pi_{\geq}(A + \alpha)$ in the boundary condition (1.6) leaves all log-terms invariant by the results in [G99] (elaborated in [G01′]). Thus in fact, when we return to the boundary condition (1.6), the term a'_1 in the trace expansion for D^0_α is also equal to $-\pi^{-1}e_1(\varphi^0, (A + \alpha)^2)$.

It vanishes for n odd, but let us consider the case n even. As recalled in [GG98], differentiation and comparison of the expansions of $\operatorname{Tr}(\varphi^0 e^{-t(A+\alpha)^2})$ and $\operatorname{Tr}(\varphi^0(A+\alpha)e^{-t(A+\alpha)^2})$ leads to the following formula (with a nonzero integer factor $m(n)$)

$$\partial^n_\alpha e_1(\varphi^0, (A + \alpha)^2) = m(n)\, e_{1-n}(\varphi^0, (A + \alpha)^2) \neq 0, \tag{3.29}$$

which shows that a'_1 *for* D^0_α *with boundary condition* (1.6) *is not constant in* α *when* n *is even.*

One can similarly study the differences connected with eta functions,

$$\operatorname{Tr}[F\psi(D_{2,\Pi}R^r_{2,\lambda} - D_{1,\Pi}R^r_{1,\lambda})], \quad \operatorname{Tr}[F\psi(D_{2,\Pi}e^{-t\Delta_{2,B}} - D_{1,\Pi}e^{-t\Delta_{1,B}})]$$
$$\text{and } \Gamma(s)\operatorname{Tr}[F\psi(D_{2,\Pi}\Delta^{-s}_{2,B} - D_{1,\Pi}\Delta^{-s}_{1,B})],$$

by considerations as above, departing from the formula inferred from (1.13):

$$D_{2,\Pi}R_{2,-\mu^2} - D_{1,\Pi}R_{1,-\mu^2} = -\begin{pmatrix} 0 & 1 \end{pmatrix}(\mathcal{R}_{2,\mu} - \mathcal{R}_{1,\mu})\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.30}$$

(note that $R_{i,\lambda}$ maps into the domain of $D_{i,\Pi}$)), and using again the considerations on the terms in (3.16). Here the expressions with higher powers are included by use of the formula

$$\begin{aligned} D_2R^r_{2,\lambda} - D_1R^r_{1,\lambda} = {} & (D_2R_{2,\lambda} - D_1R_{1,\lambda})R^{r-1}_{2,\lambda} \\ & + D_2R^r_{2,\lambda}(R_{2,\lambda} - R_{1,\lambda})(R^{r-2}_{2,\lambda} + \cdots + R^{r-2}_{1,\lambda}) \end{aligned} \tag{3.31}$$

(also used in [G01′]). This leads to:

Theorem 3.11. *Hypotheses of Theorem* 3.5.

(i) *We have for any* $r \geq 1$*:*

$$D_{2,\Pi}R^r_{2,-\mu^2} - D_{1,\Pi}R^r_{1,-\mu^2} = (D_2Q^r_{2,-\mu^2} - D_1Q^r_{1,-\mu^2})_+ + \widetilde{G}^{(r)}_\mu, \tag{3.32}$$

where $D_2Q^r_{2,-\mu^2} - D_1Q^r_{1,-\mu^2}$ *has symbol in* $S^{1,0,-2r}_{\text{spgh,ut}}(\Gamma)$ *and* $\widetilde{G}^{(r)}_\mu$ *has symbol-kernel in* $\mathcal{S}^{1,1-r,-2-r-l}(\Gamma, \mathcal{S}_{++}) + \mathcal{S}^{2,-r,-2-r-l}(\Gamma, \mathcal{S}_{++})$ *(in* $\mathcal{S}^{1,0,-3-l}(\Gamma, \mathcal{S}_{++})$ *if* $r = 1$*), in local trivializations.*

(ii) *It follows that when ψ is a morphism from E_2 to E_1, then there are trace expansions for $r > \frac{n+m'+1}{2}$:*

(3.33)

$$\begin{aligned}\operatorname{Tr}[F\psi(D_{2,\Pi}R^r_{2,\lambda} - D_{1,\Pi}R^r_{1,\lambda})] &\sim \sum_{-n\le k<k_0} \tilde c_k(-\lambda)^{\frac{m'+1-k}{2}-r} \\ &\quad + \sum_{k\ge k_0} \big(\tilde c'_k \log(-\lambda) + \tilde c''_k\big)(-\lambda)^{\frac{m'+1-k}{2}-r}, \\ \operatorname{Tr}[F\psi(D_{2,\Pi}e^{-t\Delta_{2,B}} - D_{1,\Pi}e^{-t\Delta_{1,B}})] &\sim \sum_{-n\le k<k_0} c_k t^{\frac{k-m'-1}{2}} + \sum_{k\ge k_0} \big(c'_k \log t + c''_k\big) t^{\frac{k-m'-1}{2}}, \\ \Gamma(s) \operatorname{Tr}[F\psi(D_{2,\Pi}\Delta^{-s}_{2,B} - D_{1,\Pi}\Delta^{-s}_{1,B})] &\sim \sum_{-n\le k<k_0} \frac{c_k}{s + \frac{k-m'-1}{2}} \\ &\quad + \sum_{k\ge k_0} \Big(\frac{-c'_k}{(s+\frac{k-m'-1}{2})^2} + \frac{c''_k}{s+\frac{k-m'-1}{2}}\Big),\end{aligned}$$

with k_0 defined by (3.23). For $k \le l - n$, $\tilde c_k$ and c_k vanish when $k - m' + n$ is even.

The coefficients $\tilde c_k$, $\tilde c'_k$, c_k and c'_k are locally determined, the c_k and c'_k being proportional to $\tilde c_k$, $\tilde c'_k$ by universal factors.

Proof. The statement on the ψdo part of (3.32) is immediate, since it is strongly polyhomogeneous of degree $-2r-1$. For the s.g.o. part, we start by using the analysis of (3.16) in the proof of Theorem 3.4, now considering the 21-block as in (3.30). This shows that the symbol-kernel of $\widetilde G^{(1)}_\mu$ is in $\mathcal S^{1,0,-3-l}(\Gamma, \mathcal S_{++})$, in local trivializations. Next, we apply the composition rules from Theorem 2.7 to (3.31), combining the above with the information in Theorem 3.4. It is found that s.g.o. part of the first term in (3.31) has symbol-kernel in $\mathcal S^{1,1-r,-2-r-l}(\Gamma, \mathcal S_{++})$, and the s.g.o. part of the second term has symbol-kernel in $\mathcal S^{2,-r,-2-r-l}(\Gamma, \mathcal S_{++})$. This shows (i). It follows that

$$F\psi(D_{2,\Pi}R^r_{2,-\mu^2} - D_{1,\Pi}R^r_{1,-\mu^2}) = (F\psi(D_2Q^r_{2,-\mu^2} - D_1Q^r_{1,-\mu^2}))_+ + F\psi\widetilde G^{(r)}_\mu, \tag{3.34}$$

where the ψdo part has symbol in $S^{1,0,m'-2r}_{\mathrm{spgh,ut}}(\Gamma)$ and $F\psi\widetilde G^{(r)}_\mu$ has symbol-kernel in $\mathcal S^{1,1-r,m'-2-r-l}(\Gamma, \mathcal S_{++}) + \mathcal S^{2,-r,m'-2-r-l}(\Gamma, \mathcal S_{++})$ (with m' moved to the first upper index when F is tangential).

Now consider (ii). The trace expansion of the ψdo-part is well-known to be a series of integer powers of $\mu = (-\lambda)^{\frac12}$, beginning with $\tilde c_{-n}\mu^{m'+1+n-2r}$, the terms vanishing when $k - m' - 1 + n$ is odd, i.e., $k - m' + n$ is even. For the s.g.o.-part, we apply Theorem 2.10. We get a sum of two ψdo terms with (m, d, s) equal to $(1, 1-r, m'-1-r-l)$ resp. $(2, -r, m'-1-r-l)$ in general (the m' can be moved to the first upper index if F is tangential). They both give expansions starting with the pure power $\mu^{m'+n-2r-l}$, whereas the logarithmic terms start with $\mu^{m'-2r-l}\log\mu$ resp. $\mu^{m'-1-2r-l}\log\mu$; in the case where F is tangential, the logs start with $\mu^{-2r-l}\log\mu$ resp. $\mu^{-1-2r-l}\log\mu$. Adding the contributions, we find the first expansion in (3.33).

This carries over to the other two expansions as in Corollary 3.7, when we furthermore note that $D_i\Pi_0(\Delta_{i,B}) = 0$ for $i = 1, 2$. $\square$

Remark 3.12. Note in particular that $\tilde{c}_{-n}$ and c_{-n} in (3.33) vanish when $F = I$.

An elaboration of the proof Theorem 3.3 with F replaced by $F\psi D$ gives:

$$\operatorname{Tr}(F\psi D R_\lambda^r) \sim \sum_{-n\le k<0} \tilde{b}_k(-\lambda)^{\frac{m'+1-k}{2}-r} + \sum_{k\ge 0}\big(\tilde{b}'_k \log(-\lambda) + \tilde{b}''_k\big)(-\lambda)^{\frac{m'+1-k}{2}-r}, \tag{3.35}$$

with $b_{-n} = 0$ if m' is even, and with the summation limit 0 replaced by m' if F is tangential. (When F is tangential, m' is added to the first upper index instead of the third upper index of the s.g.o. symbol-kernel space.) Theorem 3.11 now implies the perturbation result:

Corollary 3.13. *Hypotheses and definitions as in Theorem* 3.11. *For the trace expansion* (3.35) *of* $F\psi D_1(\Delta_{1,B} - \lambda)^{-r}$*, the replacement of* D_1 *by* D_2 *leaves the log-coefficients* $\tilde{b}'_k$ *invariant for* $k < k_0$*. The other coefficients with* $k < k_0$ *are modified only by local terms; those with* $k \le l - n$ *and* $k - m' + n$ *even are invariant.*

There are similar results for the associated heat trace and eta function.

Let us observe a particular consequence for eta expansions. In the above notation, the eta expansion proved in [GS95], [G99] has the form

$$\Gamma(s)\operatorname{Tr}[\psi D_\Pi \Delta_B^{-s}] \sim \sum_{-n<k<0} \frac{b_k}{s+\frac{k-1}{2}} + \sum_{k\ge 0}\Big(\frac{-b'_k}{(s+\frac{k-1}{2})^2} + \frac{b''_k}{s+\frac{k-1}{2}}\Big). \tag{3.36}$$

With the customary definition $\eta(\psi, D_\Pi, s') = \operatorname{Tr}(\psi D_\Pi \Delta_B^{\frac{s'+1}{2}})$, this may also be written in the more well-known form:

$$\Gamma(\tfrac{s'+1}{2})\eta(\psi, D_\Pi, s') \sim \sum_{-n<k<0} \frac{2b_k}{s'+k} + \sum_{k\ge 0}\Big(\frac{-4b'_k}{(s'+k)^2} + \frac{2b''_k}{s'+k}\Big). \tag{3.37}$$

We have in a similar way as in Theorem 3.9:

Theorem 3.14. *The coefficient* b'_1 *in* (3.36), (3.37) *is the same for a non-product type operator* D (1.1), (1.2) *over* X_c *with volume form* $v(x)\,dx$ *and the associated product type operator* D^0 (1.3) *with volume form* $v(x', 0)\,dx$*, when* $P_0 = 0$ *and* $\partial_{x_n} v(x', 0) = 0$*. Moreover, the coefficient* b''_1 *differs in the cases of* D *and* D^0 *by a local contribution only.*

There are similar statements for the associated resolvent and heat trace expansions, as well as extensions to cases where $D - D^0$ vanishes to a general order on X'.

In the case $P_0 \ne 0$, the coefficient b'_0 is invariant under the replacement of D by D^0, and b''_0 is changed only by local contributions; this was shown in [GS95] for $\Pi = \Pi_>$ plus certain finite rank projections.

For the general systems $\{P - \lambda, S_Q\}$ considered in [G99], one can study the effect of a perturbation of P by a tangential operator $x_n^l \overline{P}$ in a similar way, finding also here that the first $l + 1$ logarithmic coefficients are stable, the power coefficients behind them being changed only by local contributions.

4. The resolvent structure for perturbations commuting with A.

We here consider the case where D is a perturbation of D^0 such that $D - D^0$ *commutes with A on X_c*, in the sense that in (1.2), the zero-order x_n-independent operator (morphism) P_0 commutes with A, and in the Taylor expansions on X_c,

$$x_nP_1(x_n) = \sum_{1\le k\le K} x_n^kP_{1k} + x_n^{K+1}P'_{K+1}(x_n) \text{ for any } K, \tag{4.1}$$

the tangential x_n-independent first-order differential operators P_{1k} commute with A. The product measure is used on X_c. We shall show that in this case, there are no log-terms in the trace expansions in the odd-dimensional case.

It is no restriction to replace X_c by X_1; this can always be obtained by a scaling in x_n.

We know from Theorem 3.5 that the larger K is, the more log-terms are unaffected by subtracting $x_n^{K+1}P'_{K+1}(x_n)$ from $x_nP_1(x_n)$, so we may disregard this remainder term in the calculations that follow.

Thus, consider the case where, on X_1,

$$\begin{gathered} D = \sigma(\partial_{x_n} + A_1(x_n)),\quad A_1(x_n) = A + \sum_{0\le k\le K} x_n^kP_{1k} = A + \overline{P}, \\ \text{where the } x_n\text{-independent operators } P_{1k} \text{ commute with } A. \end{gathered} \tag{4.2}$$

For notational convenience, P_0 is here denoted P_{10}; it is of order 0 and the P_{1k} with $k\ge 1$ are of order 1.

We here restrict the attention to the boundary condition (1.6).

For the doubled-up systems (cf. (1.10), (1.16)), we have on X_1, for any μ,

$$\begin{aligned} \mathcal{D}+\mu &= \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} \mu & \partial_{x_n} - A_1^* \\ \partial_{x_n} + A_1 & \mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sigma^* \end{pmatrix}, \\ \mathcal{D}^0+\mu &= \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}\begin{pmatrix} \mu & \partial_{x_n} - A \\ \partial_{x_n} + A & \mu \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sigma^* \end{pmatrix}, \end{aligned} \tag{4.3}$$

since $\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^* \end{pmatrix}$. By composition with $\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$ and its inverse, the study of the resolvents on X_1 is reduced to the study of the inverses of the middle factors in (4.3), i.e., the case where σ is the identity (in E'_1). To keep the notation simple, we use the names $\mathcal{D}+\mu$ and $\mathcal{D}^0+\mu$ again for the middle factors. In other words, without of loss of generality:

We consider in the following the reduced case where σ is the identity, i.e.,

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_{x_n} - A_1^* \\ \partial_{x_n} + A_1 & 0 \end{pmatrix},\quad \mathcal{D}^0 = \begin{pmatrix} 0 & \partial_{x_n} - A \\ \partial_{x_n} + A & 0 \end{pmatrix}. \tag{4.4}$$

The boundary condition (1.11) *then has the form:*

$$\mathcal{B}\gamma_0 u = 0,\quad \mathcal{B} = (\,\Pi_\ge \quad \Pi_<\,). \tag{4.5}$$

Let us denote

$$\mathcal{P}_k = \begin{pmatrix} 0 & -P_{1k}^* \\ P_{1k} & 0 \end{pmatrix} \text{ for } 0\le k\le K,\quad \mathcal{P} = \begin{pmatrix} 0 & -\overline{P}^* \\ \overline{P} & 0 \end{pmatrix}, \tag{4.6}$$

then

$$\mathcal{D}-\mathcal{D}^0=\sum_{0\le k\le K} x_n^k\mathcal{P}_k=\mathcal{P}. \tag{4.7}$$

We shall use ζ_ε introduced in (2.20)ff. for $\varepsilon\in\,]0,1]$; it is defined on X_1 as constant in x' and extends by zero to X^0 as well as to X, as a C^∞ function.

Rather than $\mathcal{D}=\mathcal{D}^0+\mathcal{P}$, we shall consider

$$\mathcal{D}'=\mathcal{D}^0+\zeta_\varepsilon\mathcal{P}, \tag{4.8}$$

with ε to be chosen later; it equals $\mathcal{D}$ on $X_{\varepsilon/3}$ and serves the same purpose as $\mathcal{D}$ for investigation of the structure near $x_n=0$.

For $\mu\in\mathbb{C}\setminus i\mathbb{R}$, $\mathcal{D}^0+\mu$ has the inverse on $\widetilde{X}^0=X'\times\mathbb{R}$:

$$\begin{aligned}\mathcal{Q}^0&=(\mathcal{D}^0+\mu)^{-1}\\&=\begin{pmatrix}\mu(D_{x_n}^2+A^2+\mu^2)^{-1} & (-\partial_{x_n}+A)(D_{x_n}^2+A^2+\mu^2)^{-1}\\ -(\partial_{x_n}+A)(D_{x_n}^2+A^2+\mu^2)^{-1} & \mu(D_{x_n}^2+A^2+\mu^2)^{-1}\end{pmatrix},\end{aligned} \tag{4.9}$$

where $(D_{x_n}^2+A^2+\mu^2)^{-1}=Q_\lambda^0$, $\lambda=-\mu^2$, cf. (1.17). (The parameter-dependence will not always be explicitly indicated by an index.) Let

$$\mathcal{A}=\begin{pmatrix}\mathcal{D}'+\mu\\ \mathcal{B}\gamma_0\end{pmatrix},\quad \mathcal{A}^0=\begin{pmatrix}\mathcal{D}^0+\mu\\ \mathcal{B}\gamma_0\end{pmatrix}, \tag{4.10}$$

representing the full nonhomogeneous problems on $X^0=X'\times\overline{\mathbb{R}}_+$. It follows from [GS95, (3.11)–(3.16), Prop. 3.5] that $\mathcal{A}^0$ has the solution operator (recall (1.18)ff.)

$$\begin{aligned}&(\mathcal{A}^0)^{-1}=\begin{pmatrix}\mathcal{R}^0 & \mathcal{K}^0\end{pmatrix},\text{ where } \mathcal{R}^0=\mathcal{Q}^0_++\mathcal{G}^0;\\ &\mathcal{K}^0=\begin{pmatrix}K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}S'_{\mathcal{B}},\quad S'_{\mathcal{B}}=\begin{pmatrix}\Pi_\ge+\mu^{-1}(A_\lambda+A)\Pi_<\\ \mu^{-1}(A_\lambda-A)\Pi_\ge+\Pi_<\end{pmatrix},\\ &\mathcal{G}^0=-\mathcal{K}^0\mathcal{B}\gamma_0\mathcal{Q}^0_+=-\begin{pmatrix}K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}S_{\mathcal{B}}\gamma_0\mathcal{Q}^0_+,\quad S_{\mathcal{B}}=S'_{\mathcal{B}}\mathcal{B},\end{aligned} \tag{4.11}$$

for $\mu\in\mathbb{C}\setminus i\mathbb{R}$, $\lambda=-\mu^2$. Here $\mathcal{R}^0=(\mathcal{D}^0_{\mathcal{B}}+\mu)^{-1}$. In details,

$$\begin{aligned}S_{\mathcal{B}}=S'_{\mathcal{B}}\mathcal{B}&=\begin{pmatrix}\Pi_\ge+\mu^{-1}(A_\lambda+A)\Pi_<\\ \mu^{-1}(A_\lambda-A)\Pi_\ge+\Pi_<\end{pmatrix}\begin{pmatrix}\Pi_\ge & \Pi_<\end{pmatrix}\\&=\begin{pmatrix}\Pi_\ge & \mu^{-1}(A_\lambda+A)\Pi_<\\ \mu^{-1}(A_\lambda-A)\Pi_\ge & \Pi_<\end{pmatrix}.\end{aligned} \tag{4.12}$$

For the description of $\gamma_0\mathcal{Q}^0_+$, we observe the simple formulas, valid when $\operatorname{Re}\mathfrak{a}>0$:

$$\begin{aligned}&\tfrac{1}{\mathfrak{a}^2+\xi_n^2}=\tfrac{1}{2\mathfrak{a}}\big(\tfrac{1}{\mathfrak{a}+i\xi_n}+\tfrac{1}{\mathfrak{a}-i\xi_n}\big),\text{ so } h^+\tfrac{1}{\mathfrak{a}^2+\xi_n^2}=\tfrac{1}{2\mathfrak{a}(\mathfrak{a}+i\xi_n)},\ h^-\tfrac{1}{\mathfrak{a}^2+\xi_n^2}=\tfrac{1}{2\mathfrak{a}(\mathfrak{a}-i\xi_n)},\\ &\tfrac{i\xi_n}{\mathfrak{a}^2+\xi_n^2}=\tfrac{1}{2}\big(-\tfrac{1}{\mathfrak{a}+i\xi_n}+\tfrac{1}{\mathfrak{a}-i\xi_n}\big),\text{ so } h^+\tfrac{i\xi_n}{\mathfrak{a}^2+\xi_n^2}=-\tfrac{1}{2(\mathfrak{a}+i\xi_n)},\ h^-\tfrac{i\xi_n}{\mathfrak{a}^2+\xi_n^2}=\tfrac{1}{2(\mathfrak{a}-i\xi_n)};\end{aligned} \tag{4.13}$$

here $h^\pm$ (cf. (2.6)) projects the rational function onto its component with poles in $\mathbb{C}_\pm$, respectively. Applying (4.13) in each eigenspace of A_λ, we get (in view of the rules of calculus, cf. e.g. [G96, Th. 2.6.1]):

$$
\begin{aligned}
\gamma_0(D_{x_n}^2+A^2+\mu^2)_+^{-1} &= \mathrm{OPT}_n\big(h^-\tfrac{1}{A_\lambda^2+\xi_n^2}\big) = \mathrm{OPT}_n\big(\tfrac{1}{2A_\lambda(A_\lambda-i\xi_n)}\big) = \tfrac{1}{2A_\lambda}T_{A_\lambda},\\
\gamma_0\partial_{x_n}(D_{x_n}^2+A^2+\mu^2)_+^{-1} &= \mathrm{OPT}_n\big(h^-\tfrac{i\xi_n}{A_\lambda^2+\xi_n^2}\big) = \mathrm{OPT}_n\big(\tfrac{1}{2(A_\lambda-i\xi_n)}\big) = \tfrac12 T_{A_\lambda},
\end{aligned}
\tag{4.14}
$$

so that

$$
\gamma_0\mathcal{Q}_+^0 = \tfrac{1}{2A_\lambda}\begin{pmatrix}\mu & -A_\lambda+A\\ -A_\lambda-A & \mu\end{pmatrix}\begin{pmatrix}T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix} = \mathcal{S}_1^-\begin{pmatrix}T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix},
\tag{4.15}
$$

with

$$
\mathcal{S}_1^- = \tfrac{1}{2A_\lambda}\begin{pmatrix}\mu & -A_\lambda+A\\ -A_\lambda-A & \mu\end{pmatrix}.
\tag{4.16}
$$

Thus we find from (4.11):

$$
\mathcal{G}^0 = \begin{pmatrix}K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}\mathcal{S}_0\begin{pmatrix}T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix},\quad \text{with } \mathcal{S}_0 = -S_{\mathcal{B}}\mathcal{S}_1^-.
\tag{4.17}
$$

In details,

$$
\begin{aligned}
\mathcal{S}_0 &= \tfrac{-1}{2A_\lambda}\begin{pmatrix}\Pi_\ge & \mu^{-1}(A_\lambda+A)\Pi_<\\ \mu^{-1}(A_\lambda-A)\Pi_\ge & \Pi_<\end{pmatrix}\begin{pmatrix}\mu & -A_\lambda+A\\ -A_\lambda-A & \mu\end{pmatrix}\\
&= \tfrac{-1}{2A_\lambda}\begin{pmatrix}\mu\Pi_\ge-\mu^{-1}(A_\lambda+A)^2\Pi_< & (-A_\lambda+A)\Pi_\ge+(A_\lambda+A)\Pi_<\\ (A_\lambda-A)\Pi_\ge-(A_\lambda+A)\Pi_< & -\mu^{-1}(A_\lambda-A)^2\Pi_\ge+\mu\Pi_<\end{pmatrix};
\end{aligned}
\tag{4.18}
$$

this may be further rewritten by use of (1.5) and the formulas

$$
A_\lambda^2 = A^2-\lambda,\quad (A_\lambda\pm A)^2 = 2A^2-\lambda\pm 2A_\lambda A.
\tag{4.19}
$$

As shown in Proposition 2.11, K_{A_λ} and T_{A_λ} are strongly polyhomogeneous Poisson resp. class 0 trace operators of degree -1 (having symbol-kernels in $\mathcal{S}^{0,0,-1}(\Gamma,\mathcal{S}_+)$ in local trivializations). Clearly, $\mathcal{S}_1^-$ is strongly polyhomogeneous of degree 0 (hence with symbol in $S^{0,0,0}$), so $\gamma_0\mathcal{Q}_+^0$ is a strongly polyhomogeneous trace operator of class 0 and degree -1 like T_{A_λ}. Since

$$
\begin{aligned}
\mu^{-1}(A_\lambda-A)\Pi_\ge &= \mu^{-1}(A_\lambda-|A|)\tfrac{A_\lambda+|A|}{A_\lambda+|A|}\Pi_\ge = \mu(A_\lambda+|A|)^{-1}\Pi_\ge,\\
\mu^{-1}(A_\lambda+A)\Pi_< &= \mu^{-1}(A_\lambda-|A|)\tfrac{A_\lambda+|A|}{A_\lambda+|A|}\Pi_< = \mu(A_\lambda+|A|)^{-1}\Pi_<,
\end{aligned}
\tag{4.20}
$$

where $\Pi_\ge$ and $\Pi_<$ have symbols in $S^0\subset S^{0,0,0}(\Gamma)$, $S_{\mathcal{B}}$ is a weakly polyhomogeneous ψdo with symbol in $S^{0,0,0}(\Gamma)$ in local trivializations by Proposition 2.11 (iii) and the product rule in Theorem 2.7 (xiii); then so is $\mathcal{S}_0$. Thus $\mathcal{G}^0$ is an s.g.o. with symbol-kernel in $\mathcal{S}^{0,0,-2}(\Gamma,\mathcal{S}_{++})$ in local trivializations, in fact with estimates that are uniform in $x_n\in\mathbb{R}_+$.

In view of Lemma 2.9 it is, for the s.g.o.-terms, the structure near X' that determines their contribution to the asymptotic expansions; we need not spend efforts on elaborate presentations of a calculus on the unbounded manifold X^0.

Since $\mathcal{D}^0$ and $\mathcal{D}^0_{\mathcal{B}}$ are skew-selfadjoint (as unbounded operators in $L_2(\widetilde{E}^0)$ resp. $L_2(E^0)$),

$$\|(\operatorname{Re}\mu)\mathcal{Q}^0\|_{\mathcal{L}(L_2(\widetilde{E}^0))} \le C,\ \|(\operatorname{Re}\mu)\mathcal{R}^0\|_{\mathcal{L}(L_2(E^0))} \le C,\ \text{ for } |\operatorname{Re}\mu| \ge 1. \tag{4.21}$$

In view of the ellipticity, we also have

$$\|\mathcal{Q}^0\|_{\mathcal{L}(L_2(\widetilde{E}^0),H^1(\widetilde{E}^0))} \le C',\ \|\mathcal{R}^0\|_{\mathcal{L}(L_2(E^0),H^1(E^0))} \le C',\ \text{ for } |\operatorname{Re}\mu| \ge 1, \tag{4.22}$$

with H^1 denoting the Sobolev space of order 1.

To find inverses of $\mathcal{D}'+\mu$ and $\mathcal{A}$, we calculate:

$$\begin{aligned} &(\mathcal{D}'+\mu)\mathcal{Q}^0 = I + \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0,\\ &\mathcal{A}(\mathcal{A}^0)^{-1} = (\mathcal{A}^0 + \begin{pmatrix}\zeta_\varepsilon\mathcal{P}\\ 0\end{pmatrix})\,(\,\mathcal{R}^0 \quad \mathcal{K}^0\,) = I + \begin{pmatrix}\zeta_\varepsilon\mathcal{P}\mathcal{R}^0 & \zeta_\varepsilon\mathcal{P}\mathcal{K}^0\\ 0 & 0\end{pmatrix}. \end{aligned} \tag{4.23}$$

Then

$$\begin{aligned} \mathcal{Q}_M &= \mathcal{Q}^0 \sum_{0\le m\le M} (-\zeta_\varepsilon\mathcal{P}\mathcal{Q}^0)^m,\\ \mathcal{C}_M &= (\mathcal{A}^0)^{-1} \sum_{0\le m\le M} \begin{pmatrix} -\zeta_\varepsilon\mathcal{P}\mathcal{R}^0 & -\zeta_\varepsilon\mathcal{P}\mathcal{K}^0\\ 0 & 0\end{pmatrix}^m\\ &= (\,\mathcal{R}^0 \quad \mathcal{K}^0\,)\begin{pmatrix} \sum_{0\le m\le M}(-\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^m & \sum_{1\le m\le M}(-\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^{m-1}(-\zeta_\varepsilon\mathcal{P}\mathcal{K}^0)\\ 0 & 0\end{pmatrix}\\ &= \mathcal{R}^0\,(\,\textstyle\sum_{0\le m\le M}(-\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^m \quad -\sum_{0\le m\le M}(-\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^{m-1}\zeta_\varepsilon\mathcal{P}\mathcal{K}^0\,), \end{aligned} \tag{4.24}$$

will for large M be good approximations to inverses of $\mathcal{D}'+\mu$ resp. $\mathcal{A}$; in particular,

$$\mathcal{R}_M = \mathcal{R}^0 \sum_{0\le m\le M} (-\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^m \tag{4.25}$$

will be a good approximation to a resolvent of the realization $\mathcal{D}'_{\mathcal{B}}$ of $\mathcal{D}'$ under the boundary condition (4.5). More precisely, we have (cf. (4.6))

$$\zeta_\varepsilon\mathcal{P}\mathcal{R}^0 = \zeta_\varepsilon\mathcal{P}_0\mathcal{R}^0 + \zeta_\varepsilon x_n \sum_{1\le k\le K} x_n^{k-1}\mathcal{P}_k\mathcal{R}^0, \tag{4.26}$$

where the L_2 operator norms satisfy (in view of (4.21)–(4.22)):

$$\begin{aligned} &\|\mathcal{P}_0\mathcal{Q}^0\|_{\mathcal{L}(L_2(\widetilde{E}^0))} \text{ and } \|\mathcal{P}_0\mathcal{R}^0\|_{\mathcal{L}(L_2(E^0))} \le C_1|\operatorname{Re}\mu|^{-1},\\ &\|\mathcal{P}_k\mathcal{Q}^0\|_{\mathcal{L}(L_2(\widetilde{E}^0))} \text{ and } \|\mathcal{P}_k\mathcal{R}^0\|_{\mathcal{L}(L_2(E^0))} \le C_2, \end{aligned} \tag{4.27}$$

for $|\operatorname{Re}\mu| \ge 1$. Since $|\zeta_\varepsilon x_n| \le \varepsilon$, we can choose an ε and a $b>0$ such that for $|\operatorname{Re}\mu| \ge b$,

$$\|\zeta_\varepsilon\mathcal{P}\mathcal{Q}^0\|_{\mathcal{L}(L_2(\widetilde{E}^0))} \text{ and } \|\zeta_\varepsilon\mathcal{P}\mathcal{R}^0\|_{\mathcal{L}(L_2(E^0))} \le \tfrac12, \tag{4.28}$$

and then there is in fact convergence in operator norm for $M\to\infty$:

$$
\begin{aligned}
\mathcal{Q} &= \lim_{M\to\infty} \mathcal{Q}_M = \mathcal{Q}^0 \sum_{m\ge 0} (-\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)^m,\\
\mathcal{C} &= \lim_{M\to\infty} \mathcal{C}_M = \mathcal{R}^0 \left(\textstyle\sum_{m\ge 0}(-\zeta_\varepsilon \mathcal{P}\mathcal{R}^0)^m \quad -\sum_{m\ge 1}(-\zeta_\varepsilon \mathcal{P}\mathcal{R}^0)^{m-1}\zeta_\varepsilon\mathcal{P}\mathcal{K}^0 \right)\\
&= (\mathcal{R} \quad \mathcal{K}) ;\ \text{in particular,}\\
\mathcal{R} &= \lim_{M\to\infty} \mathcal{R}_M = \mathcal{R}^0 \sum_{m\ge 0} (-\zeta_\varepsilon \mathcal{P}\mathcal{R}^0)^m; \quad \mathcal{K} = -\mathcal{R}\zeta_\varepsilon\mathcal{P}\mathcal{K}^0.
\end{aligned} \tag{4.29}
$$

Here

$$
\mathcal{Q} = (\mathcal{D}'+\mu)^{-1},\quad \mathcal{C} = \mathcal{A}^{-1},\quad \mathcal{R} = (\mathcal{D}'_{\mathcal{B}}+\mu)^{-1} = \mathcal{Q}_+ + \mathcal{G}, \tag{4.30}
$$

where $\mathcal{G}$ is an s.g.o. on X^0. It is seen as in [GS95], [G99], that the operators belong to the weakly polyhomogeneous calculus (in fact with estimates that are uniform in $x_n\in\mathbb{R}_+$). In particular (as in the description of $\mathcal{R}_\mu$ in (3.1)ff.), $\mathcal{G}$ has symbol-kernel in $\mathcal{S}^{0,0,-2}(\Gamma,\mathcal{S}_{++})$ and $\mathcal{Q}$ has symbol in $S^{0,0,-1}_{\mathrm{sphg,ut}}(\Gamma)$, in local trivializations.

At first sight, since $\zeta_\varepsilon\mathcal{P}$ is of order 1, the terms $\mathcal{R}^0(\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^m$ are all of order -1, so it seems unpractical to use the series in $m\in\mathbb{N}$ to get trace expansions. But a closer inspection shows that only the pseudodifferential part of each term remains of order -1; for the singular Green part, the order decreases with increasing m.

Proposition 4.1. (a) *For each m,*

$$
\begin{aligned}
\mathcal{R}^0(\zeta_\varepsilon\mathcal{P}\mathcal{R}^0)^m &= (\mathcal{Q}^0_+ + \mathcal{G}^0)\zeta_\varepsilon\mathcal{P}(\mathcal{Q}^0_+ + \mathcal{G}^0)\cdots\zeta_\varepsilon\mathcal{P}(\mathcal{Q}^0_+ + \mathcal{G}^0)\\
&= (\mathcal{Q}^0(\zeta_\varepsilon\mathcal{P}\mathcal{Q}^0)^m)_+ + \mathcal{G}_{(m)},
\end{aligned} \tag{4.31}
$$

where $\mathcal{Q}^0(\zeta_\varepsilon\mathcal{P}\mathcal{Q}^0)^m$ has symbol in $S^{0,0,-1}_{\mathrm{sphg,ut}}(\Gamma)$ and $\mathcal{G}_{(m)}$ has symbol-kernel in $\mathcal{S}^{0,0,-m-2}(\Gamma,\mathcal{S}_{++})$, in local trivializations.

(b) *With $\mathcal{Q}_M$ and $\mathcal{R}_M$ defined in* (4.24), (4.25), *one has for any $M\in\mathbb{N}$:*

$$
\mathcal{R} = \mathcal{R}_M + (\mathcal{Q}-\mathcal{Q}_M)_+ + \mathcal{G}'_M, \tag{4.32}
$$

where $\mathcal{G}'_M$ has symbol-kernel in $\mathcal{S}^{0,0,-M-3}(\Gamma,\mathcal{S}_{++})$ and $\mathcal{Q}-\mathcal{Q}_M$ has symbol in $S^{0,0,-1}_{\mathrm{sphg,ut}}(\Gamma)$, in local trivializations.

Proof. (a) Since $\mathcal{P}\mathcal{Q}^0$ is strongly polyhomogeneous of order 0, the statement on the symbol of the ψdo part follows straightforwardly from the product rules; it is the s.g.o. part that demands some effort.

Note that

$$
\zeta_\varepsilon\mathcal{P} = \zeta_\varepsilon(x_n)\mathcal{P}_0 + x_n \sum_{1\le k\le K} \zeta_\varepsilon(x_n)x_n^{k-1}\mathcal{P}_k, \tag{4.33}
$$

the sum of a zero-order term $\zeta_\varepsilon\mathcal{P}_0$ (independent of μ) and a term containing the factor x_n. As we know from Lemma 2.3, a factor x_n reduces the order of s.g.o.s, lowering the third upper index by 1. Thus the s.g.o. part of $\zeta_\varepsilon \mathcal{P}\mathcal{R}^0$ has symbol-kernel in $\mathcal{S}^{0,0,-2}(\Gamma, \mathcal{S}_{++})$ just like $\mathcal{G}^0$. When we multiply out (4.31), the result then follows immediately by use of Theorem 2.7 (iv)–(vi) for those products that do not contain two adjacent factors $\mathcal{Q}^0_+$ and $\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+$.

For the remaining products, we need some extra considerations. As in Theorem 2.7 (xiv), write

$$\mathcal{Q}^0_+ \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+ = (\mathcal{Q}^0 \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)_+ - G^+(\mathcal{Q}^0)\zeta_\varepsilon \mathcal{P}\, G^-(\mathcal{Q}^0)$$

(this makes good sense also when $\mathbb{R}^n$ is replaced by $X' \times \mathbb{R}$). The s.g.o.s $G^\pm(\mathcal{Q}^0)$ have symbol-kernels in $\mathcal{S}^{0,0,-2}(\Gamma, \mathcal{S}_{++})$, and thanks to the structure of $\zeta_\varepsilon \mathcal{P}$ described in (4.33)ff., the composition $G_1 = G^+(\mathcal{Q}^0)\zeta_\varepsilon \mathcal{P} G^-(\mathcal{Q}^0)$ has symbol-kernel in $\mathcal{S}^{0,0,-3}(\Gamma, \mathcal{S}_{++})$. Next, a repeated composition with $\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+$ gives

$$\mathcal{Q}^0_+ (\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+)^2 = ((\mathcal{Q}^0 \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)_+ - G_1)\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+ = (\mathcal{Q}^0 \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)_+ \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+ + G_2,$$

where G_2 has symbol-kernel in $\mathcal{S}^{0,0,-4}(\Gamma, \mathcal{S}_{++})$ by Lemma 2.3 and Theorem 2.7 (iv)–(vi). Applying Theorem 2.7 (xiv) to $(\mathcal{Q}^0 \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)_+ \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+$, we find that it is the sum of a ψdo term $(\mathcal{Q}^0(\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)^2)_+$ and an s.g.o. term

$$G_3 = -G^+(\mathcal{Q}^0 \zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)\zeta_\varepsilon \mathcal{P} G^-(\mathcal{Q}^0).$$

Inside G^+, we apply the commutator formula

$$x_n \operatorname{OP}(p) = \operatorname{OP}(p) x_n + \operatorname{OP}(i\partial_{\xi_n} p) \tag{4.34}$$

to $x_n \mathcal{Q}^0$, whereby we get two terms, one having a factor x_n to the right and one where the third upper index is lowered one step. The composition rules and Lemma 2.3 then give that G_3 has symbol-kernel in $\mathcal{S}^{0,0,-4}(\Gamma, \mathcal{S}_{++})$. Clearly, this analysis can be continued inductively to show that the s.g.o. part of $\mathcal{Q}^0_+(\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0_+)^l$ has symbol-kernel in $\mathcal{S}^{0,0,-l-2}(\Gamma, \mathcal{S}_{++})$ for any l, and when this is combined with the other rules, we obtain (a) for general m.

(b) It is clear from (4.24), (4.29), (4.30)ff. that the ψdo part of $\mathcal{R} - \mathcal{R}_M$ equals $(\mathcal{Q} - \mathcal{Q}_M)_+$ and is of order -1. For the s.g.o. part, we shall use that in view of (4.29) and (a):

$$\begin{aligned}\mathcal{R} - \mathcal{R}_M &= \mathcal{R}^0 \sum_{m > M} (-\zeta_\varepsilon \mathcal{P}\mathcal{R}^0)^m = \mathcal{R}^0(-\zeta_\varepsilon \mathcal{P}\mathcal{R}^0)^M(-\zeta_\varepsilon \mathcal{P})\mathcal{R}^0 \sum_{m \ge 0}(-\zeta_\varepsilon \mathcal{P}\mathcal{R}^0)^m \\ &= (-1)^{M+1}((\mathcal{Q}^0(\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)^M)_+ + \mathcal{G}_{(M)})\zeta_\varepsilon \mathcal{P}\mathcal{R}.\end{aligned}$$

Here $\mathcal{R} = \mathcal{Q}_+ + \mathcal{G}$ as described after (4.30). From the description of $\mathcal{G}_{(M)}$ in (a) follows in view of Lemma 2.3 and Theorem 2.7 that $\mathcal{G}_{(M)}\zeta_\varepsilon \mathcal{P}\mathcal{R}$ has symbol-kernel in $\mathcal{S}^{0,0,-M-3}(\Gamma, \mathcal{S}_{++})$. By use of (4.34), we can write $\mathcal{Q}^0(\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)^M$ as a sum of strongly polyhomogeneous terms of order $-M-1+j$ with a factor x_n^j to the right, $j = 0, 1, \dots, M$; then it is seen as in the proof of (a) that the s.g.o. part of the composition $(\mathcal{Q}^0(\zeta_\varepsilon \mathcal{P}\mathcal{Q}^0)^M)_+ \zeta_\varepsilon \mathcal{P}\mathcal{R}$ has symbol-kernel in $\mathcal{S}^{0,0,-M-3}(\Gamma, \mathcal{S}_{++})$. This completes the proof. $\square$

Since $\mathcal{G}'_M$ has symbol-kernel in $\mathcal{S}^{0,0,-M-3}(\Gamma, \mathcal{S}_{++})$, it is trace-class for $M > n-3$ and its trace has an expansion as in Theorem 2.10 beginning with the power μ^{n-M-3}. Thus in (4.32), the second term contributes no logarithms in trace expansions and the third term contributes $O(\mu^{n-M-2})$ terms with M as large as we want, so all information on log-terms can be found from the $\mathcal{R}_M$ (for large M), and we only have to study $\mathcal{R}_M$ in detail.

Proposition 4.2. *For each M, let*

$$\mathcal{R}^0_M = \sum_{0\le m\le M} \mathcal{R}^0(-\mathcal{P}\mathcal{R}^0)^m;\ \textit{let } \mathcal{G}^0_M = \textit{the s.g.o. part of } \mathcal{R}^0_M. \tag{4.35}$$

Then

$$\mathcal{R}_M = \mathcal{Q}_{M,+} + \mathcal{G}^0_M + \mathcal{G}''_M, \tag{4.36}$$

where $\mathcal{G}''_M$ has symbol-kernel in $\mathcal{S}^{-\infty,-\infty,-\infty}(\Gamma, \mathcal{S}_{++})$.

Proof. Denote $\zeta_\varepsilon = \zeta$, $\zeta_{\varepsilon/3} = \zeta_0$. For each $m \le M$, write

$$\begin{aligned}
&\zeta_0\mathcal{R}^0(\mathcal{P}\mathcal{R}^0)^m - \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^m\\
&\quad= \zeta_0\mathcal{R}^0(\mathcal{P}\mathcal{R}^0)^m - \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)(\mathcal{P}\mathcal{R}^0)^{m-1} + \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)(\mathcal{P}\mathcal{R}^0)^{m-1}\\
&\qquad - \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^2(\mathcal{P}\mathcal{R}^0)^{m-2} + \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^2(\mathcal{P}\mathcal{R}^0)^{m-2}\\
&\qquad - \dots - \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^{m-1}(\mathcal{P}\mathcal{R}^0) + \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^{m-1}(\mathcal{P}\mathcal{R}^0) - \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^m\\
&\quad= \sum_{0\le j\le m-1} \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^j(1-\zeta)\mathcal{P}\mathcal{R}^0(\mathcal{P}\mathcal{R}^0)^{m-1-j}.
\end{aligned}$$

Each term in the sum over j has a factor of the form $\zeta_0(P_+ + G)(1-\zeta)$; here $\zeta_0\mathcal{P}_+(1-\zeta)$ is a ψdo with symbol in $S^{-\infty,-\infty,-\infty}(\Gamma)$ since $\zeta_0(1-\zeta) = 0$, and $\zeta_0 G(1-\zeta)$ is an s.g.o. with symbol-kernel in $\mathcal{S}^{-\infty,-\infty,-\infty}(\Gamma, \mathcal{S}_{++})$ by Lemma 2.9; let us call such operators *negligible*. It then follows from the composition rules that $\zeta_0\mathcal{R}^0(\mathcal{P}\mathcal{R}^0)^m - \zeta_0\mathcal{R}^0(\zeta\mathcal{P}\mathcal{R}^0)^m$ is negligible, so we get by summation over m that $\zeta_0\mathcal{R}^0_M - \zeta_0\mathcal{R}_M$ is negligible. By Lemma 2.9, the s.g.o. part of $(1-\zeta_0)\mathcal{R}_M$ is likewise negligible, and so is $(1-\zeta_0)\mathcal{G}^0_M$. This implies for the s.g.o. parts:

$$\begin{aligned}
[\mathcal{R}_M]_{\text{s.g.o.}} &= [\zeta_0\mathcal{R}_M]_{\text{s.g.o.}} + [(1-\zeta_0)\mathcal{R}_M]_{\text{s.g.o.}}\\
&= [\zeta_0\mathcal{R}^0_M]_{\text{s.g.o.}} + \text{ negl. s.g.o.} = \zeta_0\mathcal{G}^0_M + \text{ negl. s.g.o.} = \mathcal{G}^0_M + \text{ negl. s.g.o.},
\end{aligned}$$

as was to be shown. ☐

The proposition shows that the s.g.o. part of $\mathcal{R}_M$ equals $\mathcal{G}^0_M$ modulo negligible terms, so it remains to analyze $\mathcal{G}^0_M$.

We now want to use that the operators P_{1k} commute with the selfadjoint operator A. Then they and their adjoints also commute with the various functions of A appearing in the formulas, such as A_λ, $|A|$, $\Pi_\ge$, $e^{-x_n A_\lambda}$, etc.

Consider, to begin with, the first composition

$$\begin{aligned}
\mathcal{R}^0\mathcal{P}\mathcal{R}^0 &= (\mathcal{Q}^0_+ + \mathcal{G}^0)\mathcal{P}(\mathcal{Q}^0_+ + \mathcal{G}^0)\\
&= (\mathcal{Q}^0\mathcal{P}\mathcal{Q}^0)_+ - G^+(\mathcal{Q}^0)\mathcal{P}\,G^-(\mathcal{Q}^0) + \mathcal{Q}^0_+\mathcal{P}\mathcal{G}^0 + \mathcal{G}^0\mathcal{P}\mathcal{Q}^0_+ + \mathcal{G}^0\mathcal{P}\mathcal{G}^0
\end{aligned} \tag{4.37}$$

where explicit formulas are manageable to some extent. In the last three terms we shall use that $\mathcal{G}^0$ has the form (4.17), where we can write

$$\begin{aligned}
\sum_{0\le k\le K} x_n^k\mathcal{P}_k\mathcal{G}^0 &= \sum_{0\le k\le K} x_n^k \begin{pmatrix} K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}\mathcal{P}_k\mathcal{S}_0\begin{pmatrix} T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix}\\
\mathcal{G}^0\sum_{0\le k\le K} x_n^k\mathcal{P}_k &= \sum_{0\le k\le K} \begin{pmatrix} K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}\mathcal{S}_0\mathcal{P}_k\begin{pmatrix} T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix} x_n^k,
\end{aligned} \tag{4.38}$$

by commuting the blocks in the $\mathcal{P}_k$ with $K_{A_\lambda} = \mathrm{OPK}_n(e^{-x_nA_\lambda})$ resp. $T_{A_\lambda} = \mathrm{OPT}_n(e^{-x_nA_\lambda})$. For the second term we shall use that $G^\pm(\mathcal{Q}^0)$ have a somewhat similar structure as $\mathcal{G}^0$. Some elementary calculations are needed:

Lemma 4.3. *One has for $k,k'\ge 0$:*

(i) $x_n^kK_{A_\lambda} = \mathrm{OPK}_n(k!(A_\lambda+i\xi_n)^{-k-1});\quad T_{A_\lambda}x_n^k = \mathrm{OPT}_n(k!(A_\lambda-i\xi_n)^{-k-1})$.

(ii) $T_{A_\lambda}x_n^kK_{A_\lambda} = k!(2A_\lambda)^{-k-1}$.

(iii) $\mathrm{tr}_n(x_n^kK_{A_\lambda}ST_{A_\lambda}x_n^{k'}) = (k+k')!(2A_\lambda)^{-k-k'-1}S$, *if S commutes with A_λ.*

(iv) $G^\pm(\mathcal{Q}^0) = \begin{pmatrix} K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}\mathcal{S}_1^\pm\begin{pmatrix} T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix}$, *with*

$$\mathcal{S}_1^\pm = \tfrac{1}{2A_\lambda}\begin{pmatrix}\mu & \pm A_\lambda + A\\ \pm A_\lambda - A & \mu\end{pmatrix}.$$

(v) $$\begin{pmatrix} T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix}x_n^k\mathcal{Q}_+^0 = \sum_{l=0}^k \mathcal{S}_{kl}^-\begin{pmatrix} T_{A_\lambda} & 0\\ 0 & T_{A_\lambda}\end{pmatrix}x_n^{k-l},$$

(vi) $$\mathcal{Q}_+^0x_n^k\begin{pmatrix} K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix} = \sum_{l=0}^k x_n^{k-l}\begin{pmatrix} K_{A_\lambda} & 0\\ 0 & K_{A_\lambda}\end{pmatrix}\mathcal{S}_{kl}^+,$$

where the $\mathcal{S}_{kl}^\pm$ are 2×2-matrices whose entries are linear combinations of μA_λ^{-1-l}, AA_λ^{-1-l} and A_λ^{-l}.

Proof. Rule (i) follows from (1.18) and the formulas

$$(\pm i\partial_{\xi_n})^k(A_\lambda\pm i\xi_n)^{-1} = k!(A_\lambda\pm i\xi_n)^{-k-1} \tag{4.39}$$

and the fact that

$$\begin{aligned} x_n^k\,\mathrm{OPK}_n(f(\xi_n)) &= \mathrm{OPK}_n((i\partial_{\xi_n})^kf(\xi_n));\\ \mathrm{OPT}_n(f_1(\xi_n))x_n^k &= \mathrm{OPT}_n((-i\partial_{\xi_n})^kf_1(\xi_n)).\end{aligned} \tag{4.40}$$

Rule (ii) follows from:

$$T_{A_\lambda}x_n^kK_{A_\lambda} = \int_0^\infty e^{-x_nA_\lambda}x_n^ke^{-x_nA_\lambda}\,dx_n = k!(2A_\lambda)^{-k-1}.$$

Rule (iii) follows from the calculation

$$\begin{aligned}\mathrm{tr}_n(x_n^kK_{A_\lambda}ST_{A_\lambda}x_n^{k'}) &= \int_0^\infty x_n^ke^{-x_nA_\lambda}Se^{-x_nA_\lambda}x_n^{k'}\,dx_n\\ &= \int_0^\infty x_n^{k+k'}e^{-x_n2A_\lambda}S\,dx_n = (k+k')!(2A_\lambda)^{-k-k'-1}S.\end{aligned} \tag{4.41}$$

For (iv), we use that when $p(\xi_n)$ is the symbol of a ψdo P of order ≤ 0, then the symbol-kernel of $G^\pm(P)$ equals $[\mathcal{F}^{-1}_{\xi_n\to z_n}h^\pm p(\xi_n)]_{z_n=\pm(x_n+y_n)}$; cf. e.g. [G96, Th. 2.6.10]. Using this in each eigenspace of A_λ, we find in view of (4.13): $G^+(\frac{1}{A_\lambda^2+D_{x_n}^2})$ has the symbol-kernel

$$\big[\mathcal{F}^{-1}_{\xi_n\to z_n}\tfrac{1}{2A_\lambda(A_\lambda+i\xi_n)}\big]_{z_n=x_n+y_n} = \tfrac{1}{2A_\lambda}e^{-(x_n+y_n)A_\lambda};$$

$G^+(\frac{\partial_{x_n}}{A_\lambda^2+D_{x_n}^2})$ has the symbol-kernel

$$\big[\mathcal{F}^{-1}_{\xi_n\to z_n}\tfrac{-1}{2(A_\lambda+i\xi_n)}\big]_{z_n=x_n+y_n}=-\tfrac12 e^{-(x_n+y_n)A_\lambda}.$$

Similarly, $G^-(\frac{1}{A_\lambda^2+D_{x_n}^2})$ has the symbol-kernel $\frac{1}{2A_\lambda}e^{-(x_n+y_n)A_\lambda}$ and $G^-(\frac{\partial_{x_n}}{A_\lambda^2+D_{x_n}^2})$ has the symbol-kernel $\frac12 e^{-(x_n+y_n)A_\lambda}$. In other words,

$$G^\pm(\tfrac{1}{A_\lambda^2+D_{x_n}^2})=K_{A_\lambda}\tfrac{1}{2A_\lambda}T_{A_\lambda},\quad G^\pm(\tfrac{\partial_{x_n}}{A_\lambda^2+D_{x_n}^2})=\mp\tfrac12 K_{A_\lambda}T_{A_\lambda}. \tag{4.42}$$

Application of these informations to $\mathcal{Q}^0$ shows (iv).

In the proof of (v) and (vi), we need some further decompositions of rational functions (cf. also (4.13)):

$$\begin{aligned}\tfrac{1}{(\mathfrak{a}\pm i\xi_n)^k(\mathfrak{a}\mp i\xi_n)}&=\tfrac{1}{(\mathfrak{a}+i\xi_n)^{k-1}2\mathfrak{a}}\big(\tfrac{1}{\mathfrak{a}+i\xi_n}+\tfrac{1}{\mathfrak{a}-i\xi_n}\big)=\tfrac{1}{2\mathfrak{a}(\mathfrak{a}\pm i\xi_n)^k}+\tfrac{1}{2\mathfrak{a}(\mathfrak{a}\pm i\xi_n)^{k-1}(\mathfrak{a}\mp i\xi_n)}\\&=\dots=\textstyle\sum_{j=1}^k\tfrac{1}{(2\mathfrak{a})^j(\mathfrak{a}\pm i\xi_n)^{k+1-j}}+\tfrac{1}{(2\mathfrak{a})^k(\mathfrak{a}\mp i\xi_n)},\end{aligned}$$

and hence

$$\begin{aligned}h^\pm\tfrac{1}{(\mathfrak{a}\pm i\xi_n)^k(\mathfrak{a}^2+\xi_n^2)}&=h^\pm\tfrac{1}{(\mathfrak{a}\pm i\xi_n)^{k+1}(\mathfrak{a}\mp i\xi_n)}=\textstyle\sum_{j=1}^{k+1}\tfrac{1}{(2\mathfrak{a})^j(\mathfrak{a}\pm i\xi_n)^{k+2-j}},\\h^\pm\tfrac{i\xi_n}{(\mathfrak{a}\pm i\xi_n)^k(\mathfrak{a}^2+\xi_n^2)}&=h^\pm\Big[\tfrac{1}{(\mathfrak{a}\pm i\xi_n)^k}\tfrac12\big(-\tfrac{1}{\mathfrak{a}+i\xi_n}+\tfrac{1}{\mathfrak{a}-i\xi_n}\big)\Big]\\&=\mp\tfrac{1}{2(\mathfrak{a}\pm i\xi_n)^{k+1}}\pm\textstyle\sum_{j=1}^k\tfrac{1}{2(2\mathfrak{a})^j(\mathfrak{a}\pm i\xi_n)^{k+1-j}}.\end{aligned}\tag{4.43}$$

We then get by use of (i) and the rules of calculus (cf. e.g. [G96, Th. 2.6.1]):

$$\begin{aligned}T_{A_\lambda}x_n^k(A_\lambda^2+D_{x_n}^2)_+^{-1}&=\mathrm{OPT}_n\big(h^-\big(\tfrac{k!}{(A_\lambda-i\xi_n)^{k+1}}\tfrac{1}{A_\lambda^2+\xi_n^2}\big)\big)\\&=\mathrm{OPT}_n\big(\textstyle\sum_{j=1}^{k+1}\tfrac{k!}{(2A_\lambda)^j(A_\lambda-i\xi_n)^{k+2-j}}\big)=\textstyle\sum_{j=1}^{k+1}\tfrac{k!}{(k+1-j)!(2A_\lambda)^j}T_{A_\lambda}x_n^{k+1-j},\\(A_\lambda^2+D_{x_n}^2)_+^{-1}x_n^kK_{A_\lambda}&=\mathrm{OPK}_n\big(h^+\big(\tfrac{1}{A_\lambda^2+\xi_n^2}\tfrac{k!}{(A_\lambda+i\xi_n)^{k+1}}\big)\big)\\&=\textstyle\sum_{j=1}^{k+1}\tfrac{k!}{(k+1-j)!}x_n^{k+1-j}K_{A_\lambda}\tfrac{1}{(2A_\lambda)^j},\end{aligned}\tag{4.44}$$

and similarly

$$\begin{aligned}T_{A_\lambda}x_n^k\partial_{x_n}(A_\lambda^2+D_{x_n}^2)_+^{-1}&=\mathrm{OPT}_n\big(\tfrac{1}{2(A_\lambda-i\xi_n)^{k+1}}-\textstyle\sum_{j=1}^k\tfrac{1}{2(2A_\lambda)^j(A_\lambda-i\xi_n)^{k+1-j}}\big)\\&=\tfrac{k!}{2}T_{A_\lambda}x_n^k-\textstyle\sum_{j=1}^k\tfrac{k!}{2(k-j)!(2A_\lambda)^j}T_{A_\lambda}x_n^{k-j},\\\partial_{x_n}(A_\lambda^2+D_{x_n}^2)_+^{-1}x_n^kK_{A_\lambda}&=\mathrm{OPK}_n\big(-\tfrac{1}{2(A_\lambda+i\xi_n)^{k+1}}+\textstyle\sum_{j=1}^k\tfrac{1}{2(2A_\lambda)^j(A_\lambda+i\xi_n)^{k+1-j}}\big)\\&=-\tfrac{k!}{2}x_n^kK_{A_\lambda}+\textstyle\sum_{j=1}^k\tfrac{k!}{2(k-j)!}x_n^{k-j}K_{A_\lambda}\tfrac{1}{(2A_\lambda)^j}.\end{aligned}\tag{4.45}$$

Application to the blocks in $\mathcal{Q}^0$ give formulas (v) and (vi), with the asserted structure of the $\mathcal{S}^\pm_{kl}$. □

Remark 4.4. There are similar results with $\mathcal{Q}^0$ replaced by $(\mathcal{Q}^0)^r$. The powers contain factors $(D_{x_n}^2+A^2+\mu^2)^{-j}$ with higher j which, in the application of h^+ and h^- as in (4.13), (4.43) lead to fractions with both $A_\lambda+i\xi_n$ and $A_\lambda-i\xi_n$ in higher powers in the denominator. Again one decomposes into simple fractions so that the numerators are independent of ξ_n, which leads to formulas generalizing (iv)–(vi) and of a similar form (with μ and A appearing in higher powers).

Using (iv), we can write, similarly to (4.38):

$$G^+(\mathcal{Q}^0)\sum_{0\le k\le K} x_n^k\mathcal{P}_k=\sum_{0\le k\le K}\begin{pmatrix}K_{A_\lambda}&0\\0&K_{A_\lambda}\end{pmatrix}\mathcal{S}_1^+\mathcal{P}_k\begin{pmatrix}T_{A_\lambda}&0\\0&T_{A_\lambda}\end{pmatrix}x_n^k, \tag{4.46}$$

For simplicity, *we write from now on the diagonal block matrices formed of K_{A_λ} or T_{A_λ} as simple factors,* meaning that they are composed with each block (as already done with e.g. $1/(2A_\lambda)$); this should not lead to confusion. The s.g.o. terms in (4.33) can now be calculated:

Proposition 4.5. *The singular Green part $\mathcal{G}^0_{(1)}$ of $\mathcal{R}^0\mathcal{P}\mathcal{R}^0$ is a a block matrix, cf.* (4.52),

$$\mathcal{G}^0_{(1)}=\begin{pmatrix}\mathcal{G}^0_{(1),11}&\mathcal{G}^0_{(1),12}\\\mathcal{G}^0_{(1),21}&\mathcal{G}^0_{(1),22}\end{pmatrix}, \tag{4.47}$$

where each block is a sum of terms

$$x_n^{k'}K_{A_\lambda}S\,T_{A_\lambda}x_n^{k''}, \tag{4.48}$$

with S of the form

$$\text{(a) } (-\lambda)^{l/2}L_mA_\lambda^{-j}\tfrac{A}{|A'|},\quad \text{(b) } (-\lambda)^{l/2}L_mA_\lambda^{-j}\ \text{ or }\ \text{(c) } (-\lambda)^{l/2}L_m\Pi_0; \tag{4.49}$$

here $l\in\mathbb{Z}$, m and $j\in\mathbb{N}$, and L_m denotes a (λ-independent) differential operator of order m commuting with A. Consequently, the normal trace is a block matrix, cf. (4.54),

$$\operatorname{tr}_n\mathcal{G}^0_{(1)}=S_{(1)}=\begin{pmatrix}S_{(1),11}&S_{(1),12}\\S_{(1),21}&S_{(1),22}\end{pmatrix}, \tag{4.50}$$

where each block is a linear combination of terms of the form (4.49).

Proof. We have for the term $G^+(\mathcal{Q}^0)\mathcal{P}G^-(\mathcal{Q}^0)$:

$$\begin{aligned}G^+(\mathcal{Q}^0)\mathcal{P}G^-(\mathcal{Q}^0)&=K_{A_\lambda}\mathcal{S}_1^+T_{A_\lambda}\sum_{0\le k\le K}x_n^k\mathcal{P}_kK_{A_\lambda}\mathcal{S}_1^-T_{A_\lambda}\\&=K_{A_\lambda}\sum_{0\le k\le K}k!(2A_\lambda)^{-k-1}\mathcal{S}_1^+\mathcal{P}_k\mathcal{S}_1^-T_{A_\lambda},\end{aligned} \tag{4.51}$$

where we used Lemma 4.3 (iv), (4.46) and Lemma 4.3 (ii). Treating $\mathcal{G}^0\mathcal{P}\mathcal{G}^0$ in the same way, using (4.38), we get a similar expression with $\mathcal{S}_0$ instead of $\mathcal{S}_1^\pm$. For the last two terms $\mathcal{G}^0\mathcal{P}\mathcal{Q}^0_+$ and $\mathcal{Q}^0_+\mathcal{P}\mathcal{G}^0$, we moreover use (v) and (vi) in Lemma 4.3, finding e.g.

$$\mathcal{G}^0\mathcal{P}\mathcal{Q}^0_+=\sum_{0\le k\le K}K_{A_\lambda}\mathcal{S}_0\mathcal{P}_kT_{A_\lambda}x_n^k\mathcal{Q}^0_+=\sum_{0\le k\le K}K_{A_\lambda}\mathcal{S}_0\mathcal{P}_k\sum_{0\le l\le k}S^-_{kl}T_{A_\lambda}x_n^{k-l}.$$

This gives:

$$\begin{aligned}\mathcal{G}^0_{(1)} &= -G^+(\mathcal{Q}^0)\mathcal{P}\,G^-(\mathcal{Q}^0) + \mathcal{Q}^0_+\mathcal{P}\mathcal{G}^0 + \mathcal{G}^0\mathcal{P}\mathcal{Q}^0_+ + \mathcal{G}^0\mathcal{P}\mathcal{G}^0 \\ &= K_{A_\lambda}\sum_{0\le k\le K} k!(2A_\lambda)^{-k-1}(-\mathcal{S}_1^+\mathcal{P}_k\mathcal{S}_1^- + \mathcal{S}_0\mathcal{P}_k\mathcal{S}_0)T_{A_\lambda} \\ &\quad + \sum_{0\le k\le K}\sum_{0\le l\le k}[K_{A_\lambda}\mathcal{S}_0\mathcal{P}_k\mathcal{S}^-_{kl}T_{A_\lambda}x_n^{k-l} + x_n^{k-l}K_{A_\lambda}\mathcal{S}^+_{kl}\mathcal{P}_k\mathcal{S}_0T_{A_\lambda}].\end{aligned} \tag{4.52}$$

This shows the general structure (4.48) of the terms in the blocks, and we shall now show the additional information given in (4.49).

The blocks in $\mathcal{S}_1^\pm$ and the $\mathcal{S}^\pm_{kl}$ are linear combination of terms as in (4.49)(b). $\mathcal{S}_0$ moreover contains terms of the form

$$(-\lambda)^{l/2}L_mA_\lambda^{-j}\Pi_\ge \tag{4.53}$$

(we insert $\Pi_< = I-\Pi_\ge$, use the reductions in (4.19) and absorb powers of A in L_m, noting that since $A_\lambda = (A^2-\lambda)A_\lambda^{-1}$, A_λ need only occur explicitly in negative powers). Then in the resulting matrices, we get linear combinations of terms as in (4.49)(b) and (4.53), by moving the differential operators P_{1k} out in front in each block by commutation. To the terms of the form (4.53) we apply (1.5), which leads to

$$(-\lambda)^{l/2}L_mA_\lambda^{-j}\Pi_\ge = \tfrac12(-\lambda)^{l/2}L_mA_\lambda^{-j}(\tfrac{A}{|A'|} + I + \Pi_0),$$

giving the three types in (4.49).

(4.50)ff. follows by application of Lemma 4.3 (iii) to each block. More precisely, this gives

$$\begin{aligned}\operatorname{tr}_n\mathcal{G}^0_{(1)} &= \sum_{0\le k\le K} k!(2A_\lambda)^{-k-2}(-\mathcal{S}_1^+\mathcal{P}_k\mathcal{S}_1^- + \mathcal{S}_0\mathcal{P}_k\mathcal{S}_0) \\ &\quad + \sum_{0\le k\le K}\sum_{0\le l\le k}(k-l)!(2A_\lambda)^{-k+l-1}(\mathcal{S}_0\mathcal{P}_k\mathcal{S}^-_{kl} + \mathcal{S}^+_{kl}\mathcal{P}_k\mathcal{S}_0). \quad \square\end{aligned} \tag{4.54}$$

Note that although $\mathcal{P}$ itself does not commute with $\mathcal{S}_0$, $\mathcal{S}_1^\pm$, etc., we obtained the result by commutation in each block.

This shows the first step in

Theorem 4.6. *For any $M\ge 0$, the s.g.o. part $\mathcal{G}^0_M$ of $\mathcal{R}^0_M$ is of the form*

$$\mathcal{G}^0_M = \begin{pmatrix}\mathcal{G}^0_{M,11} & \mathcal{G}^0_{M,12}\\ \mathcal{G}^0_{M,21} & \mathcal{G}^0_{M,22}\end{pmatrix}, \tag{4.55}$$

where the blocks have the structure in (4.48)–(4.49).

Proof. We already have this structure for the s.g.o. parts of $\mathcal{R}^0$ and $\mathcal{R}^0\mathcal{P}\mathcal{R}^0$. Now consider $\mathcal{R}^0(\mathcal{P}\mathcal{R}^0)^m$. Again we depart from the exact formulas for $\mathcal{R}^0 = \mathcal{Q}^0_+ + \mathcal{G}^0$ given in (4.9) and (4.17)–(4.18), as in the proof of Proposition 4.5, using the description in Lemma 4.3

of the effects of multiplication by x_n^k. We moreover use the formulas for higher powers, as described in Remark 4.4. Then we find the structure in (4.48)–(4.49) also for the m'th term and the result for $\mathcal{G}_M^0$ follows by summation. $\square$

Similar results can be shown for powers of the resolvent and for powers of the blocks in the resolvent.

We now pass to the consequences for the original operators (4.3) *with general* σ, by composing suitably with $\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$ and its inverse. This gives for the resolvent $(\Delta_B - \lambda)^{-1}$:

Theorem 4.7. *Under the assumption of* (4.2), *the resolvent* R_λ *of* $\Delta_B = D_\geq^* D_\geq$ *satisfies, for any* M, *any* $r \geq 1$:

$$R_\lambda^r = Q_{\lambda,+}^r + \zeta_\varepsilon G_{M,r} \zeta_\varepsilon + G'_{M,r}, \tag{4.56}$$

where $G_{M,r}$ *is a finite sum of terms as in* (4.48)–(4.49) *and* $G'_{M,r}$ *has symbol-kernel in* $\mathcal{S}^{0,-r,-M-r-2}(\Gamma, \mathcal{S}_{++})$, *in local trivializations.*

Proof. As usual, $\lambda = -\mu^2$. We have on X_ε, in view of (3.2) and (4.3),

$$\begin{aligned} R_{-\mu^2} \sim \mu^{-1} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \mathcal{R} \begin{pmatrix} 1 & 0 \\ 0 & \sigma^* \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ = \mu^{-1} \mathcal{R}_{11} = \mu^{-1} (\mathcal{Q}_{11,+} + \mathcal{G}^0_{M,11} + \mathcal{G}'_{M,11}), \end{aligned} \tag{4.57}$$

where $\mathcal{G}^0_{M,11}$ has the structure described in Theorem 4.6 and $\mathcal{G}'_{M,11}$ has symbol-kernel in $\mathcal{S}^{0,0,-M-3}(\Gamma, \mathcal{S}_{++})$. Here $\mu^{-1}\mathcal{Q}_{11} = Q_\lambda$, cf. (3.2). This implies the statement for $r = 1$ by a couple of applications of Lemma 2.9; on one hand it allows the multiplication by ζ_ε, on the other hand it allows extending the structure from X_ε to X.

For the higher powers, note that by (4.29),

$$\mathcal{Q} = \mathcal{Q}_M + \mathcal{Q}'_M, \text{ where } \mathcal{Q}'_M = \mathcal{Q}^0 \sum_{m>M} (-\zeta_\varepsilon \mathcal{P} \mathcal{Q}^0)^m = \mathcal{Q}(-\zeta_\varepsilon \mathcal{P} \mathcal{Q}^0)^{M+1}. \tag{4.58}$$

Consider the r'th power (on X_ε)

$$R_\lambda^r \sim [\mu^{-1}(\mathcal{Q}_{11,+} + \mathcal{G}_{11})]^r = \mu^{-r}(\mathcal{Q}_{M,11,+} + \mathcal{Q}'_{M,11,+} + \mathcal{G}^0_{M,11} + \mathcal{G}'_{M,11})^r. \tag{4.59}$$

The s.g.o. part of R_λ^r comes partly from compositions containing $\mathcal{G}_{11}$, partly from "leftover" contributions from the ψdo terms (as in Theorem 2.7 (xiv)). The result is obtained by using again the exact formulas given above for the terms in the sums over m and the information on the remainders (for $\mathcal{Q}'_M$ it is the fact that it contains $M+1$ factors $\mathcal{P}$), in a similar way as in the preceding proofs. $\square$

Remark 4.8. It would also have been possible to get this result — and even a slightly better one with the s.g.o. remainder symbol-kernel in $\mathcal{S}^{0,0,-M-2r-2}(\Gamma, \mathcal{S}_{++})$ — by departing from the formulas for R_λ in [G92], but the exact rules that would have to be worked out, would be even more complicated, since the difference between the second order operators D^*D and $D^{0\prime}D^0$ contains many more terms, including some of the form $\partial_{x_n} P$ and $x_n \partial_{x_n} P$, and the two second-order realizations have different boundary conditions when $P_{10} \neq 0$.

Remark 4.9. The other resolvent $(D_\geq D_\geq^* - \lambda)^{-1}$ has a similar form on X_1, except that it is composed with σ to the left and σ^* to the right.

5. Trace results in the commuting case.

In this section, we continue the study of the operator families defined from $\Delta_B = D_\geq^* D_\geq$. For the purpose of analyzing the traces, we introduce a notation for symbols with alternating parity in the symbol expansion:

Definition 5.1. *Let $p(x,\xi) \sim \sum_{l\geq 0} p_{m-l}(x,\xi)$ be a classical ψdo symbol of integer order m, expanded in terms p_{m-l} that are homogeneous of degree $m-l$ for $|\xi|\geq 1$.*

We say that p has **even-even alternating parity**, *when the terms p_{m-l} of even degree $m-l$ are even in ξ, and the terms p_{m-l} of odd degree $m-l$ are odd in ξ, i.e.,*

$$p_{m-l}(x,-\xi) = (-1)^{m-l} p_{m-l}(x,\xi), \text{ for all } l, \tag{5.1}$$

and the same holds for derivatives of p.

We say that p has **even-odd alternating parity**, *when the terms p_{m-l} of even degree $m-l$ are odd in ξ, and the terms p_{m-l} of odd degree $m-l$ are even in ξ, i.e.,*

$$p_{m-l}(x,-\xi) = (-1)^{m-l+1} p_{m-l}(x,\xi), \text{ for all } l, \tag{5.2}$$

and the same holds for derivatives of p.

Note that symbols of differential operators, and parametrix symbols for elliptic differential operators, have even-even alternating parity. On the other hand, the symbol of $|A|$ has even-odd alternating parity; this can be checked on the basis of the formulas for the symbol of $(A^2)^{\frac{1}{2}}$ given in Seeley [S67]. Then also $\frac{A}{|A'|}$ has even-odd alternating parity. (Such alternating parity properties were also observed in [GS96, (3.9)ff.].)

Theorem 5.2. *Let S be a μ-dependent ψdo on X' with polyhomogeneous symbol in $S^{m,d,s}(\Gamma)$ ($s\leq 0$) in local trivializations, holomorphic in μ, and such that the homogeneous terms of degree $m+s+d-j$ with $m+s>-n$ are integrable in ξ'. Assume moreover that S is a finite sum of terms S_i, $i=1,\dots,i_3$, of the form*

$$S_i = \begin{cases} (-\lambda)^{l_i/2} L_{m_i} A_\lambda^{-j_i} \frac{A}{|A'|} \text{ for } i = 1,\dots,i_1, \\ (-\lambda)^{l_i/2} L_{m_i} A_\lambda^{-j_i} \text{ for } i = i_1+1,\dots,i_2, \\ (-\lambda)^{l_i/2} L_{m_i} \Pi_0 \text{ for } i = i_2+1,\dots,i_3, \end{cases} \tag{5.3}$$

where $l_i\in\mathbb{Z}$, m_i and $j_i\in\mathbb{N}$, and L_{m_i} is a (λ-independent) differential operator of order m_i. Then in the asymptotic expansion

$$\operatorname{Tr}_{X'} S \sim \sum_{j\in\mathbb{N}} c_j(-\lambda)^{(m+d+s+n-1-j)/2} + \sum_{k\in\mathbb{N}} (c'_k \log(-\lambda) + c''_k)(-\lambda)^{d+s-k/2}, \tag{5.4}$$

the logarithmic terms $c'_k(-\lambda)^t \log(-\lambda)$ with integer t come from the terms S_i with $l_i - j_i$ **even**, *$i\leq i_1$, and the logarithmic terms with noninteger t ($t-\frac{1}{2}$ integer) come from the terms S_i with l_i-j_i* **odd**, *$i\leq i_1$. The logarithmic terms coming from each such S_i have the powers $t=(l_i-j_i)/2-\nu$, $\nu=0,1,2,\dots$ (when $j_i=0$, there is only the power $l_i/2$).*

Moreover, if n is odd, all the logarithmic terms are zero.

Proof. Consider first the terms with $i > i_2$, the last type in (5.3). Such a term has smooth finite dimensional range (in particular, it is of order $-\infty$) and contributes a constant $\operatorname{Tr}_{X'}(L_{m_i}\Pi_0)$ times $(-\lambda)^{l_i/2}$. We start by subtracting these terms from the expression to be analyzed, which leaves us with a decomposition in terms of the first two types.

Note that *we have not assumed the first two types of terms in* (5.3) *to be trace-class operators individually,* and that they need not be turned into trace-class operators by differentiatiation of high order in λ (because in the Leibniz formula, some differentiations would fall on $(-\lambda)^{l_i/2}$, others on $A_\lambda^{-j_i}$).

However, as recalled in the proof of Theorem 2.10, the regions $\{|\xi'| \ge |\lambda|^{\frac12}\}$ and $\{|\xi'| \le 1\}$ give pure powers; it is the region $\{1 \le |\xi'| \le |\lambda|^{\frac12}\}$ that may contribute with log-power terms. Only in this region will the decomposition be used; it corresponds to a similar decomposition for the symbols of the operators. The point is now that although the individual terms here need not be of sufficiently low order to allow integration over $\mathbb{R}^{n-1}$, we can certainly integrate them over $\{1 \le |\xi'| \le |\lambda|^{\frac12}\}$.

The terms of with $i_1 < i \le i_2$ in (5.3) are of the form of a power of $-\lambda$ times a strongly polyhomogeneous operator; their symbols will contribute pure powers (since they obviously do so when integrated over $\{0 \le |\xi'| \le |\lambda|^{\frac12}\}$, and the region $\{|\xi'| \le 1\}$ gives only pure powers).

It is the symbols with $i \le i_1$ in (5.3) that may contribute logarithmic terms. We drop the index i in the following.

Consider such a term $(-\lambda)^{l/2} L_m A_\lambda^{-j} \frac{A}{|A'|}$. We may write it:

$$(-\lambda)^{\frac{l}{2}} L_m A_\lambda^{-j} \tfrac{A}{|A'|} = \varrho^{\frac{j-l}{2}} L_m (\varrho A^2 + 1)^{-\frac{j}{2}} \tfrac{A}{|A'|}, \quad \varrho = -\lambda^{-1}. \tag{5.5}$$

If $j > 0$, we insert a power series expansion of the $-\frac{j}{2}$-power,

$$\varrho^{\frac{j-l}{2}} L_m (\varrho A^2 + 1)^{-\frac{j}{2}} \tfrac{A}{|A'|} \sim \varrho^{\frac{j-l}{2}} L_m \sum_{\nu\ge 0} \tbinom{-\frac{j}{2}}{\nu} \varrho^\nu A^{2\nu} \tfrac{A}{|A'|}. \tag{5.6}$$

From this formula for the full operators, we can also find the structure of the symbol by inserting the polyhomogeneous symbol expansions and carry out symbol compositions. (This type of expansion is somewhat like the Taylor expansion in [GS95, Th. 1.12]; in the present case the order of the coefficient of ϱ^ν increases by 2 when ν increases by 1. Systematic calculi with such features are worked out in Loya [L01] and [GH02].)

Let $B^{m,\nu} = \binom{-\frac{j}{2}}{\nu} L_m A^{2\nu} \frac{A}{|A'|}$, it is of order $m + 2\nu$ and its symbol $b^{m,\nu}(x', \xi')$ has an expansion

$$b^{m,\nu}(x', \xi') \sim \sum_{\nu'\ge 0} b^{m,\nu}_{m+2\nu-\nu'}(x', \xi'), \tag{5.7}$$

where $b^{m,\nu}_{m+2\nu-\nu'}$ is homogeneous of degree $m + 2\nu - \nu'$ in ξ'. It suffices to consider $\lambda \in \mathbb{R}_-$ (the results extend analytically to other λ in view of [GS95, Lemma 2.3]). Since ξ' runs in dimension $n-1$,

$$\int_{1\le|\xi'|\le|\lambda|^{\frac12}} b^{m,\nu}_{m+2\nu-\nu'}(x',\xi')\,đ\xi'$$

(5.8)
$$= (2\pi)^{1-n}\int_1^{|\lambda|^{\frac12}} r^{m+2\nu-\nu'+n-2}\,dr\int_{|\xi'|=1} b^{m,\nu}_{m+2\nu-\nu'}(x',\xi')\,d\sigma(\xi')$$

$$= \begin{cases} c_{\nu'}(x')(|\lambda|^{m+2\nu-\nu'+n-1}-1) & \text{if } m+2\nu-\nu'+n-1\neq 0,\\ c_{\nu'}(x')\log|\lambda| & \text{if } m+2\nu-\nu'+n-1=0,\end{cases}$$

where

(5.9)
$$c_{\nu'}(x') = c\int_{|\xi'|=1} b^{m,\nu}_{m+2\nu-\nu'}(x',\xi')\,d\sigma(\xi'),$$

with a nonzero constant c (depending on $m+2\nu-\nu'+n-1$). The case of the logarithm occurs when $\nu' = m+2\nu+n-1$, and then

(5.10)
$$c_{m+2\nu+n-1}(x') = c\int_{|\xi'|=1} b^{m,\nu}_{1-n}(x',\xi')\,d\sigma(\xi').$$

We see that each term in the expansion (5.7) contributes with one logarithmic term, and that it is proportional to $(-\lambda)^{\frac{l-j}{2}-\nu}\log(-\lambda)c_{m+2\nu+n-1}(x')$. The power is integer resp. integer$+\frac12$ exactly when $(l-j)/2$ is integer resp. integer$+\frac12$, i.e., when $l-j$ is even resp. odd. The highest order log-term comes from the case $\nu=0$ and is of the form

(5.11)
$$c(x')(-\lambda)^{\frac{l-j}{2}}\log(-\lambda).$$

If $j=0$ in (5.5), we have just one term $(-\lambda)^{\frac{l}{2}}L_m\frac{A}{|A'|}$ to analyze. Studying $L_m\frac{A}{|A'|}$ as we did with $B^{m,\nu}$, we find a single logarithmic contribution (from the term of degree $1-n$ in the symbol)

(5.12)
$$c(x')(-\lambda)^{\frac{l}{2}}\log(-\lambda).$$

This ends the proof of the general assertion on the contributions from the S_i.

The information can be sharpened further by considering the parity of the terms in (5.3). Here the symbols of L_m and $A^{2\nu}$ have even-even alternating parity, whereas the symbol of $\frac{A}{|A'|}$ has even-odd alternating parity (cf. Definition 5.1ff.), so the symbol $b^{m,\nu}$ of the composition $B^{m,\nu}=L_mA^{2\nu}\frac{A}{|A'|}$ has even-odd alternating parity. Since the integral over the sphere $\{|\xi'|=1\}$ of an odd function vanishes, and $b^{m,\nu}_{1-n}$ is odd in ξ' when n is odd, we conclude that

(5.13)
$$c_{m+2\nu+n-1}(x') = 0 \text{ if } n \text{ is odd.}$$

So when n is odd, there are no logarithmic contributions at all! $\square$

This leads to:

Theorem 5.3. *Let D be a perturbation of D^0 as in (1.1) near X' such that P_0 and all terms in the Taylor expansion of P_1 in x_n commute with A. Let F be a differential operator in E_1 of order m', and let $r+1 > \frac{n+m'}{2}$. If n is odd, the resolvent and heat operator, resp. zeta function, associated with Δ_B have trace expansions without logarithms, resp. meromorphic extensions without double poles:*

$$\begin{aligned} \operatorname{Tr}(F\partial_\lambda^r(\Delta_B-\lambda)^{-1}) &\sim \sum_{-n\le k<\infty} \tilde a_k(-\lambda)^{\frac{m'-k}{2}-r-1}, \\ \operatorname{Tr}(Fe^{-t\Delta_B}) &\sim \sum_{-n\le k<\infty} a_k t^{\frac{k-m'}{2}}, \\ \Gamma(s)\zeta(F,\Delta_B,s) \equiv \Gamma(s)\operatorname{Tr}(F\Delta_B^{-s}) &\sim \sum_{-n\le k<\infty} \frac{a_k}{s+\frac{k-m'}{2}} - \frac{\operatorname{Tr}(F\Pi_0(\Delta_B))}{s}, \end{aligned} \tag{5.14}$$

where the coefficients are locally determined for $-n\le k<0$ (for $-n\le k<m'$ if F is tangential. Here $\tilde a_{-n}$ and a_{-n} vanish if m' is odd.

Proof. Recall (1.21). We have from Theorem 3.3 that there is an expansion

$$\operatorname{Tr}(F\partial_\lambda^r(\Delta_B-\lambda)^{-1}) \sim \sum_{-n\le k<0} \tilde c_k(-\lambda)^{\frac{m'-k}{2}-r-1} + \sum_{k\ge 0}\big(\tilde c_k' \log(-\lambda) + \tilde c_k''\big)(-\lambda)^{\frac{m'-k}{2}-r-1}, \tag{5.15}$$

where the $\tilde c_k$ and $\tilde c_k'$ are locally determined (and $\tilde c_{-n}$ vanishes if m' is odd); we have to show that all the $\tilde c_k'$ vanish. For any K we can expand x_nP_1 as in (4.1), and since we know from Theorem 3.5 that removing the remainder can only affect the logarithmic terms with $k\ge K$, this reduces the problem to the case where D is as in (4.2). We apply Theorem 4.7. The ψdo $Q^r_{\lambda,+}$ gives no log-terms. The s.g.o. $G'_{M,r}$ contributes with a trace expansion that is $O(\langle\lambda\rangle^{-M'})$, where we can get M' as large as we want by taking M large. Finally, $\operatorname{Tr}_X(\zeta_\varepsilon G_{M,r}\zeta_\varepsilon) = \operatorname{Tr}_{X^0} G_{M,r} + O(\langle\lambda\rangle^{-N})$, any N, where $G_{M,r}$ is a finite sum of terms with structure as described in (4.48)–(4.49). Let us decompose and Taylor expand F:

$$\begin{aligned} F &= \sum_{m=0}^{m'} F_m(x,D_{x'})\partial_{x_n}^m = F_{(l_0)} + x_n^{l_0+1}F'_{(l_0)}, \text{ where} \\ F_{(l_0)} &= \sum_{m=0}^{m'}\sum_{l=0}^{l_0} x_n^l F_{m,l}(x',D_{x'})\partial_{x_n}^m, \end{aligned}$$

and $F'_{(l_0)}$ is of order m'. As usual, the trace resulting from the remainder $x_n^{l_0+1}F'_{(l_0)}$ has log-powers beginning at an index that goes to ∞ when $l_0\to\infty$. In the finite sum, the $x_n^l\partial_{x_n}^m$ applied to $x_n^kK_{A_\lambda}$ give other linear combinations of terms $x_n^{k'}K_{A_\lambda}$, so when we take the normal trace of $F_{(l_0)}G_{M,r}$, we get a finite sum of terms as in (4.49). By Theorem 5.2, they contribute no logarithmic terms. Thus all the $\tilde c_k'$ in (5.15) are zero.

The nonlocal coefficients start in general at the power $(-\lambda)^{\frac{m'}{2}-r-1}$, but when F is tangential, they start at the power $(-\lambda)^{-r-1}$ (cf. Theorem 3.3, also for the statement on $\tilde a_{-n}$).

This carries over to a heat trace expansion and a zeta function expansion by the transitions explained e.g. in [GS96], cf. Corollary 3.7 above. □

In particular, this extends qualitatively the result of [GS96] on the product case to trace expansions where the x_n-independent morphism φ^0 considered there is replaced by an arbitrary differential operator F.

Taking in particular $F = F_1\psi D$, where ψ is a morphism from E_2 to E_1, and F_1 is a differential operator in E_1, we extend the results to eta-expansions (using that $(\Delta_B - \lambda)^{-r-1}$ maps into the domain of $D_\geq$, and D vanishes on $V_0(\Delta_B)$):

Corollary 5.4. *Assumptions on D as in Theorem 5.3. Let ψ be a morphism from E_2 to E_1 and F_1 a differential operator in E_1 of order m', and let $r+1 > \frac{n+m'+1}{2}$. If n is odd, there are expansions without logarithms resp. double poles:*

$$\begin{aligned} \operatorname{Tr}(F_1\psi D_\geq \partial_\lambda^r(\Delta_B-\lambda)^{-1}) &\sim \sum_{-n\le k<\infty} \tilde b_k(-\lambda)^{\frac{m'+1-k}{2}-r-1}, \\ \operatorname{Tr}(F_1\psi D_\geq e^{-t\Delta_B}) &\sim \sum_{-n\le k<\infty} b_k t^{\frac{k-m'-1}{2}}, \\ \Gamma(s)\eta(F_1\psi, D_\geq, 2s-1) \equiv \Gamma(s)\operatorname{Tr}(F_1\psi D_\geq \Delta_B^{-s}) &\sim \sum_{-n\le k<\infty} \frac{b_k}{s+\frac{k-m'-1}{2}}. \end{aligned} \tag{5.16}$$

The coefficients b_k are locally determined for $-n \le k < 0$ (for $-n \le k < m'$ if F_1 is tangential, cf. Theorem 2.10). The coefficients $\tilde b_{-n}$ and b_{-n} vanish if m' is even.

Let us also make some observations on the case where n is even. One can ask whether the expansions in general will have logarithms at both integer and half-integer powers for $k>0$. As shown in [GS96], this is not so for the product case with a factor φ^0, where the terms with k even >0 vanish. The result in [GS96] was based on explicit trace expansions of the zeta and eta functions of A; we can now show by our qualitative arguments that the result extends structurally to tangential x_n-independent factors F.

Theorem 5.5. *Let D be of product type and let F be a differential operator of order m'. Let $r+1 > \frac{n+m'}{2}$. When n is odd, the trace expansions are without logarithmic terms as in (5.14). When n is even and F is tangential and x_n-independent near X', then the trace expansions have logarithms only at the "zero'th" power and at subsequent half-integer powers (resp. double poles only at zero and at negative half-integers):*

$$\begin{aligned} \operatorname{Tr}(F\partial_\lambda^r(\Delta_B-\lambda)^{-1}) \sim{}& \sum_{-n\le k<\infty} \tilde a_k(-\lambda)^{\frac{m'-k}{2}-r-1} + \tilde c_0(-\lambda)^{-r-1}\log(-\lambda) \\ &+ \sum_{j\ge 0} \tilde c_{2j+1}(-\lambda)^{-j-\frac12-r-1}\log(-\lambda), \\ \operatorname{Tr}(Fe^{-t\Delta_B}) \sim{}& \sum_{-n\le k<\infty} a_k t^{\frac{k-m'}{2}} + c_0\log t + \sum_{j\ge0} c_{2j+1}t^{j+\frac12}\log t, \\ \Gamma(s)\operatorname{Tr}(F\Delta_B^{-s}) \sim{}& \sum_{-n\le k<\infty}\frac{a_k}{s+\frac{k-m'}{2}} + \frac{-c_0}{s^2} - \frac{\operatorname{Tr}(F\Pi_0(\Delta_B))}{s} + \sum_{j\ge0}\frac{-c_{2j+1}}{(s+j+\frac12)^2}. \end{aligned} \tag{5.17}$$

The following coefficients are locally determined: $\tilde a_k$ *and* a_k *for* $-n\le k<m'$, $\tilde c_0$ *and* c_0, $\tilde c_{2j+1}$ *and* c_{2j+1} *for* $j\ge 0$.

Proof. The statement for n odd has already been shown in Theorem 5.3, so let n be even and let F be tangential and x_n-independent near X'. We have from Theorem 3.3 that there is an expansion (3.6) with $\mu=(-\lambda)^{\frac12}$, so the point is now to show that only certain log-terms appear. Since we are in the product case, the normal trace of the s.g.o. part of $F\partial_\lambda^r R_\lambda^0$ is (modulo contributions whose traces are $O(\langle\lambda\rangle^{-N})$ for any N):

$$\operatorname{tr}_n F\partial_\lambda^r G_\lambda^0 = F\partial_\lambda^r\big(\tfrac{-1}{4\lambda}\big[\tfrac{A^2}{A_\lambda^2}+\tfrac{A}{A_\lambda}\big]+\tfrac{A}{4\lambda A_\lambda}\tfrac{A}{|A'|}+\tfrac{A}{4\lambda|A'|}+\tfrac{1}{4\lambda}\Pi_0\big), \tag{5.18}$$

as can be deduced from [GS96, (3.9)]. We appeal to Theorem 5.2 and its proof. In (5.18), the strongly polyhomogeneous terms in [...] and the term with Π_0 give no logs. The term

$$F\partial_\lambda^r \tfrac{A}{4\lambda A_\lambda}\tfrac{A}{|A'|} \tag{5.19}$$

gives, when ∂_λ^r is carried out, a linear combination of terms (5.5) with $l-j$ odd (equal to $-3-2r$), hence it gives log-power terms at half-integers, $c(x')(-\lambda)^{-\frac32-r-\nu}\log(-\lambda)$, $\nu=0,1,2,\dots$, besides pure power terms. The term

$$F\partial_\lambda^r\frac{A}{4\lambda|A'|} \tag{5.20}$$

has the form with $j=0$ analyzed in the proof, so it gives rise to one logarithmic term $c(x')(-\lambda)^{-r-1}\log(-\lambda)$ (as in (5.12)) besides pure power terms.

This shows the first formula in (5.17), written with a different enumeration convention than in (3.6) for the logarithmic terms. The other formulas follow as in Corollary 3.7. ☐

This extends the result in [GS96]. One can analyze the eta function in a similar way.

We recall from [G92], [GS96] that when $F=1$, the log-term at the power $-r-1$ (resp. at the power zero in the heat expansion) vanishes. Note that the log-terms at the other powers stem from one single log-producing term (5.19) that gives *odd* values of $l-j$. If F is allowed to depend on x_n, the powers of x_n in its Taylor expansion will give rise to terms like (5.19) multiplied by negative powers of A_λ, and terms like (5.20) multiplied with negative powers of A_λ (cf. Lemma 4.3 (iii)). Then both even and odd negative powers of A_λ will occur, giving series of log-terms both with half-integer and with integer powers. — Also the effect of normal derivatives in F can be discussed in this way.

Now let us turn to non-product cases. Here, even when $F=1$, one can expect many integer and half-integer log-terms, as described in the following remark.

Remark 5.6. (Even n.) Consider the case of a nonzero perturbation as in (4.2) and let just $F=1$. A thorough analysis seems unmanageable at this point, but we can get some evidence for what to expect by analyzing the second term in the expansion of $\mathcal R_M^0$ in (4.35). By circular permutation,

$$\operatorname{Tr}(\partial_\lambda^r(\mathcal R^0\zeta_\varepsilon\mathcal P\mathcal R^0)) = \operatorname{Tr}(\partial_\lambda^r(\zeta_\varepsilon\mathcal P(\mathcal R^0)^2)) = -\operatorname{Tr}(\zeta_\varepsilon\mathcal P\partial_\lambda^{r+1}\mathcal R^0). \tag{5.21}$$

Consider the log-terms produced by its singular Green part.

If $\mathcal{P} = \mathcal{P}_0$, we can apply Theorem 5.5 directly to see that there is an expansion with logs at half-integer powers $-l - \frac{1}{2} - r - 2$ and a single log-term at the integer power $-r-2$.

If $\mathcal{P}$ contains a term $x_n^k \mathcal{P}_k$ with $k \geq 1$, the normal trace of the s.g.o. contribution from this term will be similar to (5.18), of the form:

$$\text{(5.22)} \quad \operatorname{tr}_n L x_n^k \partial_\lambda^{r+1} G_\lambda^0 = k! L \partial_\lambda^{r+1} \big(\tfrac{1}{(2A_\lambda)^k} \big(\tfrac{-1}{4\lambda} \big[\tfrac{A^2}{A_\lambda^2} + \tfrac{A}{A_\lambda} \big] + \tfrac{A}{4\lambda A_\lambda} \tfrac{A}{|A'|} + \tfrac{A}{4\lambda |A'|} + \tfrac{1}{4\lambda} \Pi_0 \big) \big),$$

in view of Lemma 4.3 (iii). Again, the terms in [...] produce no logs, but the interesting fact is that now the terms generalizing (5.19) and (5.20) together contain A_λ in both even and odd negative powers (and λ in only integer powers), so that by Theorem 5.2, log-terms are produced at both integer and half-integer powers from a certain step on. So already the second term in (4.35) will then contribute sequences of nontrivial log-power terms both with integer and half-integer powers; a strong indication that such a structure will be found in general.

References

[APS75] M. F. Atiyah, V. K. Patodi and I. M. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.

[BM71] Boutet de Monvel, L., *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.

[DGK99] J. S. Dowker, P. B. Gilkey and K. Kirsten, *Heat asymptotics with spectral boundary conditions*, AMS Contemporary Math. **242** (1999), 107–124.

[GG98] P. B. Gilkey and G. Grubb, *Logarithmic terms in asymptotic expansions of heat operator traces*, Comm. Part. Diff. Eq. **23** (1998), 777-792.

[GK02] P. B. Gilkey and K. Kirsten, *Heat asymptotics with spectral boundary conditions II*, Proc. Royal Soc. Edinb. A (to appear).

[G92] G. Grubb, *Heat operator trace expansions and index for general Atiyah-Patodi-Singer problems*, Comm. P. D. E. **17** (1992), 2031–2077.

[G96] ———, *Functional Calculus of Pseudodifferential Boundary Problems, 2nd Edition*, Progress in Mathematics, vol. 65, Birkhäuser, Boston, 1996, 522 pp.

[G99] ———, *Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems*, Arkiv f. Mat. **37** (1999), 45–86.

[G01] ———, *A weakly polyhomogeneous calculus for pseudodifferential boundary problems*, J. Funct. Analysis **184** (2001), 19–76.

[G01′] ———, *Poles of zeta and eta functions for perturbations of the Atiyah-Patodi-Singer problem*, Comm. Math. Phys. **215** (2001), 583–589.

[G02] ———, *Spectral boundary conditions for generalizations of Laplace and Dirac operators*, Comm. Math. Phys. (to appear), arXiv:math.AP/0302286.

[GH02] G. Grubb and L. Hansen, *Complex powers of resolvents of pseudodifferential operators*, Comm. Part. Diff. Eq. **27** (2002), 2333–2361.

[GS95] G. Grubb and R. Seeley, *Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems*, Inventiones Math. **121** (1995), 481–529.

[GS96] ———, *Zeta and eta functions for Atiyah-Patodi-Singer operators*, J. Geom. Analysis **6** (1996), 31–77.

[L01] P. Loya, *The structure of the resolvent of elliptic pseudodifferential operators*, J. Funct. Analysis **184** (2001), 77–134.

[S67] R. T. Seeley, *Complex powers of an elliptic operator*, Amer. Math. Soc. Proc. Symp. Pure Math. **10** (1967), 288–307.